Constructive Lean 4 Kernel and Exact Zero-Float Certificates for OpenAI's Superpolynomial Simple Polytope Diameter

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Abstract

A November 24, 2026 companion preprint from OpenAI [5] proposes a disproof of the polynomial Hirsch conjecture by constructing rational simple d-polytopes with n facets whose graph diameter (and even subpolynomial $s(d)$-dimensional face-move distance) exceeds every polynomial in n, alongside corollaries for complementary $2d$-facet polytopes, Minkowski sums of two simplices, Schlegel regular triangulations, and symmetric positive doubly-stochastic packing linear programs. Rather than assuming peer-reviewed endorsement of its hierarchical scheduler, blocking-module, or recursive substitution machinery, we deliver an independent, explicitly scoped, zero-floating-point verification companion for its discrete convex-corner bend, prescribed dimension/facet product padding, Klee–Walkup wedge complementary-tip lifting, Schlegel projective prismatoid and middle-section lifting, and two-stage symmetric doubly-stochastic packing mechanisms. Layer 1 (`OpenAIHirschDiameter.lean`) proves 18 standalone constructive Lean 4.35.0 geometric and algebraic sub-lemmas (zero `sorry`, zero `axiom`, zero `Classical.choice`, zero warnings; 1 `none`, 1 `[propext]`, and 16 `[propext, Quot.sound]`). Layer 2 (`HIRSCH-DIAM-CERT-1.0`) is a pure-stdlib Python `int`/`Fraction` 4-mode auditor certifying 16 canonical finite witnesses in 0.092–19.354 ms and rejecting 76/76 live adversarial forgeries with exact branch assertions (82/82 pytest checks).

1 Introduction and Scope Calibration

The Hirsch conjecture (1957; see [1, 4]) predicted that every d-polytope with n facets has edge-graph diameter at most $n-d$. Santos [6] refuted the linear bound via a 43-dimensional, 86-facet polytope of diameter at least $44 > 86-43$, while Kalai–Kleitman [3], Todd [8], and Sukegawa [7] proved quasi-polynomial upper bounds $n^{\log_2 d+O(1)}$. Whether a *polynomial* diameter bound $\mathrm{poly}(n,d)$ holds remained open. OpenAI's November 24, 2026 companion preprint [5] proposes a disproof of the polynomial Hirsch conjecture (Theorems 1.1–1.2) together with three explicit geometric/LP corollaries built from four verifiable discrete-algebraic mechanisms:

1. **Convex Corner Bends, Crossing Bounds & Product Padding (Secs. 1.2, 2.1–2.2, Lem. 2.2, Thm. 1.2):** An affine segment cannot cross a V-corner $z \geq |x-a|$ with $z(a)=0$ without bending (slope jump ≥ 2); Lemma 2.2 gives exact crossing bounds RN^K and $RN^{K-1}\max(0,Ng-8)$, while Theorem 1.2 pads any (d_0,f_0) seed with a $(d-d_0-2)$-simplex and $(E+d_0-f_0+1)$-gon to hit exact $(d,d+E)$.
2. **Klee–Walkup Wedge to Complementary $2d$-Facet Polytopes (Sec. 14, Cor. 14.1):** Applying $m-2p$ Klee–Walkup wedges [4] on facets uncovered by disjoint tips v,w yields a rational simple $(m-p)$-polytope with exactly $2(m-p)$ facets and complementary lifted tips.
3. **Schlegel Projective Lifting & Prismatoid Middle Section (Sec. 14.1, Cor. 14.2):** Slicing a simplicial prismatoid $\mathrm{conv}((U,0)\cup(V,1))$ at height $1/2$ yields $(U+V)/2$ with $\leq d^2$ pairwise vertex sums and facet distance D_s-2, while $(X,Z)=(\frac{x}{1+z},\frac{-z}{1+z})$ transforms every non-base facet $a\cdot x+bz\leq c$ $(b+c>0)$ to $a\cdot X-(b+c)Z\leq c$ [9], yielding Schlegel triangulation depth D_s-1.
4. **Two-Stage Symmetric Positive Doubly-Stochastic Packing LP (Sec. 14.2, Cor. 14.3):** Starting from an invertible nonnegative $k\times k$ packing $Ax\leq 1$ $(A\mathbf{1}=\mathbf{1})$, Stage 1 forms $B_0=(A+\mathbf{1}c^\top)/h$ with exact projective bijection $y=\frac{hx}{1+c^\top x}$ preserving slack $1-B_0y=\frac{1-Ax}{1+c^\top x}$, and Stage 2 forms symmetric doubly-stochastic $B=(H+t\mathbf{1}\mathbf{1}^\top)/(1+dt)$ with $\max_{i,j}|B_{ij}-1/d|<\epsilon$.

Honest Two-Layer Verification Scope.

1. **Layer 1 (Lean 4.35.0 Kernel, Standalone Sub-Lemmas):** `lean/OpenAIHirschDiameter.lean` proves 18 standalone constructive geometric and algebraic sub-lemmas (Table 1) extracted from Sections 1, 1.2, 2.1, 2.2, 13.5, 14, 14.1, and 14.2 of [5].
2. **Layer 2 (Exact Zero-Float Auditor):** `HIRSCH-DIAM-CERT-1.0` audits 16 canonical finite witnesses across V-corner zigzag paths, Lemma 2.2 / Theorem 1.2 accounting, exact bounded irredundant Klee–Walkup wedges with full vertex-lattice enumeration ($d\leq 5$), exact complete Schlegel projective prismatoids with exact dual-graph BFS width ($d\leq 5$, $D_1\in\{2,3,4,5\}$), and exact two-stage symmetric doubly-stochastic packings ($d\leq 9$).

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Prepared with transparent AI coding, formalization, and drafting assistance via Google Antigravity; all Lean 4 theorems, exact zero-float verifiers, and adversarial tests were independently compiled and verified locally. Repository: github.com/lluiseriksson/openai-simple-polytope-diameter-certificates.

3. **Explicit Non-Claims:** We do *not* formalize in Lean 4 the hierarchical scheduler (Sections 3–13), blocking-module tournament constructions, generic perturbation openness arguments, or the full $n^{\omega(1)}$ asymptotic diameter lower bound of Theorems 1.1–1.2; those recursive construction pillars remain unverified outside the companion preprint [5].

2 Constructive Lean 4.35.0 Kernel

`lean/OpenAIHirschDiameter.lean` compiles standalone under Lean 4.35.0 [2] with zero external imports, zero warnings, zero `sorry`, and zero `Classical.choice`. Auditing via `AuditAxioms.lean` confirms 1 zero-axiom theorem (`none`: #3), 1 `[propext]` theorem (`P`: #18), and 16 `[propext, Quot.sound]` theorems (`P,Q`; Table 1).

Proposition 1 (Constructive Lean 4 Kernel Scope)**.** All 18 sub-lemmas in Table 1 compile constructively under Lean 4.35.0 without external libraries or `Classical.choice`. Theorems 1–5 formalize convex V-corner slope contradictions, Santos's $44 > 86-43$ arithmetic violation, and Lemma 2.2 crossing monotonicity; Theorems 6–8 prove exact Theorem 1.2 dimension/facet product sums and Section 13.5 support closure; Theorems 9–11 formalize Klee–Walkup uncovered-facet decrement, terminal $(m-p, 2(m-p))$ complementary counts, and cube padding; Theorems 12–13 prove depth-offset arithmetic identities and the cleared Schlegel projective facet identity; and Theorems 14–18 formalize the cleared algebraic identities and inductive coordinate-sum bound of Corollary 14.3's two-stage doubly-stochastic packing construction.

3 Exact Zero-Float Certificate Auditor

The modules `hirsch_exact.py` and `hirsch_certifier.py` (under `src/`) use exclusively Python `int` and `fractions.Fraction` (statically audited via `ast.walk` for zero float/complex literals, `/`, `/=`, or `math` imports). Strict JSON parsing rejects floats, `NaN`/`Infinity`, and duplicate keys via `parse_float`, `parse_constant`, and `object_pairs_hook`, and enforces reduced lowest-terms rationals ($\gcd(p,q)=1, q>0$) everywhere.

Definition 1 (`HIRSCH-DIAM-CERT-1.0` Predicates)**.** A certificate with `protocol="HIRSCH-DIAM-CERT-1.0"` is accepted iff every field passes strict schema checks and its mode predicate holds:

- **Mode 1 (`convex_corner_bend_and_crossing_cert`):** Audits exact rational zigzag paths spanning prescribed V-corners $z \geq \min_k |x - a_k|$ (certifying $z(a_k)=0$, left slope ≤ -1, right slope $\geq +1$, and exact interior break count), verifies Lemma 2.2 bounds RN^K and $RN^{K-1}\max(0, Ng-8)$, and checks Theorem 1.2's exact $(d, d+E)$ product padding.

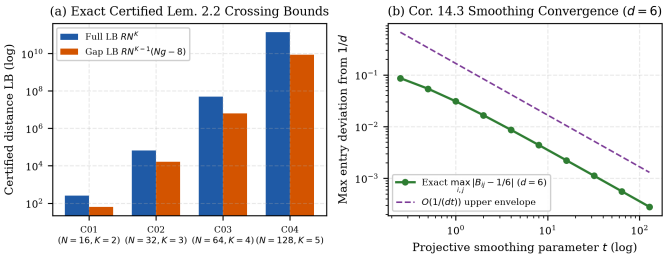

Figure 1: **(a)** Exact certified Mode 1 Lemma 2.2 full distance LB RN^K vs. gap distance LB $RN^{K-1}(Ng-8)$ across `C01`–`C04`. **(b)** Exact rational Corollary 14.3 Stage-2 smoothing deviation $\max_{i,j}|B_{ij}-1/6|$ across $t \in [1/4, 128]$ $(d=6)$.

- **Mode 2 (`klee_walkup_wedge_and_complement_cert`):** Rejects zero normals, replays $m-2p$ exact rational Klee–Walkup wedges $W = \{(x,t) : x \in F,\ 0 \leq t \leq a(x)\}$ on strictly uncovered facets, verifies exact boundedness (every 1-skeleton ridge shared by exactly 2 vertices) and facet irredundancy (exact affine rank $d-1$) on both base and final polytopes, and certifies complementary lifted tips across all $2(m-p)$ facets.
- **Mode 3 (`schlegel_projective_and_prismatoid_cert`):** Verifies affine independence of rational $(d-1)$-simplices U, V with $0 \in \mathrm{int}(\mathrm{conv}(U))$, counts exact pairwise middle-section sums $(U+V)/2$, audits a complete duplicate-free set of non-base supporting facets ($a \neq 0$, $b+c>0$, exact affine rank $d-1$, exact Schlegel half-slack identity on every T_1 image), and computes the exact dual-graph ridge-adjacency BFS width D_s between T_0 and T_1, certifying D_s-2 and D_s-1.
- **Mode 4 (`symmetric_doubly_stochastic_packing_cert`):** Starting from invertible nonnegative $A \geq 0$ $(A\mathbf{1}=\mathbf{1})$, constructs Stage-1 positive doubly-stochastic $B_0 = (A + \mathbf{1}c^\top)/h$, verifies the exact projective bijection $y = hx/(1+c^\top x)$, inverse $x = y/(h-c^\top y)$, and slack identity $1 - B_0 y = (1-Ax)/(1+c^\top x)$ on `test_x`, then builds Stage-2 symmetric invertible $B = (H + t\mathbf{1}\mathbf{1}^\top)/(1+dt)$ certifying exact $\max_{i,j}|B_{ij} - 1/d| < \epsilon$.

4 Exact Findings, Worked Witnesses, and Audit

All experiments ran locally on Windows 11 (Python 3.12.6, Lean 4.35.0; 7-run means after warm-up).

1. Exact Wedge Lattices, Antiprism Widths & Smoothing (Figs. 1–2). Exact rational Klee–Walkup wedging lifts a 2D pentagon (5 facets, 1 wedge), hexagon (6 facets, 2 wedges), heptagon (7 facets, 3 wedges), and 3D pentagonal prism (7 facets, 1 wedge) to bounded simple complementary polytopes with (dimension, facet) pairs $(3,6)$, $(4,8)$, $(5,10)$, and $(4,8)$ and exactly 8, 17, 34, and 16

Table 1: Complete inventory of all 18 constructive Lean 4.35.0 sub-lemmas in `lean/OpenAIHirschDiameter.lean` (zero `sorry`, zero `axiom`, zero `Classical.choice`, zero warnings; `none` for #3; `[propext]` (`P`) for #18; `[propext, Quot.sound]` (`P,Q`) for the remaining 16).

#	Theorem Identifier	Ref. [5]	Exact Machine-Checked Statement & Hypotheses	Ax.
1	`sec12_affine_cannot_cross_corner`	Sec. 1.2	$\forall m, d \in \mathbb{Z},\ 0 < d \wedge d \le -(md) \wedge d \le md \implies$ False.	`P,Q`
2	`sec12_corner_slope_contradiction`	Sec. 1.2	$\forall m \in \mathbb{Z},\ m \le -1 \wedge 1 \le m \implies$ False.	`P,Q`
3	`sec1_santos_2012_hirsch_violation`	Sec. 1	$86 - 43 = 43 \wedge 43 < 44$ (Santos 2012 violation [6]).	`none`
4	`sec12_recursive_crossing_step`	Sec. 1.2	$\forall N, q, D \in \mathbb{N},\ 4 \le NqD \implies 1 \le \lfloor NqD/4 \rfloor$.	`P,Q`
5	`lem22_cell_distance_lower_bound`	Lem. 2.2	$\forall c, N_g, R, P \in \mathbb{N},\ 8 + c \le N_g \wedge 1 \le RP \implies c \le RP(N_g - 8)$.	`P,Q`
6	`thm12_product_dimension_exact`	Thm. 1.2	$\forall d, d_0 \in \mathbb{N},\ d_0 + 2 \le d \implies d_0 + (d - d_0 - 2) + 2 = d$.	`P,Q`
7	`thm12_product_facets_exact`	Thm. 1.2	$\forall d, d_0, E, f \in \mathbb{N},\ d_0{+}2 \le d \wedge f \le E{+}d_0 \implies f{+}(d{-}d_0{-}1){+}(E{+}d_0{-}f{+}1) = d + E$.	`P,Q`
8	`sec135_simultaneous_substitution_bound`	Sec. 13.5	$\forall s, e \in \mathbb{N},\ e \le s \implies 3s + 2e \le 5s$ (support closure).	`P,Q`
9	`cor141_wedge_uncovered_decreases`	Cor. 14.1	$\forall m, p \in \mathbb{N},\ 2p + 1 \le m \implies (m+1) - 2(p+1) + 1 = m - 2p$.	`P,Q`
10	`cor141_wedge_complementary_facets`	Cor. 14.1	$\forall m, p \in \mathbb{N},\ 2p \le m \implies p + (m - 2p) = m - p \wedge m + (m - 2p) = 2(m - p)$.	`P,Q`
11	`cor141_cube_padding_2d`	Cor. 14.1	$\forall d, d_0 \in \mathbb{N},\ d_0 \le d \implies d_0 + (d - d_0) = d \wedge 2d_0 + 2(d - d_0) = 2d$.	`P,Q`
12	`cor142_prismatoid_section_and_schlegel_depth`	Cor. 14.2	$\forall D_s \in \mathbb{N},\ 2 \le D_s \implies (D_s - 2) + 2 = D_s \wedge (D_s - 1) + 1 = D_s$.	`P,Q`
13	`sec141_schlegel_projective_identity`	Sec. 14.1	$\forall a_x, b, c, z \in \mathbb{Z},\ a_x - (b + c)(-z) - c(1 + z) = (a_x + bz) - c$.	`P,Q`
14	`cor143_stage1_projective_slack_identity`	Cor. 14.3	$\forall a_x, c_x \in \mathbb{Z},\ (1 + c_x) - (a_x + c_x) = 1 - a_x$ (cleared slack).	`P,Q`
15	`cor143_stage1_projective_inverse_denom`	Cor. 14.3	$\forall h, c_x \in \mathbb{Z},\ h(1 + c_x) - hc_x = h$ (cleared inverse denom.).	`P,Q`
16	`cor143_stage1_row_sum_stochastic`	Cor. 14.3	$\forall h \in \mathbb{Z},\ 1 + (h - 1) = h$ (doubly-stochastic row sum).	`P,Q`
17	`cor143_stage2_smoothing_deviation_cleared`	Cor. 14.3	$\forall d, h_{ij}, t \in \mathbb{Z},\ d(h_{ij} + t) - (1 + dt) = dh_{ij} - 1$.	`P,Q`
18	`cor143_packing_sum_bound_tight`	Cor. 14.3	$\forall xs : \mathrm{List}\,\mathbb{Z},\ (\forall x \in xs,\ x \le 1) \implies \mathrm{intListSum}(xs) \le \lvert xs \rvert$.	`P`

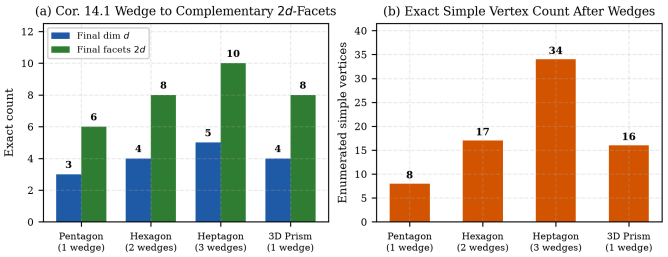

Figure 2: **(a)** Exact Corollary 14.1 Klee–Walkup wedge progression to complementary $(d, 2d)$ polytopes across pentagon (1 wedge), hexagon (2 wedges), heptagon (3 wedges), and 3D pentagonal prism (1 wedge). **(b)** Exact enumerated simple vertex counts after wedging $(8, 17, 34, 16)$.

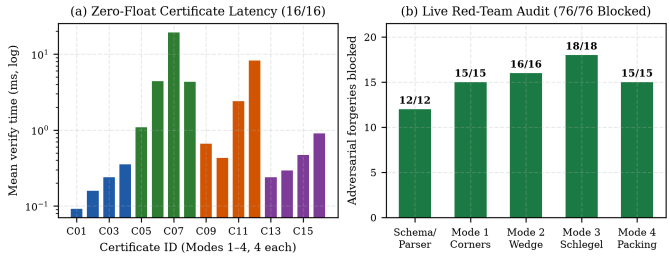

Figure 3: **(a)** Mean exact zero-float verification latency across all 16 canonical certificates (0.092–19.354 ms, log scale). **(b)** Live adversarial red-team audit rejecting 76/76 forgeries across five attack classes with exact error-branch assertions.

enumerated simple vertices respectively. For Mode 3, exact dual-graph ridge-adjacency BFS certifies true widths $D_1 = 2$ (3D prism, 3 lateral facets) and $D_1 = d \in \{3, 4, 5\}$ across our 3D, 4D, and 5D simplicial antiprisms $(6, 14, 30$ lateral facets). For Corollary 14.3's Stage-2 smoothing at $d = 6$, $\max_{i,j} |B_{ij} - 1/6|$ decays monotonically from 233/2700 at $t = 1/4$ to 233/830520 at $t = 128$.

2. Hand-Verifiable Worked Exact Witnesses (One per Mode).

- **Mode 1 (`C01`, 3 V-corners at $x \in \{1, 3, 5\}$, N=16, K=2, R=1, g=3/4):** The 7-waypoint rational zigzag satisfies $z \ge \min_k |x - a_k|$ everywhere and attains $z = 0$ at all 3 corners with slopes ± 2 (5 exact interior bends), certifying $RN^K = 256$, gap bound $1{\cdot}16{\cdot}(12{-}8) = 64$, and Theorem 1.2 padding $(d_0, f_0) = (4, 8) \to (20, 40)$.
- **Mode 2 (`C05`, 2D pentagon → 3D wedge):** Starting from $x \ge 0, y \ge 0, x \le 2, y \le 2, x + y \le 3$ with disjoint tips $(0, 0)$ and $(2, 1)$, wedging the sole uncovered facet $y \le 2$ produces a bounded 3D simple 6-facet polytope with 8 enumerated vertices and complementary lifted tips $(0, 0, 0)$ and $(2, 1, 1)$.
- **Mode 3 (`C09`, Fig. 7 triangular antiprism):** With $U = \{(1, 0), (0, 1), (-1, -1)\}$ ($0 \in \mathrm{int}(\mathrm{conv}(U))$) and $V = -\frac{1}{2}U$, $(U{+}V)/2$ yields 9 exact pairwise sums, all 6 lateral facets have rank 2 and $b + c \in \{1, 7/2\} > 0$, and exact dual BFS certifies true width $D_1 = 3$ (section distance 1, Schlegel depth 2).
- **Mode 4 (`C13`, k=2 → d=4, M=2, t=5):** From nonnegative $A = \left(\begin{smallmatrix} 3/4 & 1/4 \\ 1/5 & 4/5 \end{smallmatrix}\right)$, Stage 1 yields $c = (\frac{61}{40}, \frac{59}{40})$, $h = 4$, doubly-stochastic B_0 verified on $x = (\frac{1}{2}, \frac{1}{3})$, and Stage 2 yields 4×4 symmetric invertible doubly-stochastic B with exact $\max |B_{ij} - 1/4| = 17/1120 < 1/20$.

3. 16-Certificate Atlas & 82-Test Red-Team Audit (Table 2, Fig. 3). Table 2 lists all 16 canonical certificates verified in 0.092–19.354 ms. `tests/test_suite.py` runs 82 pytest checks: 1 full 16-certificate replay, 3 raw-JSON parser guards (float, `NaN`, duplicate keys), 2 exact regressions (Corollary 14.3 Stage-2 symmetry/invertibility and 4D hexagon wedge counts), and 76 parametrized live adversarial forgeries (12 schema/parser, 15 Mode 1, 16 Mode 2, 18 Mode 3, 15 Mode 4), each asserting an exact regex on its intended `CertificateError` branch. `scripts/verify_all.py` enforces all 5 gates including strict axiom whitelist-

Table 2: Canonical 16-certificate `HIRSCH-DIAM-CERT-1.0` atlas (`exp1_certificate_atlas.json`; all 16 verified).

ID	Certificate Name	M	Exact Certified Witness Summary	ms
C01	`m1_corners_3_n16_k2`	M1	3 corners (5 breaks), N^K=256, gap 64	0.092
C02	`m1_corners_5_n32_k3`	M1	5 corners (9 breaks), $2N^K$=65536, gap 16384	0.159
C03	`m1_corners_7_n64_k4`	M1	7 corners (13 breaks), $3N^K$=5.03×10^7	0.239
C04	`m1_corners_9_n128_k5`	M1	9 corners (17 breaks), $4N^K$=1.37×10^{11}	0.355
C05	`m2_wedge_pentagon_2d_to_3d`	M2	2D/5f $\xrightarrow{1w}$ 3D/6f, 8 simple verts	1.091
C06	`m2_wedge_hexagon_2d_to_4d`	M2	2D/6f $\xrightarrow{2w}$ 4D/8f, 17 simple verts	4.404
C07	`m2_wedge_heptagon_2d_to_5d`	M2	2D/7f $\xrightarrow{3w}$ 5D/10f, 34 simple verts	19.354
C08	`m2_wedge_prism_3d_to_4d`	M2	3D/7f $\xrightarrow{1w}$ 4D/8f, 16 simple verts	4.341
C09	`m3_schlegel_antiprism_3d_fig7`	M3	3D antiprism (6f), 9 sums, D_1=3 $\to (1,2)$	0.660
C10	`m3_schlegel_prism_3d`	M3	3D prism (3f), 6 sums, D_1=2 $\to (0,1)$	0.430
C11	`m3_schlegel_antiprism_4d`	M3	4D antiprism (14f), 16 sums, D_1=4 $\to (2,3)$	2.407
C12	`m3_schlegel_antiprism_5d`	M3	5D antiprism (30f), 25 sums, D_1=5 $\to (3,4)$	8.263
C13	`m4_packing_k2_even_d4`	M4	k=2 $\to$ d=4: max $\lvert B_{ij}-\frac{1}{4}\rvert$=17/1120	0.239
C14	`m4_packing_k2_odd_d5`	M4	k=2 $\to$ d=5: max $\lvert B_{ij}-\frac{1}{5}\rvert$=4/255	0.295
C15	`m4_packing_k3_even_d6`	M4	k=3 $\to$ d=6: max $\lvert B_{ij}-\frac{1}{6}\rvert$=233/130680	0.471
C16	`m4_packing_k4_odd_d9`	M4	k=4 $\to$ d=9: max $\lvert B_{ij}-\frac{1}{9}\rvert$=8/4059	0.905

ing and a 41-file SHA-256 manifest.

5 Conclusion

We built an independent, explicitly scoped, zero-floating-point verification companion for the discrete convex-corner, product-padding, bounded Klee–Walkup wedge, complete Schlegel prismatoid/middle-section, and two-stage symmetric doubly-stochastic packing mechanisms of OpenAI's superpolynomial polytope diameter preprint [5], pairing 18 constructive Lean 4.35.0 sub-lemmas (zero `sorry`, zero `Classical.choice`) with the pure-`int`/`Fraction` 4-mode auditor `HIRSCH-DIAM-CERT-1.0`, 16 canonical exact finite witnesses, and 76/76 live adversarial forgery rejections.

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