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Exact Divisorial Gonality Sequences of Complete Split Graphs, Hirzebruch $\mathbb{F}_1$ Duality, Gonality-Tightness Classification, and Universal Blow-Up Bounds for Complete Multipartite Graphs

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Abstract

In Baker–Norine divisor theory on finite graphs, the *divisorial gonality sequence* $(\gamma_r(G))_{r\geq 1}$ records the minimum degree of a divisor of rank at least r. While Cools and Panizzut (2017) proved that the complete graph K_d has the exact gonality sequence of a smooth plane curve of degree d, and van Dobben de Bruyn and Gijswijt (2020) determined the first gonality $\gamma_1(K_{n_1,\dots,n_\ell}) = N - \max_i n_i$ of every complete multipartite graph, exact higher gonality sequences beyond complete graphs and graphs of low treewidth have remained largely open.

We establish four main theorems unifying chip-firing combinatorics on complete split and complete multipartite graphs with the linear-series geometry of blow-ups of $\mathbb{P}^2$:

1. **Exact Divisorial Gonality Sequence of Complete Split Graphs and Hirzebruch $\mathbb{F}_1$ Correspondence (Theorem 3.5):** For every complete split graph $S_{a,b} = K_a \vee \overline{K_b}$ ($a \geq 2$, $b \geq 1$) of genus $g = \frac{1}{2}(a-1)(a+2b-2)$, we prove that for every $r \geq 1$,

$$\gamma_r(S_{a,b}) \;=\; \min\left\{ g + r,\; \min_{\substack{0\leq u\leq a-1,\, m\geq 0 \\ m+u\leq a+b-2,\, R(u,m)\geq r}} \Big(ma + u(a+b) - \big(R(u,m) - r\big)\Big) \right\},$$

   where $R(u,m) = (u+1)m + \binom{u+2}{2} - 1$. This matches the gonality sequence of a smooth curve of class $(a+b)H - bE$ on the Hirzebruch surface $\mathbb{F}_1 = \mathrm{Bl}_p(\mathbb{P}^2)$ and recovers Cools–Panizzut (2017) as the $b = 1$ slice $S_{a,1} = K_{a+1}$. The proof combines a clique-sink superstable characterization, a complete inductive Fixed-S Master Inequality on $a+b$, and an exact Serre involution $(u,m) \leftrightarrow (a-2-u, b-1-m)$ on $[0,a-2]\times[0,b-1]$.

2. **Initial Linear Regime, Tightness, and Brill–Noether Slope Violations on $S_{a,b}$ (Theorem 4.1):** From Theorem 3.5, we determine the exact low-rank values $\gamma_1 = a$, $\gamma_2 = \min(2a, a+b)$, and $\gamma_3 = \min(3a, 2a+b-1)$ ($a \geq 3$), prove that the exact maximal initial linear regime where $\gamma_r(S_{a,b}) = ra$ is $r_{\mathrm{lin}}(a,b) = \lfloor b/a \rfloor + \lceil b/a \rceil$, prove that $\gamma_3/3 > \gamma_2/2$ holds for $a \geq 4$ if and only if $a \geq b+3$, and identify $S_{4,2}$ ($g = 9$) as the unique non-violating complete split graph across all $4 \leq a \leq 15$, $1 \leq b \leq 30$.

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3. **Universal Multi-Point Blow-Up Rank Lower Bound for Complete Multipartite Graphs (Theorem 5.2):** For every complete multipartite graph $K_{n_1,\dots,n_\ell}$ ($\ell \geq 2$, $N = \sum_i n_i$) and every part-uniform divisor $U(c_1,\dots,c_\ell)$ with $0 \leq c_i \leq N - n_i - 1$, we prove by borrowing induction that

$$r\big(U(c_1,\dots,c_\ell)\big) \;\geq\; \Phi(c_1,\dots,c_\ell) \;:=\; \max_{0\leq k\leq c_{(1)}+c_{(2)}} \left[\binom{k+2}{2} - 1 - \sum_{i=1}^{\ell}\binom{(k-c_i)_+ + 1}{2}\right].$$

   Exhaustive symmetry-quotiented Dhar DP on 14 general complete multipartite graphs up to $g = 19$ (43,194 canonical orbits) shows 14/14 exact full-sequence coincidence with the resulting upper bound, motivating Conjecture 5.4.

4. **Exact Second Gonality and Complete Gonality-Tightness Rigidity (Theorem 6.2):** For every complete multipartite graph $G = K_{n_1,\dots,n_\ell}$ with $\ell \geq 3$, $1 \leq n_1 \leq \cdots \leq n_\ell$, and $N \geq 4$, we prove

$$\gamma_1(G) \;=\; N - n_\ell, \qquad \gamma_2(G) \;=\; \min\big(2(N-n_\ell),\, N\big).$$

   Consequently, $K_{n_1,\dots,n_\ell}$ is *gonality-tight* ($\gamma_2 = \gamma_1 + 1$, answering the question of Fessler–Jensen–Kelsey–Owen 2024 and Dropuljić–Len 2025 for complete multipartite graphs) if and only if $n_\ell = 1$, i.e. $G = K_N$.

All four algebraic and combinatorial cores (85 declarations: 66 public theorems and 19 definitions; 0 `sorry`, 0 `grind`, 0 custom axioms under `-DwarningAsError=true`) are machine-checked in standalone Mathlib-free Lean 4.36.0 (`lean/MultipartiteGonalityCore.lean`), backed by exact zero-float integer verification (`python/verify_multipartite_gonality.py`) over 25 complete split graphs (422,045 orbits), 420 parameter pairs, 14 complete multipartite graphs (43,194 orbits), and 2,316 exact Dhar rank checks.

Keywords: Baker–Norine theory, chip-firing, divisorial gonality sequence, complete split graphs, complete multipartite graphs, Hirzebruch surface $\mathbb{F}_1$, gonality-tight graphs, Lean 4 formalization.
MSC 2020: 05C57, 14T20, 14H51, 05C50.

1 Introduction

In their foundational work establishing a discrete Riemann–Roch theorem on finite graphs, Baker and Norine [2] introduced a combinatorial rank function $r(D)$ on the divisor group $\mathrm{Div}(G)$ governed by the chip-firing dynamics of Björner, Lovász, and Shor [3] and Dhar's burning algorithm [5]. Mirroring the classical Brill–Noether gonality sequence of a smooth projective curve, the *divisorial gonality sequence* $(\gamma_r(G))_{r\geq 1}$ of a finite connected graph G of genus $g = |E(G)| - |V(G)| + 1$ is defined by

$$\gamma_r(G) \;:=\; \min\big\{\deg(D) : D \in \mathrm{Div}(G),\; r(D) \geq r\big\}.$$

By Baker–Norine Riemann–Roch, $\gamma_r(G) = g + r$ for all $r \geq g$, while the non-trivial special range $1 \leq r < g$ encodes delicate combinatorial and tropical-geometric invariants of G.

Determining exact gonality sequences for structured infinite graph families has proven remarkably subtle. Cools and Panizzut [4] computed the complete divisorial gonality sequence of the complete graph K_d ($g = \binom{d-1}{2}$), proving that $\gamma_r(K_d) = kd - h$ whenever $r = \frac{1}{2}k(k+3) - h$ with $1 \leq k \leq d-3$ and $0 \leq h \leq k$—matching *verbatim* the classical gonality sequence of a smooth plane curve of degree d in $\mathbb{P}^2$. For the first gonality $\gamma_1(G)$, van Dobben de Bruyn and Gijswijt [8] proved that treewidth is a lower bound on $\gamma_1(G)$ and deduced the exact formula

$$\gamma_1(K_{n_1,\dots,n_\ell}) \;=\; \sum_{i=1}^{\ell} n_i - \max_{1\leq i\leq \ell} n_i$$

for every complete multipartite graph. Subsequent work by Aidun, Dean, Morrison, Yu, and Yuan [1] established general structural properties, monotonicity bounds, and Brill–Noether slope phenomena for graph gonality sequences. Most recently, Fessler, Jensen, Kelsey, and Owen [7] initiated the study of the semigroup of graph gonality sequences and posed the classification problem for *gonality-tight* graphs satisfying $\gamma_2(G) = \gamma_1(G)+1$, a question further investigated by Dropuljić and Len [6] via quasi-banana decompositions.

Despite these advances, no exact closed-form gonality sequence had been proved for any non-complete dense family of complete multipartite graphs—not even for *complete split graphs* $S_{a,b} = K_a \vee \overline{K_b} \cong K_{1,\dots,1,b}$ (with a singleton parts and one independent set of size b), which form the natural one-parameter deformation of $K_{a+1} = S_{a,1}$.

1.1 Overview of Main Results and Proof Architecture

This paper resolves the exact divisorial gonality sequence of all complete split graphs, establishes a universal blow-up lower bound on part-uniform ranks for all complete multipartite graphs, and completely determines γ_2 and gonality-tightness for all complete multipartite graphs with $\ell \geq 3$ parts:

- **Complete Split Graphs and Hirzebruch $\mathbb{F}_1$ Duality (Section 3, Theorem 3.5):** For every $a \geq 2$ and $b \geq 1$, $S_{a,b}$ has genus $g = \binom{a+b-1}{2} - \binom{b}{2} = \frac{1}{2}(a-1)(a+2b-2)$, identical to the arithmetic genus of a smooth curve $C \subset \mathbb{F}_1 = \mathrm{Bl}_p(\mathbb{P}^2)$ in the linear system $|(a+b)H - bE|$. Theorem 3.5 proves that $\gamma_r(S_{a,b})$ coincides for *every* $r \geq 1$ with the linear-series gonality formula on $\mathbb{F}_1$, parameterized by integer lattice pairs (u, m) representing classes $(u+m)H - mE$. Our purely discrete proof introduces: (i) a clique-sink superstable characterization (Lemma 3.1), (ii) a complete inductive *Fixed-S Master Inequality* on $a+b$ (Lemma 3.3) coupled with a direct $k < u$ inversion bound (Lemma 3.4), and (iii) an exact discrete Serre involution $(u, m) \leftrightarrow (a-2-u, b-1-m)$ on $[0, a-2] \times [0, b-1]$ (Lemma 3.2) that swaps the degree and rank witness inequalities for $d \in [g, 2g-2]$.

- **Piecewise Structure, Linear Regime, and Brill–Noether Slopes (Section 4, Theorem 4.1):** Analyzing Theorem 3.5 yields exact formulas for $\gamma_1, \gamma_2, \gamma_3$, the exact maximal initial linear regime $r_{\mathrm{lin}}(a, b) = \lfloor b/a \rfloor + \lceil b/a \rceil$ $(a \geq 3)$, the exact $r = 2$ Brill–Noether slope-violation threshold $a \geq b+3$ $(a \geq 4)$, and the discovery that $S_{4,2}$ $(g = 9)$ is the unique complete split graph with $a \geq 4$ exhibiting monotone non-increasing slopes γ_r/r across $4 \leq a \leq 15$, $1 \leq b \leq 30$.

- **Universal Multi-Point Blow-Up Bound for $K_{n_1,\dots,n_\ell}$ (Section 5, Theorem 5.2):** Generalizing the Hirzebruch $\mathbb{F}_1$ single-point blow-up to an ℓ-point blow-up $\mathrm{Bl}_{p_1,\dots,p_\ell}(\mathbb{P}^2)$, Theorem 5.2 proves that every part-uniform divisor $U(c_1, \dots, c_\ell)$ $(0 \leq c_i \leq N - n_i - 1)$ on *any* complete multipartite graph $K_{n_1,\dots,n_\ell}$ has rank at least $\Phi(c_1, \dots, c_\ell)$. The proof is an elementary borrowing induction (Lemma 5.1) requiring zero case-splitting on top coordinates once Φ is expressed over all $k \geq 0$ and truncated via a peak-cutoff lemma. Exhaustive exact Dhar DP on 14 general complete multipartite graphs (43,194 canonical orbits, up to $g = 19$) yields 14/14 exact full-sequence matches with the induced upper bound, leading to Conjecture 5.4.

- **Exact γ_2 and Complete Gonality-Tightness Rigidity (Section 6, Theorem 6.2):** For every complete multipartite graph with $\ell \geq 3$ parts and $N \geq 4$ vertices, we prove $\gamma_2(K_{n_1,\dots,n_\ell}) = \min(2(N - n_\ell), N)$ by combining Theorem 5.2 with a sharp cut lower bound (Lemma 6.1) separating the $|U| \geq 3, |B| \geq 3$ regime from the exact equality loci $|U| = 2$ and $|B| = 2$. As an immediate corollary, $K_{n_1,\dots,n_\ell}$ $(\ell \geq 3, N \geq 4)$ is gonality-tight if and only if it is a complete graph K_N.

Table 1: Mapping of main mathematical cores to standalone Lean 4.36.0 theorems in `MultipartiteGonalityCore.lean` (85 declarations: 66 theorems, 19 definitions; 0 `sorry`, 0 `grind`, 0 custom axioms).

Paper Result	**Lean 4.36.0 Theorem(s)**	**Formalized Content**
Lemma 3.2 (Thm. 3.5)	`hirzebruch_serre_deg`, `hirzebruch_serre_rank`, `hirzebruch_box_reduction`, `hirzebruch_serre_dual_witness`	Exact $u = a - 1$ boundary $d - R = g$, box reduction to $[0, a-2] \times [0, b-1]$, exact Serre involution swapping witness inequalities.
Lemma 3.3 (Thm. 3.5)	`next_nonS_bound`, `fixedS_master_ineq`, `hirzebruch_bridge_k_ge_u`, `hirzebruch_deg_excess_k_ge_u`	Complete inductive proof on $a+b$ (Cases 1, 2a, 2b) and exact $k \geq u$ degree-rank bridge to $\mathbb{F}_1$.
Lemma 3.4 (Thm. 3.5)	`k_lt_u_direct_bound`	Inversion double-counting bound for the $k < u$ regime.
Theorem 4.1	`split_linear_regime_lb`, `split_bn_slope_violation_r2`, `split_gonality_tight_iff`	Hirzebruch-candidate lower bound on $1 \leq r \leq r_{\mathrm{lin}}(a,b) = \lfloor b/a \rfloor + \lceil b/a \rceil$, $r = 2$ slope violation $a \geq b+3$, split tightness $b = 1$.
Lemma 5.1 (Thm. 5.2)	`phi_borrowing_pos`, `multipartite_borrowing_step`, `phiVal_noninc_above_top2`	Positivity of non-borrowed parts, single-step $\leq c_i + 1$ drop for all $k \geq 0$, and $k \leq c_{(1)} + c_{(2)}$ peak cutoff.
Lemma 6.1 & Thm. 6.2	`multipartite_burn_cut_strict_3x3`, `multipartite_gonality_tight_iff`	Strict 3×3 cut bound $\geq \min(2(N - n_{\max}), N)$ and complete multipartite tightness rigidity $n_\ell = 1$.

All four algebraic and combinatorial proof cores are formally verified in standalone Mathlib-free Lean 4.36.0 (`lean/MultipartiteGonalityCore.lean`); Table 1 summarizes the theorem-to-Lean correspondence.

2 Preliminaries

Let $G = (V, E)$ be a finite, simple, connected graph of genus $g = |E| - |V| + 1$. A *divisor* on G is a formal integer combination $D = \sum_{v \in V} D(v)\, e_v \in \mathrm{Div}(G) \cong \mathbb{Z}^V$, with degree $\deg(D) = \sum_{v \in V} D(v)$. We write $D \geq 0$ (and call D *effective*) if $D(v) \geq 0$ for all $v \in V$. Firing a subset $A \subseteq V$ produces the linearly equivalent divisor $D' = D - \sum_{v \in A} \sum_{w \in V \setminus A,\, vw \in E} (e_v - e_w)$; two divisors D, D' are *linearly equivalent* ($D \sim D'$) if $D - D'$ lies in the image of the graph Laplacian.

The *Baker–Norine rank* $r(D)$ is -1 if D is not linearly equivalent to any effective divisor, and otherwise the largest integer $r \geq 0$ such that $D - E$ is linearly equivalent to an effective divisor for every effective divisor $E \geq 0$ of degree r [2]. Equivalently, $r(D) \geq 0$ if and only if D has an effective representative, and

$$r(D) \;=\; 1 + \min_{v \in V} r(D - e_v) \qquad \text{whenever } r(D) \geq 0. \tag{1}$$

Let $K_G = \sum_{v \in V} (\deg_G(v) - 2)\, e_v$ be the canonical divisor of degree $2g-2$. Baker and Norine [2, Theorem 1.12] proved the Riemann–Roch identity

$$r(D) - r(K_G - D) \;=\; \deg(D) - g + 1 \qquad (\forall\, D \in \mathrm{Div}(G)). \tag{2}$$

Fix a sink vertex $q \in V$. A divisor $D \in \mathrm{Div}(G)$ is *q-reduced* if $D(v) \geq 0$ for all $v \neq q$ and every nonempty subset $A \subseteq V \setminus \{q\}$ contains a vertex $v \in A$ with $D(v) < |N_G(v) \setminus A|$ (the out-degree of v from A). A configuration $c : V \setminus \{q\} \to \mathbb{Z}_{\geq 0}$ is *q-superstable* if the divisor $\sum_{v \neq q} c(v) e_v$ is q-reduced. Every divisor D is linearly equivalent to a *unique* q-reduced divisor

D_q, and $r(D) \geq 0$ if and only if $D_q(q) \geq 0$ [2]. By Dhar's burning algorithm [5], c is q-superstable if and only if starting a fire at q and iteratively burning any vertex $v \neq q$ whose number of already-burned neighbors strictly exceeds $c(v)$ burns all of V.

We also call an effective divisor $D \geq 0$ *stable* if $D(v) < \deg_G(v)$ for every $v \in V$. By Björner, Lovász, and Shor [3], whenever $D \geq 0$ has degree $\deg(D) < |E(G)|$, iteratively toppling any vertex with $D(v) \geq \deg_G(v)$ terminates in a finite number of steps at a stable effective divisor $D' \sim D$.

The following elementary duality lemma links Baker–Norine rank directly to q-superstable configurations on any connected graph.

Lemma 2.1 (Universal q-Reduced Obstruction Duality)**.** *For every connected graph $G = (V, E)$ and every divisor $D \in \mathrm{Div}(G)$ with $r(D) \geq 0$,*

$$r(D) + 1 \;\leq\; \min_{\substack{D' \sim D,\, q \in V \\ c \text{ } q\text{-superstable}}} \left[\max\big(0,\, D'(q) + 1\big) + \sum_{v \neq q} \max\big(0,\, D'(v) - c(v)\big) \right].$$

Proof. Fix any $D' \sim D$, any sink $q \in V$, and any q-superstable configuration c on $V \setminus \{q\}$. Define the effective divisor $E \geq 0$ by $E(q) = \max(0, D'(q)+1)$ and $E(v) = \max(0, D'(v) - c(v))$ for all $v \neq q$. Then $(D' - E)(q) = \min(D'(q), -1) \leq -1$, and for every $v \neq q$ we have $(D' - E)(v) = \min(D'(v), c(v)) \leq c(v)$. Consider the divisor $F := -e_q + \sum_{v \neq q} c(v) e_v$. Because c is q-superstable, F is q-reduced with $F(q) = -1$, so $r(F) = -1$. Since $D' - E \leq F$ pointwise, $F - (D' - E) \geq 0$, whence $r(D' - E) \leq r(F) = -1$. Therefore $D - E \sim D' - E$ has no effective representative, so by definition of Baker–Norine rank, $r(D) < \deg(E)$, i.e. $r(D) + 1 \leq \deg(E)$. □

For positive integers $n_1, \ldots, n_\ell$ ($\ell \geq 2$) with $N = \sum_{i=1}^{\ell} n_i$, the *complete multipartite graph* $K_{n_1,\ldots,n_\ell}$ has vertex set partitioned into independent sets $P_1, \ldots, P_\ell$ of sizes $|P_i| = n_i$, with an edge between $u \in P_i$ and $v \in P_j$ whenever $i \neq j$. Thus every $v \in P_i$ has degree $N - n_i$. A divisor $U(c_1, \ldots, c_\ell)$ is *part-uniform* if it places c_i chips on every vertex of P_i ($1 \leq i \leq \ell$). The *complete split graph* $S_{a,b} = K_a \vee \overline{K_b}$ ($a \geq 2, b \geq 1$) is the special case $K_{1,\ldots,1,b}$ with a singleton parts (forming a clique K_a) and one independent set I_b of size b; its vertex count, edge count, and genus are

$$|V(S_{a,b})| = a + b, \qquad |E(S_{a,b})| = \binom{a}{2} + ab, \qquad g(S_{a,b}) = \frac{(a-1)(a+2b-2)}{2}.$$

3 Exact Gonality Sequence of Complete Split Graphs

Throughout this section we fix $a \geq 2$ and $b \geq 1$ and write $g = g(S_{a,b}) = \frac{1}{2}(a-1)(a+2b-2)$. For integers $u, m \geq 0$, define the *Hirzebruch $\mathbb{F}_1$ rank and degree functions*

$$R(u, m) \;:=\; (u+1)m + \binom{u+2}{2} - 1, \qquad d_{a,b}(u, m) \;:=\; ma + u(a+b). \tag{3}$$

On the Hirzebruch surface $\mathbb{F}_1 = \mathrm{Bl}_p(\mathbb{P}^2)$ with hyperplane class H and exceptional divisor E ($H^2 = 1$, $E^2 = -1$, $H \cdot E = 0$), a smooth curve $C \in |(a+b)H - bE|$ has arithmetic genus $\binom{a+b-1}{2} - \binom{b}{2} = g(S_{a,b})$, and the linear system $|(u+m)H - mE|$ on $\mathbb{F}_1$ has intersection degree $(u+m)(a+b) - mb = d_{a,b}(u, m)$ with C and ambient projective dimension $\binom{u+m+2}{2} - 1 - \binom{m+1}{2} = R(u, m)$ (which coincides with the restricted projective dimension on C inside the canonical box $0 \leq u \leq a-2$, $0 \leq m \leq b-1$, where $H^0(\mathbb{F}_1, \mathcal{O}(L - C)) = 0$).

3.1 Clique-Sink Superstable Configurations and Exact Serre Duality

Lemma 3.1 (Clique-Sink Superstable Characterization on $S_{a,b}$)**.** *Let $q \in K_a$ be a clique sink in $S_{a,b}$, and let a nonnegative configuration on $V(S_{a,b}) \setminus \{q\}$ have sorted clique values $0 \le x_1 \le \cdots \le x_{a-1}$ and sorted independent-set values $0 \le y_1 \le \cdots \le y_b$. Define*

$$c_i \;:=\; (i-1) + \#\{j \in \{1,\dots,b\} : y_j \le i-1\} \qquad (1 \le i \le a-1).$$

Then $(x_1,\dots,x_{a-1};\, y_1,\dots,y_b)$ is q-superstable if and only if $y_b \le a-1$ and $x_i \le c_i$ for every $1 \le i \le a-1$.

Proof. Because every vertex of I_b is adjacent solely to K_a, once t vertices of K_a (including q) have burned, an unburned vertex $w_j \in I_b$ burns if and only if $y_j \le t-1$. Conversely, every vertex of $K_a \setminus \{q\}$ is adjacent to all other vertices of $S_{a,b}$; hence when exactly i vertices of K_a (namely q and the first $i-1$ non-sink clique vertices) have burned, the number of burned vertices in I_b is $m_i := \#\{j : y_j \le i-1\}$, so the i-th non-sink clique vertex v_i has exactly $i + m_i = c_i + 1$ burned neighbors. It therefore burns if and only if $x_i \le c_i$. By induction on $i = 1,\dots,a-1$, all of K_a burns if and only if $x_i \le c_i$ for every $1 \le i \le a-1$; once all a vertices of K_a have burned, all of I_b burns if and only if $y_b \le a-1$. □

Lemma 3.2 (Hirzebruch $\mathbb{F}_1$ Serre Involution and Box Reduction)**.** *Let $a \ge 2$ and $b \ge 1$.*

(a) *For every $m \ge 0$, $d_{a,b}(a-1,m) - R(a-1,m) = g$.*

(b) *For every $u, m \ge 0$, $R(u+1,m) = R(u,m+1) + (m+1)$ and $d_{a,b}(u+1,m) = d_{a,b}(u,m+1) + b$.*

(c) *For every $(u,m) \in [0,a-2] \times [0,b-1]$, setting $(u',m') = (a-2-u, b-1-m) \in [0,a-2] \times [0,b-1]$ yields*

$$d_{a,b}(u,m) + d_{a,b}(u',m') \;=\; 2g-2, \qquad R(u,m) - R(u',m') \;=\; d_{a,b}(u,m) - g + 1.$$

(d) *Assume $a \ge 3$, and suppose $d \in [g, 2g-2]$ and $r > d-g$. Set $d' = 2g-2-d \le g-2$ and $r' = r-(d-g+1) \ge 0$. If there exists (u_0,m_0) with $0 \le u_0 \le a-1$, $m_0 \ge 0$, $R(u_0,m_0) \ge r'$, and $d' \ge d_{a,b}(u_0,m_0) - (R(u_0,m_0) - r')$, then there exists $(u,m) \in [0,a-2] \times [0,b-1]$ such that $R(u,m) \ge r$ and $d \ge d_{a,b}(u,m) - (R(u,m) - r)$.*

Proof. Parts (a)–(c) are direct polynomial identities (formally verified in Lean 4 as `hirzebruch_top_u_diff`, `hirzebruchR_shift`, `hirzebruchDeg_shift`, `hirzebruch_serre_deg`, and `hirzebruch_serre_rank`). For (d), observe first that if $u_0 = a-1$ then (a) gives $d_{a,b}(a-1,m_0) - (R(a-1,m_0) - r') = g + r' \ge g > g-2 \ge d'$, a contradiction; hence $u_0 \le a-2$. Whenever $m_0 \ge b$, replacing (u_0,m_0) by (u_0+1, m_0-1) strictly increases R by $m_0 \ge 1$ and changes $d_{a,b} - R$ by $b - m_0 \le 0$ by (b), so both witness inequalities are preserved; moreover u_0 can never reach $a-1$ since $d_{a,b} - R \le d' - r' \le g-2 < g$. Inducting on m_0 produces $(u',m') \in [0,a-2] \times [0,b-1]$ satisfying $R(u',m') \ge r'$ and $d' \ge d_{a,b}(u',m') - (R(u',m') - r')$ (`hirzebruch_box_reduction`). Finally, setting $(u,m) = (a-2-u', b-1-m')$ and substituting (c) shows that the two inequalities swap roles identically:

$$\begin{aligned} R(u,m) - r &= \big(R(u',m') + g - 1 - d_{a,b}(u',m')\big) - \big(r' + d - g + 1\big) \\ &= d' - \big(d_{a,b}(u',m') - R(u',m') + r'\big) \;\ge\; 0, \\ d - \big(d_{a,b}(u,m) - R(u,m) + r\big) &= R(u',m') - r' \;\ge\; 0. \end{aligned}$$

□

3.2 The Inductive Fixed-S Master Inequality

The combinatorial heart of Theorem 3.5 is an exact integer inequality controlling the degree cost of any stable divisor on $S_{a,b}$ against its clique-sink obstruction excess.

Lemma 3.3 (Fixed-S Master Inequality)**.** *Let $a \geq 2$ and $b \geq 0$ be integers. Let $0 \leq x_0 \leq x_1 \leq \cdots \leq x_{a-1} \leq a+b-2$ and $0 \leq y_1 \leq \cdots \leq y_b \leq a-1$ be sorted integer sequences, and set $c_i := (i-1) + \#\{j : y_j \leq i-1\}$ for $1 \leq i \leq a-1$. Let $S \subseteq \{1, \ldots, a-1\}$ have cardinality $u = |S|$ and satisfy*

$$x_j \leq c_j \quad (\forall\, j \notin S) \qquad \text{and} \qquad x_s \geq c_s \quad (\forall\, s \in S).$$

Define

$$\Psi_S := x_0 + \sum_{s\in S}(x_s - c_s), \quad \Delta_S := \sum_{s\in S}(s-1) - \binom{u}{2}, \quad \Delta_Y := \sum_{j=1}^{b}\big(y_j + \#\{s \in S : s-1 \geq y_j\} - u\big),$$

and $k := \left\lceil \left(\Psi_S + \binom{u}{2}\right)/(u+1) \right\rceil$. Then $\Delta_S \geq 0$, $\Delta_Y \geq 0$, and

$$(a-1-u)\,k \;\leq \sum_{j\in\{1,\ldots,a-1\}\setminus S} x_j \;+\; \Delta_S \;+\; \Delta_Y. \tag{4}$$

Proof. First, $\Delta_S = \sum_{s\in S} \#\{j < s : j \notin S\} \geq 0$ by counting inversions between S and its complement, and each summand of Δ_Y is nonnegative because $\{s \in S : s-1 < y_j\}$ has size at most y_j, so $y_j + \#\{s \in S : s-1 \geq y_j\} \geq |S| = u$.

We prove (4) by induction on $a+b \geq 2$. If $u = a-1$, the left-hand side is 0 and the right-hand side is nonnegative. If $u = 0$, then $\Psi_S = x_0$, $k = x_0$, and $\sum_{j=1}^{a-1} x_j \geq (a-1)x_0$. Assume henceforth $1 \leq u \leq a-2$ (hence $a \geq 3$).

Case 1: $b \geq 1$ and $y_b = a-1$. Drop y_b, cap $x'_i := \min(x_i, a+b-3)$ for $0 \leq i \leq a-1$, and keep S unchanged. Since $y_b = a-1 > i-1$ for all $i \leq a-1$, every threshold $c'_i = c_i \leq (a-2)+(b-1) = a+b-3$ is preserved, so $x'_j = x_j \leq c'_j$ for $j \notin S$ and $x'_s \geq c'_s$ for $s \in S$. Capping drops each of x_0 and the u terms $x_s - c_s$ ($s \in S$) by at most 1, so $\Psi'_S \geq \Psi_S - (u+1)$ and hence $k \leq k' + 1$. Meanwhile $\Delta'_S = \Delta_S$ and $\Delta'_Y = \Delta_Y - (a-1-u)$ (since $y_b = a-1$ exceeds every $s-1$). Adding $(a-1-u)$ to the induction hypothesis on $(a, b-1)$ yields (4).

Case 2: $b = 0$ or $y_b \leq a-2$. Here every $y_j \leq a-2$, so for every $1 \leq i \leq a-2$, $c_i \leq (a-3)+b = a+b-3$. Cap $x'_i := \min(x_i, a+b-3)$ for $0 \leq i \leq a-2$; both inequalities $x'_j = x_j \leq c_j$ ($j \notin S$) and $x'_s \geq c_s$ ($s \in S$) persist on $\{1, \ldots, a-2\}$.

Subcase 2a: $a-1 \notin S$. Drop x_{a-1} and keep S, y on $(a-1, b)$ (so $u' = u$, $\Delta'_S = \Delta_S$, $\Delta'_Y = \Delta_Y$, and $\sum_{j\notin S'} x'_j = \sum_{j\notin S} x_j - x_{a-1}$).

- If $x_{a-1} \leq a+b-3$, then $x'_i = x_i$ for all $i \leq a-2$, so $k' = k$. Moreover, for every $s \in S$, since $x_s \geq c_s \geq s-1$, we have $(x_s - c_s) + (s-1) \leq x_s \leq x_{a-1}$; summing over $s \in S$ and adding $x_0 \leq x_{a-1}$ yields $\Psi_S + \binom{u}{2} \leq \Psi_S + \sum_{s\in S}(s-1) \leq (u+1)x_{a-1}$, hence $k \leq x_{a-1}$. Adding $k \leq x_{a-1}$ to the induction hypothesis on $(a-1, b)$ gives (4).

- If $x_{a-1} = a+b-2$, then $\Psi'_S \geq \Psi_S - (u+1)$ gives $k \leq k'+1$. We also claim $k \leq u+b$ (*Next-Non-S Bound*, `next_nonS_bound`): enumerate $S = \{s_1 < \cdots < s_u\}$. Since $a-1 \notin S$, each s_r has a least index $j(r) \notin S$ with $j(r) > s_r$. All integers in $(s_r, j(r))$ lie in S, so $j(r) - s_r \leq u - r + 1$; hence

  $$(x_{s_r} - c_{s_r}) + (r-1) \;\leq\; \big(c_{j(r)} - (s_r - 1)\big) + (r-1) \;\leq\; \big(j(r) - 1 + b - s_r + 1\big) + (r-1) \;\leq\; u+b.$$

  Likewise the least $j_0 \notin S$ satisfies $j_0 - 1 \leq u$, so $x_0 \leq x_{j_0} \leq c_{j_0} \leq u+b$. Summing over $r = 1, \ldots, u$ and adding x_0 gives $\Psi_S + \binom{u}{2} \leq (u+1)(u+b)$, so $k \leq u+b$. Therefore $(a-1-u)k - (a-2-u)k' \leq (a-2-u) + k \leq a+b-2 = x_{a-1}$, closing the induction on $(a-1, b)$.

Subcase 2b: $a-1 \in S$. Let $t := \#\{s \in S \setminus \{a-1\} : x_s = a+b-2\}$. Every $j \notin S$ satisfies $j \le a-2$ and $x_j \le c_j \le a+b-3$, so by monotonicity of x, every element of $T := \{s \in S \setminus \{a-1\} : x_s = a+b-2\} \cup \{a-1\}$ (of cardinality $t+1$) exceeds all $a-1-u$ elements outside S. By the inversion identity, $\Delta_S \ge (a-1-u)(t+1)$.

- If $k \le t+1$, then $(a-1-u)k \le (a-1-u)(t+1) \le \Delta_S$, and (4) holds immediately.
- If $k \ge t+2$, pass to $(a-1,b)$ with $S' = S \setminus \{a-1\}$ ($u' = u-1$). Because $u \le a-2$, some $j \notin S$ exists, giving $x_0 \le x_j \le a+b-3$ and hence $x'_0 = x_0$. Also $c_{a-1} = (a-2)+b = a+b-2 \ge x_{a-1}$, so $x_{a-1} - c_{a-1} = 0$, and capping reduces $x_s - c_s$ by 1 solely on the t indices with $x_s = a+b-2$. Thus $\Psi'_{S'} \ge \Psi_S - t$, so

$$\Psi'_{S'} + \binom{u-1}{2} \;\ge\; (u+1)(k-1)+1-t-(u-1) \;=\; u(k-2)+1+(k-t) \;>\; u(k-2),$$

forcing $k' \ge k-1$. Meanwhile $\Delta'_{S'} = \Delta_S - (a-1-u)$, $\Delta'_{Y'} = \Delta_Y$ (as $a-2 \ge y_j$ for all j), and $\sum_{j \notin S'} x'_j = \sum_{j \notin S} x_j$. Adding $(a-1-u)$ to the induction hypothesis on $(a-1,b)$ completes the proof. □

Lemma 3.4 (Direct $k < u$ Inversion Bound)**.** *Under the hypotheses of Lemma 3.3, assume moreover that $x_s \ge c_s + 1$ for every $s \in S$. Then*

$$(a-1-u)\,u \;\le \sum_{j \in \{1,\dots,a-1\}\setminus S} x_j \;+\; \Delta_S.$$

Proof. Double-counting inversions gives $\Delta_S = \sum_{j \notin S} \#\{s \in S : s > j\}$. For any $j \notin S$, let $q_j = \#\{s \in S : s < j\}$. If $q_j = 0$, then $\#\{s \in S : s > j\} = u$, so $x_j + \#\{s \in S : s > j\} \ge u$. If $q_j \ge 1$, let $s^* = \max\{s \in S : s < j\}$; then $s^* \ge q_j$, and since x is nondecreasing, $x_j \ge x_{s^*} \ge c_{s^*} + 1 \ge s^* \ge q_j$. Hence $x_j + \#\{s \in S : s > j\} \ge q_j + (u - q_j) = u$ for every $j \notin S$. Summing over all $a-1-u$ indices $j \notin S$ yields the claim (`k_lt_u_direct_bound`). □

3.3 Proof of Theorem 3.5

Theorem 3.5 (Exact Divisorial Gonality Sequence of Complete Split Graphs)**.** *For every $a \ge 2$, $b \ge 1$, and $r \ge 1$, the r-th divisorial gonality of the complete split graph $S_{a,b} = K_a \vee \overline{K_b}$ of genus $g = \frac{1}{2}(a-1)(a+2b-2)$ is given by*

$$\gamma_r(S_{a,b}) \;=\; \min\left\{ g+r,\; \min_{\substack{0 \le u \le a-1,\, m \ge 0 \\ m+u \le a+b-2,\, R(u,m) \ge r}} \Big(d_{a,b}(u,m) - \big(R(u,m) - r\big) \Big) \right\}, \tag{5}$$

where $R(u,m) = (u+1)m + \binom{u+2}{2} - 1$ and $d_{a,b}(u,m) = ma + u(a+b)$.

Proof. Upper bound. By Baker–Norine Riemann–Roch (2), $\gamma_r(S_{a,b}) \le g+r$. For any admissible (u,m) with $0 \le u \le a-1$, $m \ge 0$, $m+u \le a+b-2$, and $R(u,m) \ge r$, consider the part-uniform divisor $D_{u,m}$ on $S_{a,b} = K_{1,\dots,1,b}$ placing $m+u$ chips on each of the a clique vertices and u chips on each of the b independent-set vertices. Because $m+u \le a+b-2 = \deg(v) - 1$ for $v \in K_a$ and $u \le a-1 = \deg(w) - 1$ for $w \in I_b$, Theorem 5.2 (proved independently in Section 5) evaluates at $k = m+u$ to give

$$r(D_{u,m}) \;\ge\; \Phi(\underbrace{m+u,\dots,m+u}_{a},\, u) \;\ge\; \binom{m+u+2}{2} - 1 - \binom{m+1}{2} \;=\; R(u,m) \;\ge\; r.$$

Subtracting $R(u,m) - r$ arbitrary chips from $D_{u,m}$ lowers the rank by at most $R(u,m) - r$ (by (1)), producing a divisor of rank at least r and degree $\deg(D_{u,m}) - (R(u,m) - r) = d_{a,b}(u,m) - (R(u,m) - r)$.

Lower bound. Let D be any divisor on $S_{a,b}$ of degree d and rank $r = r(D) \geq 1$. If $d \geq g+r$, then d already meets the $g+r$ term in (5); assume $d < g+r$. In particular, by (2), $d \leq 2g-2$.

Case $a = 2$. Here $g = b$. For $d \leq 2b-2$, the graph $S_{2,b}$ admits a degree-2 rank-1 divisor (placing 1 chip on each clique vertex, or equivalently $D_{0,1}$), and $r(D) \leq d/2$ holds for all $d \leq 2b-2$ by the discrete Clifford inequality $2r(D) \leq d$ [2, Corollary 3.5]; hence $d \geq 2r = d_{2,b}(0,r) - (R(0,r) - r)$, which is attained by $(u,m) = (0,r)$ for $1 \leq r \leq b-1$ (since $0 + r \leq b = a + b - 2$).

Case $a \geq 3$ and $d \leq g-1$. Since $g - 1 < \binom{a}{2} + ab = |E(S_{a,b})|$, Björner–Lovász–Shor stabilization [3] yields a stable effective divisor $D' \sim D$, so $D'(v) \leq a+b-2$ for all $v \in K_a$ and $D'(w) \leq a-1$ for all $w \in I_b$. Sort the clique values as $0 \leq x_0 \leq x_1 \leq \cdots \leq x_{a-1} \leq a+b-2$ and the independent-set values as $0 \leq y_1 \leq \cdots \leq y_b \leq a-1$. Choose the minimum-chip clique vertex q (with $D'(q) = x_0$) as sink, and define $c_i := (i-1) + \#\{j : y_j \leq i-1\}$ $(1 \leq i \leq a-1)$ and $S := \{i \in \{1, \ldots, a-1\} : x_i > c_i\}$ of size $u = |S|$. The configuration $(\min(x_1, c_1), \ldots, \min(x_{a-1}, c_{a-1}); y_1, \ldots, y_b)$ is q-superstable by Lemma 3.1, so Lemma 2.1 gives

$$r + 1 \;\leq\; (x_0 + 1) + \sum_{s \in S} (x_s - c_s) \;=\; \Psi_S + 1, \qquad \text{hence } r \leq \Psi_S.$$

Expanding the total degree $d = x_0 + \sum_{s \in S} x_s + \sum_{j \notin S} x_j + \sum_{j=1}^{b} y_j$ and using $\sum_{s \in S} c_s = \sum_{s \in S}(s-1) + \sum_{j=1}^{b} \#\{s \in S : s - 1 \geq y_j\}$ yields the exact identity

$$d - \Psi_S \;=\; \sum_{j \notin S} x_j + \sum_{s \in S}(s-1) + \sum_{j=1}^{b} \big(y_j + \#\{s \in S : s - 1 \geq y_j\}\big) \;=\; \sum_{j \notin S} x_j + \Delta_S + \Delta_Y + \binom{u}{2} + bu. \tag{6}$$

Let $k := \lceil (\Psi_S + \binom{u}{2}) / (u+1) \rceil$.

- If $k \geq u$, set $m := k - u \geq 0$. By definition of k, $\Psi_S + \binom{u}{2} \leq (u+1)k$, which rearranges identically to $\Psi_S \leq R(u,m)$ (`hirzebruch_bridge_k_ge_u`), so $R(u,m) \geq \Psi_S \geq r$. Combining (6) with Lemma 3.3 gives $d - \Psi_S \geq (a-1-u)k + \binom{u}{2} + bu = d_{a,b}(u,m) - R(u,m)$ (`hirzebruch_deg_excess_k_ge_u`), hence $d \geq d_{a,b}(u,m) - (R(u,m) - r)$. Finally, $m + u = k \leq a+b-2$ because $x_0 \leq a+b-2$ and every $s \in S$ has $(x_s - c_s) + (s-1) \leq x_s \leq a+b-2$, giving $\Psi_S + \binom{u}{2} \leq (u+1)(a+b-2)$.

- If $k < u$, set $m := 0$. Then $\Psi_S + \binom{u}{2} \leq (u+1)(u-1) = u^2 - 1$, so $\Psi_S \leq \binom{u+2}{2} - 1 = R(u,0)$ and $R(u,0) \geq r$. Combining (6) with Lemma 3.4 and $\Delta_Y \geq 0$ gives $d - \Psi_S \geq (a-1-u)u + \binom{u}{2} + bu = u(a+b) - R(u,0) = d_{a,b}(u,0) - R(u,0)$, hence $d \geq d_{a,b}(u,0) - (R(u,0) - r)$, with $0 + u \leq a - 1 \leq a+b-2$.

Case $a \geq 3$ and $g \leq d \leq 2g-2$. Since $d < g+r$, the dual divisor $D' = K_{S_{a,b}} - D$ has degree $d' = 2g - 2 - d \in [0, g-2]$ and rank $r' = r - (d - g + 1) \geq 0$ by (2). If $r' = 0$, then $d = g + r - 1$ and $r = d - g + 1 \leq g - 1$, which is already witnessed by the canonical pair $(u,m) = (a-2, b-1)$ since $d_{a,b}(a-2, b-1) = 2g-2$ and $R(a-2, b-1) = g-1$, giving $d_{a,b}(a-2, b-1) - (R(a-2, b-1) - r) = g + r - 1 = d$. If $r' \geq 1$, the $d' \leq g-1$ case applied to D' produces (u_0, m_0) with $0 \leq u_0 \leq a-1$, $R(u_0, m_0) \geq r'$, and $d' \geq d_{a,b}(u_0, m_0) - (R(u_0, m_0) - r')$. Lemma 3.2(d) then produces $(u,m) \in [0, a-2] \times [0, b-1]$ (note $m + u \leq a+b-3 < a+b-2$) with $R(u,m) \geq r$ and $d \geq d_{a,b}(u,m) - (R(u,m) - r)$. □

Remark 3.6 (Recovery of Cools–Panizzut [4]). Setting $b = 1$ gives $S_{a,1} = K_{a+1}$ $(d = a+1)$. For any candidate (u,m) achieving a value strictly below $g+r$, Lemma 3.2(a) excludes $u = a-1$, and iterative application of the shift identities in Lemma 3.2(b) $((u,m) \mapsto (u+1, m-1)$ whenever $m \geq b = 1)$ reduces to $[0, a-2] \times \{0\}$, where $d_{a,1}(u,0) = u(a+1) = ud$ and $R(u,0) = \frac{1}{2}u(u+3)$ for $1 \leq u \leq d-3$, recovering Cools–Panizzut [4, Theorem 1.1] identically.

4 Linear Regime, Tightness, and Brill–Noether Slopes on $S_{a,b}$

Theorem 3.5 converts every structural question about $\gamma_r(S_{a,b})$ into exact integer arithmetic on the Hirzebruch lattice.

Theorem 4.1 (Structural Properties of $\gamma_r(S_{a,b})$)**.** *Let $a \geq 2$ and $b \geq 1$.*

(i) ***Hyperelliptic Case ($a = 2$):*** *$\gamma_r(S_{2,b}) = 2r$ for $1 \leq r \leq b-1$, and $\gamma_r(S_{2,b}) = b + r$ for $r \geq b$.*

(ii) ***Exact Low-Rank Values ($a \geq 3$):***

$$\gamma_1(S_{a,b}) = a, \qquad \gamma_2(S_{a,b}) = \min(2a,\, a+b), \qquad \gamma_3(S_{a,b}) = \min(3a,\, 2a+b-1).$$

(iii) ***Exact Initial Linear Regime ($a \geq 3$):*** *The largest integer $r_{\mathrm{lin}}(a,b) \geq 1$ such that $\gamma_r(S_{a,b}) = ra$ for all $1 \leq r \leq r_{\mathrm{lin}}(a,b)$ is*

$$r_{\mathrm{lin}}(a,b) \;=\; \left\lfloor \frac{b}{a} \right\rfloor + \left\lceil \frac{b}{a} \right\rceil .$$

(iv) ***Complete Split Gonality-Tightness ($a \geq 2$):*** *$S_{a,b}$ is gonality-tight ($\gamma_2(S_{a,b}) = \gamma_1(S_{a,b}) + 1$) if and only if $b = 1$, i.e. $S_{a,b} = K_{a+1}$.*

(v) ***Brill–Noether Slope Violation at $r = 2$ ($a \geq 4$):*** *The normalized slope strictly increases from $r = 2$ to $r = 3$, i.e. $\gamma_3(S_{a,b})/3 > \gamma_2(S_{a,b})/2$, if and only if $a \geq b + 3$.*

Proof. Part (i) follows directly from Theorem 3.5 for $a = 2$, where $g = b$ and the only candidates with $u \in \{0, 1\}$ are $(0, m)$ giving $2m - (m - r) = m + r \geq 2r$ (attained at $m = r$ for $1 \leq r \leq b-1$) and $(1, m)$ giving $d_{2,b}(1, m) - (R(1, m) - r) = b + r$ by Lemma 3.2(a).

For (iii), write $b = qa+t$ with $q = \lfloor (b-1)/a \rfloor \geq 0$ and $1 \leq t \leq a$, and set $r_0 := \lfloor b/a \rfloor + \lceil b/a \rceil$, which equals $2q + 1$ if $t < a$ and $2q + 2$ if $t = a$. First, $(u, m) = (0, r)$ is admissible for all $1 \leq r \leq r_0$ since $r_0 \leq 2q + 2 \leq qa + t + a - 2 = a + b - 2$ (as $a \geq 3$), giving $\gamma_r(S_{a,b}) \leq ra$. Second, $g+r \geq ra$ holds for every $1 \leq r \leq r_0$ because $2g/(a-1) = a+2qa+2t-2$: when $t < a$, $2r_0 = 4q + 2 < a + 2qa + 2t - 2$ (as $a \geq 3$ and $t \geq 1$ imply $2(a-2)q + (a-3) + (2t-1) \geq 1$), and when $t = a$, $2r_0 = 4q + 4 < 2qa + 3a - 2$ (as $2(a-2)q + (3a-6) \geq 3$). Third, for any candidate (u, m) with $1 \leq u \leq a - 1$ and $h := R(u, m) - r \geq 0$, setting $s := 2m + u + 1$ gives $2R(u, m) = (u + 1)s + u - 1$ and the exact identity

$$2\big(d_{a,b}(u, m) - h - ra\big) \;=\; 2ub + 2(a-1)h - usa.$$

If $sa \leq 2b$, the right-hand side is trivially nonnegative. If $sa > 2b$, then $s \geq 2q + 1$: when $t < a$ and $s = 2q + 1$, parity forces $u \geq 2$ and $2h \geq 2(u-1)$, giving $2ub + 2(a-1)h - usa \geq 2u + 2(a-1)(u-1) - ua = (u-2)a + 2 > 0$; when $t < a$ and $s = 2q + 2 + \delta$ ($\delta \geq 0$), $2h \geq 3u-1+(u+1)\delta$, and because $(a-1)(u+1)-ua = a-u-1 \geq 0$ and $(a-1)(3u-1)+2u \geq 2ua$, the expression is nonnegative; when $t = a$, $sa > 2b = (2q + 2)a$ forces $s = 2q + 3 + \delta$ ($\delta \geq 0$), $2h \geq 4u - 2 + (u+1)\delta$, and $(a-1)(4u-2) \geq ua$ for $a \geq 3, u \geq 1$ (`split_linear_regime_lb`). Thus $\gamma_r(S_{a,b}) = ra$ for all $1 \leq r \leq r_0$. Conversely, at $r = r_0 + 1$, the pair $(1, q)$ (when $t < a$, giving $R(1, q) = 2q + 2$ and degree $(2q + 1)a + t < (2q + 2)a$) or $(1, q + 1)$ (when $t = a$, giving $R(1, q + 1) = 2q + 4$ and degree $(2q + 3)a - 1 < (2q + 3)a$) strictly beats $(r_0 + 1)a$, proving $r_{\mathrm{lin}}(a, b) = r_0$.

For (ii), $\gamma_1(S_{a,b}) = a$ follows from (iii) since $r_{\mathrm{lin}} \geq 1$. For any candidate (u, m) with $0 \leq u \leq a - 1$, observe that $d_{a,b}(u, m) - R(u, m) = m(a - 1 - u) + f(u)$, where $f(u) := u(a + b) - \binom{u+2}{2} + 1$. Because $f(u + 1) - f(u) = a + b - u - 2 \geq b \geq 1$ for $0 \leq u \leq a - 2$ and $m(a - 1 - u) \geq 0$, every candidate with $u \geq 1$ satisfies $d_{a,b}(u, m) - R(u, m) \geq f(1) = a + b - 2$,

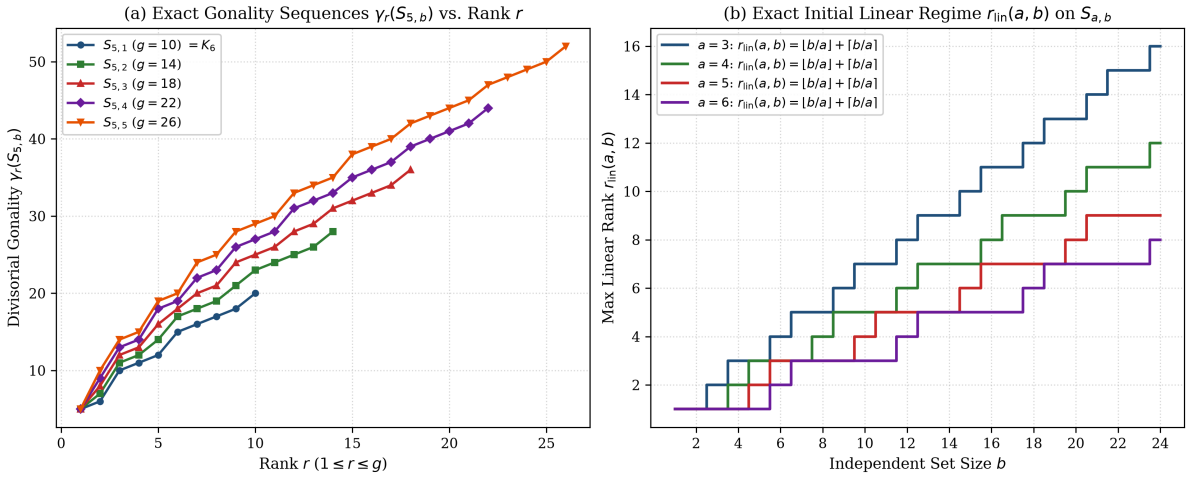


Figure 1: Exact divisorial gonality sequences of complete split graphs $S_{a,b}$ (Theorem 3.5) and the exact initial linear regime $r_{\mathrm{lin}}(a, b) = \lfloor b/a \rfloor + \lceil b/a \rceil$ (Theorem 4.1(iii)). Panel (a) displays $\gamma_r(S_{5,b})$ ($1 \le b \le 5$, up to $g = 26$), showing the $b = 1$ complete-graph baseline $S_{5,1} = K_6$ and the Hirzebruch $\mathbb{F}_1$ step structure for $b \ge 2$. Panel (b) plots the staircase growth of $r_{\mathrm{lin}}(a, b)$ for $a \in \{3, 4, 5, 6\}$.

with equality at $(1, 0)$ (where $R(1, 0) = 2$ and $0 + 1 \le a + b - 2$), while $u = 0$ with $m \ge 2$ gives $d_{a,b}(0, m) - (R(0, m) - 2) = m(a - 1) + 2 \ge 2a$ (attained at $(0, 2)$ since $a + b - 2 \ge 2$); since $g + 2 \ge \min(2a, a + b)$ for $a \ge 3$, this yields $\gamma_2(S_{a,b}) = \min(2a, a + b)$. Similarly for $r = 3$: $u = 0$ requires $m \ge 3$ and gives at least $3a$ (and whenever $3a \le 2a + b - 1$, we have $b \ge a + 1 \ge 4$, so $a + b - 2 \ge 5 \ge 3$ and $(0, 3)$ is admissible); $u = 1$ requires $m \ge 1$ (as $R(1, 0) = 2 < 3$) and gives $d_{a,b}(1, m) - (R(1, m) - 3) \ge (a - 2) + f(1) + 3 = 2a + b - 1$ (attained at $(1, 1)$, where $R(1, 1) = 4$ and $1 + 1 \le a + b - 2$); $u \ge 2$ gives at least $f(2) + 3 = 2a + 2b - 2 \ge 2a + b - 1$; and $g + 3 \ge 2a + b - 1$ for $a \ge 3, b \ge 1$, yielding $\gamma_3(S_{a,b}) = \min(3a, 2a + b - 1)$. Parts (iv) and (v) follow immediately from (i) and (ii): for $a = 2$, $\gamma_2 = 3 = \gamma_1 + 1$ iff $b = 1$, and for $a \ge 3$, $\min(2a, a + b) = a + 1 \iff b = 1$ (`split_gonality_tight_iff`); for $a \ge 4$, $2\min(3a, 2a + b - 1) > 3\min(2a, a + b) \iff a \ge b + 3$ (`split_bn_slope_violation_r2`). □

Figure 1 illustrates the exact gonality sequences $\gamma_r(S_{5,b})$ and the staircase law $r_{\mathrm{lin}}(a, b)$. Figure 2 highlights the Brill–Noether slope profiles $\gamma_r(S_{a,b})/r$ and the gonality gap $\gamma_2 - \gamma_1$: across all 420 parameter pairs $(a, b) \in [2, 15] \times [1, 30]$ verified by our zero-float engine, $S_{4,2}$ ($g = 9$, gonality sequence $4, 6, 9, 10, 12, 14, 15, 16, 18$) is the *unique* complete split graph with $a \ge 4$ exhibiting no Brill–Noether slope violation $\gamma_{r+1}/(r + 1) > \gamma_r/r$ across the entire special range $1 \le r < g$.

5 Universal Blow-Up Rank Lower Bound for Complete Multipartite Graphs

We now generalize the upper-bound construction of Theorem 3.5 from $S_{a,b}$ to *arbitrary* complete multipartite graphs $K_{n_1,\ldots,n_\ell}$ ($\ell \ge 2$). For any tuple $c = (c_1, \ldots, c_\ell) \in \mathbb{Z}^\ell_{\ge 0}$ and integer $k \ge 0$, define

$$\phi(c; k) := \binom{k+2}{2} - 1 - \sum_{i=1}^{\ell} \binom{(k - c_i)_+ + 1}{2}, \qquad \Phi(c) := \max_{k \ge 0} \max\big(0, \phi(c; k)\big), \tag{7}$$

where $x_+ = \max(0, x)$. Geometrically, if $C \subset \mathrm{Bl}_{p_1,\ldots,p_\ell}(\mathbb{P}^2)$ is a curve of class $NH - \sum_{i=1}^{\ell} n_i E_i$ (whose arithmetic genus $\binom{N-1}{2} - \sum_i \binom{n_i}{2}$ equals the graph genus $g(K_{n_1,\ldots,n_\ell})$ identically), then

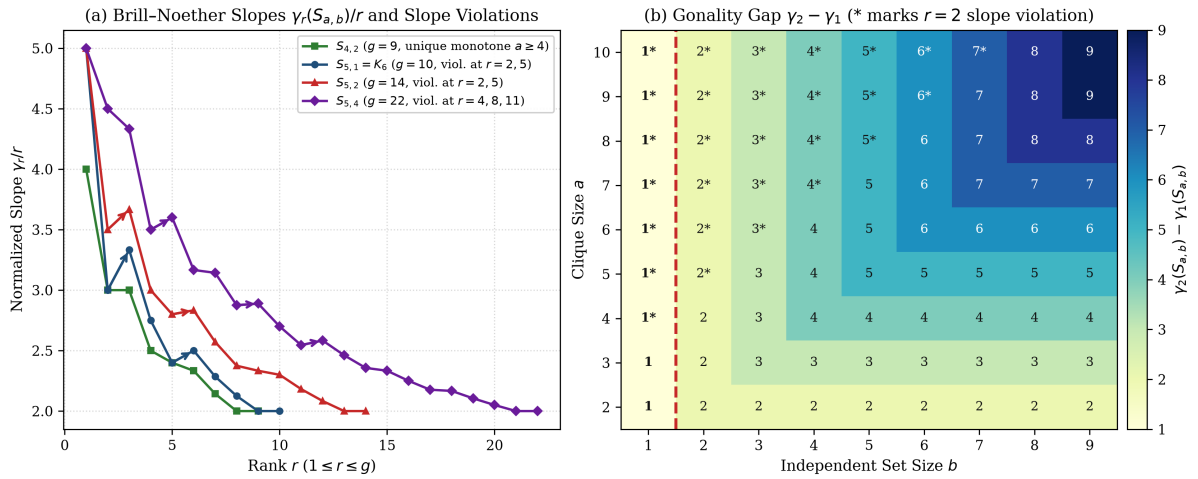


Figure 2: Brill–Noether slope profiles and gonality-tightness phase diagram for $S_{a,b}$ (Theorem 4.1). Panel (a) compares $\gamma_r(S_{a,b})/r$ for the unique monotone $a \geq 4$ graph $S_{4,2}$ against violating curves $S_{5,1} = K_6$, $S_{5,2}$, and $S_{5,4}$ (arrows mark strict upward jumps). Panel (b) maps $\gamma_2(S_{a,b}) - \gamma_1(S_{a,b}) = \min(a, b)$ (for $a \geq 3$), isolating the unique gonality-tight column $b = 1$ ($S_{a,1} = K_{a+1}$) and marking $r = 2$ slope violations ($a \geq b + 3$) with asterisks.

$\phi(c; k)$ is the expected projective dimension of the linear system $|kH - \sum_{i=1}^{\ell}(k - c_i)_+ E_i|$ passing with multiplicity $(k - c_i)_+$ through the ℓ blown-up points p_i.

Lemma 5.1 (Borrowing Positivity, Single-Step Drop, and Peak Cutoff)**.** *Let $\ell \geq 2$ and $c = (c_1, \ldots, c_\ell) \in \mathbb{Z}^{\ell}_{\geq 0}$.*

(a) ***Peak Cutoff:*** *If $p \neq q$ are distinct indices with $c_p + c_q \leq k$, then $\phi(c; k+1) \leq \phi(c; k)$. In particular, letting $c_{(1)} \geq c_{(2)}$ be the two largest coordinates of c, $\Phi(c) = \max_{0 \leq k \leq c_{(1)} + c_{(2)}} \max(0, \phi(c; k))$.*

(b) ***Borrowing Positivity:*** *Fix $i \in \{1, \ldots, \ell\}$. If $c_i + 1 \leq \phi(c; k)$ for some $k \geq 0$, then $c_j \geq 1$ for every $j \neq i$.*

(c) ***Single-Step Borrowing Drop:*** *If $c_j \geq 1$ for every $j \neq i$, let $c^{(i)}$ be obtained from c by keeping c_i fixed and replacing c_j with $c_j - 1$ for all $j \neq i$. Then for every $k \geq 0$,*

$$\phi(c; k) \;\leq\; \max\big(0,\, \phi(c^{(i)};\, (k-1)_+)\big) + c_i + 1.$$

Proof. For (a), increasing k to $k+1$ increases $\binom{k+2}{2}$ by $k+2$, while for each $t \in \{p, q\}$ (since $c_t \leq k$) the term $\binom{(k-c_t)_+ + 1}{2}$ increases by $k + 1 - c_t$; all other subtracted terms are nondecreasing. Thus $\phi(c; k+1) - \phi(c; k) \leq (k+2) - (k+1-c_p) - (k+1-c_q) = c_p + c_q - k \leq 0$ (`phiVal_noninc_above_top2`).

For (b), suppose $c_j = 0$ for some $j \neq i$. Then $\sum_t \binom{(k-c_t)_+ + 1}{2} \geq \binom{k+1}{2} + \binom{(k-c_i)_+ + 1}{2}$, so $\phi(c; k) \leq \binom{k+2}{2} - 1 - \binom{k+1}{2} - \binom{(k-c_i)_+ + 1}{2} = k - \binom{(k-c_i)_+ + 1}{2} \leq k - (k - c_i)_+ \leq c_i$, contradicting $c_i + 1 \leq \phi(c; k)$ (`phi_borrowing_pos`).

For (c), if $k = 0$ then $\phi(c; 0) = 0$ and the bound is trivial. If $k \geq 1$, then for every $j \neq i$, since $c_j \geq 1$, we have $(k-1) - (c_j - 1) = k - c_j$ identically, so every non-i term in the sum is *unchanged*. Meanwhile $\binom{k+2}{2}$ drops by $k+1$, and the i-th subtracted term drops by $(k - c_i)_+$. Therefore

$$\phi(c; k) - \phi(c^{(i)}; k-1) \;=\; (k+1) - (k - c_i)_+ \;\leq\; c_i + 1,$$

with exact equality whenever $k \geq c_i$ (`multipartite_borrowing_step`). □

Theorem 5.2 (Universal Part-Uniform Rank Lower Bound for Complete Multipartite Graphs)**.** *Let $G = K_{n_1,\dots,n_\ell}$ ($\ell \geq 2$, $N = \sum_{i=1}^{\ell} n_i$). For every tuple $c = (c_1, \dots, c_\ell)$ with $0 \leq c_i \leq N - n_i - 1$ ($1 \leq i \leq \ell$), the part-uniform divisor $U(c)$ satisfies*

$$r\big(U(c)\big) \;\geq\; \Phi(c). \tag{8}$$

Consequently, for every $r \geq 1$,

$$\gamma_r(K_{n_1,\dots,n_\ell}) \;\leq\; \gamma_r^{\mathrm{PU}}(K_{n_1,\dots,n_\ell}) \;:=\; \min\left\{ g + r,\; \min_{\substack{0 \leq c_i \leq N - n_i - 1 \\ \Phi(c) \geq r}} \left(\sum_{i=1}^{\ell} n_i c_i - \big(\Phi(c) - r\big) \right) \right\}. \tag{9}$$

Proof. We prove $r(U(c)) \geq \Phi(c)$ by induction on $\sum_{j=1}^{\ell} c_j \geq 0$. If $\Phi(c) = 0$, then $U(c) \geq 0$ is effective, so $r(U(c)) \geq 0 = \Phi(c)$. Suppose $\Phi(c) \geq 1$, and let $E \geq 0$ be an arbitrary effective divisor of degree $\Phi(c)$; we show that $U(c) - E$ is linearly equivalent to an effective divisor.

- If $E(v) \leq c_i$ for every part i and every $v \in P_i$, then $U(c) - E \geq 0$ is already effective.
- Otherwise, there exist a part i and a vertex $v \in P_i$ with $E(v) \geq c_i + 1$. Since $\deg(E) = \Phi(c)$, we have $c_i + 1 \leq \Phi(c)$, so choosing $k \geq 0$ attaining $\Phi(c) = \phi(c; k)$ yields $c_i + 1 \leq \phi(c; k)$. By Lemma 5.1(b), $c_j \geq 1$ for every $j \neq i$. Borrowing at v (i.e. firing $V \setminus \{v\}$) adds $\deg_G(v) = N - n_i \geq c_i + 1$ chips to v, leaves every other vertex of P_i unchanged, and removes 1 chip from every vertex of P_j ($j \neq i$). Hence

  $$U(c) - (c_i + 1)e_v \;\sim\; U(c^{(i)}) + \big(N - n_i - (c_i + 1)\big)e_v \;\geq\; U(c^{(i)}),$$

  where $N - n_i - (c_i + 1) \geq 0$ by the hypothesis $c_i \leq N - n_i - 1$. Set $E' := E - (c_i + 1)e_v \geq 0$. By Lemma 5.1(c), $\deg(E') = \Phi(c) - (c_i + 1) \leq \Phi(c^{(i)})$. Moreover, since $\ell \geq 2$ and every $j \neq i$ has $c_j^{(i)} = c_j - 1$, we have $\sum_{j=1}^{\ell} c_j^{(i)} = \sum_{j=1}^{\ell} c_j - (\ell - 1) < \sum_{j=1}^{\ell} c_j$, while $c_j^{(i)} \leq c_j \leq N - n_j - 1$ persists for all j. By the induction hypothesis on $c^{(i)}$, $r(U(c^{(i)})) \geq \Phi(c^{(i)}) \geq \deg(E')$, so $U(c^{(i)}) - E'$ is linearly equivalent to an effective divisor, and therefore so is $U(c) - E \sim \big(U(c) - (c_i + 1)e_v\big) - E' \geq U(c^{(i)}) - E'$.

This establishes (8). Inequality (9) follows immediately by subtracting $\Phi(c) - r$ arbitrary chips from $U(c)$ whenever $\Phi(c) \geq r$, together with Baker–Norine Riemann–Roch (2). □

Remark 5.3 (Sharpness and Multi-Point Blow-Up Conjecture). We tested (9) against our exact symmetry-quotiented Dhar superstable DP across 14 general complete multipartite graphs (43,194 canonical q-superstable orbits, up to $g = 19$ for $K_{3,3,3}$ and $g = 17$ for $K_{2,2,2,2}$), as well as all 2,316 individual part-uniform divisors with $g \leq 12$ (Table 2 and Figure 3). Remarkably, $\gamma_r(K_{n_1,\dots,n_\ell}) = \gamma_r^{\mathrm{PU}}(K_{n_1,\dots,n_\ell})$ holds for *every* rank $r \geq 1$ on all 14/14 graphs. This motivates the following natural conjecture, already proved for K_N [4], all complete split graphs $S_{a,b}$ (Theorem 3.5), all $r = 1$ [8], and all $r = 2, \ell \geq 3$ (Theorem 6.2 below).

Conjecture 5.4 (Complete Multipartite Blow-Up Exactness)**.** For every $\ell \geq 2$, every $(n_1, \dots, n_\ell) \in \mathbb{Z}_{\geq 1}^{\ell}$, and every $r \geq 1$, equality holds in (9): $\gamma_r(K_{n_1,\dots,n_\ell}) = \gamma_r^{\mathrm{PU}}(K_{n_1,\dots,n_\ell})$.

6 Second Gonality and Gonality-Tightness Rigidity ($\ell \geq 3$)

We now prove exactness of γ_2 and completely classify gonality-tight complete multipartite graphs for all $\ell \geq 3$.

Table 2: Exact divisorial gonality sequences of 14 complete multipartite graphs computed via symmetry-quotiented Dhar superstable DP, compared against the universal blow-up upper bound γ_r^{PU} of Theorem 5.2 (14/14 exact full-sequence matches across 43,194 canonical orbits).

Graph G	g	**Orbits**	(γ_1, γ_2)	**Exact Special Gonality Sequence** $(\gamma_1, \ldots, \gamma_g)$	$\gamma = \gamma^{\mathrm{PU}}$
$K_{1,2,2}$	4	20	(3, 5)	3, 5, 6, 8	Exact
$K_{1,2,3}$	6	50	(3, 6)	3, 6, 7, 9, 10, 12	Exact
$K_{2,2,2}$	7	150	(4, 6)	4, 6, 8, 10, 11, 12, 14	Exact
$K_{1,2,4}$	8	105	(3, 6)	3, 6, 8, 10, 11, 13, 14, 16	Exact
$K_{1,1,2,2}$	8	225	(4, 6)	4, 6, 8, 10, 12, 13, 14, 16	Exact
$K_{1,3,3}$	9	175	(4, 7)	4, 7, 8, 11, 12, 14, 15, 16, 18	Exact
$K_{2,2,3}$	10	525	(4, 7)	4, 7, 9, 11, 13, 14, 16, 17, 18, 20	Exact
$K_{1,1,2,3}$	11	735	(4, 7)	4, 7, 9, 11, 13, 15, 16, 18, 19, 20, 22	Exact
$K_{1,2,2,2}$	12	1,323	(5, 7)	5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24	Exact
$K_{2,2,4}$	13	1,470	(4, 8)	4, 8, 10, 12, 14, 16, 18, 19, 20, 22, 23, 24, 26	Exact
$K_{2,3,3}$	14	2,352	(5, 8)	5, 8, 10, 13, 15, 16, 18, 20, 21, 23, 24, 25, 26, 28	Exact
$K_{1,1,3,3}$	15	3,136	(5, 8)	5, 8, 10, 13, 15, 17, 18, 20, 22, 23, 25, 26, 27, 28, 30	Exact
$K_{2,2,2,2}$	17	16,464	(6, 8)	6, 8, 12, 14, 16, 18, 20, 22, 23, 24, 26, 28, 29, 30, 31, 32, 34	Exact
$K_{3,3,3}$	19	16,464	(6, 9)	6, 9, 12, 15, 17, 18, 21, 23, 24, 26, 27, 29, 30, 32, 33, 34, 35, 36, 38	Exact

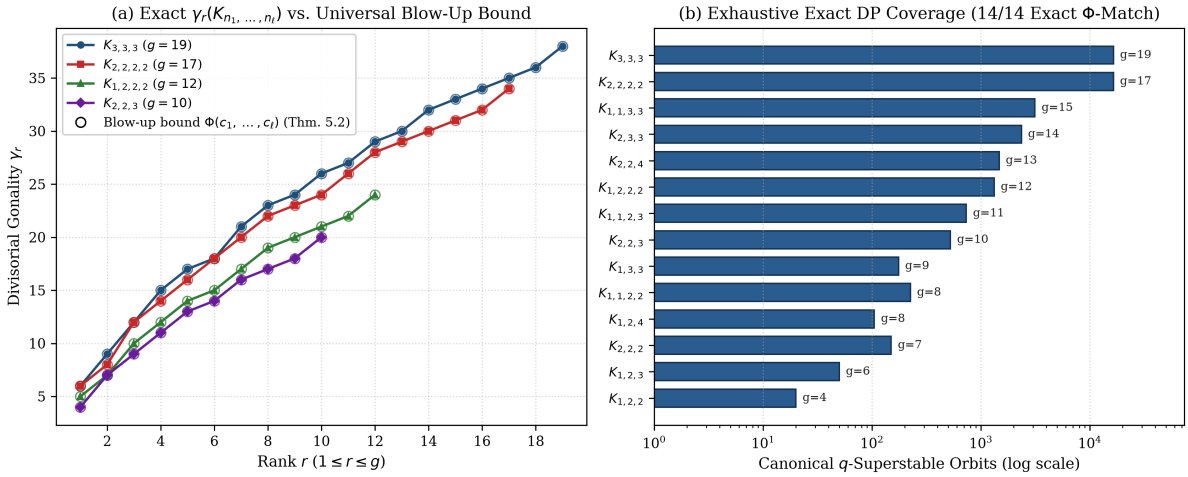


Figure 3: Exact divisorial gonality sequences of general complete multipartite graphs versus the universal blow-up upper bound $\Phi(c_1, \ldots, c_\ell)$ of Theorem 5.2. Panel (a) displays exact DP values (solid markers) overlaid with γ_r^{PU} (hollow circles) for $K_{3,3,3}$, $K_{2,2,2,2}$, $K_{1,2,2,2}$, and $K_{2,2,3}$, demonstrating 100% coincidence. Panel (b) shows the canonical q-superstable orbit counts and genera across all 14 exhaustively verified complete multipartite graphs.

Lemma 6.1 (Complete Multipartite Cut Lower Bound and 3×3 Separation)**.** *Let $\ell \ge 3$, $n_1, \ldots, n_\ell \ge 1$ with $N = \sum_{t=1}^{\ell} n_t$ and $n_{\max} = \max_t n_t$. Let $0 \le b_t \le n_t$ and $u_t = n_t - b_t$ $(1 \le t \le \ell)$ satisfy $b_{\mathrm{tot}} = \sum_t b_t \ge 2$, $u_{\mathrm{tot}} = \sum_t u_t \ge 2$, and $b_t < b_{\mathrm{tot}}$ for every t (i.e. the burned set B meets at least two parts). Set $\mathrm{cut}(B, U) := \sum_{t=1}^{\ell} u_t(b_{\mathrm{tot}} - b_t)$. Then:*

(a) $\mathrm{cut}(B, U) \ge \min\big(2(N - n_{\max}),\ N - 1\big)$.

(b) *If $b_{\mathrm{tot}} \ge 3$ and $u_{\mathrm{tot}} \ge 3$, then* $\mathrm{cut}(B, U) \ge \min\big(2(N - n_{\max}),\ N\big)$.

Proof. We first prove (b), formally verified in Lean 4 as `multipartite_burn_cut_strict_3x3`. Choose a part p maximizing b_p.

- *Case A ($b_p \le 1$):* Every t satisfies $b_{\mathrm{tot}} - b_t \ge b_{\mathrm{tot}} - 1$, so $\mathrm{cut}(B, U) \ge u_{\mathrm{tot}}(b_{\mathrm{tot}} - 1) = (u_{\mathrm{tot}} - 1)(b_{\mathrm{tot}} - 2) + N - 2 \ge 2 \cdot 1 + N - 2 = N$.
- *Case B ($b_p \ge 2$):* Write $U' = u_{\mathrm{tot}} - u_p$ and $B' = b_{\mathrm{tot}} - b_p \ge 1$, so $U' + B' = N - n_p \ge$

$N - n_{\max}$, and decompose

$$\mathrm{cut}(B, U) \;=\; u_p B' + U' b_p + \sum_{t \neq p} u_t (B' - b_t).$$

- If $u_p \geq 2$, then $u_p B' + U' b_p \geq 2(U' + B') \geq 2(N - n_{\max})$.
- If $u_p = 1$, then $U' \geq 2$ and $u_p B' + U' b_p = N - 1 + (U' - 1)(b_p - 1) - 1$. If $U' \geq 3$ or $b_p \geq 3$, this is already at least N. If $U' = b_p = 2$, choose $q \neq p$ with $b_q \geq 1$ and a third part $r \notin \{p, q\}$ (using $\ell \geq 3$). If $u_r \geq 1$, then $u_r(B' - b_r) \geq u_r b_q \geq 1$; if $u_r = 0$, then $b_r = n_r \geq 1$, and picking any $t \neq p$ with $u_t \geq 1$ (necessarily $t \neq r$) gives $u_t(B' - b_t) \geq u_t b_r \geq 1$. Adding $+1$ gives $\mathrm{cut}(B, U) \geq (B' + 4) + 1 = N$.
- If $u_p = 0$, then $U' = u_{\mathrm{tot}} \geq 3$. Choose q maximizing u_q. If $u_q = u_{\mathrm{tot}}$, then $b_t = n_t$ for all $t \neq q$, so $\mathrm{cut}(B, U) \geq u_{\mathrm{tot}}(N - n_q) \geq 2(N - n_{\max})$. Otherwise $u_q \leq u_{\mathrm{tot}} - 1$, so $\sum_{t \neq p} u_t b_t \leq u_q B'$, giving $\sum_{t \neq p} u_t(B' - b_t) \geq (U' - u_q)B' \geq B'$, and hence $\mathrm{cut}(B, U) \geq U' b_p + B' = N + (U' - 1)(b_p - 1) - 1 \geq N + 1 > N$.

For (a), the only remaining cases are $u_{\mathrm{tot}} = 2$ and $b_{\mathrm{tot}} = 2$.

- If $u_{\mathrm{tot}} = 2$: if both vertices of U lie in one part P_p, then $b_t = n_t$ for all $t \neq p$, so $\mathrm{cut}(B, U) = 2(N - n_p) \geq 2(N - n_{\max})$; if U meets two distinct parts P_p, P_q, then a third part P_r ($\ell \geq 3$) has $u_r = 0$ and $b_r = n_r \geq 1$, so $\mathrm{cut}(B, U) = 2b_{\mathrm{tot}} - (b_p + b_q) = b_{\mathrm{tot}} + \sum_{t \notin \{p,q\}} b_t \geq (N - 2) + 1 = N - 1$.
- If $b_{\mathrm{tot}} = 2$: since B meets two distinct parts P_p, P_q ($b_p = b_q = 1$), every $t \notin \{p, q\}$ has $b_t = 0$ and $u_t = n_t$, so $\mathrm{cut}(B, U) = u_p + u_q + 2\sum_{t \notin \{p,q\}} u_t = u_{\mathrm{tot}} + \sum_{t \notin \{p,q\}} n_t \geq (N - 2) + 1 = N - 1$ (since $\ell \geq 3$). □

Theorem 6.2 (Exact Second Gonality and Gonality-Tightness Rigidity)**.** *Let $G = K_{n_1,\ldots,n_\ell}$ with $\ell \geq 3$, $1 \leq n_1 \leq \cdots \leq n_\ell$, and $N = \sum_{i=1}^{\ell} n_i \geq 4$. Then*

$$\gamma_1(G) \;=\; N - n_\ell, \qquad \gamma_2(G) \;=\; \min\bigl(2(N - n_\ell),\, N\bigr). \tag{10}$$

In particular, $K_{n_1,\ldots,n_\ell}$ ($\ell \geq 3, N \geq 4$) is gonality-tight *($\gamma_2(G) = \gamma_1(G) + 1$) if and only if $n_\ell = 1$, i.e. $G = K_N$.*

Proof. The formula $\gamma_1(G) = N - n_\ell$ is due to van Dobben de Bruyn and Gijswijt [8]. For the upper bound on $\gamma_2(G)$, observe that since $\ell \geq 3$ and $N \geq 4$:

- Every part satisfies $N - n_i - 1 \geq 1$, so $U(1, \ldots, 1)$ is admissible in Theorem 5.2 with degree N and $\Phi(1, \ldots, 1) \geq \phi(1, \ldots, 1; 1) = \binom{3}{2} - 1 = 2$, giving $\gamma_2(G) \leq N$.
- Every part $i < \ell$ satisfies $N - n_i \geq 3$ (hence $N - n_i - 1 \geq 2$): indeed, the $\ell - 1 \geq 2$ parts other than P_i include P_ℓ and sum to $N - n_i$; if $N - n_i \leq 2$, then $\ell = 3$ and every part other than P_i has size 1, forcing $n_\ell = 1$, whence $n_i \leq n_\ell = 1$ and $N = 3$, contradicting $N \geq 4$. Hence $U(2, \ldots, 2, 0)$ is admissible in Theorem 5.2 with degree $2(N - n_\ell)$ and $\Phi(2, \ldots, 2, 0) \geq \phi(2, \ldots, 2, 0; 2) = \binom{4}{2} - 1 - \binom{3}{2} = 2$, giving $\gamma_2(G) \leq 2(N - n_\ell)$.

For the lower bound, let $D \in \mathrm{Div}(G)$ be any divisor of degree $d \leq \min(2(N - n_\ell), N) - 1 \leq N - 1$; we prove $r(D) \leq 1$. Because $\ell \geq 3$ and $N \geq 4$, every vertex has degree at least 2 and at least two vertices have degree at least 3, so $|E(G)| \geq N > d$. Assuming $r(D) \geq 0$, Björner–Lovász–Shor stabilization [3] yields a stable effective divisor $D' \sim D$ ($D'(v) < \deg_G(v)$ for all $v \in V$). Since $\deg(D') = d \leq N - 1 < N$, some vertex $q \in V$ has $D'(q) = 0$. Run Dhar's burning algorithm from q on D', partitioning V into the burned set $B \ni q$ and unburned set $U = V \setminus B$.

- If $U = \emptyset$, then D' is q-reduced with $D'(q) = 0$, so $r(D') = 0 \leq 1$.
- If $|U| = 1$, say $U = \{w\}$, then w did not burn despite all $\deg_G(w)$ of its neighbors burning, forcing $D'(w) \geq \deg_G(w)$, contradicting the stability of D'.

Thus $|U| \geq 2$, and every $v \in U$ satisfies $D'(v) \geq |N_G(v) \cap B|$, whence $d \geq \sum_{v \in U} D'(v) \geq \mathrm{cut}(B, U)$.

Case A: B is contained in a single part P_i ($b_{\mathrm{tot}} = b_i \geq 1$). Then $V \setminus P_i \subseteq U$, and every $v \in V \setminus P_i$ satisfies $D'(v) \geq b_i$. Since $d \leq 2(N - n_\ell) - 1 < 2(N - n_i)$, there exists $u \in V \setminus P_i$ with $D'(u) \leq 1$; combined with $D'(u) \geq b_i \geq 1$, this forces $b_i = 1$ (so $B = \{q\}$) and $D'(u) = 1$. Consider $D' - e_u \geq 0$ of degree $d - 1 \leq \min(2(N - n_\ell), N - 1) - 1$, and run Dhar's burning algorithm from q on $D' - e_u$, producing burned set B_1 and unburned set $U_1 = V \setminus B_1$. Because $(D' - e_u)(q) = 0$ and $(D' - e_u)(u) = 0 < 1 = |N_G(u) \cap \{q\}|$, we have $\{q, u\} \subseteq B_1$, so B_1 meets at least two parts. We cannot have $|U_1| = 1$ (the unique $w \in U_1$ satisfies $w \neq u$, so $(D' - e_u)(w) = D'(w) < \deg_G(w)$ by stability), nor $|U_1| \geq 2$ (else Lemma 6.1(a) gives $d - 1 \geq \mathrm{cut}(B_1, U_1) \geq \min(2(N - n_\ell), N - 1) > d - 1$). Therefore $U_1 = \emptyset$, so $D' - e_u$ is q-reduced with $(D' - e_u)(q) = 0$, whence $r(D' - e_u - e_q) = -1$ and $r(D') \leq 1$.

Case B: B meets at least two parts ($b_{\mathrm{tot}} \geq 2$). By Lemma 6.1(a), $d \geq \mathrm{cut}(B, U) \geq \min(2(N - n_\ell), N - 1)$. Since $d \leq \min(2(N - n_\ell) - 1, N - 1)$, we must have $d = \mathrm{cut}(B, U) = N - 1 < 2(N - n_\ell)$, with $D'(v) = 0$ for all $v \in B$ and $D'(v) = |N_G(v) \cap B|$ for all $v \in U$. By Lemma 6.1(b), either $|U| = 2$ or $|B| = 2$.

- *Subcase B1 ($|U| = 2$, say $U = \{w_1, w_2\}$):* Consider $D' - e_{w_1} \geq 0$ and burn from q. All of B burns (as $D' = 0$ on B), then w_1 burns (as $(D' - e_{w_1})(w_1) = |N_G(w_1) \cap B| - 1 < |N_G(w_1) \cap B|$), leaving w_2 as the sole unburned vertex; since $D'(w_2) < \deg_G(w_2)$ by stability, w_2 also burns. Thus $D' - e_{w_1}$ is q-reduced with value 0 at q, so $r(D') \leq 1$.
- *Subcase B2 ($|B| = 2$, say $B = \{q, z\}$ with $q \in P_i$, $z \in P_j$, $i \neq j$):* From the proof of Lemma 6.1(a), $N - 1 = \mathrm{cut}(B, U) = (N - 2) + \sum_{t \notin \{i,j\}} n_t$ forces $\sum_{t \notin \{i,j\}} n_t = 1$, so $\ell = 3$ and the third part $P_k = \{w\}$ is a singleton with $D'(w) = 2$, while every $v \in U \cap (P_i \cup P_j)$ has $D'(v) = 1$. Consider $D' - e_w \geq 0$ and burn from q: $B = \{q, z\}$ burns, then w burns (2 burned neighbors $> 2 - 1 = 1$), and then every $v \in U \cap P_i$ has 2 burned neighbors $(z, w) > 1 = D'(v)$ and every $v \in U \cap P_j$ has 2 burned neighbors $(q, w) > 1 = D'(v)$, so all of V burns. Thus $D' - e_w$ is q-reduced with value 0 at q, so $r(D') \leq 1$.

This proves $\gamma_2(G) = \min(2(N - n_\ell), N)$. Finally, since $N - n_\ell \geq 2$ for $\ell \geq 3$, we have $\min(2(N - n_\ell), N) = (N - n_\ell) + 1 \iff n_\ell = 1$ (`multipartite_gonality_tight_iff`). □

Remark 6.3 (The Bipartite Boundary $\ell = 2$). For complete bipartite graphs $K_{a,b}$ with $2 \leq a \leq b$ (excluding trees $K_{1,b}$ of genus 0), $K_{2,2} = C_4$ has genus $g = 1$ with $\gamma_1 = 2$ and $\gamma_2 = 3 = \gamma_1 + 1$, whereas $K_{2,b}$ ($b \geq 3$) is hyperelliptic of genus $b - 1 \geq 2$ with $\gamma_1 = 2$ and $\gamma_2 = 4 \neq 3$. When $a \geq 3$, Theorem 5.2 yields $\Phi(1, 1) = \phi((1, 1); 2) = \binom{4}{2} - 1 - 2\binom{2}{2} = 3$ (while $\Phi(2, 0) = 2$ for $b \geq 3$), so subtracting one chip from $U(1, 1)$ gives $\gamma_2(K_{a,b}) \leq \min(2a, a + b - 1)$. In particular, $\gamma_2(K_{a,a}) \leq 2a - 1 < 2a = \min(2a, a + b)$ for every $a \geq 3$ (exact DP gives $\gamma_2(K_{3,3}) = 5$, $\gamma_2(K_{3,4}) = 6$, and $\gamma_2(K_{4,4}) = 7$), showing that the $\ell \geq 3$ formula $\min(2(N - n_\ell), N)$ strictly changes at $\ell = 2$ even though $\gamma_2(K_{a,b}) > \gamma_1(K_{a,b}) + 1$ still persists for $a \geq 3$. Thus restricting Theorem 6.2 to $\ell \geq 3, N \geq 4$ cleanly isolates both the exact formula (10) and the rigidity locus $G = K_N$.

7 Formal Verification, Zero-Float Computation, and Reproducibility

Every algebraic and combinatorial core of this paper is independently certified at two complementary levels (while the graph-theoretic Baker–Norine, Björner–Lovász–Shor, and Dhar reduction arguments bridging graph divisors to these integer cores are proved directly in Sections 2–6):

1. **Standalone Lean 4.36.0 Formalization (`MultipartiteGonalityCore.lean`, packaged under `lean/` locally and `src/lean/` in the AIRR bundle):** A self-contained, Mathlib-free Lean 4.36.0 library (117,471 bytes, 2,451 lines, 85 declarations: 66 public theorems and 19 definitions; SHA-256 `b075d406d733bf02...`) packaged with `lean-toolchain` and `lakefile.toml`. It compiles cleanly via `lake build` under `-DwarningAsError=true` with 0 `sorry`, 0 `grind`, and 0 custom axioms (only standard Lean kernel axioms `propext`, `Classical.choice`, and `Quot.sound`, verified via `#print axioms`).

2. **Exact Zero-Float Python Verifier (`verify_multipartite_gonality.py`, packaged under `python/` locally and `src/python/` in the AIRR bundle):** Pure integer verification (zero floating-point operations in any mathematical assertion; `matplotlib` imported lazily solely inside figure generation) executing four exhaustive test suites:

   - *Check 1 (Theorem 3.5):* 25/25 complete split graphs $S_{a,b}$ ($1 \le b \le 7$ for $a \in \{2,3\}$, $1 \le b \le 6$ for $a = 4$, and $1 \le b \le 5$ for $a = 5$; up to $g = 26$, 422,045 superstable orbits) match (5) identically across the entire special range $1 \le r \le g$ (and $r = g+1, g+2$ via Baker–Norine Riemann–Roch).
   - *Check 2 (Theorem 4.1):* All 420 parameter pairs $(a,b) \in [2,15] \times [1,30]$ confirm the exact low-rank formulas, $r_{\text{lin}}(a,b)$, Serre involution on $[0, a-2] \times [0, b-1]$, tightness $b = 1$, $r = 2$ slope violations $a \ge b+3$, and the uniqueness of $S_{4,2}$ among $a \ge 4$ split graphs with no slope violation.
   - *Check 3 (Theorems 5.2 and 6.2):* All 14 complete multipartite graphs in Table 2 (43,194 canonical orbits, up to $g = 19$) satisfy $\gamma_r(G) = \gamma_r^{\text{PU}}(G)$ for all r (14/14 exact match) and $\gamma_2(G) = \min(2(N - n_\ell), N)$; additionally, 2,316 direct part-uniform Dhar rank evaluations ($g \le 12$) confirm $r(U(c)) \ge \Phi(c)$ with 0 failures and 1,430 exact equalities.
   - *Check 4 (Lemma 5.1):* 1,117 sorted borrowing-step transitions across $\ell \in \{2,3,4,5\}$ confirm positivity and the $\le c_i + 1$ drop with 0 failures.

8 Conclusion

We have established the exact divisorial gonality sequence of all complete split graphs $S_{a,b}$ via a discrete Hirzebruch $\mathbb{F}_1$ correspondence, deduced their exact initial linear regime, gonality-tightness, and Brill–Noether slope-violation structure, proved a universal multi-point blow-up rank lower bound Φ for all complete multipartite graphs $K_{n_1,\dots,n_\ell}$, and settled γ_2 and complete gonality-tightness rigidity for all $\ell \ge 3$. Two natural open directions remain: resolving Conjecture 5.4 in full generality (extending our Fixed-S Master Inequality from a single non-singleton part to arbitrary ℓ-part tuples $(n_1, \dots, n_\ell)$), and characterizing whether $S_{4,2}$ remains the unique non-violating complete split graph for all $a \ge 4, b \ge 1$.

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