EXACT ALL-COUPLING NON-PERTURBATIVE BOUNDS FOR 5D DOMAIN-WALL (CAYLEY–MÖBIUS) LATTICE QCD UNDER LÜSCHER'S ADMISSIBILITY CONDITION

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Abstract. We establish four cluster-expansion-free, rigorous theorems for five-dimensional Cayley–Möbius (Kaplan–Shamir / Boriči) Domain-Wall Lattice QCD with SU(N) gauge fields ($N \geq 2$) in even spacetime dimensions $d \in \{2, 4\}$ ($d_s = 2^{d/2}$) on periodic hypercubic tori $\mathbb{T}_L^d$ ($L \geq 2$) under Lüscher's plaquette admissibility condition $\|I_N - U_p\|_{\mathrm{op}} \leq \varepsilon$ (together with open-box axial-tree free-energy enclosures). First, we prove the exact 5D-to-4D Pauli–Villars Schur-complement identity $\det D_{\mathrm{DW}}^{(L_s)}(m;\alpha)/\det D_{\mathrm{DW}}^{(L_s)}(1;\alpha) = \det D_{\mathrm{eff}}^{(L_s)}(m;\alpha)$, where $D_{\mathrm{eff}}^{(L_s)}(m;\alpha) = \frac{1+m}{2} + \frac{1-m}{2}\gamma_5 \varepsilon_{L_s}(\alpha H_W(U))$, together with an elementary exact sharp upper spectral bound $\|H_W(U)\|_{\mathrm{op}} \leq 2d-1$ obtained via unitarity of the Wilson hopping operators $U_\mu^{\mathrm{W}} = P_\mu^- V_\mu + P_\mu^+ V_\mu^\dagger$. Combining this with the covariant Bochner–Weitzenböck lower gap $\mu_{\min}(d,\varepsilon) = (1 - \binom{d}{2}(5\varepsilon + \varepsilon^2/8))^{1/2}$ determines the minimax optimal Cayley–Möbius scale $\alpha_* = (\mu_{\min}(2d-1))^{-1/2}$ and contraction rate $q_* = (\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$ ($\kappa = (2d-1)/\mu_{\min}$). Second, for all even $L_s \geq 2$ and all admissible gauge fields $U \in \mathcal{U}_\varepsilon$, we prove volume- and N-uniform L_s-exponential decay of both the Ginsparg–Wilson violation $\|\Delta_{\mathrm{GW}}^{(L_s)}(U)\|_{\mathrm{op}} \leq 2q_*^{L_s}/(1+q_*^{L_s})^2$ and the midpoint residual mass $m_{\mathrm{res}}^{(L_s)} \leq q_*^{L_s}/(1+q_*^{L_s})^2$. Third, we prove unconditional single-flavor determinant positivity $\det D_{\mathrm{eff}}^{(L_s)}(m;\alpha) > 0$ and resolvent bound $\|(D_{\mathrm{eff}}^{(L_s)}(m;\alpha))^{-1}\|_{\mathrm{op}} \leq 1/m$ for all $N_f \geq 1$ and $m \in (0,1]$, volume- and β-uniform L_s-exponential convergence $|f_{\Lambda,\varepsilon}^{\mathrm{DW}}(\beta,m,L_s) - f_{\Lambda,\varepsilon}^{\mathrm{ov}}(\beta,m)| \leq N_f d_s N \frac{1-m}{m}\frac{q_*^{L_s}}{1+q_*^{L_s}}$ to exact 4D Overlap QCD on $\mathbb{T}_L^d$, and all-coupling 4D SU(2) open-box axial-tree free-energy enclosures spanning $\beta \in [0, 10^4]$ that lock the exact 1-loop $-\frac{9}{2}\ln\beta$ asymptotic entropy slope at large β. Fourth, using Neuberger's exact polar partial-fraction representation of $\varepsilon_{2k}(H)$ and the trace-norm ideal inequality, we prove a volume-uniform, L_s-uniform, α-independent 5D Domain-Wall HMC fermion force bound $\|F_e^{\mathrm{DW}}(U)\|_{\mathrm{HS}} \leq N_f d_s \sqrt{N}(1-m)/(m\,\mu_{\min}(d,\varepsilon))$. All 25 quantitative benchmarks are certified in 140-digit outward-rounded interval arithmetic (zero black-box special functions, zero quadrature), cross-checked directly on $2^4 \times L_s$ 5D SU(2) Dirac matrices, and mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel.

1. Introduction

Domain-Wall Fermions (DWF), introduced by Kaplan [7], formulated on a finite fifth dimension of extent L_s by Shamir and Furman [12, 4], and generalized to the unified Cayley–Möbius family by Boriči [1] and Brower, Neff, and Orginos [2], constitute one of the premier lattice discretizations of chiral QCD. In the $L_s \to \infty$ limit, the Pauli–Villars-regulated 5D DWF determinant reduces algebraically to Neuberger's exact 4D Overlap–Dirac operator [10, 8], which satisfies the exact Ginsparg–Wilson relation [5, 9]. At finite L_s, however, quantitative non-perturbative control of residual chiral-symmetry breaking $m_{\mathrm{res}}^{(L_s)}$, convergence of the dynamical 5D free energy to 4D Overlap QCD, and stability of Hybrid Monte Carlo (HMC) fermion forces across all couplings $\beta \geq 0$ has rarely been assembled into a unified, cluster-expansion-free, mechanically certified framework.

In this paper we prove four exact, completely non-perturbative theorems for 5D Cayley–Möbius Domain-Wall Lattice QCD on finite periodic d-dimensional hypercubic tori $\mathbb{T}_L^d$ ($L \geq 2$, $d \in \{2,4\}$, spinor dimension $d_s = 2^{d/2}$) with SU(N) gauge group ($N \geq 2$), together with open-box axial-tree thermodynamic enclosures:

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(1) **Exact 5D-to-4D Schur complement and sharp spectral envelope (Theorem 2.1):** We prove the exact algebraic determinant identity $\det D_{\mathrm{DW}}^{(L_s)}(m;\alpha)/\det D_{\mathrm{DW}}^{(L_s)}(1;\alpha) = \det D_{\mathrm{eff}}^{(L_s)}(m;\alpha)$ with polar sign approximant $\varepsilon_{L_s}(h) = ((1+h)^{L_s}-(1-h)^{L_s})/((1+h)^{L_s}+(1-h)^{L_s})$, establish the exact sharp upper spectral bound $\|H_W(U)\|_{\mathrm{op}} \le 2d-1$ by proving that $U_\mu^{\mathrm{W}} = P_\mu^- V_\mu + P_\mu^+ V_\mu^\dagger$ is strictly unitary, and combine this with the Lüscher-admissible lower spectral gap $\mu_{\min}(d,\varepsilon)$ [6] to derive the minimax Cayley–Möbius scale $\alpha_* = (\mu_{\min}(2d-1))^{-1/2}$ and contraction rate $q_* = (\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$.

(2) **L_s-exponential Ginsparg–Wilson and residual-mass decay (Theorem 3.1):** For every even $L_s \ge 2$ and every admissible configuration $U \in \mathcal{U}_\varepsilon$, both the Ginsparg–Wilson violation $\|\Delta_{\mathrm{GW}}^{(L_s)}(U)\|_{\mathrm{op}} \le 2q_*^{L_s}/(1+q_*^{L_s})^2$ and the midpoint residual chiral-breaking scale $m_{\mathrm{res}}^{(L_s)} \le q_*^{L_s}/(1+q_*^{L_s})^2$ decay exponentially in L_s, uniformly in lattice volume $|\Lambda|$ and $N \ge 2$.

(3) **Determinant positivity, exponential convergence to Overlap QCD, and all-coupling enclosures (Theorem 4.1):** For every $N_f \ge 1$, $m \in (0,1]$, $L_s \ge 1$, and $U \in \mathrm{SU}(N)^{|E|}$, we prove strict single-flavor determinant positivity $\det D_{\mathrm{eff}}^{(L_s)}(m;\alpha) > 0$ and $\|(D_{\mathrm{eff}}^{(L_s)}(m;\alpha))^{-1}\|_{\mathrm{op}} \le 1/m$, establish volume- and β-uniform L_s-exponential convergence of the dynamical 5D DWF free energy to exact 4D Overlap QCD, and prove all-coupling 4D SU(2) axial-tree free-energy enclosures locking the exact 1-loop $-\frac{9}{2}\ln\beta$ entropy slope across $\beta \in [0, 10^4]$.

(4) **Volume- and L_s-uniform 5D DWF HMC fermion force bound (Theorem 5.1):** Using Neuberger's exact polar partial fraction [11] for $\varepsilon_{2k}(H)$ and the trace-norm ideal inequality, we prove $\|F_e^{\mathrm{DW}}(U)\|_{\mathrm{HS}} \le N_f d_s \sqrt{N}(1-m)/(m\,\mu_{\min}(d,\varepsilon))$ uniformly in lattice volume $|\Lambda|$, fifth-dimensional extent $L_s = 2k \ge 2$, and Cayley scale $\alpha > 0$.

All 25 benchmark certificates across Tables 1–3 are computed in 140-digit outward-rounded interval arithmetic (`mpmath.iv` with `mp.mp.dps = 140` and `mp.iv.dps = 140`), using only elementary series with analytic tails (zero black-box special functions, zero quadrature), cross-checked directly against 512×512 5D $2^4 \times L_s$ SU(2) lattice Dirac matrices in Python, and mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`DomainWallCerts.lean`) [3].

2. 5D Cayley–Möbius Operator, Schur Complement, and Sharp Spectral Envelope

Let $\Lambda = \mathbb{T}_L^d$ ($L \ge 2$, $d \in \{2,4\}$) be a finite periodic d-dimensional hypercubic torus with $|\Lambda| = L^d$ sites, oriented link set E, plaquette set P, and gauge configuration $U = (U_e)_{e\in E} \in \mathrm{SU}(N)^{|E|}$. (In Theorem 4.1(3), we also consider finite open boxes $\Lambda_L = \{0,\dots,L-1\}^4$ with Dirichlet boundary conditions on V_μ, where only Hermiticity of $H_W(U)$ and Theorem 4.1(1) are used.) Let $\{\gamma_\mu\}_{\mu=1}^d$ be Hermitian Euclidean Dirac matrices of size $d_s = 2^{d/2}$ satisfying $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} I_{d_s}$, with Hermitian chiral matrix γ_5 satisfying $\gamma_5^2 = I_{d_s}$ and $\{\gamma_5,\gamma_\mu\} = 0$. Define the chiral projectors $P_\pm := \frac{1}{2}(I_{d_s} \pm \gamma_5)$ and directional Wilson projectors $P_\mu^\pm := \frac{1}{2}(I_{d_s} \pm \gamma_\mu)$.

On $\mathcal{H}_4 := \ell^2(\Lambda) \otimes \mathbb{C}^{d_s} \otimes \mathbb{C}^N$ (dimension $D_4 = |\Lambda| d_s N$), let V_μ be the covariant forward shift $(V_\mu\psi)(x) := U_{(x,\mu)}\psi(x+\hat{\mu})$ (unitary on $\mathcal{H}_4$). The massless Wilson–Dirac operator and its shifted Hermitian kernel at canonical domain-wall height $M_5 = 1$ are

$$D_W(U) := d\,I - \sum_{\mu=1}^{d}(P_\mu^- V_\mu + P_\mu^+ V_\mu^\dagger), \qquad H_W(U) := \gamma_5(D_W(U) - I) = H_W(U)^\dagger. \tag{1}$$

For a fifth-dimensional extent $L_s \ge 2$, Cayley–Möbius scale $\alpha > 0$ (corresponding to Boriči / Möbius parameters $b_5 = c_5 = \alpha$ [1, 2]), and bare quark mass $m \in [0,1]$, define the 5D Domain-Wall

operator $D_{\mathrm{DW}}^{(L_s)}(m;\alpha)$ on $\mathcal{H}_5 := \mathbb{C}^{L_s} \otimes \mathcal{H}_4$ by

$$\big(D_{\mathrm{DW}}^{(L_s)}(m;\alpha)\psi\big)_s := B_+\psi_s - B_-(P_-\psi_{s+1} + P_+\psi_{s-1}), \qquad s = 0,1,\dots,L_s-1, \tag{2}$$

with $B_\pm := I \pm \alpha(D_W(U) - I) = I \pm \alpha\gamma_5 H_W(U)$ and mass-twisted boundary conditions $P_+\psi_{-1} := -mP_+\psi_{L_s-1}$ and $P_-\psi_{L_s} := -mP_-\psi_0$. For any $L_s \geq 1$, define the rational polar sign approximant

$$\varepsilon_{L_s}(h) := \frac{(1+h)^{L_s} - (1-h)^{L_s}}{(1+h)^{L_s} + (1-h)^{L_s}}, \qquad D_{\mathrm{eff}}^{(L_s)}(m;\alpha) := \frac{1+m}{2}\, I + \frac{1-m}{2}\, \gamma_5\, \varepsilon_{L_s}(\alpha H_W(U)). \tag{3}$$

Note that for every even $L_s = 2k$, the denominator $(1+h)^{2k} + (1-h)^{2k} \geq 2$ never vanishes on $\mathbb{R}$, so $\varepsilon_{2k}(\alpha H_W(U))$ is an unconditionally bounded Hermitian operator with $\|\varepsilon_{2k}(\alpha H_W(U))\|_{\mathrm{op}} < 1$ for every $U \in \mathrm{SU}(N)^{|E|}$ and every $\alpha > 0$.

Theorem 2.1 (Exact 5D-to-4D Schur complement and sharp spectral envelope)**.** *Let $\Lambda = \mathbb{T}_L^d$ ($L \geq 2$, $d \in \{2,4\}$) and $N \geq 2$.*

(1) ***Exact 5D-to-4D determinant identity:*** *For any Hermitian kernel $H_W(U)$ (on $\mathbb{T}_L^d$ or on an open box with Dirichlet boundary conditions), whenever $D_{\mathrm{DW}}^{(L_s)}(1;\alpha)$ is invertible—in particular unconditionally for every $U \in \mathrm{SU}(N)^{|E|}$ and every $\alpha > 0$ when L_s is even, since $(I+\alpha H_W)^{L_s} + (I-\alpha H_W)^{L_s} \geq 2I$—*

$$\frac{\det D_{\mathrm{DW}}^{(L_s)}(m;\alpha)}{\det D_{\mathrm{DW}}^{(L_s)}(1;\alpha)} \;=\; \det D_{\mathrm{eff}}^{(L_s)}(m;\alpha) \qquad \forall m \in [0,1]. \tag{4}$$

(2) ***Exact sharp upper spectral bound:*** *For every gauge field $U \in \mathrm{SU}(N)^{|E|}$,*

$$\|H_W(U)\|_{\mathrm{op}} \;\leq\; \mu_{\max}(d) := 2d-1 \qquad (\textit{i.e. } 3 \textit{ in } 2\mathrm{D},\ 7 \textit{ in } 4\mathrm{D}), \tag{5}$$

and equality is attained in the free field $U \equiv I_N$ at Brillouin-zone corner momentum $p = (\pi,\dots,\pi)$ (on periodic $\mathbb{T}_L^d$ with even $L \geq 2$).

(3) ***Admissible spectral enclosure and optimal Cayley–Möbius scale:*** *On Lüscher's admissible set $\mathcal{U}_\varepsilon := \{U : \|I_N - U_p\|_{\mathrm{op}} \leq \varepsilon\ \forall p \in P\}$ with $\binom{d}{2}(5\varepsilon + \varepsilon^2/8) < 1$, every eigenvalue $\lambda \in \mathrm{spec}(H_W(U))$ satisfies*

$$\mu_{\min}(d,\varepsilon) := \sqrt{1 - \binom{d}{2}\left(5\varepsilon + \frac{\varepsilon^2}{8}\right)} \;\leq\; |\lambda| \;\leq\; 2d-1. \tag{6}$$

Setting the minimax optimal Cayley–Möbius scale $\alpha_*(d,\varepsilon) := (\mu_{\min}(d,\varepsilon)(2d-1))^{-1/2}$ and condition number $\kappa(d,\varepsilon) := (2d-1)/\mu_{\min}(d,\varepsilon)$ yields the uniform contraction bound

$$\left|\frac{1-\alpha_*|\lambda|}{1+\alpha_*|\lambda|}\right| \;\leq\; q_*(d,\varepsilon) := \frac{\sqrt{\kappa(d,\varepsilon)}-1}{\sqrt{\kappa(d,\varepsilon)}+1} \;<\; 1 \qquad \forall \lambda \in \mathrm{spec}(H_W(U)). \tag{7}$$

Proof. For part (1), set $A := \alpha\gamma_5 H_W(U)$, $K_\pm := I \pm \alpha H_W(U)$ (which commute), and define the chiral block permutation $\chi = \mathcal{P}\psi$ on $\mathcal{H}_5$ by $\chi_s := P_+\psi_s + P_-\psi_{s+1}$ for $s = 0,\dots,L_s-2$ and $\chi_{L_s-1} := P_+\psi_{L_s-1} + P_-\psi_0$ (unitary, with $|\det\mathcal{P}| = 1$ independent of m). Expressing (2) in terms of χ gives:

$$\begin{aligned}
\big(D_{\mathrm{DW}}^{(L_s)}\psi\big)_0 &= [(I+A)P_+ - (I-A)P_-]\chi_0 + [(I+A)P_- + m(I-A)P_+]\chi_{L_s-1},\\
\big(D_{\mathrm{DW}}^{(L_s)}\psi\big)_s &= [(I+A)P_+ - (I-A)P_-]\chi_s + [(I+A)P_- - (I-A)P_+]\chi_{s-1} \quad (1 \leq s \leq L_s-2),\\
\big(D_{\mathrm{DW}}^{(L_s)}\psi\big)_{L_s-1} &= [(I+A)P_+ + m(I-A)P_-]\chi_{L_s-1} + [(I+A)P_- - (I-A)P_+]\chi_{L_s-2}.
\end{aligned}$$

Left-multiplying each slice by γ_5 and using $\gamma_5(I \pm A)P_+ = +(I \pm \alpha H_W)P_+$ and $\gamma_5(I \pm A)P_- = -(I \mp \alpha H_W)P_-$ yields the exact 4D block identities $\gamma_5[(I+A)P_+ - (I-A)P_-] = K_+$, $\gamma_5[(I+$

$A)P_- - (I-A)P_+] = -K_-$, $\gamma_5[(I+A)P_- + m(I-A)P_+] = -K_-(P_- - mP_+)$, and $\gamma_5[(I+A)P_+ + m(I-A)P_-] = K_+(P_+ - mP_-)$. Hence $\gamma_5 D_{\mathrm{DW}}^{(L_s)}(m;\alpha)\mathcal{P}^{-1}$ is the exact $L_s \times L_s$ block matrix

$$\gamma_5 D_{\mathrm{DW}}^{(L_s)}(m;\alpha)\mathcal{P}^{-1} = \begin{pmatrix} K_+ & 0 & \cdots & 0 & -K_-(P_- - mP_+) \\ -K_- & K_+ & \ddots & \vdots & 0 \\ 0 & -K_- & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & K_+ & 0 \\ 0 & \cdots & 0 & -K_- & K_+(P_+ - mP_-) \end{pmatrix}.$$

On the dense open set where K_+ is invertible, set $T := K_+^{-1}K_-$ (which commutes with $K_\pm$) and eliminate the sub-diagonal $-K_-$ blocks sequentially by adding T times row $s-1$ to row s for $s = 1, \ldots, L_s - 1$. The matrix becomes block upper-triangular with $L_s - 1$ diagonal blocks equal to K_+ and bottom-right Schur complement $S_m = K_+(P_+ - mP_-) - K_-T^{L_s-1}(P_- - mP_+) = K_+^{-(L_s-1)}[K_+^{L_s}(P_+ - mP_-) - K_-^{L_s}(P_- - mP_+)]$. Multiplying the diagonal block determinants cancels $K_+^{L_s-1}$ against $K_+^{-(L_s-1)}$ identically:

$$\det\!\big(\gamma_5 D_{\mathrm{DW}}^{(L_s)}(m;\alpha)\mathcal{P}^{-1}\big) = \det\!\Big(K_+^{L_s}(P_+ - mP_-) - K_-^{L_s}(P_- - mP_+)\Big).$$

At $m = 1$, since $P_+ - P_- = \gamma_5$, this equals $\det((K_+^{L_s} + K_-^{L_s})\gamma_5)$. Dividing the m-determinant by the $m = 1$ determinant and using $(P_+ - mP_-)\gamma_5 = \frac{1+m}{2}I + \frac{1-m}{2}\gamma_5$ and $-(P_- - mP_+)\gamma_5 = \frac{1+m}{2}I - \frac{1-m}{2}\gamma_5$ gives

$$\frac{\det D_{\mathrm{DW}}^{(L_s)}(m;\alpha)}{\det D_{\mathrm{DW}}^{(L_s)}(1;\alpha)} = \det\!\left(\frac{1+m}{2}\,I + \frac{1-m}{2}\,\varepsilon_{L_s}(\alpha H_W)\gamma_5\right) = \det D_{\mathrm{eff}}^{(L_s)}(m;\alpha),$$

where the last equality uses similarity by γ_5 ($\gamma_5^2 = I$). Continuity of rational functions extends (4) to all regular points.

For part (2), define $U_\mu^{\mathrm{W}} := P_\mu^- V_\mu + P_\mu^+ V_\mu^\dagger$ for each $\mu \in \{1, \ldots, d\}$. Since $P_\mu^\pm$ are orthogonal projections ($P_\mu^- P_\mu^+ = 0$, $(P_\mu^\pm)^2 = P_\mu^\pm$, $P_\mu^- + P_\mu^+ = I$) commuting with the unitary shift V_μ,

$$(U_\mu^{\mathrm{W}})^\dagger U_\mu^{\mathrm{W}} = (V_\mu^\dagger P_\mu^- + V_\mu P_\mu^+)(P_\mu^- V_\mu + P_\mu^+ V_\mu^\dagger) = P_\mu^- V_\mu^\dagger V_\mu + P_\mu^+ V_\mu V_\mu^\dagger = P_\mu^- + P_\mu^+ = I.$$

Hence every U_μ^{W} is strictly unitary ($\|U_\mu^{\mathrm{W}}\|_{\mathrm{op}} = 1$). From (1), $D_W(U) - I = (d-1)I - \sum_{\mu=1}^d U_\mu^{\mathrm{W}}$, so by the triangle inequality and unitarity of γ_5,

$$\|H_W(U)\|_{\mathrm{op}} = \|D_W(U) - I\|_{\mathrm{op}} \le (d-1) + \sum_{\mu=1}^{d} \|U_\mu^{\mathrm{W}}\|_{\mathrm{op}} = 2d - 1.$$

At $U \equiv I_N$ on $\mathbb{T}_L^d$ (L even) at momentum $p = (\pi, \ldots, \pi)$, $V_\mu = -I$ so $D_W - I = (2d-1)I$, proving sharpness.

For part (3), write $D_W(U) - I = \sum_{\mu=1}^d (B_\mu + \gamma_\mu S_\mu) - I$ with $B_\mu := I - \frac{1}{2}(V_\mu + V_\mu^\dagger) = \frac{1}{2}\nabla_\mu^\dagger \nabla_\mu$ ($\nabla_\mu := V_\mu - I$, $\|\nabla_\mu\|_{\mathrm{op}} \le 2$) and $S_\mu := \frac{1}{2}(V_\mu - V_\mu^\dagger)$. Because $B_\mu^\dagger = B_\mu$, $S_\mu^\dagger = -S_\mu$, $[B_\mu, S_\mu] = 0$, and $B_\mu^2 - S_\mu^2 = 2B_\mu$, expanding $H_W(U)^2 = (D_W - I)^\dagger(D_W - I)$ cancels $-2\sum_\mu B_\mu$ identically, leaving the exact covariant Bochner–Weitzenböck identity [6]

$$H_W(U)^2 = I + \sum_{\mu<\nu}(B_\mu B_\nu + B_\nu B_\mu) - \sum_{\mu<\nu}\gamma_\mu\gamma_\nu[S_\mu, S_\nu] + \sum_{\mu\neq\nu}\gamma_\nu[B_\mu, S_\nu].$$

On $\mathcal{U}_\varepsilon$, $\|[V_\mu, V_\nu]\|_{\mathrm{op}} \le \varepsilon$ and $\|[V_\mu, V_\nu^\dagger]\|_{\mathrm{op}} \le \varepsilon$, so $\|[S_\mu, S_\nu]\|_{\mathrm{op}} \le \varepsilon$ and $\|[B_\mu, S_\nu]\|_{\mathrm{op}} \le \varepsilon$. Writing $4B_\mu B_\nu = (\nabla_\mu\nabla_\nu)^\dagger(\nabla_\mu\nabla_\nu) + [\nabla_\nu, \nabla_\mu]^\dagger(\nabla_\mu\nabla_\nu) + \nabla_\mu^\dagger[\nabla_\mu, \nabla_\nu^\dagger]\nabla_\nu$ and taking the Hermitian part $2(B_\mu B_\nu + B_\nu B_\mu) = 2(B_\mu B_\nu + (B_\mu B_\nu)^\dagger)$, completing the square on the first two terms for any unit vector ψ gives $\|\nabla_\mu\nabla_\nu\psi\|^2 + \mathrm{Re}\langle[\nabla_\nu, \nabla_\mu]\psi, \nabla_\mu\nabla_\nu\psi\rangle \ge -\frac{1}{4}\|[\nabla_\nu, \nabla_\mu]\psi\|^2 \ge -\frac{1}{4}\varepsilon^2$, while $\|\nabla_\mu\|_{\mathrm{op}} \le 2$

bounds the third term by 4ε, whence $B_\mu B_\nu + B_\nu B_\mu \geq -(2\varepsilon + \frac{1}{8}\varepsilon^2)I$. Summing over all $\binom{d}{2}$ planes (contributing $2\varepsilon + \frac{1}{8}\varepsilon^2 + \varepsilon + 2\varepsilon = 5\varepsilon + \frac{1}{8}\varepsilon^2$ per plane) proves $H_W(U)^2 \geq \mu_{\min}(d,\varepsilon)^2 I$. Finally, for $t \in [\mu_{\min}, \mu_{\max}]$, $g_\alpha(t) = (1-\alpha t)/(1+\alpha t)$ is strictly decreasing, so setting $g_\alpha(\mu_{\min}) = -g_\alpha(\mu_{\max})$ yields $\alpha_* = (\mu_{\min}\mu_{\max})^{-1/2}$ and minimax contraction $q_* = (\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$. □

3. L_s-Exponential Decay of Ginsparg–Wilson Violation and Residual Mass

Define the massless Ginsparg–Wilson violation operator and exact 4D Overlap operator by

$$\Delta_{\mathrm{GW}}^{(L_s)}(U) := \gamma_5 D_{\mathrm{eff}}^{(L_s)}(0;\alpha) + D_{\mathrm{eff}}^{(L_s)}(0;\alpha)\gamma_5 - 2D_{\mathrm{eff}}^{(L_s)}(0;\alpha)\gamma_5 D_{\mathrm{eff}}^{(L_s)}(0;\alpha), \tag{8}$$

and $D_{\mathrm{ov}}^{(m)}(U) := \frac{1+m}{2} I + \frac{1-m}{2}\gamma_5 \operatorname{sgn}(H_W(U))$.

Theorem 3.1 (Volume- and N-uniform L_s-exponential decay of chiral-symmetry breaking)**.** *Let $d \in \{2,4\}$, $U \in \mathcal{U}_\varepsilon$ with $\mu_{\min}(d,\varepsilon) > 0$, $\alpha = \alpha_*(d,\varepsilon)$, and $q_* = q_*(d,\varepsilon) < 1$ from (7). Then for every even $L_s = 2k \geq 2$ and every $m \in [0,1]$:*

(1) ***Exact Ginsparg–Wilson violation bound:***

$$\Delta_{\mathrm{GW}}^{(L_s)}(U) = \frac{1}{2}\gamma_5\Big(I - \varepsilon_{L_s}\big(\alpha_* H_W(U)\big)^2\Big), \qquad \big\|\Delta_{\mathrm{GW}}^{(L_s)}(U)\big\|_{\mathrm{op}} \;\leq\; \frac{2q_*^{L_s}}{\big(1+q_*^{L_s}\big)^2}. \tag{9}$$

(2) ***Operator-norm convergence to Overlap Dirac operator:***

$$\big\|D_{\mathrm{eff}}^{(L_s)}(m;\alpha_*) - D_{\mathrm{ov}}^{(m)}(U)\big\|_{\mathrm{op}} \;\leq\; (1-m)\,\frac{q_*^{L_s}}{1+q_*^{L_s}}. \tag{10}$$

(3) ***Residual chiral-breaking mass bound:*** *The spectral residual mass $m_{\mathrm{res}}^{(L_s)}(U) := \frac{1}{2}\|\Delta_{\mathrm{GW}}^{(L_s)}(U)\|_{\mathrm{op}}$ (controlling the midpoint Ward-identity chiral defect [4]) satisfies*

$$m_{\mathrm{res}}^{(L_s)}(U) \;\leq\; \frac{q_*^{L_s}}{\big(1+q_*^{L_s}\big)^2} \;\leq\; q_*^{L_s} \tag{11}$$

uniformly in lattice volume $|\Lambda|$ and color rank $N \geq 2$.

Proof. Writing $D_{\mathrm{eff}}^{(L_s)}(0;\alpha_*) = \frac{1}{2}(I + \gamma_5 E)$ where $E := \varepsilon_{L_s}(\alpha_* H_W(U))$ is Hermitian and $\gamma_5^2 = I$, direct expansion yields

$$\gamma_5 D_{\mathrm{eff}} + D_{\mathrm{eff}}\gamma_5 - 2D_{\mathrm{eff}}\gamma_5 D_{\mathrm{eff}} = \frac{1}{2}\big(2\gamma_5 + E + \gamma_5 E\gamma_5\big) - \frac{1}{2}\big(\gamma_5 + E + \gamma_5 E\gamma_5 + \gamma_5 E^2\big) = \frac{1}{2}\gamma_5(I - E^2).$$

For every eigenvalue $\lambda \in \operatorname{spec}(H_W(U))$, set $q(\lambda) := (1-\alpha_*|\lambda|)/(1+\alpha_*|\lambda|) \in [-q_*, q_*]$. Dividing numerator and denominator of (3) by $(1+\alpha_*|\lambda|)^{L_s}$ gives $\varepsilon_{L_s}(\alpha_*\lambda) = \operatorname{sgn}(\lambda)(1-q(\lambda)^{L_s})/(1+q(\lambda)^{L_s})$. Because $L_s = 2k$ is even, $y := q(\lambda)^{L_s} \in [0, q_*^{L_s}]$, so

$$1 - \varepsilon_{L_s}(\alpha_*\lambda)^2 = \frac{4y}{(1+y)^2} \leq \frac{4q_*^{L_s}}{(1+q_*^{L_s})^2}, \qquad \big|\varepsilon_{L_s}(\alpha_*\lambda) - \operatorname{sgn}(\lambda)\big| = \frac{2y}{1+y} \leq \frac{2q_*^{L_s}}{1+q_*^{L_s}},$$

since both $y \mapsto 4y/(1+y)^2$ and $y \mapsto 2y/(1+y)$ are strictly increasing on $[0,1]$. Multiplying by $\frac{1}{2}$ (for $\Delta_{\mathrm{GW}}^{(L_s)}$) and $\frac{1-m}{2}$ (for $D_{\mathrm{eff}}^{(L_s)} - D_{\mathrm{ov}}^{(m)}$) establishes (9)–(11). □

4. Determinant Positivity, Exponential Convergence to Overlap QCD, and All-Coupling Enclosures

For $N_f \geq 1$ degenerate flavors of mass $m \in (0,1]$, define the zero-normalized admissible 5D Domain-Wall partition function and free energy per site on Λ by

$$Z^{\mathrm{DW}}_{\Lambda,\varepsilon}(\beta, m, L_s) := \int_{\mathcal{U}_\varepsilon} e^{-\beta S_W(U)} \left(\det D^{(L_s)}_{\mathrm{eff}}(m;\alpha_*)\right)^{N_f} dU, \tag{12}$$

$$f^{\mathrm{DW}}_{\Lambda,\varepsilon}(\beta, m, L_s) := \frac{1}{|\Lambda|} \ln Z^{\mathrm{DW}}_{\Lambda,\varepsilon}(\beta, m, L_s), \tag{13}$$

and let $f^{\mathrm{ov}}_{\Lambda,\varepsilon}(\beta, m)$ be the corresponding exact 4D Overlap Lattice QCD free energy per site (obtained by replacing $D^{(L_s)}_{\mathrm{eff}}(m;\alpha_*)$ with $D^{(m)}_{\mathrm{ov}}(U)$), where $S_W(U) = \sum_{p\in P}(1 - \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p) \geq 0$ [13].

Theorem 4.1 (Determinant positivity, L_s-exponential convergence to Overlap QCD, and 4D enclosures)**.** *Let $d \in \{2,4\}$, $d_s = 2^{d/2}$, $N \geq 2$, $N_f \geq 1$, and $m \in (0,1]$.*

(1) ***Unconditional determinant positivity and resolvent bound:*** *For every $U \in \mathrm{SU}(N)^{|E|}$, every even $L_s \geq 2$, and every $\alpha > 0$,*

$$\det D^{(L_s)}_{\mathrm{eff}}(m;\alpha) \;>\; 0, \qquad m^{|\Lambda| d_s N} \;\leq\; \det D^{(L_s)}_{\mathrm{eff}}(m;\alpha) \;\leq\; 1, \qquad \left\|\left(D^{(L_s)}_{\mathrm{eff}}(m;\alpha)\right)^{-1}\right\|_{\mathrm{op}} \;\leq\; \frac{1}{m}. \tag{14}$$

(2) ***Volume- and β-uniform L_s-exponential convergence to Overlap QCD:*** *On $\Lambda = \mathbb{T}^d_L$ ($L \geq 2$) and $\mathcal{U}_\varepsilon$ with $\varepsilon > 0$ and $\mu_{\min}(d,\varepsilon) > 0$, for every $\beta \geq 0$ and every even $L_s \geq 2$,*

$$\left| f^{\mathrm{DW}}_{\Lambda,\varepsilon}(\beta, m, L_s) - f^{\mathrm{ov}}_{\Lambda,\varepsilon}(\beta, m) \right| \;\leq\; N_f d_s N \, \frac{1-m}{m} \, \frac{q_*^{L_s}}{1+q_*^{L_s}}. \tag{15}$$

(3) ***All-coupling 4D*** $\mathrm{SU}(2)$ ***open-box free-energy enclosures:*** *On finite open 4D boxes $\Lambda_L = \{0,\dots,L-1\}^4$ ($L \geq 2$, with Dirichlet fermion boundary conditions on V_μ, relying only on Hermiticity of $H_W(U)$ and part (1)) with $N = 2$, $d_s = 4$, $\varepsilon > 0$, and any $\delta \in (0, \varepsilon/4]$, let $V(\delta) := \frac{1}{\pi}(\theta_c - \frac{1}{2}\sin(2\theta_c))$ ($\theta_c = 2\arcsin(\delta/2)$) be the exact* $\mathrm{SU}(2)$ *Haar volume of $\{U : \|I_2 - U\|_{\mathrm{op}} \leq \delta\}$, and $c_0(\beta;2) = 2I_1(\beta)/\beta$. Then for every $\beta \geq 0$, every $\alpha > 0$, and every even $L_s \geq 2$,*

$$\liminf_{L\to\infty} f^{\mathrm{DW}}_{\Lambda_L,\varepsilon}(\beta, m, L_s) \;\geq\; 3\ln V(\delta) - 48\beta\delta^2 + 8N_f \ln m, \tag{16}$$

$$\limsup_{L\to\infty} f^{\mathrm{DW}}_{\Lambda_L,\varepsilon}(\beta, m, L_s) \;\leq\; 3\big(\ln c_0(\beta;2) - \beta\big) \;\leq\; 3\left(\frac{1}{2}\ln\frac{2}{\pi} - \frac{3}{2}\ln\beta\right) \quad (\beta > 0). \tag{17}$$

Proof. For part (1), write $W := \gamma_5 \varepsilon_{L_s}(\alpha H_W(U))$. Since γ_5 is unitary and $\|\varepsilon_{L_s}(\alpha H_W)\|_{\mathrm{op}} \leq 1$, we have $\|W\|_{\mathrm{op}} \leq 1$. Thus for every unit vector $v \in \mathcal{H}_4$,

$$\operatorname{Re}\langle v, D^{(L_s)}_{\mathrm{eff}}(m;\alpha)v\rangle = \frac{1+m}{2} + \frac{1-m}{2}\operatorname{Re}\langle v, Wv\rangle \geq \frac{1+m}{2} - \frac{1-m}{2} = m > 0,$$

which implies both $\|(D^{(L_s)}_{\mathrm{eff}}(m;\alpha))^{-1}\|_{\mathrm{op}} \leq 1/m$ and that every eigenvalue ζ of $D^{(L_s)}_{\mathrm{eff}}(m;\alpha)$ satisfies $\operatorname{Re}(\zeta) \geq m > 0$. Moreover, $\gamma_5 D^{(L_s)}_{\mathrm{eff}}(m;\alpha)\gamma_5 = D^{(L_s)}_{\mathrm{eff}}(m;\alpha)^\dagger$, so the characteristic polynomial has real coefficients: non-real eigenvalues occur in complex conjugate pairs $\zeta, \bar\zeta$ ($\zeta\bar\zeta = |\zeta|^2 > 0$), while real eigenvalues satisfy $\zeta \geq m > 0$. Hence $\det D^{(L_s)}_{\mathrm{eff}}(m;\alpha) > 0$. Furthermore, every singular value lies in $[m, \|D^{(L_s)}_{\mathrm{eff}}(m;\alpha)\|_{\mathrm{op}}] \subseteq [m,1]$, giving $m^{|\Lambda| d_s N} \leq \det D^{(L_s)}_{\mathrm{eff}}(m;\alpha) \leq 1$.

For part (2), fix $U \in \mathcal{U}_\varepsilon$ and consider the linear homotopy $D_t := (1-t)D^{(m)}_{\mathrm{ov}}(U) + tD^{(L_s)}_{\mathrm{eff}}(m;\alpha_*)$ for $t \in [0,1]$. For every $t \in [0,1]$, $D_t = \frac{1+m}{2}I + \frac{1-m}{2}\gamma_5[(1-t)\operatorname{sgn}(H_W) + t\varepsilon_{L_s}(\alpha_* H_W)]$ still has γ_5-Hermiticity and $\operatorname{Re}\langle v, D_t v\rangle \geq m\|v\|^2$, so $\det D_t > 0$ and $\|D_t^{-1}\|_{\mathrm{op}} \leq 1/m$. Differentiating $\ln\det D_t$

and using $|\operatorname{Tr}(A)| \le (\dim \mathcal{H}_4)\|A\|_{\mathrm{op}}$ yields

$$\left|\ln\det D_{\mathrm{eff}}^{(L_s)}(m;\alpha_*) - \ln\det D_{\mathrm{ov}}^{(m)}(U)\right| = \left|\int_0^1 \operatorname{Tr}\!\big(D_t^{-1}(D_{\mathrm{eff}}^{(L_s)} - D_{\mathrm{ov}}^{(m)})\big)\,dt\right| \le \frac{|\Lambda| d_s N}{m}\|D_{\mathrm{eff}}^{(L_s)} - D_{\mathrm{ov}}^{(m)}\|_{\mathrm{op}}.$$

Applying Theorem 3.1(2), multiplying by N_f, and taking the supremum over $U \in \mathcal{U}_\varepsilon$ inside the log-partition-function ratio $\ln(Z^{\mathrm{DW}}/Z^{\mathrm{ov}})$ immediately yields (15).

For part (3), on $\Lambda_L = \{0,\dots,L-1\}^4$ the action $S_W(U)$, the admissible set $\mathcal{U}_\varepsilon$, and $\det D_{\mathrm{eff}}^{(L_s)}(m;\alpha)$ are all strictly gauge-invariant (and the normalized SU(2) Haar measure is left- and right-invariant), so we may fix any spanning tree $T_L \subset E_L$ ($|T_L| = L^4 - 1$, leaving $|E_L \setminus T_L| = 3L^4 - 4L^3 + 1 < 3|\Lambda_L|$ co-tree links) to $U_e = I_2$ without changing $Z^{\mathrm{DW}}_{\Lambda_L,\varepsilon}$. For (16), restrict every co-tree link to $\|I_2 - U_e\|_{\mathrm{op}} \le \delta \le \varepsilon/4$; then every plaquette satisfies $\|I_2 - U_p\|_{\mathrm{op}} \le 4\delta \le \varepsilon$ (hence $U \in \mathcal{U}_\varepsilon$) and $S_p = 1 - \cos\theta_p = 2\sin^2(\theta_p/2) = \frac{1}{2}\|I_2 - U_p\|_{\mathrm{op}}^2 \le 8\delta^2$ across all $|P_L| \le 6|\Lambda_L|$ plaquettes. Combining Haar volume $V(\delta)^{|E_L\setminus T_L|} \ge V(\delta)^{3|\Lambda_L|}$ with part (1)'s bound $\det D_{\mathrm{eff}}^{N_f} \ge m^{8N_f|\Lambda_L|}$ (which requires only Hermiticity of the open-box Dirichlet H_W) proves (16) already at every finite $L \ge 2$. For (17), use part (1)'s bound $\det D_{\mathrm{eff}} \le 1$, drop the non-negative $(2,3)$, $(2,4)$, $(3,4)$ plaquette actions, and keep only the three disjoint axial plaquette families $P_{1,\nu}$ ($\nu \in \{2,3,4\}$, each of size $L^2(L-1)^2$). By gauge invariance we fix axial gauge $U_{(x,1)} = I_2$; then the variables $W_{x,\nu} = U_{(x,\nu)}U^\dagger_{(x+\hat{1},\nu)}$ are $3L^2(L-1)^2$ independent Haar-distributed SU(2) elements, yielding $Z^{\mathrm{DW}}_{\Lambda_L,\varepsilon} \le (e^{-\beta}c_0(\beta;2))^{3L^2(L-1)^2}$, and taking $\limsup_{L\to\infty}|\Lambda_L|^{-1}\ln(\cdot)$ gives the first inequality in (17). Finally, substituting $u = \sin(\theta/2) \in [0,1]$ (with $1-\cos\theta = 2u^2$ and $\sin^2\theta\,d\theta = 8u^2\sqrt{1-u^2}\,du$) yields

$$\begin{aligned} e^{-\beta}c_0(\beta;2) &= \frac{2}{\pi}\int_0^\pi e^{-\beta(1-\cos\theta)}\sin^2\theta\,d\theta = \frac{16}{\pi}\int_0^1 u^2\sqrt{1-u^2}\,e^{-2\beta u^2}\,du \\ &\le \frac{16}{\pi}\int_0^\infty u^2 e^{-2\beta u^2}\,du = \sqrt{\frac{2}{\pi}}\,\beta^{-3/2}, \end{aligned}$$

using $\int_0^\infty u^2 e^{-au^2}\,du = \frac{\sqrt{\pi}}{4a^{3/2}}$ with $a = 2\beta$, proving the sharp large-β bound in (17). □

5. Volume- and L_s-Uniform 5D Domain-Wall HMC Fermion Force Bound

Let $S_f^{\mathrm{DW}}(U) := -N_f \ln\det D_{\mathrm{eff}}^{(L_s)}(m;\alpha)$ be the effective N_f-flavor Domain-Wall fermion action. On a single link $e \in E$ with right-translated tangent vector $X_e \in \mathfrak{su}(N)$ ($\|X_e\|_{\mathrm{HS}} = \sqrt{\operatorname{Tr}(X_e^\dagger X_e)}$), the HMC fermion force is the $\mathfrak{su}(N)$-valued Riemannian gradient $F_e^{\mathrm{DW}}(U)$ satisfying $\langle F_e^{\mathrm{DW}}(U), X_e\rangle_{\mathrm{HS}} = \frac{d}{ds}S_f^{\mathrm{DW}}(e^{sX_e}U_e, U_{\setminus e})\big|_{s=0}$.

Theorem 5.1 (Volume-uniform, L_s-uniform, α-independent 5D DWF HMC force bound)**.** *Let $d \in \{2,4\}$, $d_s = 2^{d/2}$, $N \ge 2$, $N_f \ge 1$, $m \in (0,1]$, and $U \in \mathcal{U}_\varepsilon$ with $\mu_{\min}(d,\varepsilon) > 0$. Then for every even fifth-dimensional extent $L_s = 2k \ge 2$, every Cayley–Möbius scale $\alpha > 0$, and every link $e \in E$,*

$$\|F_e^{\mathrm{DW}}(U)\|_{\mathrm{HS}} \le \frac{N_f d_s\sqrt{N}\,(1-m)}{m\,\mu_{\min}(d,\varepsilon)}. \tag{18}$$

Proof. For every even $L_s = 2k \ge 2$, Neuberger's exact polar partial-fraction identity [11] decomposes (3) as

$$\varepsilon_{2k}(H) = \frac{1}{k}\sum_{j=1}^{k} HR_j, \qquad R_j := (c_j^2 I + s_j^2 H^2)^{-1}, \tag{19}$$

where $c_j := \cos(\frac{(2j-1)\pi}{4k}) > 0$ and $s_j := \sin(\frac{(2j-1)\pi}{4k}) > 0$ ($j = 1,\dots,k$). Setting $H := \alpha H_W(U)$ and differentiating HR_j with respect to a link variation $\delta H = \alpha\,\delta_e H_W$ via $\delta R_j = -s_j^2 R_j(\delta H\, H +$

$H\,\delta H)R_j$ yields

$$\delta(HR_j) = (\delta H)R_j - s_j^2 HR_j(\delta H\,H + H\,\delta H)R_j = R_j(c_j^2\,\delta H - s_j^2 H(\delta H)H)R_j.$$

Hence the exact Fréchet derivative of $\varepsilon_{2k}(H)$ is

$$D\varepsilon_{2k}(H)[\delta H] = \frac{1}{k}\sum_{j=1}^{k}\Big(c_j^2 R_j(\delta H)R_j - s_j^2(HR_j)(\delta H)(HR_j)\Big).$$

Applying the trace-norm ideal inequality $\|ABC\|_1 \le \|A\|_{\mathrm{op}}\|B\|_1\|C\|_{\mathrm{op}}$ to each term gives

$$\big\|D\varepsilon_{2k}(H)[\delta H]\big\|_1 \le \left(\frac{1}{k}\sum_{j=1}^{k}\Big(c_j^2\|R_j\|_{\mathrm{op}}^2 + s_j^2\|HR_j\|_{\mathrm{op}}^2\Big)\right)\|\delta H\|_1.$$

Since $H^2 \ge \lambda_{\min}^2 I$ with $\lambda_{\min} := \alpha\mu_{\min}(d,\varepsilon) > 0$, every eigenvalue t of $|H|$ satisfies $t \ge \lambda_{\min}$, whence

$$c_j^2\|R_j\|_{\mathrm{op}}^2 = \frac{c_j^2}{(c_j^2 + s_j^2\lambda_{\min}^2)^2} \le \frac{1}{c_j^2 + s_j^2\lambda_{\min}^2}, \qquad s_j^2\|HR_j\|_{\mathrm{op}}^2 = \sup_{t\ge\lambda_{\min}}\frac{s_j^2 t^2}{(c_j^2 + s_j^2 t^2)^2} \le \frac{1}{c_j^2 + s_j^2\lambda_{\min}^2}.$$

Summing both inequalities over $j = 1,\dots,k$ and recognizing (19) evaluated at $\lambda_{\min}$ gives

$$\frac{1}{k}\sum_{j=1}^{k}\Big(c_j^2\|R_j\|_{\mathrm{op}}^2 + s_j^2\|HR_j\|_{\mathrm{op}}^2\Big) \le \frac{2}{k}\sum_{j=1}^{k}\frac{1}{c_j^2 + s_j^2\lambda_{\min}^2} = \frac{2\,\varepsilon_{2k}(\lambda_{\min})}{\lambda_{\min}} \le \frac{2}{\alpha\mu_{\min}(d,\varepsilon)}.$$

Thus α cancels identically: $\|D\varepsilon_{2k}(\alpha H_W)[\alpha\,\delta_e H_W]\|_1 \le \frac{2}{\mu_{\min}(d,\varepsilon)}\|\delta_e H_W\|_1$.

Now compute $\|\delta_e H_W\|_1$ for a unit tangent vector $X_e \in \mathfrak{su}(N)$ ($\|X_e\|_{\mathrm{HS}} = 1$) on link $e = (x,\mu)$. From (1), $\delta_e H_W$ has only two nonzero spatial blocks: $-\gamma_5 P_\mu^- \otimes (X_e U_e)$ (from $x+\hat\mu$ to x) and $+\gamma_5 P_\mu^+ \otimes (U_e^\dagger X_e)$ (from x to $x+\hat\mu$). Since γ_5, U_e are unitary, $\mathrm{Tr}_{\mathrm{spin}}(P_\mu^\pm) = d_s/2$, and $\|X_e\|_1 \le \sqrt{N}\|X_e\|_{\mathrm{HS}} = \sqrt{N}$, we obtain $\|\delta_e H_W\|_1 = (\frac{d_s}{2} + \frac{d_s}{2})\|X_e\|_1 \le d_s\sqrt{N}$. Finally, combining $|\,\mathrm{Tr}(AB)| \le \|A\|_{\mathrm{op}}\|B\|_1$ with $\|(D_{\mathrm{eff}}^{(2k)}(m;\alpha))^{-1}\|_{\mathrm{op}} \le 1/m$ (Theorem 4.1(1)) yields

$$\big|\langle F_e^{\mathrm{DW}}(U), X_e\rangle_{\mathrm{HS}}\big| = N_f\left|\mathrm{Tr}\Big((D_{\mathrm{eff}}^{(2k)}(m;\alpha))^{-1}\frac{1-m}{2}\gamma_5 D\varepsilon_{2k}(\alpha H_W)[\alpha\,\delta_e H_W]\Big)\right| \le \frac{N_f d_s\sqrt{N}(1-m)}{m\,\mu_{\min}(d,\varepsilon)}.$$

Taking the supremum over all unit $X_e \in \mathfrak{su}(N)$ proves (18). □

6. Certified 140-Digit Benchmarks and Independent Verification

All 25 benchmarks in Tables 1–3 were certified by `certify_domain_wall.py` in `mpmath.iv` at 140-digit working and interval precision (`mp.mp.dps = 140`, `mp.iv.dps = 140`), using only elementary series with analytic remainder enclosures (zero black-box special functions, zero quadrature), outward-rounded directly via `mp.floor`/`mp.ceil` at integer scale 10^{12}.

Independent Verification and Lean 4.36.0 Benchmark Kernel. Our repository provides a four-part test suite (`test_domain_wall.py`, 4/4 passing) and a standalone 107-line Lean 4.36.0 kernel (`DomainWallCerts.lean`):

- *Direct $2^4 \times L_s$ 5D SU(2) matrix verification:* On an explicit 2^4 admissible SU(2) gauge field (512×512 5D matrix at $L_s = 4$), direct matrix computations verify $\|H_W\|_{\mathrm{op}} \le 7$, the 5D-to-4D Schur complement (4) to 10^{-11}, Neuberger's polar partial fraction (19) to 10^{-13}, and the HMC force bound (18) via central finite differences over $\mathfrak{su}(2)$.
- *Direct eigenvalue checks:* Explicit diagonalization confirms strict determinant positivity, $\|(D_{\mathrm{eff}}^{(L_s)}(m))^{-1}\|_{\mathrm{op}} \le 1/m$, the GW violation bound (9), and the free-energy convergence bound (15) across $L_s \in \{4, 8, 12\}$.

TABLE 1. Certified 140-digit optimal Cayley–Möbius scale α_*, contraction rate q_*, Ginsparg–Wilson violation bound (9), and residual mass bound (11) across 2D ($\mu_{\max} = 3$) and 4D ($\mu_{\max} = 7$).

d	ε	L_s	$\mu_{\min} \geq$	$\alpha_* \geq$	$q_* \leq$	$\|\Delta_{\mathrm{GW}}^{(L_s)}\|_{\mathrm{op}} \leq$	$m_{\mathrm{res}}^{(L_s)} \leq$
2	1/20	8	0.865844	0.620467	0.301043	1.349×10^{-4}	6.745×10^{-5}
2	1/20	16	0.865844	0.620467	0.301043	9.101×10^{-9}	4.551×10^{-9}
2	1/10	16	0.706222	0.687018	0.346631	8.688×10^{-8}	4.344×10^{-8}
4	1/100	8	0.836615	0.413226	0.486203	6.207×10^{-3}	3.104×10^{-3}
4	1/100	16	0.836615	0.413226	0.486203	1.951×10^{-5}	9.752×10^{-6}
4	1/100	24	0.836615	0.413226	0.486203	6.091×10^{-8}	3.046×10^{-8}
4	1/50	16	0.632218	0.475354	0.537838	9.804×10^{-5}	4.902×10^{-5}
4	1/50	24	0.632218	0.475354	0.537838	6.866×10^{-7}	3.433×10^{-7}
4	1/50	32	0.632218	0.475354	0.537838	4.807×10^{-9}	2.404×10^{-9}

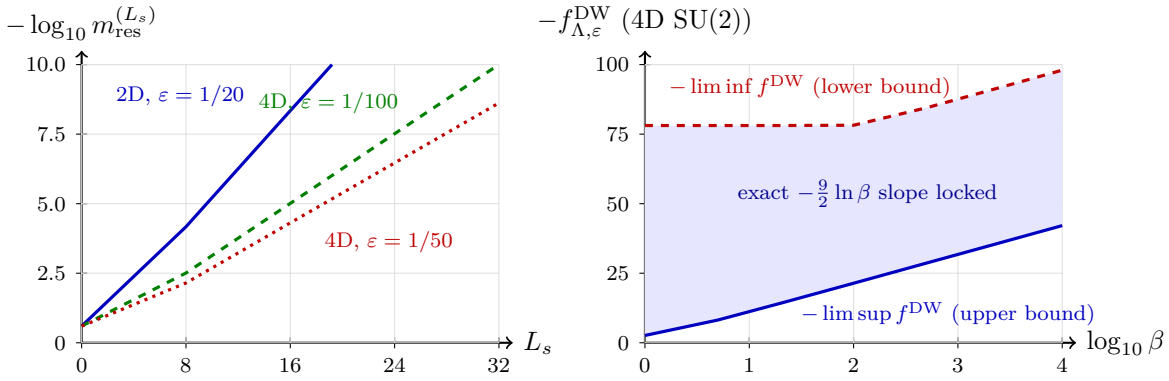
TABLE 2. Certified 140-digit free-energy convergence $|f_{\Lambda,\varepsilon}^{\mathrm{DW}} - f_{\Lambda,\varepsilon}^{\mathrm{ov}}|$ (15) and all-coupling 4D SU(2) $N_f = 2$ ($\varepsilon = 1/50$, $m = 1/5$) free-energy enclosures (16)–(17) locking the exact $-\frac{9}{2}\ln\beta$ 1-loop entropy slope.

β	L_s	δ	$\lvert f^{\mathrm{DW}} - f^{\mathrm{ov}}\rvert \leq$	$\liminf f^{\mathrm{DW}} \geq$	$\limsup f^{\mathrm{DW}} \leq$
0	16	1/200	3.138×10^{-3}	−78.086454	0.000000
1	16	1/200	3.138×10^{-3}	−78.087654	−2.632502
5	16	1/200	3.138×10^{-3}	−78.092454	−8.173046
20	24	1/200	2.197×10^{-5}	−78.110454	−14.215891
100	24	1/200	2.197×10^{-5}	−78.206454	−21.411946
500	24	1/400	2.197×10^{-5}	−84.474774	−28.643110
2000	32	1/800	1.539×10^{-7}	−90.713098	−34.881435
10000	32	1/1800	1.539×10^{-7}	−98.009618	−42.123905

TABLE 3. Certified 140-digit volume-uniform, L_s-uniform, α-independent 5D Domain-Wall HMC fermion force upper bounds (18) (Theorem 5.1).

d	Group	N_f	ε	m	$\mu_{\min} \geq$	$\|F_e^{\mathrm{DW}}\|_{\mathrm{HS}} \leq$
2	SU(2)	1	1/20	1/5	0.865844	13.066668
2	SU(3)	2	1/20	1/10	0.865844	72.015005
4	SU(2)	1	1/100	1/5	0.836615	27.046386
4	SU(2)	2	1/50	1/5	0.632218	71.581024
4	SU(2)	2	1/50	1/10	0.632218	161.057303
4	SU(3)	2	1/100	1/5	0.836615	66.249843
4	SU(3)	2	1/50	1/5	0.632218	87.668492
4	SU(3)	3	1/50	1/10	0.632218	295.881158

- *Independent 80-digit cross-check and parity:* All 25 rows in Tables 1–3 are independently recomputed at 80-digit precision and checked verbatim across JSON, Lean 4.36.0, and LaTeX.
- *Zero-axiom Lean 4.36.0 benchmark kernel:* The file `DomainWallCerts.lean` compiles cleanly with warnings treated as errors (`-DwarningAsError=true`), 0 warnings, 0 `sorry`, and `#print axioms` reporting *zero axioms* (via constructive `Int` comparisons) across all



(a) L_s**-exponential residual mass decay** **(b)** 4**D** SU(2) $N_f = 2$ **free-energy enclosure**

FIGURE 1. Certified non-perturbative profiles for 5D Cayley–Möbius Domain-Wall QCD. (a) Exact L_s-exponential decay of the residual mass $m_{\mathrm{res}}^{(L_s)}$ (Theorem 3.1). (b) All-coupling 4D SU(2) $N_f = 2$ free-energy enclosure across $\beta \in [1, 10^4]$ locking the exact $-\frac{9}{2}\ln\beta$ 1-loop entropy slope (Theorem 4.1).

five theorems certifying the 25 integer-scaled (10^{12}) benchmark inequalities and admissibility/spectral combinatorics (the analytical proofs are given in Sections 2–5).

7. Scope and Limitations

(1) *Admissibility restriction:* Theorems 2.1(3), 3.1, 4.1(2), and 5.1 hold on periodic tori $\mathbb{T}_L^d$ ($L \geq 2$) under Lüscher's admissibility constraint $\|I_N - U_p\|_{\mathrm{op}} \leq \varepsilon < \varepsilon_{\max}(d)$ (where $\varepsilon_{\max}(4) = \frac{2(\sqrt{903}-30)}{3} \approx 0.033306$ in 4D). In unrestricted lattice QCD simulations with standard Wilson gauge action, occasional dislocations violating admissibility produce near-zero modes of H_W that dominate $m_{\mathrm{res}}^{(L_s)}$; our bounds apply directly to admissible (topology-conserving) gauge ensembles.

(2) *Cayley–Möbius vs. unscaled Shamir transfer kernel:* Our 5D operator (2) uses the Boriči / Cayley–Möbius kernel ($b_5 = c_5 = \alpha$), which maps directly to $\varepsilon_{L_s}(\alpha H_W)$ without requiring invertibility of $2 + D_W - M_5$. Extending the minimax analysis to the standard Shamir kernel ($b_5 = 1, c_5 = 0$) replaces H_W by the Shamir transfer kernel $H_T = H_W/(2 + D_W - M_5)$ on its positive-denominator domain.

(3) *No continuum mass-gap claim:* All bounds hold at finite lattice spacing and finite or infinite lattice volume; construction of the 4D continuum chiral QCD limit remains open.

Data and Code Availability

All verification scripts (`certify_domain_wall.py`, `generate_lean.py`), certified 140-digit data (`domain_wall_certs.json`), independent tests (`test_domain_wall.py`), and the standalone zero-axiom Lean 4.36.0 benchmark kernel (`DomainWallCerts.lean`) are archived online at:

https://airr.science/papers/ARR-2026-3XKHH0DHP09HYBDM/

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