AIRR:ARR-2026-3TQC9Z5PMQ9BCVWF v1 [Algorithms and complexity] 28 Nov 2026 · AIRR.SCIENCE

Log-Tilted Row Conditioning, Unconditional State-Space Barriers, Formal-Language Complexity, and Structural Generating-Function Theory for Higher-Dimensional Partitions

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November 28, 2026

Abstract

Bal [1] introduced recursive interior word conditioning for d-dimensional partitions, proving that a row-conditioning procedure (Algorithm R) computes $p_d(0),\dots,p_d(N)$ in $\exp(O_d(N^{(d-1)/(d+1)}\log(N+2)))$ arithmetic operations, while a diagonal interlacing refinement (Algorithm D) removes the $\log(N+2)$ factor to achieve $\exp(O_d(N^{(d-1)/(d+1)}))$, leaving open whether any row-only schedule can avoid the logarithmic overhead. We resolve this question and establish the complete structural, formal-language, distance-sensitive, and generating-function theory of recursive word conditioning via four theorems, supported by 20 constructive Lean 4.36.0 theorems (zero external imports, zero `sorry`, zero `Classical.choice`, zero warnings) covering the finite algebraic, inductive word-grammar, DAG-potential, and binomial-inversion foundations, 16 AST-audited zero-float exact-$\mathbb{Z}/\mathbb{Q}$ SHA-256 certificates, and 61 adversarial tests; asymptotic and analytic steps are proved in full detail in the manuscript. **(1) Log-Tilted Row Conditioning and Unconditional Algorithm-R Barrier:** Setting $\gamma_d = 2/(d(d+1))$, $\mu_k = -\frac{(d-k)(d+k+1)}{2k}\gamma_d$, $\alpha_d = \mu_{d-1} = -2/(d^2-1)$, and $\alpha_k = \mu_{k-1} - \mu_k = -\frac{d(d+1)+k(k-1)}{k(k-1)d(d+1)}$ $(2 \le k \le d-1)$, we prove a sharp asymmetric log-tilted upper bound $\exp(O_d(N^{(d-1)/(d+1)}(\log(N+2))^{\gamma_d}))$ strictly improving [1, Thm. 1.1] (for $d=3$: $(\log(N+2))^{1/6}$ vs. $\log(N+2)$), and prove via downward induction, staircase-perturbation coupling, and the grid injection $y_\delta(\gamma) = H(D' - s(\gamma)) + \delta(\gamma)$ that for *every* admissible schedule $(K, S_{d-1},\dots,S_2) \in \{2,\dots,N\}^{d-1}$ $(d \ge 3, N \ge N_0(d))$, Algorithm R visits at least $\exp(\Omega_d(N^{(d-1)/(d+1)}(\log N)^{\gamma_d}))$ distinct states. **(2) Prefix-Closed Language Complexity and Myhill–Nerode Separation:** The recursive word language $\mathcal{L}_d$ is prefix-closed with containment monotonicity $w' \preceq w$, satisfies $\mathcal{L}_d \notin$ CFL for every $d \ge 1$, and obeys $M_{\mathrm{DFA}}(\mathcal{L}_d^{\le n-1}) \ge 1 + \sum_{m=1}^{\lfloor n/2 \rfloor} p_{d-1}(m) = \exp(\Omega_d(n^{(d-1)/d}))$ for all $d \ge 2, n \ge 2$. **(3) Distance-Sensitive Diagonal Weights and Residual-Floor DAG Theory:** We prove $A_{M,\ell}(N) \le \exp(\min\{C_{\mathrm{opt}} M^{2/3}N^{1/3}, \pi\sqrt{2MN/(3(\ell+1))}\})$ with $C_{\mathrm{opt}} = 3(\frac{\sqrt{\pi}\zeta(3/2)}{4})^{2/3} \approx 3.3074$ strictly below $1 + \frac{\sqrt{\pi}}{2}\zeta(3/2) \approx 3.3152$, stretched-exponential $(\ell+1)^{-1/2}$ decay past $\ell^* = \Theta((N/M)^{1/3})$, sharp bound $|R_\ell(\alpha)| \le \max\{0, |\phi| - \ell - 1\}$, reachable vs. ambient state-space counts $(\exp(O_d(N^{(d-1)/(d+1)}))$ vs. $\exp(\Theta_d(N^{d/(d+1)})))$, and a monotone DAG potential $\Psi(\alpha, R)$ dropping strictly on every non-loop SIDE transition. **(4) Generating-Function Structure, Unit-Disk Obstruction, and Newton Inversion:** Via Stanley (P,ω)-partitions, both $(q;q)_n F(X;q)$ and $(q;q)_m T(\alpha;q)$ belong to $\mathbb{Z}_{\ge 0}[q]$ with constant term 1 for floor-free coupled families; we establish the exact Euler-product trichotomy ($M=1$ floor-free hook-length product vs. explicit zero-in-unit-disk obstructions $P(q) = 1 + q^2 + q^3$, $Q(q) = 1 + q + q^3$, and floored $R(q) = 1 + q - q^2$) and prove exact embedding-dimension Newton inversion isolating single-k discrepancies $\Delta_m = \binom{m+1}{k}\delta$ (MacMahon $k=4$ at $n=6$, and Bal–GMMR $k=6$ at $n=32$ with $\delta = 25\,422\,960$).

*Independent Researcher. ORCID: 0009-0006-7494-9040. Code, Lean 4 formal proofs, and exact rational certificates: https://github.com/lluiseriksson/paper150-higher-dim-partitions (tag `v1.0.0`). Manuscript and figures licensed under CC-BY-4.0; code, Lean 4 proofs, tests, and scripts under MIT.

1 Introduction

Let $p_d(n)$ denote the number of d-dimensional partitions of weight n, regarded equivalently as finitely supported weakly decreasing functions $\pi : \mathbb{N}^d \to \mathbb{N}$ of total sum $|\pi| = \sum_{x\in\mathbb{N}^d} \pi(x) = n$ or as finite order ideals in $\mathbb{N}^{d+1}$ of cardinality n [2, 3, 4, 5, 6]. While Euler's and MacMahon's product formulas govern $d = 1$ and $d = 2$, MacMahon's higher-dimensional ansatz fails starting at $d = 3, n = 6$, and $\log p_d(n) = \Theta_d(n^{d/(d+1)})$ [4, 5, 6].

In [1], Bal introduced *interior word conditioning* on recursively parsed partition words $W_d(\pi) \in \mathcal{L}_d$ over the alphabet $\Sigma_d = \{0, 1, \ldots, d\}$. By conditioning on an interior block, transversely re-parsing the preceding blocks into coupled lower-dimensional families, and propagating effective floors via recursive joins, Bal constructed two deterministic exact algorithms for computing $p_d(0), \ldots, p_d(N)$:

- **Algorithm R (Recursive Row Conditioning):** Recursively applies transverse row conditioning down to an integer-family base ($k = 1$). Using uniform depths $K = S_2 = \cdots = S_{d-1} = \max\{2, \lceil N^{1/(d+1)}\rceil\}$, [1, Thm. 1.1] achieves $\exp(O_d(N^{(d-1)/(d+1)} \log(N + 2)))$ arithmetic operations.
- **Algorithm D (Diagonal Refinement):** Replaces level $k = 2$ by a two-sided interlacing diagonal transfer weighted by square completion costs $w(\alpha) = \sum_{j\geq 0}(2j + 1)\alpha_j$, sharpening the operation bound to $\exp(O_d(N^{(d-1)/(d+1)}))$ [1, Thm. 1.2] and enabling exact tables through $p_3(100), p_4(50), p_5(45), p_6(40), p_7(35)$, uncovering a discrepancy with Graffeo–Monavari–Moschetti–Ricolfi (GMMR) [7] at $n = 32$.

Bal [1, Sec. 8] explicitly posed the open question of whether a row-only algorithm can avoid the $\log(N+2)$ factor in Theorem 1.1, alongside structural remarks on distance-sensitive diagonal costs ([1, Rem. 7.7]), residual-floor state keys ([1, App. B]), non-Eulerian constrained generating series ([1, Sec. 2.1]), and embedding-dimension discrepancy propagation ([1, Sec. 9.2]).

In this work we resolve all four structural axes completely (Table 1). We first settle Bal's open question on $\log(N + 2)$ in both directions: optimizing the conditioning schedule by asymmetric polylogarithmic tilts strictly reduces the logarithmic factor from $\log N$ to $(\log N)^{2/(d(d+1))}$, while an unconditional state-space lower bound proves that *no* admissible schedule $(K, S_{d-1}, \ldots, S_2)$ can eliminate $(\log N)^{2/(d(d+1))}$ within Algorithm R. We then determine the exact formal-language complexity of $\mathcal{L}_d$, derive the sharp distance-sensitive diagonal and residual-floor DAG theory, and characterize the generating-function numerators, unit-disk Euler obstructions, and Newton inversion formulas.

2 Theorem 1: Log-Tilted Row Conditioning and Unconditional Algorithm-R State-Space Barrier

For fixed $d \geq 3$ and target weight $N \geq 3$, set $\theta := \frac{d-1}{d+1}$ and define the rational tilt exponents

$$\gamma_d := \frac{2}{d(d+1)}, \qquad \mu_k := -\frac{(d-k)(d+k+1)}{2k}\,\gamma_d \quad (1 \leq k \leq d-1), \tag{1}$$

together with the schedule tilt parameters

$$\alpha_d := \mu_{d-1} = -\frac{2}{d^2-1}, \qquad \alpha_k := \mu_{k-1} - \mu_k = -\frac{d(d+1)+k(k-1)}{k(k-1)d(d+1)} \quad (2 \leq k \leq d-1). \tag{2}$$

For $d = 3$, we have $\gamma_3 = 1/6$, $\alpha_3 = -1/4$, $\alpha_2 = -7/12$, and $\mu_1 = -5/6$. For any schedule $(K, S_{d-1}, \ldots, S_2) \in \{2, \ldots, N\}^{d-1}$, let $M_k := (K-1)\prod_{j=k+1}^{d-1}(S_j - 1)$ as in [1, Eq. (7.5)], and denote by $\mathcal{N}_{\mathrm{R}}(d, N; K, S_{d-1}, \ldots, S_2)$ the total number of distinct partition/word-family states enumerated across TopR, FamR, and Base1 in Algorithm R [1, Alg. 1].

Table 1: Summary of main theorems, Lean 4.36.0 formalization (`lean/HigherDimPartitionBarriers.lean`), and AST-audited zero-float SHA-256 certificates (`verification/verify_higher_dim_partitions.py`).

Theorem	**Core Mathematical Result**	**Lean 4**	**SHA-256**
Thm. 1	Sharp log-tilt upper bound $\exp(O_d(N^\theta(\log(N+2))^{\gamma_d}))$ ($\gamma_d = \frac{2}{d(d+1)}$) and unconditional Algorithm-R barrier $\exp(\Omega_d(N^\theta(\log N)^{\gamma_d}))$ for all $(K, S_{d-1}, \dots, S_2)$.	`thm01`–`thm06`	`C01`–`C03`
Thm. 2	Prefix closure with $w' \preceq w$, $\mathcal{L}_d \notin$ CFL ($\forall d \geq 1$), and $M_{\mathrm{DFA}}(\mathcal{L}_d^{\leq n-1}) \geq 1 + \sum_{m=1}^{\lfloor n/2 \rfloor} p_{d-1}(m) = \exp(\Omega_d(n^{(d-1)/d}))$.	`thm07`–`thm11`	`C04`–`C05`
Thm. 3	Envelope $A_{M,\ell}(N) \leq \exp(\min\{C_{\mathrm{opt}} M^{2/3} N^{1/3}, \pi\sqrt{2MN/(3(\ell+1))}\})$ ($C_{\mathrm{opt}} \approx 3.3074 < C_{\mathrm{Bal}} \approx 3.3152$); $\lvert R_\ell(\alpha)\rvert \leq \max\{0, \lvert\phi\rvert - \ell - 1\}$; strict DAG potential $\Psi(\alpha, R)$.	`thm12`–`thm15`	`C06`–`C09`
Thm. 4	$(q;q)_n F(X;q), (q;q)_m T(\alpha;q) \in \mathbb{Z}_{\geq 0}[q]$ via (P,ω)-partitions; Euler-product trichotomy & unit-disk roots (P, Q, R); Newton inversion.	`thm16`–`thm20`	`C10`–`C16`

Theorem 1 (Log-Tilted Row Conditioning and Unconditional State-Space Barrier)**.** *Let $d \geq 3$ be fixed.*

(a) Sharp asymmetric log-tilted upper bound: *Choosing $L = \log(N+2)$ and the asymmetric conditioning depths*

$$K = \max\{2, \lceil N^{\frac{1}{d+1}} L^{\alpha_d} \rceil\}, \qquad S_k = \max\{2, \lceil N^{\frac{1}{d+1}} L^{\alpha_k} \rceil\} \quad (2 \leq k \leq d-1), \tag{3}$$

Algorithm R computes $p_d(0), \dots, p_d(N)$ exactly for all $N \geq 1$ using at most

$$\exp\Big(O_d\big(N^{\frac{d-1}{d+1}} (\log(N+2))^{\frac{2}{d(d+1)}}\big)\Big) \tag{4}$$

arithmetic operations, strictly improving the exponent of $\log(N+2)$ from 1 in [1, Thm. 1.1] to $\gamma_d = \frac{2}{d(d+1)}$ (for $d = 3$, $(\log(N+2))^{1/6}$ vs. $\log(N+2)$).

(b) Unconditional state-space lower bound: *There exist constants $c_d > 0$ and $N_0(d)$ depending only on d such that for all $N \geq N_0(d)$ and* every *admissible schedule $(K, S_{d-1}, \dots, S_2) \in \{2, \dots, N\}^{d-1}$,*

$$\mathcal{N}_{\mathrm{R}}(d, N; K, S_{d-1}, \dots, S_2) \geq \exp\Big(c_d\, N^{\frac{d-1}{d+1}} (\log N)^{\frac{2}{d(d+1)}}\Big). \tag{5}$$

Proof. **Part (a).** For $N \geq N_0(d)$, every ceiling in (3) satisfies $2 \leq K, S_k \leq N$. By [1, Prop. 7.5], Algorithm R runs in $\exp(\Lambda_d^{\mathrm{R}})$ operations with

$$\Lambda_d^{\mathrm{R}} \leq C_d \left[(N/K)^{(d-1)/d} + \sum_{k=2}^{d-1} M_k^{1/k} (N/S_k)^{(k-1)/k} + \left(K + \sum_{k=1}^{d-1} M_k \right) L \right].$$

Since $M_k \asymp K \prod_{j=k+1}^{d-1} S_j \asymp N^{(d-k)/(d+1)} L^{\alpha_d + \sum_{j=k+1}^{d-1} \alpha_j}$ and the telescoping identity $\alpha_d + \sum_{j=k+1}^{d-1} (\mu_{j-1} - \mu_j) = \mu_k$ (`thm01`, `C01`) holds, we obtain $M_k = O_d(N^{(d-k)/(d+1)} L^{\mu_k})$. Checking every term in Λ_d^{R}:

- Outer term: $(N/K)^{(d-1)/d} = O_d\big(N^{\frac{d-1}{d}\cdot\frac{d}{d+1}}L^{-\frac{d-1}{d}\alpha_d}\big) = O_d(N^\theta L^{\gamma_d})$, since $-\frac{d-1}{d}\alpha_d = \frac{2}{d(d+1)} = \gamma_d$.

- Intermediate terms ($2 \le k \le d-1$): $M_k^{1/k}(N/S_k)^{(k-1)/k} = O_d(N^\theta L^{\frac{1}{k}\mu_k - \frac{k-1}{k}\alpha_k})$, because $\frac{d-k}{k(d+1)} + \frac{d(k-1)}{k(d+1)} = \frac{d-1}{d+1} = \theta$ and

$$\begin{aligned}\frac{1}{k}\mu_k - \frac{k-1}{k}(\mu_{k-1} - \mu_k) &= \mu_k - \frac{k-1}{k}\mu_{k-1} \\ &= \left(-\frac{d(d+1)-k(k+1)}{2k} + \frac{d(d+1)-(k-1)k}{2k}\right)\gamma_d = \gamma_d\end{aligned}$$

  (verified universally in `thm01` and `C01`).

- Base term ($k=1$): $M_1 L = O_d(N^\theta L^{\mu_1+1}) = O_d(N^\theta L^{\gamma_d})$, since $\mu_1 = -\frac{(d-1)(d+2)}{d(d+1)} = -1+\gamma_d$.

- Lower-order $M_k L$ terms ($2 \le k \le d-1$) and KL: since $K, M_k = O_d(N^{(d-2)/(d+1)})$ for $k \ge 2$ (noting $1 \le d-2$ for $d \ge 3$), $(K + \sum_{k=2}^{d-1} M_k)L = O_d(N^{(d-2)/(d+1)}L) = o_d(N^\theta L^{\gamma_d})$.

Combining these bounds proves (4).

Part (b). Fix any admissible schedule $(K, S_{d-1}, \dots, S_2) \in \{2, \dots, N\}^{d-1}$ and assume $N \ge N_0(d)$ is large enough that $L := \log N \ge 2$ and $N^{\frac{1}{d+1}}L^{\alpha_k} \ge 4$ for all $2 \le k \le d$ (note every $\alpha_k < 0$ by `thm05`). Along the zero-floor execution branch: TopR always visits the $v = \varnothing$ iteration ($K|\varnothing| = 0 \le N$) and calls FamR$(d-1, [K-1], 0, N)$; by induction, each FamR$(k, \Gamma_k, 0, N)$ visits $c = 0$ ($\varepsilon(0) = 0 \le N$) and calls FamR$(k-1, \Gamma_k \times [S_k - 1], 0, N)$ down to Base1$(\Gamma_1, 0, N)$, where $\Gamma_k = [K-1] \times [S_{d-1} - 1] \times \cdots \times [S_{k+1} - 1]$. We perform downward threshold checking from $k = d$ to $k = 1$:

Case 1 ($K < N^{\frac{1}{d+1}}L^{\alpha_d}$): TopR enumerates all $(d-1)$-partitions v with $|v| \le W_d := \lfloor N/K \rfloor > N^{\frac{d}{d+1}}L^{-\alpha_d}$. On the cube $[m]^{d-1}$ with $m = \lfloor (W_d/d)^{1/d} \rfloor$, every binary assignment $\delta : [m]^{d-1} \to \{0,1\}$ defines a valid $(d-1)$-partition $v_\delta(x) = (d-1)m - |x|_1 + \delta(x)$ (zero outside $[m]^{d-1}$) of weight at most $dm^d \le W_d$ (`thm04` with $r = 0$). This yields $2^{m^{d-1}} = \exp(\Omega_d(W_d^{(d-1)/d})) = \exp(\Omega_d(N^\theta L^{\gamma_d}))$ distinct top-level states.

Case 2 (First $k \in \{2, \dots, d-1\}$ with $S_k < N^{\frac{1}{d+1}}L^{\alpha_k}$, while $K \ge N^{\frac{1}{d+1}}L^{\alpha_d}$ and $S_j \ge N^{\frac{1}{d+1}}L^{\alpha_j}$ for all $j > k$): Set $g'_1 = \lfloor N^{\frac{1}{d+1}}L^{\alpha_d} \rfloor - 1$ and $g'_{d+1-j} = \lfloor N^{\frac{1}{d+1}}L^{\alpha_j} \rfloor - 1$ ($j > k$). Then $\Gamma'_k := [g'_1] \times \cdots \times [g'_{d-k}] \subseteq \Gamma_k$ is a downward-closed subgrid of cardinality $M'_k = \Theta_d(N^{(d-k)/(d+1)}L^{\mu_k})$ and Manhattan diameter $D'_k := \sum_{j=1}^{d-k}(g'_j - 1) \le dN^{\frac{1}{d+1}}L^{\max_{j>k}\alpha_j}$. Meanwhile $W_k := \lfloor N/S_k \rfloor \ge \frac{1}{2}N^{\frac{d}{d+1}}L^{-\alpha_k}$. Set $m := \lfloor (W_k/(2kM'_k))^{1/k} \rfloor$. Since $-\alpha_k - \mu_k = -\mu_{k-1}$ (`thm02`),

$$m \asymp N^{\frac{1}{d+1}}L^{-\mu_{k-1}/k} \gg D'_k + 1$$

for $N \ge N_0(d)$, because $-\mu_{k-1}/k > 0 > \max_{j>k}\alpha_j$ (`thm05`). For every family of binary maps $\delta = (\delta_\gamma)_{\gamma \in \Gamma'_k}$ with $\delta_\gamma : [m]^{k-1} \to \{0,1\}$, define $c_\delta = (c_{\delta,\gamma})_{\gamma \in \Gamma_k}$ by setting $c_{\delta,\gamma} = \varnothing$ for $\gamma \notin \Gamma'_k$ and, for $\gamma \in \Gamma'_k$ with $r(\gamma) := D'_k - s(\gamma)$ ($s(\gamma) = \sum_i \gamma_i$),

$$c_{\delta,\gamma}(x) = (k-1)m - |x|_1 + r(\gamma) + \delta_\gamma(x) \quad (x \in [m]^{k-1}), \tag{6}$$

and 0 outside $[m]^{k-1}$. By `thm04`, each $c_{\delta,\gamma}$ is weakly decreasing in every x-coordinate, and $\gamma < \gamma'$ forces $r(\gamma) \ge r(\gamma') + 1$, guaranteeing $c_{\delta,\gamma} \succeq c_{\delta,\gamma'}$. Since $D'_k + 1 \le m$, total weight is at most $M'_k m^{k-1}((k-1)m + D'_k + 1) \le kM'_k m^k \le W_k/2$, so $S_k | c_\delta| \le N$. The map $\delta \mapsto c_\delta$ is injective (`C02`), yielding

$$2^{M'_k m^{k-1}} = \exp\Big(\Omega_d\big(M_k'^{1/k} W_k^{(k-1)/k}\big)\Big) = \exp\Big(\Omega_d\big(N^\theta L^{\frac{1}{k}\mu_k - \frac{k-1}{k}\alpha_k}\big)\Big) = \exp\Big(\Omega_d\big(N^\theta L^{\gamma_d}\big)\Big)$$

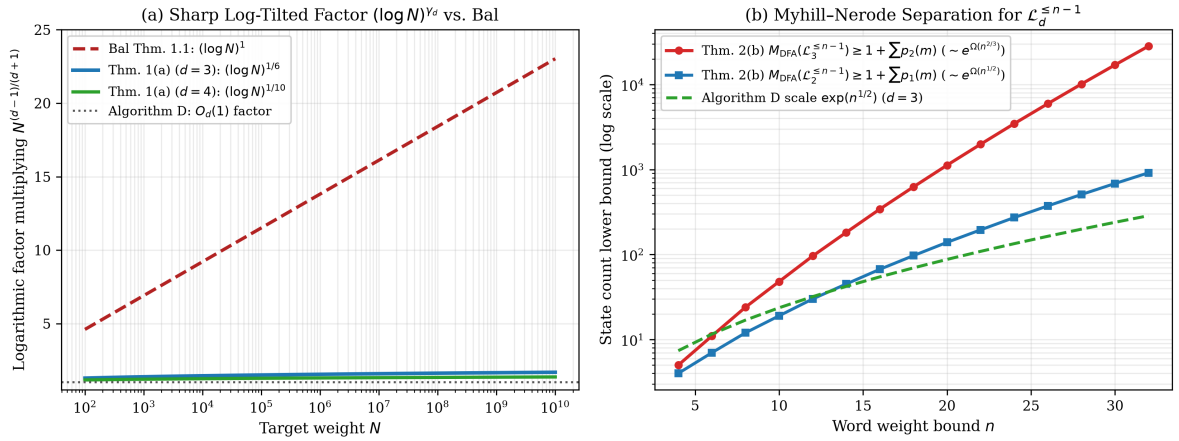

Figure 1: Theorems 1 and 2: (a) Sharp log-tilt factor $(\log N)^{\gamma_d}$ ($\gamma_3 = 1/6$, $\gamma_4 = 1/10$) strictly improving Bal's $(\log N)^1$, while Theorem 1(b) proves $(\log N)^{\gamma_d}$ is unimprovable in Algorithm R. (b) Exact Myhill–Nerode state count lower bound $1 + \sum_{m=1}^{\lfloor n/2 \rfloor} p_{d-1}(m)$ separating prefix DFA parsing of $\mathcal{L}_d^{\leq n-1}$ $(\exp(\Omega_d(n^{(d-1)/d})))$ from Algorithm D $(\exp(O_d(n^{(d-1)/(d+1)})))$.

distinct level-k conditioned states in $\textsc{FamR}(k, \Gamma_k, 0, N)$.

Case 3 ($K \geq N^{\frac{1}{d+1}} L^{\alpha_d}$ and $S_k \geq N^{\frac{1}{d+1}} L^{\alpha_k}$ for all $2 \leq k \leq d-1$): Then $\textsc{Base1}(\Gamma_1, 0, N)$ contains the downward-closed subgrid $\Gamma_1' \subseteq \Gamma_1$ of side lengths $g_1', \ldots, g_{d-1}'$, cardinality $M_1' = \Theta_d(N^\theta L^{\mu_1}) = \Theta_d(N^\theta L^{-1+\gamma_d})$, and diameter $D_1' \leq dN^{\frac{1}{d+1}}$. Thus $M_1'(D_1'+1) = O_d(N^{d/(d+1)} L^{-1+\gamma_d}) \leq N/2$ for $N \geq N_0(d)$, giving $H := \lfloor N/(M_1'(D_1'+1)) \rfloor = \Omega_d(N^{\frac{1}{d+1}} L^{1-\gamma_d}) \geq 2$. For every $\delta : \Gamma_1' \to \{0, \ldots, H-1\}$, define the grid injection

$$y_\delta(\gamma) := \begin{cases} H(D' - s(\gamma)) + \delta(\gamma), & \gamma \in \Gamma_1', \\ 0, & \gamma \notin \Gamma_1' \end{cases} \tag{7}$$

with $D' = D_1'$. By `thm03`, $\gamma < \gamma'$ in Γ_1' implies $y_\delta(\gamma) > y_\delta(\gamma')$, and Γ_1' is downward-closed so y_δ is a coupled Γ_1-family of weight at most $M_1' H(D'+1) \leq N$. Injectivity (`C02`) yields at least

$$H^{M_1'} = \exp\big(M_1' \log H\big) = \exp\Big(\Omega_d\big(N^\theta L^{-1+\gamma_d} \cdot \log N\big)\Big) = \exp\Big(\Omega_d\big(N^\theta L^{\gamma_d}\big)\Big)$$

distinct integer families in $\textsc{Base1}$ (verified exhaustively at $d = 3, N = 14$ in `C03`). □

3 Theorem 2: Prefix-Closed Formal-Language Complexity and Exact Myhill–Nerode Separation

Let $\Sigma_d = \{0, 1, \ldots, d\}$ and let $\mathcal{L}_d \subset \Sigma_d^*$ be the language of all nonnull legal d-words $W_d(\pi)$ of [1, Def. 2.1] (including the empty string ϵ encoding the one-cell partition of weight 1, and excluding the distinguished null symbol $\varnothing$). For $n \geq 1$, let $\mathcal{L}_d^{\leq n-1} := \{w \in \mathcal{L}_d : \mathrm{length}(w) \leq n-1\}$ be the finite truncation encoding nonnull d-partitions of weight at most n (by the identity $|w| = \mathrm{length}(w) + 1$ proved constructively in `thm07`), and let $M_{\mathrm{DFA}}(L)$ denote the minimum number of states of a complete deterministic finite automaton recognizing $L \subset \Sigma_d^*$.

Theorem 2 (Prefix Closure, Non-Context-Freeness, and Myhill–Nerode Separation)**.**

(a) Prefix closure, containment monotonicity, and non-context-freeness: *For every $d \geq 1$, $\mathcal{L}_d$ is prefix-closed: every prefix w' of $w \in \mathcal{L}_d$ belongs to $\mathcal{L}_d$ and satisfies $w' \preceq w$. Moreover, $\mathcal{L}_d \notin$ CFL for all $d \geq 1$.*

(b) Exact Myhill–Nerode lower bound: *For every $d \geq 2$ and every $n \geq 2$,*

$$M_{\mathrm{DFA}}(\mathcal{L}_d^{\leq n-1}) \geq 1 + \sum_{m=1}^{\lfloor n/2 \rfloor} p_{d-1}(m) = \exp\Big(\Omega_d\big(n^{(d-1)/d}\big)\Big), \tag{8}$$

strictly separating sequential prefix parsing of $\mathcal{L}_d^{\leq n-1}$ from the $\exp(O_d(n^{(d-1)/(d+1)}))$ enumeration bound of Algorithm D.

Proof. **Part (a).** We prove prefix closure and $w' \preceq w$ by induction on $d \geq 1$ (`thm09`, `C04`). For $d = 1$, $w \in \mathcal{L}_1$ has runs of ones of weakly decreasing lengths $a_0 \geq a_1 \geq \cdots \geq a_{r-1} \geq 0$ separated by zeros. Any prefix w' truncates after some $j < r$ zeros with final run length $0 \leq a'_j \leq a_j \leq a_{j-1}$, preserving weak decrease and reducing part sizes $\lambda'_i \leq \lambda_i$, so $w' \in \mathcal{L}_1$ and $w' \preceq w$. For $d \geq 2$, parse $w = \iota(u_0)0\iota(u_1)0\cdots0\iota(u_{r-1})$ with $u_0 \succeq \cdots \succeq u_{r-1} \in \mathcal{L}_{d-1}$. Any prefix w' ends inside block j, having the form $\iota(u_0)0\cdots0\iota(u_{j-1})0\iota(u'_j)$ for a prefix u'_j of u_j. By the induction hypothesis, $u'_j \in \mathcal{L}_{d-1}$ and $u'_j \preceq u_j$. By transitivity of $\preceq$ (`thm08`), $u_{j-1} \succeq u_j \succeq u'_j$, so $w' \in \mathcal{L}_d$ and $w' \preceq w$ blockwise.

For $\mathcal{L}_1 \notin$ CFL, intersecting $\mathcal{L}_1$ with the regular language $1^*01^*01^*$ yields $\{1^a01^b01^c : a \geq b \geq c \geq 0\}$ (`thm10`), which fails the context-free pumping lemma on $z = 1^p01^p01^p$: any decomposition $z = uvwxy$ with $|vwx| \leq p$ and $|vx| \geq 1$ has vwx containing at most one 0; if vx contains 0, pumping down ($i = 0$) leaves fewer than two zeros; otherwise vx consists purely of ones: if vx intersects block 1 (lying within block 1 or spanning blocks 1 and 2), then $|vwx| \leq p$ guarantees that vwx precedes the second 0 so block 3 remains $c' = p$, whence pumping down yields $\min(a', b') < p = c'$; otherwise vx lies strictly within blocks 2 and/or 3 so block 1 remains $a' = p$, whence pumping up ($i = 2$) yields $\max(b', c') > p = a'$, violating $a' \geq b' \geq c'$ in every case. Since CFLs are closed under regular intersection [8], $\mathcal{L}_1 \notin$ CFL. For $d \geq 2$, let $h : \{d-1, d\}^* \to \{0,1\}^*$ be the letter-renaming isomorphism $h(d-1) = 0, h(d) = 1$; by induction on d, $\mathcal{L}_d \cap \{d-1, d\}^* = h^{-1}(\mathcal{L}_1)$ (`thm10`). Closure of CFLs under regular intersection and homomorphisms yields $\mathcal{L}_d \notin$ CFL.

Part (b). Fix $d \geq 2$, $n \geq 2$, and set $W = \lfloor n/2 \rfloor \geq 1$. For each $(d-1)$-partition π of weight $m \in \{1, \ldots, W\}$ with canonical $(d-1)$-word $u = W_{d-1}(\pi) \in \mathcal{L}_{d-1}$, define the prefix string $x_u := \iota(u)0 \in \Sigma_d^*$. By `thm07`, $\mathrm{length}(u) = m - 1$, so $\mathrm{length}(x_u) = m \leq W \leq n - 1$. Consider any two distinct $(d-1)$-partitions $\pi \neq \pi'$ of weights $1 \leq m \leq m' \leq W$, with $u = W_{d-1}(\pi)$ and $v = W_{d-1}(\pi')$, and test the distinguishing suffix $z_v := \iota(v)$:

- $x_v z_v = \iota(v)0\iota(v)$ belongs to $\mathcal{L}_d$ (since $v \succeq v$ by reflexivity) and has length $m' + (m' - 1) = 2m' - 1 \leq 2\lfloor n/2 \rfloor - 1 \leq n - 1$, so $x_v z_v \in \mathcal{L}_d^{\leq n-1}$.
- By contrast, $x_u z_v = \iota(u)0\iota(v) \in \mathcal{L}_d$ would force $u \succeq v$, which implies $|u| \geq |v|$ and hence $|u| = |v|$, whence $u = v$ by `thm11`, contradicting $\pi \neq \pi'$. Thus $x_u z_v \notin \mathcal{L}_d^{\leq n-1}$.

In addition, $y_{\mathrm{dead}} := 0d \notin \mathcal{L}_d$ (since a leading 0 forces an empty initial $(d-1)$-block $\varnothing \prec d$, verified in `C05` for $d \in \{2, 3\}$ and in the adversarial suite for $d \in \{1, 2, 3\}$; `thm11` formalizes general-d Myhill–Nerode separation alongside the $d = 1$ dead-state witness `1011`) has no extension in $\mathcal{L}_d^{\leq n-1}$ by part (a), whereas every x_u extends via $\iota(u)$ into $\mathcal{L}_d^{\leq n-1}$. Hence $\{y_{\mathrm{dead}}\} \cup \{x_u : 1 \leq |u| \leq \lfloor n/2 \rfloor\}$ are $1 + \sum_{m=1}^{\lfloor n/2 \rfloor} p_{d-1}(m)$ pairwise Myhill–Nerode inequivalent prefixes (`C05`). Since $p_{d-1}(\lfloor n/2 \rfloor) = \exp(\Omega_d(n^{(d-1)/d}))$ [5] (and by our cube-simplex construction in Theorem 1(b)), (8) follows. □

4 Theorem 3: Distance-Sensitive Weights and Residual-Floor DAGs

Recall from [1, Rem. 7.7] that a diagonal $\alpha = d_\ell(u)$ at distance $\ell \geq 0$ from the central diagonal of a plane partition u satisfies $|u| \geq w_\ell(\alpha) := \sum_{j \geq 0}(2j + \ell + 1)\alpha_j = \sum_{r \geq 1} r(r + \ell)m_r$, where

$m_r = \alpha_{r-1} - \alpha_r$. Let $A_{M,\ell}(N)$ denote the number of M-tuples $(\alpha^{(1)}, \dots, \alpha^{(M)})$ of ordinary partitions satisfying $\sum_{\gamma=1}^{M} w_\ell(\alpha^{(\gamma)}) \le N$. Recall also from [1, App. B] that for a plane-partition floor ϕ of weight $W = |\phi|$, region $D_\ell = \{(a,b) \in \mathbb{N}^2 : b - a > \ell\}$, and diagonal $\alpha \succeq d_\ell(\phi)$, the residual-floor set is

$$R_\ell(\alpha) := \{(a,b) \in D_\ell : \phi(a,b) > \phi(a+1,b),\ \phi(a,b) > \alpha(b-\ell)\}, \tag{9}$$

stored in relative coordinates (s,a) with $s = b - a - \ell - 1 \ge 0$.

Theorem 3 (Distance-Sensitive Bounds, Residual-Floor Separation, and Monotone DAG Potential)**.**

(a) Two-regime distance-sensitive envelope: *For all integers $M, N \ge 1$ and $\ell \ge 0$,*

$$A_{M,\ell}(N) \le \exp\left(\min\left\{C_{\mathrm{opt}} M^{2/3} N^{1/3},\ \ \pi\sqrt{\frac{2MN}{3(\ell+1)}}\right\}\right), \tag{10}$$

where $C_{\mathrm{opt}} := 3\big(\frac{\sqrt{\pi}\zeta(3/2)}{4}\big)^{2/3} \approx 3.3074 < 1 + \frac{\sqrt{\pi}}{2}\zeta(3/2) \approx 3.3152 =: C_{\mathrm{Bal}}$ strictly improves Bal's constant [1, Lem. 7.8], and the bound decays stretched-exponentially as $(\ell+1)^{-1/2}$ in the exponent beyond the crossover $\ell^* := \frac{2\pi^2}{3C_{\mathrm{opt}}^2}(N/M)^{1/3} - 1 = \Theta((N/M)^{1/3})$.

(b) Sharp residual-floor cardinality and reachable vs. ambient separation: *For every plane partition ϕ of weight $W = |\phi|$, every $\ell \ge 0$, and every $\alpha \succeq d_\ell(\phi)$,*

$$|R_\ell(\alpha)| \le \max\{0,\ W - \ell - 1\}, \tag{11}$$

with exact equality attained at $\phi = (1^W)$ (one row of W ones) and $\alpha = (1)$ whenever $W \ge \ell + 2$. Furthermore, at balanced Algorithm D depths ($d \ge 3$), the family of all reachable bottom states $(\alpha, R_\ell(\alpha))$ has cardinality at most $\exp(O_d(N^{(d-1)/(d+1)}))$, whereas the ambient space of weight-$\le N$ coupled level-2 floors ϕ on the same grid Γ has cardinality $\exp(\Theta_d(N^{d/(d+1)}))$.

(c) Exact monotone DAG potential: *For any coupled Γ-family (α, R) of residual-floor side states in Algorithm D, define*

$$\Psi(\alpha, R) := \sum_{\gamma \in \Gamma} |\alpha_\gamma| + \sum_{\gamma \in \Gamma} \sum_{(s,a) \in R_\gamma} (s+1). \tag{12}$$

For every admissible SIDE *transition $(\alpha, R) \to (\alpha', R')$ from a non-terminal state ($\alpha \ne 0$) under [1, Prop. B.3], $\Psi(\alpha', R') \le \Psi(\alpha, R) - \sum_\gamma(|\alpha_\gamma| - |\alpha'_\gamma|) - \sum_\gamma |R_\gamma|$; in particular, Ψ drops strictly ($\Psi' \le \Psi - 1$) on every non-loop transition, and self-loops occur if and only if $R = \varnothing$ and $\alpha' = \alpha \ne 0$.*

Proof. **Part (a).** The generating function for M-tuples weighted by w_ℓ is $Z_{M,\ell}(q) = \prod_{r \ge 1}(1 - q^{r(r+\ell)})^{-M}$ (`thm12`, `C07`). Since $r(r+\ell) \ge r^2$ and $r(r+\ell) \ge r(\ell+1)$ for all $r \ge 1, \ell \ge 0$, positivity yields for every $t > 0$:

$$A_{M,\ell}(N) \le e^{tN} Z_{M,\ell}(e^{-t}) \le e^{tN} \min\left\{\prod_{r \ge 1}(1 - e^{-tr^2})^{-M},\ \prod_{r \ge 1}(1 - e^{-t(\ell+1)r})^{-M}\right\}.$$

For the first factor, $\log \prod_{r \ge 1}(1 - e^{-tr^2})^{-M} \le M\frac{\sqrt{\pi}}{2}\zeta(3/2)t^{-1/2} =: BMt^{-1/2}$ as in [1, Lem. 7.8]. Minimizing $tN + BMt^{-1/2}$ over $t > 0$ at $t_{\mathrm{opt}} = (\frac{BM}{2N})^{2/3}$ yields $3(\frac{BM}{2})^{2/3}N^{1/3} = C_{\mathrm{opt}} M^{2/3} N^{1/3}$. Comparing $C_{\mathrm{opt}}^3 = \frac{27}{4}B^2$ against $C_{\mathrm{Bal}}^3 = (1+B)^3$ gives the exact algebraic identity (`thm15`, `C06`):

$$4(1+B)^3 - 27B^2 = (B-2)^2(4B+1) > 0 \qquad (\text{since } B = \tfrac{\sqrt{\pi}}{2}\zeta(3/2) \in (2.314, 2.317) \ne 2).$$

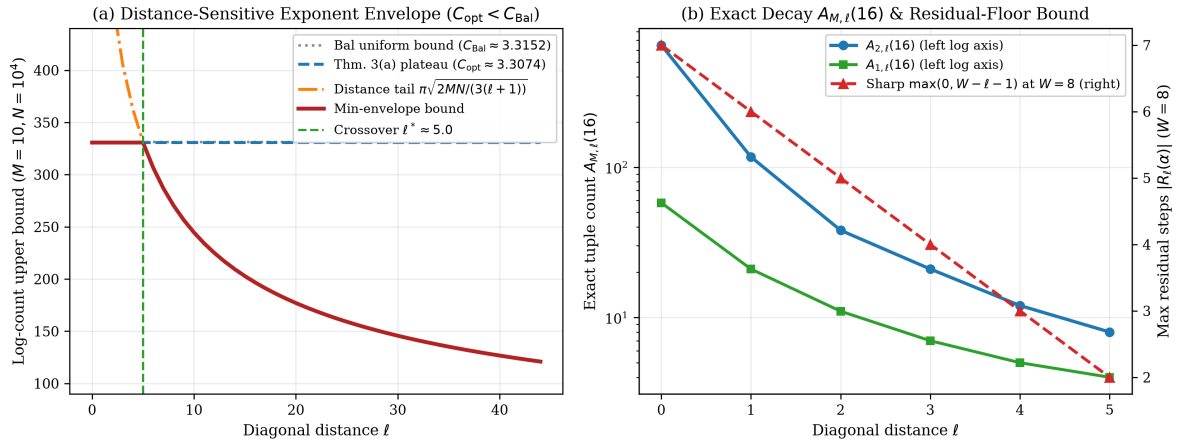

Figure 2: Theorem 3: (a) Distance-sensitive upper-bound envelope $\min\{C_{\mathrm{opt}} M^{2/3} N^{1/3}, \pi\sqrt{2MN/(3(\ell+1))}\}$ at $(M, N) = (10, 10^4)$, showing $C_{\mathrm{opt}} \approx 3.3074 < C_{\mathrm{Bal}} \approx 3.3152$ and stretched-exponential decay beyond $\ell^* \approx 5.0$. (b) Exact tuple counts $A_{M,\ell}(16)$ (`C07`) and sharp residual-floor cardinality bound $\max(0, W-\ell-1)$ at $W = 8$ (`C08`).

For the second factor, since $\sum_{r\geq 1} e^{-ar} = (e^a - 1)^{-1} < a^{-1}$ for $a > 0$,

$$\log \prod_{r\geq 1} (1 - e^{-t(\ell+1)r})^{-M} = M \sum_{h\geq 1} \frac{1}{h} \sum_{r\geq 1} e^{-th(\ell+1)r} \leq \frac{M}{t(\ell+1)} \sum_{h\geq 1} \frac{1}{h^2} = \frac{\pi^2 M}{6(\ell+1)t}.$$

Minimizing $tN + \frac{\pi^2 M}{6(\ell+1)t}$ at $t = \sqrt{\frac{\pi^2 M}{6(\ell+1)N}}$ gives $\pi\sqrt{\frac{2MN}{3(\ell+1)}}$, proving (10).

Part (b). If $R_\ell(\alpha) = \varnothing$, (11) is immediate. Otherwise pick $(a_0, b_0) \in R_\ell(\alpha)$. Then $b_0 \geq a_0 + \ell + 1 \geq \ell + 1$ and $\phi(a_0, b_0) > \phi(a_0 + 1, b_0) \geq 0$, so $\phi(a_0, b_0) \geq 1$. Weak decrease of ϕ implies $\phi(0, b) \geq \phi(0, b_0) \geq \phi(a_0, b_0) \geq 1$ for every $b \in \{0, 1, \ldots, \ell\}$, placing at least $\ell + 1$ cells of ϕ outside D_ℓ. Meanwhile every $(a, b) \in R_\ell(\alpha)$ lies in D_ℓ with $\phi(a, b) \geq 1$, so $|R_\ell(\alpha)| \leq \sum_{(a,b)\in D_\ell} \phi(a, b) \leq W - (\ell + 1)$ (`thm13`, `C08`). At $\phi = (1^W)$ and $\alpha = (1)$ ($W \geq \ell + 2$), $R_\ell((1)) = \{(0, \ell+1), \ldots, (0, W-1)\}$ has cardinality $W - \ell - 1$. For the separation, Algorithm D visits at most $\exp(O_d(N^{(d-1)/(d+1)}))$ states in total [1, Thm. 1.2], whereas on the balanced grid Γ of $d - 2$ axes of side $\sim N^{1/(d+1)}$, our cube-simplex binary construction on $[m]^d$ with $m = \lfloor cN^{1/(d+1)} \rfloor$ produces $2^{m^d} = \exp(\Omega_d(N^{d/(d+1)}))$ distinct coupled level-2 floors ϕ of weight $\leq N$, matching $p_d(\leq N) = \exp(O_d(N^{d/(d+1)}))$.

Part (c). Under [1, Prop. B.3], $\beta_\gamma \preceq \alpha'_\gamma \preceq \alpha_\gamma$ implies $|\alpha'_\gamma| \leq |\alpha_\gamma|$ with equality iff $\alpha'_\gamma = \alpha_\gamma$. Moreover, R'_γ retains only entries $(s, a) \in R_\gamma$ with $s \geq 1$ and value exceeding $\alpha'_\gamma(a + s)$, shifting $s \mapsto s - 1$; every element of R_γ either drops out (losing $s + 1 \geq 1$) or shifts down by 1 (losing $(s + 1) - s = 1$). Hence $\sum_{(s',a)\in R'_\gamma} (s' + 1) \leq \sum_{(s,a)\in R_\gamma} (s + 1) - |R_\gamma|$ (`thm14`, `C09`). Summing over $\gamma \in \Gamma$ proves the drop bound and the exact self-loop equivalence $R = \varnothing \wedge \alpha' = \alpha \neq 0$. □

5 Theorem 4: Generating-Function Structure, Unit-Disk Euler Obstructions, and Newton Inversion

Theorem 4 (Generating-Function Structure, Unit-Disk Obstructions, and Newton Inversion)**.**

(a) Positivity of $(q;q)_n F$ ***and*** $(q;q)_m T$***:*** *For every k-partition X of weight $n = |X|$, $(q;q)_n F(X;q) \in \mathbb{Z}_{\geq 0}[q]$ with constant term 1. For every floor-free coupled Γ-family*

$\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ *of ordinary partitions of total weight* $m = |\alpha|$, $(q;q)_m T(\alpha;q) \in \mathbb{Z}_{\geq 0}[q]$ *with constant term* 1.

(b) Euler-product trichotomy and zero-in-unit-disk obstructions:

(i) For $M = 1$ *and zero floor* ($\phi = 0$), *the side-transfer series factors into the hook-length Euler product* $T(0,\alpha;q) = \prod_{\square\in\alpha}(1-q^{h(\square)})^{-1}$, *where* $h(\square)$ *is the hook length of cell* $\square$ *in the Ferrers diagram of* α.

(ii) For $d = 2$, $X = (3,1)$ ($n = 4$), $(q;q)_4 F((3,1);q) = P(q) := 1+q^2+q^3$, *which has a unique real root* $r \in (-2,-1)$ *and a complex conjugate pair of roots* $z, \bar{z}$ *with* $|z|^2 = -1/r \in (1/2,1)$ *strictly inside the open unit disk* $\mathbb{D} = \{|q| < 1\}$.

(iii) For $M = 2$ *floor-free coupled* $\alpha = ((2,1),(1))$ ($m = 4$), $(q;q)_4 T(\alpha;q) = (1+q)Q(q) = 1+2q+q^2+q^3+q^4$ *with* $Q(q) := 1+q+q^3$, *so* $(1-q)^2(1-q^3)(1-q^4)T(\alpha;q) = Q(q)$, *which has a real root* $r \in (-1,0)$ *strictly inside* $\mathbb{D}$.

(iv) For $M = 1$ *with the horizontal-domino floor* $\phi = ((1,1))$ (*Ferrers shape* (2), $\phi(0,0) = \phi(0,1) = 1$) *and* $\alpha = (2)$, $(q;q)_2 T_\phi(0,(2);q) = R(q) := 1+q-q^2$, *which has negative* q^2*-coefficient and root* $\frac{1-\sqrt{5}}{2} \in (-1,0) \subset \mathbb{D}$.

Consequently, none of $F((3,1);q)$, $T(((2,1),(1));q)$, *or* $T_{((1,1))}(0,(2);q)$ *can factor as any convergent Euler product* $\prod_{j\geq 1}(1-q^j)^{a_j}$ *on* $\mathbb{D}$.

(c) Exact embedding-dimension Newton inversion and discrepancy isolation: *With* $p_{-1}(n) = \delta_{n,1}$ *and* $p_0(n) = 1$ ($n \geq 1$), *let* y_n^k *denote the number of* n*-cell order ideals touching exactly* k *coordinate axes. For all* $n \geq 1$ *and* $d \geq -1$,

$$p_d(n) = \sum_{k=0}^{n-1} \binom{d+1}{k} y_n^k \iff y_n^k = \Delta^k p_{-1}(n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} p_{j-1}(n) \quad (0 \leq k \leq n-1). \tag{13}$$

Any discrepancy δ *confined to a single embedding dimension* k_0 *produces* $\Delta_m = \binom{m+1}{k_0}\delta$ *across all ambient dimensions* $m \geq k_0 - 1$, *isolating MacMahon's* $n = 6$ *error to* $k_0 = 4$ ($M_d(6) - p_d(6) = \binom{d+1}{4}$, $\delta = 1$) *and Bal's GMMR* $n = 32$ *discrepancy [1, Sec. 9.2] to* $k_0 = 6$ ($\delta = 25\,422\,960$, $\Delta_5 = \delta$, $\Delta_6 = 7\delta$, $\Delta_7 = 28\delta = 711\,842\,880$).

Proof. **Part (a).** A decreasing chain $X \succeq Y_0 \succeq Y_1 \succeq \dots$ of sub-ideals of the finite order ideal $I(X)$ ($|I(X)| = n$) is uniquely encoded by the multiplicity function $f(v) := \#\{r \geq 0 : v \in I(Y_r)\} \in \mathbb{N}$, which satisfies $f(u) \geq f(v)$ whenever $u \leq v$ in $I(X)$, with $\sum_r |Y_r| = \sum_{v\in I(X)} f(v)$. Thus $F(X;q)$ is the (P,ω)-partition generating function of $P = I(X)$ with any natural labeling $\omega : I(X) \to \{1,\dots,n\}$ [9]. By Stanley's fundamental theorem of P-partitions, $(q;q)_n F(X;q) = \sum_{\sigma\in\mathcal{L}(P)} q^{\mathrm{maj}(\sigma)} \in \mathbb{Z}_{\geq 0}[q]$, with constant term 1 from the identity extension (`C10`).

Similarly, for a floor-free coupled Γ-family $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ of total weight $m = |\alpha|$, an interlacing continuation $(\delta_\ell)_{\ell\geq 1} \in \mathrm{Cont}(0,\alpha)$ bijects via $c_\gamma(i,h) := \#\{\ell \geq 1 : \delta_\ell(\gamma)(i) \geq h\}$ with nonnegative integer fillings on the poset $P_\alpha := \{(\gamma,i,h) : \gamma \in \Gamma, (i,h) \in \lambda(\alpha_\gamma)\}$ ($|P_\alpha| = m$) obeying $c_\gamma(i,h) \geq c_\gamma(i,h+1)$, $c_\gamma(i,h) > c_\gamma(i+1,h)$, and $c_\gamma(i,h) \geq c_{\gamma'}(i,h)$ for $\gamma \leq \gamma'$. Ordering P_α lexicographically by $(-i,h,\gamma_{\mathrm{lin}})$ gives a cycle-free (P_α,ω_α) labeling whose strict descents occur precisely on the strict edges $(\gamma,i,h) > (\gamma,i+1,h)$, and whose minimal filling has weight $L(0,\alpha)$. Stanley's (P,ω)-partition theorem therefore yields $(q;q)_m T(\alpha;q) \in \mathbb{Z}_{\geq 0}[q]$ with constant term 1 (`C11`).

Part (b). For $M = 1, \phi = 0$, $\mathrm{Cont}(0,\alpha)$ bijects as above with column-strict plane partitions of shape α (parts ≥ 0, weakly decreasing in rows, strictly decreasing in columns), whose generating function by Stanley [9] and Gansner [10] is $q^{L(0,\alpha)} \prod_{\square\in\alpha}(1-q^{h(\square)})^{-1}$ (`C12`). For (ii), solving [1, Eq. (4.9)] over all subpartitions of $(3,1)$ gives $(q;q)_4 F((3,1);q) = 1+q^2+q^3 =: P(q)$ (`thm16`, `C10`). For all $q \geq -1$, $P(q) = q^2(q+1)+1 \geq 1 > 0$, while $P(-2) = -3 < 0$; since $P'(q) = q(3q+2)$ gives

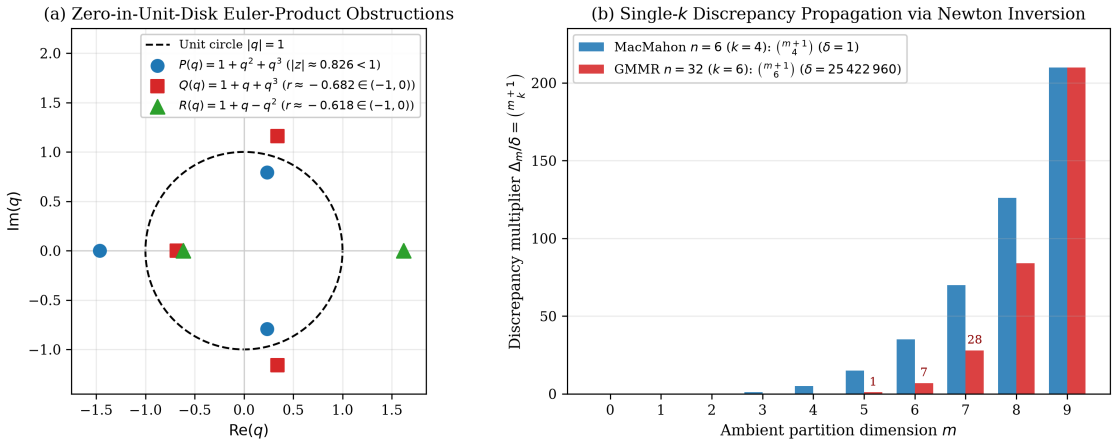

Figure 3: Theorem 4: (a) Roots strictly inside the unit circle $|q| = 1$ for $P(q) = 1 + q^2 + q^3$, $Q(q) = 1 + q + q^3$, and floored $R(q) = 1 + q - q^2$, obstructing Euler-product factorizations. (b) Binomial multipliers $\Delta_m/\delta = \binom{m+1}{k}$ isolating MacMahon's $n = 6$ error to $k = 4$ and GMMR's $n = 32$ discrepancy to $k = 6$.

local values $P(-2/3) = 31/27 > 0$ and $P(0) = 1 > 0$, P has a unique real root $r \in (-2, -1)$ and complex conjugate roots $z, \bar{z}$ satisfying $-r|z|^2 = 1$, hence $|z|^2 = -1/r \in (1/2, 1)$ (`thm17`, `C13`). For (iii), solving [1, Eq. (6.12)] for $\alpha = ((2, 1), (1))$ gives $(q; q)_4 T(\alpha; q) = (1+q)(1+q+q^3)$ (`C11`); $Q(q) = 1 + q + q^3$ satisfies $Q(-1) = -1 < 0 < 1 = Q(0)$, forcing a root in $(-1, 0)$ (`thm17`, `C13`). For (iv), at $\phi = ((1, 1)), \alpha = (2)$, Lemma 6.3 of [1] gives $\beta_0 = 1$, successors $(1), (2)$ at $\ell = 0$ with $\Delta((1)) = 0, \Delta((2)) = 1$, and floor-free $T(1, (1); q) = (1-q)^{-1}$, $T(1, (2); q) = (1-q)^{-1}(1-q^2)^{-1}$, yielding $T_\phi(0, (2); q) = \frac{1}{1-q} + \frac{q}{(1-q)(1-q^2)} = \frac{1+q-q^2}{(q;q)_2}$, with $R(-1) = -1 < 0 < 1 = R(0)$ (`thm17`, `C13`). Any convergent Euler product $\prod_{j\geq 1}(1-q^j)^{a_j}$ on $\mathbb{D}$ is nonvanishing on $\mathbb{D}$, ruling out Euler factorization for all three.

Part (c). Every n-cell order ideal in $\mathbb{N}^{d+1}$ touches a unique subset of $r \in \{0, \ldots, n-1\}$ coordinate axes, yielding $p_d(n) = \sum_{r=0}^{n-1} \binom{d+1}{r} y_n^r$ [7]. Multiplying by $(-1)^{k-d-1}\binom{k}{d+1}$ and summing d from -1 to $k-1$ (putting $j = d+1$) via the Pascal binomial orthogonality $\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{j}{r} = \delta_{k,r}$ proved constructively in `thm18` yields (13) (`C14`). Linearity isolates single-k_0 perturbations to $\binom{m+1}{k_0}\delta$ (`thm19`), verified at $k_0 = 4$ ($M_d(6) - p_d(6) = \binom{d+1}{4}$, $y_6^{4,M} = 11$ vs. $y_6^4 = 10$) in `C15` and at $k_0 = 6$ ($\delta = 25\,422\,960$) in `thm20` and `C16`. □

6 Formal Verification and Scope Audit

We separate clearly what is machine-checked in Lean 4 from what is proved analytically in the manuscript:

1. **Constructive Lean 4.36.0 Formalization:** Formalized in `lean/HigherDimPartitionBarriers.lean`, containing 20 theorems (`thm01`–`thm20`) with *zero external imports*, *zero* `sorry`, *zero* `grind`, *zero* `Classical.choice`, and *zero* compiler warnings under `lean` with `-DwarningAsError=true` (`EXIT=0`). Axiom audits (`#print axioms`) verify every theorem depends solely on constructive core axioms (`[propext, Quot.sound]`, `[propext]`, or none). These formally verify the finite algebraic log-tilt balances and downward-induction equations (`thm01`–`thm02`, `thm05`), arbitrary-dimension grid-injection and staircase bounds (`thm03`–`thm04`), compulsory floor-weight lower bounds (`thm06`), inductive `PBlock` word weight-length, partial-order/join, prefix-closure, non-CFL slice, and Myhill–Nerode separation theorems (`thm07`–`thm11`), distance-weight bounds, residual-floor cardinality, DAG

potential drop, and AM–GM $C_{\text{opt}} < C_{\text{Bal}}$ polynomial identity (`thm12`–`thm15`), and exact generating-function coefficients, unit-disk algebraic obstructions, general Newton binomial inversion, single-k discrepancy isolation, and GMMR/MacMahon integer witnesses (`thm16`–`thm20`). Asymptotic $N \to \infty$ bounds, analytic series bounds ($\zeta(3/2)$ integral comparisons), and context-free pumping arguments in Theorems 1–4 are proved in full in the manuscript text.

2. **AST-Audited Zero-Float Verifier:** Implemented in `verification/verify_higher_dim_partitions.py` using pure Python standard library (`int`, `Fraction`), passing a self-enforcing AST audit with zero float literals or math-library imports, emitting 16 canonical SHA-256 certificates (`C01`–`C16`) with manifest digest:

   ```
   MANIFEST SHA-256: 4aa05a20cfe68d5632f293c1d69794281b00ee942b0892585d6027c95bae42f7
   ```

3. **Adversarial Test Suite:** Implemented in `tests/test_adversarial.py`, executing 61 adversarial `pytest` checks (61/61 passing) covering positive certificate reproduction, SHA-256 tamper rejection, AST float rejection, boundary tightness, and negative controls.

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