Reflection Positivity and Exact Gap Certificates from Forgotten Quantum Order

Physical transfer matrices, Hausdorff localizers, and an exact tetrahedral spectrum

Lluis Eriksson

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Abstract

A balanced sequence of short Hamiltonian kicks has no first-order drift, but its order-dependent commutator holonomy survives when the order record is discarded. We show that an order/reverse physical echo has a second structure which is invisible in the continuous-time limit alone: if the distribution of orders is invariant under reversal, the averaged channel is a self-adjoint random-unitary transfer operator. For quotient action $S = \sum_j \operatorname{dist}(h_j, Z(\mathcal{A}))$, it obeys the nonperturbative operator inequality

$$\cos(4|y|S)I \leq T_y \leq I, \qquad |y|S < \pi/8.$$

Consequently both site- and link-reflection forms are positive and $H_y = -(4y^4)^{-1}\log T_y$ is a finite-dimensional Osterwalder–Schrader Hamiltonian. The physical transfer has the quantitative expansion

$$T_y = I + 4y^4\mathcal{Q} + \mathcal{R}_y, \qquad \|\mathcal{R}_y\|_{2\to 2}\,,\ \|\mathcal{R}_y\|_{\diamond} \leq (4S)^6|y|^6/720.$$

Thus its mass converges at order y^2 to the curvature-frame gap of $-\mathcal{Q}$, with an explicit finite-pulse error bound.

For the regular tetrahedral qubit loop we then solve the complete physical problem at finite pulse. The transfer is exactly depolarizing for every y, with nontrivial eigenvalue

$$\lambda(y) = \frac{4(c^2-s^2)^2(c^6+13c^4s^2+35c^2s^4+79s^6)^2-1}{3}, \quad c=\cos(\sqrt{3}y), \quad s=\frac{\sin(\sqrt{3}y)}{\sqrt{3}}.$$

Its first positive zero is isolated at $0.45569 < y_0 < 0.45570$, and its OS mass begins $m_y = 64/3 - 128y^2 + (46528/45)y^4 + O(y^6)$.

Finally, we turn the gap question into a complete finite-dimensional Hausdorff certificate. If $T \geq 0$ has r distinct visible eigenvalues, moments through degree $2r-1$ certify support below a proposed edge exactly; depth $r-1$ is uniformly sharp. The resolvent version is rational whenever the frame matrix is rational. A balanced four-kick example certifies the exact gap $10/3$, while the false claim 4 is refuted by the rational witness $(-8/3, 88/9)$, whose quadratic value is $-1/64$. A simultaneous Wishart–Loewner band adds finite-sample type-I control for independent Gaussian correlator sketches, including witnesses selected after seeing the data. Symbolic enumeration of all 24 orders and deterministic numerical tests accompany the manuscript.

1 Question, answer, and novelty boundary

Reflection positivity is the bridge from Euclidean data to a positive Hilbert space and Hamiltonian [1, 2]. On a lattice it is closely tied to a positive self-adjoint transfer matrix [3, 4, 5]. Detailed balance, random-unitary channels, the Hausdorff moment problem, and spectral inference from Euclidean correlators are also established subjects [6, 7, 11, 12, 13, 14, 15]. Our question is narrower:

> Can a complete, finite-pulse forgotten-order experiment itself generate a positive transfer operator, and can the resulting gap be bounded or falsified by exact, finite data?

The answer is affirmative under explicit hypotheses. Central balance removes first-order dynamics. Reversal symmetry pairs every physical echo with its Hilbert–Schmidt adjoint. A phase-diameter argument then promotes self-adjointness to strict positivity on a nonperturbative pulse window. This creates site and link reflection forms before taking any scaling limit. The same physical words approach the double-commutator frame generator with an explicit error, so the transfer mass is connected quantitatively to the previously derived forgotten-order diffusion.

The second answer is algebraic. Once positivity is known, a proposed transfer edge is a compact-interval support statement. A Hausdorff localizer detects every offending finite-dimensional spectral atom at a known finite depth. Applied to a rational curvature frame through its resolvent, the test becomes exact rational arithmetic rather than floating-point diagonalization. Under a separate Gaussian sketch model, one simultaneous covariance band preserves the significance level even if the final separating witness is optimized after observing the sample.

We do not claim the general OS reconstruction theorem, the general existence of positive transfer matrices, detailed balance, moment localizers, or Gaussian covariance concentration. Cho et al. already combine reflection positivity and semidefinite constraints for Euclidean two-point correlators [14]; Wagman develops transfer-matrix spectral extraction through Lanczos methods [15]. The contribution here is the architecture-specific chain

$$\begin{aligned}\text{physical forgotten-order echo} &\Longrightarrow \text{positive transfer} \Longrightarrow \text{controlled mass}\\ &\Longrightarrow \text{finite exact gap certificate,}\end{aligned}$$

together with the exact tetrahedral finite-pulse solution. Our literature statement is scoped, not a claim of exhaustive priority.

Relation to the nine-paper source chain

This is a new tenth manuscript, not a replacement for the nine source papers. The first two source manuscripts study architecture-dependent decoherence and noncommuting filters. The next seven progressively develop operational transport of the cut, information loss under noncommuting record projections, block-record diffusion, dissipative holonomy, operational curvature, matrix isoperimetry, and sharp synthesis/physical-word limits. The ninth paper supplies the balanced echo expansion and identifies the tetrahedron as the sharp four-kick mixer. None of the nine proves physical-transfer reflection positivity, the exact finite-pulse tetrahedral spectrum, the OS mass estimate, or the complete rational Hausdorff certificate derived here. The statistical layer is adapted from the companion finite-sample spectral-certificate program and is not claimed as new in isolation.

2 Retained kicks and physical echoes

Let

$$\mathcal{A} = \bigoplus_{a=1}^{q} M_{d_a}(\mathbb{C})$$

with faithful tracial Hilbert–Schmidt product $\langle X, Y\rangle = \mathrm{Tr}(X^*Y)$. A pinching $\mathcal{E}$ onto $\mathcal{A}$ may be inserted after every physical kick. Since the Hamiltonians below are retained, $h_j = h_j^* \in \mathcal{A}$, the restriction to $\mathcal{A}$ is simply unitary conjugation. We assume central balance

$$B := \sum_{j=1}^{m} h_j \in Z(\mathcal{A}). \tag{1}$$

Central parts do not affect conjugation on $\mathcal{A}$. Choose central z_j attaining the quotient norm and put

$$r_j = h_j - z_j, \qquad S = \sum_{j=1}^{m} \|r_j\|_\infty = \sum_{j=1}^{m} \mathrm{dist}(h_j, Z(\mathcal{A})). \tag{2}$$

The sum of the r_j remains central.

For an order $\sigma = (\sigma_1, \ldots, \sigma_m)$, with σ_1 acting first, define

$$U_{\sigma,y} = e^{-iyr_{\sigma_m}} \cdots e^{-iyr_{\sigma_1}}, \tag{3}$$

$$V_{\sigma,y} = U_{\sigma,y} U^*_{\sigma^R,y}, \qquad \sigma^R = (\sigma_m, \ldots, \sigma_1), \tag{4}$$

$$R_{\sigma,y} = \mathrm{Ad}_{V_{\sigma,y}}, \qquad \mathrm{Ad}_V(X) = VXV^*. \tag{5}$$

The echo contains $2m$ physical kicks. In the full space, each factor may be written $\mathcal{E} \circ \mathrm{Ad}_{e^{-iyr_j}} \circ \mathcal{E}$; no inverse of a dissipative pinching is used.

Let μ be a law on orders such that

$$\mu(\sigma) = \mu(\sigma^R), \tag{6}$$

and define the averaged physical transfer

$$T_y = \sum_\sigma \mu(\sigma) R_{\sigma,y}. \tag{7}$$

It is unital, trace preserving, and completely positive. The new point is its Hilbert-space order structure.

3 A nonperturbative reflection-positive window

Theorem 3.1 (Physical transfer positivity). *Under (1)–(6), T_y is Hilbert–Schmidt self-adjoint. If $4|y|S < \pi/2$, then*

$$\boxed{\cos(4|y|S)I \;\leq\; T_y \;\leq\; I} \tag{8}$$

as operators on $L^2(\mathcal{A}, \mathrm{Tr})$. In particular, T_y is strictly positive.

Proof. Directly from (5),

$$R_{\sigma^R,y} = R^*_{\sigma,y} = R^{-1}_{\sigma,y}.$$

Thus reversal symmetry writes T_y as a convex combination of pairs $(R + R^*)/2$, proving self-adjointness.

For each branch, the product path constructing $V_{\sigma,y}$ has operator-length at most $2|y|S$. In the stated window it admits a Hermitian logarithm $V_{\sigma,y} = e^{-iK_{\sigma,y}}$ with $\|K_{\sigma,y}\| \leq 2|y|S$. Its spectral oscillation is therefore at most $4|y|S$. In an eigenbasis of $K_{\sigma,y}$, the paired channel

$$\frac{1}{2}\left(\mathrm{Ad}_{V_{\sigma,y}} + \mathrm{Ad}^*_{V_{\sigma,y}}\right)$$

has eigenvalues $\cos(\kappa_p - \kappa_q)$. They lie between $\cos(4|y|S)$ and 1. Convex averaging proves (8). $\square$

This is tracial detailed balance in elementary form: $T_y = T_y^*$ for the tracial GNS product. It should not be confused with KMS detailed balance at a nontrivial Gibbs state [6, 7].

Corollary 3.2 (Site and link reflection forms). *For arbitrary $X_0, \ldots, X_N \in L^2(\mathcal{A})$,*

$$\sum_{p,q=0}^{N} \left\langle X_p, T_y^{p+q} X_q \right\rangle = \left\| \sum_{p=0}^{N} T_y^p X_p \right\|_2^2 \geq 0, \tag{9}$$

$$\sum_{p,q=0}^{N} \left\langle X_p, T_y^{p+q+1} X_q \right\rangle = \left\| T_y^{1/2} \sum_{p=0}^{N} T_y^p X_p \right\|_2^2 \geq 0. \tag{10}$$

Hence

$$H_y = -\frac{1}{4y^4} \log T_y \geq 0, \qquad T_y = e^{-4y^4 H_y}, \tag{11}$$

is a well-defined finite-dimensional OS Hamiltonian.

Remark 3.3 (Scope). Equations (9)–(10) are operatorial transfer-reflection positivity. They do not by themselves construct a local Euclidean field measure or an infinite-volume relativistic QFT. In particular, the mass below is a finite transfer mass, not the Yang–Mills mass gap.

4 From the physical transfer to the curvature gap

The order-dependent second Magnus term is

$$H_\sigma = -\frac{i}{2}\sum_{s>r}[h_{\sigma_s}, h_{\sigma_r}]. \tag{12}$$

Define the negative semidefinite frame generator

$$\mathcal{Q}_\mu = -\frac{1}{2}\sum_{\sigma}\mu(\sigma)\,\mathrm{ad}^2_{H_\sigma}\,. \tag{13}$$

It is a tracial GKLS generator written as a sum of Hermitian double commutators [8, 9]. For the uniform law on all permutations,

$$\mathcal{Q} = -\frac{1}{24}\sum_{j<k}\mathrm{ad}^2_{C_{jk}}, \qquad C_{jk} = i[h_j, h_k]. \tag{14}$$

Theorem 4.1 (Physical-word expansion in two norms)**.** *Under the hypotheses of Theorem 3.1,*

$$\boxed{T_y = I + 4y^4\mathcal{Q}_\mu + \mathcal{R}_y, \qquad \|\mathcal{R}_y\|_\diamond,\ \|\mathcal{R}_y\|_{2\to 2} \le \frac{(4S)^6}{720}|y|^6.} \tag{15}$$

Proof. The Magnus expansion of $U_{\sigma,y}$ is

$$\log U_{\sigma,y} = -iyB + y^2\Omega_{2,\sigma} + y^3\Omega_{3,\sigma} + O(y^4).$$

Reversal changes the sign of the even Magnus terms and preserves the odd ones [10]. Since B is central, the relative echo satisfies

$$\log V_{\sigma,y} = -2iy^2H_\sigma + O(y^4).$$

Pairing the channel with its inverse cancels degrees one, two, three, and five and gives

$$\frac{1}{2}(R_{\sigma,y} + R_{\sigma,y}^{-1}) = I - 2y^4\,\mathrm{ad}^2_{H_\sigma} + O(y^6).$$

Averaging yields the displayed fourth-order term. Each echo is a product of $2m$ unitary channels, and the sum of their generator norms is at most $4S$. The multinomial Leibniz rule bounds the sixth derivative by $(4S)^6$. Taylor's theorem gives (15). The argument is run independently in diamond norm and in Hilbert–Schmidt induced norm; no dimension-free comparison between those norms is assumed. □

Now specialize to $\mathcal{A} = M_d(\mathbb{C})$ and assume the uniform frame is irreducible,

$$\{C_{jk} : j<k\}' = \mathbb{C}I. \tag{16}$$

Let $\mathcal{A}_0 = \{X : \operatorname{Tr} X = 0\}$ and

$$\gamma = \lambda_{\min}((-\mathcal{Q})|_{\mathcal{A}_0}) > 0, \qquad \varepsilon_y = \frac{64}{45}S^6y^2. \tag{17}$$

Theorem 4.2 (Finite-pulse mass control)**.** *Assume the positivity window of Theorem 3.1 and $\varepsilon_y < \gamma$. The identity is then the only fixed vector of T_y. If $\lambda_*(T_y)$ denotes its largest eigenvalue on $\mathcal{A}_0$,*

$$\tilde{\gamma}_y = \frac{1-\lambda_*(T_y)}{4y^4} \quad \textit{satisfies} \quad |\tilde{\gamma}_y - \gamma| \le \varepsilon_y. \tag{18}$$

If also $4y^4(\gamma+\varepsilon_y) < 1$, the OS mass

$$m_y = -\frac{1}{4y^4}\log\lambda_*(T_y) \tag{19}$$

obeys

$$\boxed{|m_y - \gamma| \le \varepsilon_y + \frac{2y^4(\gamma+\varepsilon_y)^2}{1-4y^4(\gamma+\varepsilon_y)}.} \tag{20}$$

Proof. On $\mathcal{A}_0$, put $A_y = (I - T_y)/(4y^4)$. Equation (15) gives $\|A_y + \mathcal{Q}\|_{2\to 2} \le \varepsilon_y$. Weyl's inequality proves (18); $\varepsilon_y < \gamma$ excludes an additional fixed vector. Finally use $-\log(1-x) = x + \rho(x)$, with $0 \le \rho(x) \le x^2/[2(1-x)]$ for $0 \le x < 1$, and enlarge the harmless factor to the stated bound. □

For $A, B \in \mathcal{A}_0$, the spectral theorem immediately gives

$$|\left\langle A, T_y^n B \right\rangle| \le e^{-4y^4 m_y n} \|A\|_2 \|B\|_2 . \tag{21}$$

If $\Omega(X) = \mathrm{Tr}(X)I/d$, define the finite nuclear mixing index

$$\nu_y(n) = \left\| T_y^n - \Omega \right\|_{S_1(L^2)} . \tag{22}$$

Positivity yields

$$\nu_y(n) = \sum_{r=2}^{d^2} \lambda_r(T_y)^n \le (d^2-1)e^{-4y^4 m_y n}. \tag{23}$$

This is a finite-dimensional nuclearity proxy, not a Type III split-inclusion theorem.

Proposition 4.3 (Gaussian Hamiltonian realization of the limit)**.** *Let $L_{jk} = C_{jk}/\sqrt{12}$, and let independent real Brownian motions drive the Stratonovich unitary equation*

$$dU_t = -i \sum_{j<k} L_{jk} U_t \circ dB_t^{jk}.$$

Then

$$\mathbb{E}[U_t X U_t^*] = e^{t\mathcal{Q}} X. \tag{24}$$

Proof. Converting the conjugation process to Itô form contributes $-\frac{1}{2}\sum_{j<k} \mathrm{ad}_{L_{jk}}^2 = \mathcal{Q}$ to its drift; taking expectations removes the martingale term. □

Thus the same frame appears in three layers: physical finite-pulse echoes, the positive OS Hamiltonian H_y, and a continuous random-Hamiltonian diffusion.

5 Exact tetrahedral physical transfer

Let X, Y, Z be the Pauli matrices and choose

$$h_1 = X + Y + Z, quad h_2 = X - Y - Z, quad h_3 = -X + Y - Z, quad h_4 = -X - Y + Z. \tag{25}$$

These kicks are balanced, each has operator norm $\sqrt{3}$, and $S = 4\sqrt{3}$. Let μ be uniform on all 24 orders.

Theorem 5.1 (Exact finite-pulse tetrahedral spectrum)**.** *For every real y, the complete physical echo transfer is depolarizing:*

$$T_y(I) = I, \qquad T_y(P) = \lambda(y)P, quad P \in \{X, Y, Z\}, \tag{26}$$

where, with $c = \cos(\sqrt{3}y)$ and $s = \sin(\sqrt{3}y)/\sqrt{3}$,

$$p(c,s) = c^6 + 13c^4s^2 + 35c^2s^4 + 79s^6, \tag{27}$$

$$\boxed{\lambda(y) = \frac{4(c^2-s^2)^2 p(c,s)^2 - 1}{3}.} \tag{28}$$

Its small-pulse expansion is

$$\lambda(y) = 1 - \frac{256}{3}y^4 + 512y^6 - \frac{7424}{15}y^8 - \frac{1274368}{105}y^{10} + \frac{53662208}{525}y^{12} + O(y^{14}). \tag{29}$$

The first positive zero is unique and satisfies

$$0.45569 < y_0 < 0.45570, \qquad y_0 = 0.455698535295322\ldots. \tag{30}$$

For $0 < y < y_0$*, the exact OS mass and nuclear index are*

$$m_y = -\frac{\log \lambda(y)}{4y^4} = \frac{64}{3} - 128y^2 + \frac{46528}{45}y^4 - \frac{276096}{35}y^6 + O(y^8), \tag{31}$$

$$\nu_y(n) = 3\lambda(y)^n. \tag{32}$$

Proof. Write each pulse as $e^{-iyh_j} = cI - isy_j$, where $y_j = h_j$ and $y_j^2 = 3I$. Expanding the eight factors of $V_{\sigma,y}$ in the Pauli algebra shows that every order has the common trace

$$\operatorname{Tr} V_{\sigma,y} = 2(c^2 - s^2)p(c,s).$$

For completeness, this calculation needs no matrix entries. If a partial word is $aI - i\boldsymbol{b}\cdot\boldsymbol{\sigma}$, left multiplication by $cI - is\boldsymbol{v}\cdot\boldsymbol{\sigma}$ updates

$$(a, \boldsymbol{b}) \longmapsto (ca - s\boldsymbol{v}\cdot\boldsymbol{b}, c\boldsymbol{b} + sa\boldsymbol{v} + s\boldsymbol{v}\times\boldsymbol{b}). \tag{33}$$

Iterating (33) over the four tetrahedral vectors, then over the sign-reversed word, and using $\boldsymbol{v}_j \cdot \boldsymbol{v}_k = -1$ for $j \neq k$ gives the displayed scalar for all 24 permutations. The set of all orders is invariant under the tetrahedral rotation group. Hence its average on the irreducible three-dimensional Bloch sector is scalar. For a qubit unitary with trace t, the corresponding rotation has trace $|t|^2 - 1$; division by three gives (28). The ancillary symbolic certificate expands all 24 orders and independently verifies that every off-diagonal Bloch entry vanishes.

Modulo $c^2 + 3s^2 = 1$, (28) factors as

$$\lambda = \frac{(A-3)(A-1)}{3}, \qquad A = 512s^8 - 256s^6 + 64s^4. \tag{34}$$

For $z = s^2 > 0$, $dA/dz = 128z(1 - 6z + 16z^2) > 0$. Since s^2 is increasing up to the first root, the first crossing is the unique solution of $A = 1$. Direct interval evaluation gives (30). Series expansion proves (29) and (31); (32) follows from the spectrum $\{1, \lambda, \lambda, \lambda\}$. □

The universal window from Theorem 3.1 is $y < \pi/(32\sqrt{3}) = 0.056681\ldots$, whereas the symmetry-resolved exact positivity window extends to y_0. The difference quantifies the price of an architecture-independent action bound.

6 Complete finite-depth Hausdorff certificates

Let $T = T^*$ act on a D-dimensional Hilbert space and let $W \succeq 0$. Define scalar moments

$$m_n = \operatorname{Tr}(WT^n). \tag{35}$$

Write the visible spectral decomposition as

$$T = \sum_{a=1}^{s} v_a P_a, \qquad w_a = \operatorname{Tr}(WP_a) > 0, \qquad m_n = \sum_{a=1}^{s} w_a v_a^n. \tag{36}$$

For $N \geq 0$, set

$$H_N = (m_{i+j})_{i,j=0}^N, \quad S_N = (m_{i+j+1})_{i,j=0}^N, \quad U_N^\theta = \theta H_N - S_N. \tag{37}$$

Theorem 6.1 (Exact visible-edge certificate)**.** *The visible support lies in* $[0, \theta]$ *if and only if*

$$S_{s-1} \succeq 0, \qquad U_{s-1}^\theta \succeq 0. \tag{38}$$

If $T \succeq 0$ *is already known, the first condition is automatic. If* s *is unknown, depth* $D - 1$ *always suffices. Moments through degree* $2s - 1$ *are required. The depth* $s - 1$ *is uniformly sharp.*

Proof. For $q(v) = \sum_{i=0}^{N} q_i v^i$,

$$q^* S_N q = \sum_a w_a v_a |q(v_a)|^2, \tag{39}$$

$$q^* U_N^\theta q = \sum_a w_a (\theta - v_a) |q(v_a)|^2. \tag{40}$$

Support in $[0, \theta]$ implies positivity. Conversely, if a visible node lies above θ, the degree-$(s-1)$ Lagrange polynomial which is one at that node and zero at all others makes the second form negative. A negative node is isolated similarly by the first form.

For sharpness, take $s-1$ distinct nodes strictly inside $[0, \theta]$. Their upper localizer at degree $s-2$ is positive definite. Add one node above θ with sufficiently small weight; degree $s-2$ remains positive by openness, whereas the degree-$(s-1)$ Lagrange polynomial detects it. □

For the OS transfer $T_y = e^{-4y^4 H_y}$, a visible mass lower bound $m_y \geq \delta$ is exactly the edge claim

$$v_a \leq \theta = e^{-4y^4\delta}. \tag{41}$$

A negative localizer therefore refutes an overly large mass claim. If $W > 0$, every eigenspace is detected and "visible" may be removed. Without this condition it cannot: for $T = \operatorname{diag}(1/4, 3/4)$, $W = \operatorname{diag}(1, 0)$, and $\theta = 1/2$, every observed localizer passes although the full edge is $3/4$.

For exact arithmetic it is preferable to certify the positive frame operator $K = -\mathcal{Q}$ directly. Let $x_0 > 0$ and define

$$b_n = x_0^n \operatorname{Tr}(K + x_0 I)^{-(n+1)}. \tag{42}$$

This is (35) for

$$T = x_0 (K + x_0 I)^{-1}, \qquad W = (K + x_0 I)^{-1}.$$

If r is the number of distinct eigenvalues of K, Theorem 6.1 gives

$$\boxed{K \geq cI \quad \Longleftrightarrow \quad U_{r-1}^\theta \succeq 0, \qquad \theta = \frac{x_0}{c + x_0}.} \tag{43}$$

When a known kernel of dimension z has to be removed, subtract z/x_0 from every b_n. Rational K, x_0, c produce rational moments and localizers. Exact LDL^T proves positivity; a rational vector with negative quadratic value proves failure.

7 An exact rational four-kick certificate

Consider the balanced Bloch vectors

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = (0,0,1), \quad v_4 = (-1,-1,-1). \tag{44}$$

For $h_j = v_j \cdot \boldsymbol{\sigma}$, the uniform curvature frame $K = -\mathcal{Q}$ on the Bloch sector is exactly

$$G = \begin{pmatrix} 4 & 2/3 & 2/3 \\ 2/3 & 4 & 2/3 \\ 2/3 & 2/3 & 4 \end{pmatrix}, \qquad \operatorname{spec} G = \{10/3, 10/3, 16/3\}. \tag{45}$$

Choose $x_0 = 2$. The first five resolvent moments are

$$\left(\frac{45}{88}, \frac{1377}{7744}, \frac{42849}{681472}, \frac{1351809}{59969536}, \frac{43116705}{5277319168} \right). \tag{46}$$

The true claim $c = 10/3$, hence $\theta = 3/8$, gives

$$U_1^{3/8} = \begin{pmatrix} 27/1936 & 81/21296 \\ 81/21296 & 243/234256 \end{pmatrix} \succeq 0, \qquad \det U_1^{3/8} = 0. \tag{47}$$

The larger claim $c = 4$, hence $\theta = 1/3$, is false. The exact witness

$$q = \begin{pmatrix} -8/3 \\ 88/9 \end{pmatrix} \quad \text{satisfies} \quad \boxed{q^T U_1^{1/3} q = -1/64 < 0.} \tag{48}$$

No eigensolver or tolerance enters this certificate. It is also sharp at the true threshold: the zero determinant is expected because the lowest spectral atom lies exactly on the proposed edge.

8 Finite-sample Gaussian sketches

Exact moments are available in symbolic models, but estimated correlators are correlated across time. The following layer gives simultaneous coverage in one explicit observation model. It is included to state exactly what survives statistical noise, not to identify generic laboratory noise with a Gaussian law.

Choose real probe vectors ψ_a, $a = 1, \dots, c$, and let $u_{k,a} = T^{k/2}\psi_a$ for $k = 0, \dots, 2N+1$. Draw independent standard Gaussian ambient vectors $g_1, \dots, g_n$, observe

$$z_{r,k,a} = \langle g_r, u_{k,a} \rangle, \qquad z_r \sim \mathcal{N}(0, K), \tag{49}$$

and put $S_n = n^{-1} \sum_r z_r z_r^T$. The diagonal time blocks of K are the matrix moments $B_k[a, b] = \left\langle \psi_a, T^k \psi_b \right\rangle$. Thus one joint covariance contains every block needed by the localizer and preserves their correlations.

Theorem 8.1 (Wishart–Loewner confidence band)**.** *Let $d = (2N+2)c$, $0 < \alpha < 1$, and*

$$\eta = \frac{\sqrt{d} + \sqrt{2\log(2/\alpha)}}{\sqrt{n}} < 1, \quad a = (1+\eta)^{-2}, \quad b = (1-\eta)^{-2}. \tag{50}$$

Then, including singular K,

$$\mathbb{P}\{aS_n \preceq K \preceq bS_n\} \geq 1 - \alpha. \tag{51}$$

Proof. Write the data matrix as $Z = GK^{1/2}$, with G standard Gaussian. The extreme-singular-value inequality gives, except on probability $2e^{-t^2/2}$,

$$\sqrt{n} - \sqrt{d} - t \leq s_{\min}(G) \leq s_{\max}(G) \leq \sqrt{n} + \sqrt{d} + t.$$

Take $t = \sqrt{2\log(2/\alpha)}$, square, divide by n, and apply congruence by $K^{1/2}$ [16, 17]. Invert the scalar factors to obtain (51). Invertibility of K is never used. ☐

To test a proposed edge, ask whether there exists any covariance Q in the band $aS_n \preceq Q \preceq bS_n$ whose extracted diagonal blocks form a positive localizer. Reject if this semidefinite feasibility problem is empty.

Corollary 8.2 (Adaptive level control)**.** *Under the Gaussian, independent, known-zero-mean model (49), the preceding rejection rule has type-I error at most α. The bound remains valid if the separating localizer witness is optimized after seeing all data.*

Proof. Under the null, the true K has a positive localizer. On the simultaneous event (51), that same K is feasible, so rejection is impossible. The rule can reject only on the complement, of probability at most α. Post-data witness selection does not alter this event containment. ☐

Non-rejection at a truncated depth is not a positive gap proof unless the number of visible atoms or an adequate rank/flatness condition is known. Likewise, the theorem does not cover Markov-correlated sketches, non-Gaussian primitives, or an estimated mean. Its role is one-sided: it falsifies an overly large visible gap with controlled level.

9 Certificates and numerical audit

Two independent scripts accompany the paper. The exact script uses SymPy to enumerate all 24 tetrahedral orders in the Pauli algebra, verifies isotropy and the common trace, reduces the formula modulo $c^2 + 3s^2 = 1$, checks the series through y^{12}, and exports the rational localizers (47)–(48). Its JSON digest is

```
8c6c32b6356a604a65fba83054fa6213
539c7bc841ca19ffa32bcdf00e2de757.
```

The deterministic numerical script evaluates every physical order at 48 pulse values and performs 128 random balanced-loop trials. The maximum operator error between the direct physical transfer and (28) is 1.89×10^{-15}; the maximum self-adjoint error in the random campaign is 1.66×10^{-16}. The least site and link Gram eigenvalues are -4.50×10^{-15} and -2.12×10^{-15}, respectively, at roundoff scale. A log–log fit gives mass error slope 1.997, consistent with the proved $O(y^2)$ law. Its JSON digest is

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667ead5425cede360bd117ab5919a34.
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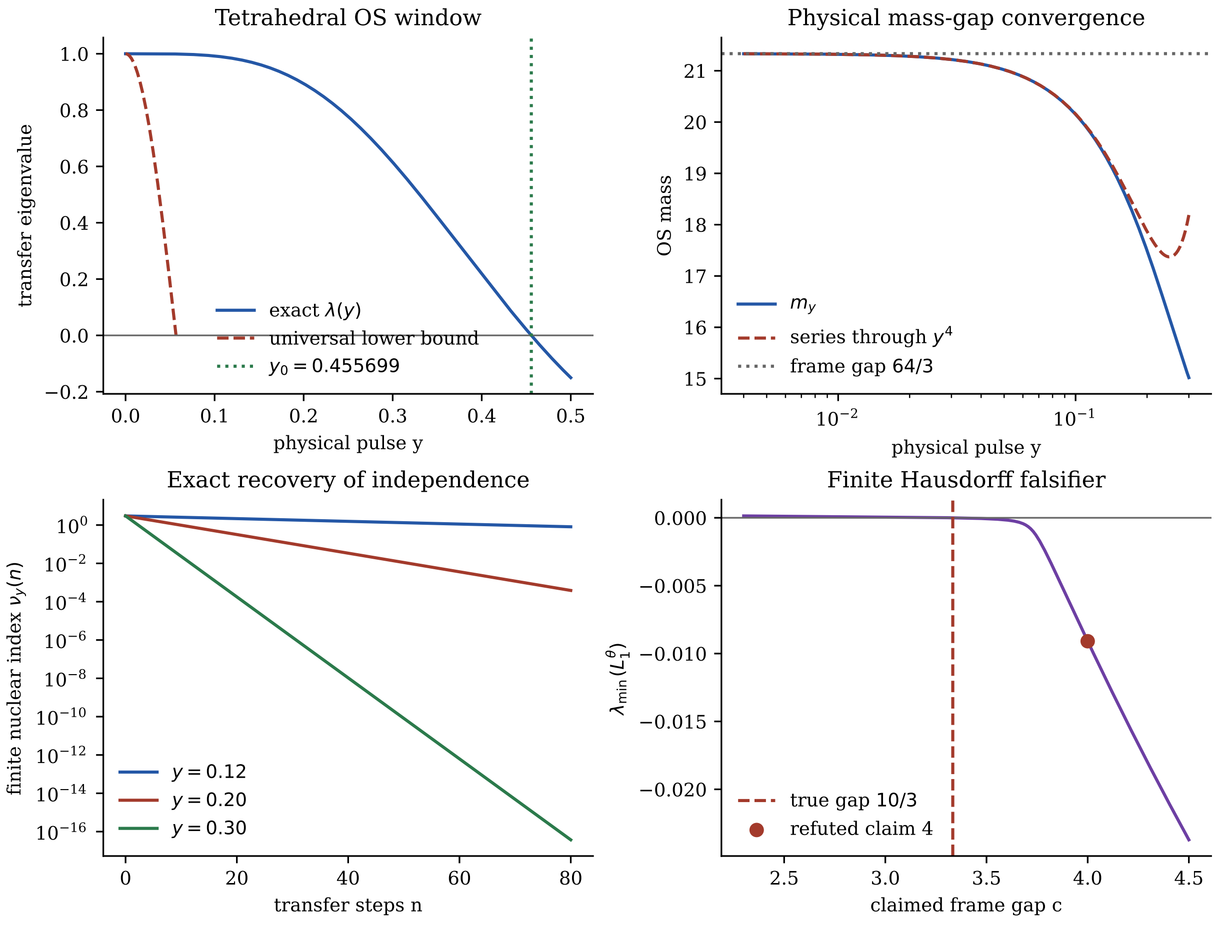

Figure 1: Finite-pulse and certificate audit. Top left: exact tetrahedral transfer eigenvalue, the universal action bound, and the first positivity zero. Top right: the OS mass and its small-pulse expansion. Bottom left: exact decay of the finite nuclear index. Bottom right: the degree-one Hausdorff localizer changes sign when a claimed rational-frame gap exceeds the true value 10/3.

The generated figure PDF has SHA-256 digest

```
90d2cd906bb1a02a493641de7ca8462d8
4f9446d176b5ad13abe026b321233fd.
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10 Limitations and falsifiable boundary

The theorem chain has several hard edges.

1. The cosine bound requires fully retained kicks. Off-block kicks make the reduced word nonunitary, so inverse pairing and the phase-diameter proof fail.
2. Reversal symmetry is essential. A self-adjoint average is not obtained from an arbitrary order law.
3. Strict transfer positivity does not alone exclude additional fixed observables. The mass theorem also requires frame irreducibility and $\varepsilon_y < \gamma$.
4. The Hausdorff conclusion is visible to W. Full-gap language requires $W > 0$, a complete trace, or separately certified detection of every eigenspace.
5. Localizer non-rejection below the complete finite depth is not evidence for a positive gap. Negative localizers are exact falsifiers; a positive certificate needs the known finite-dimensional atom bound.
6. Resolvent traces do not control nonnormal generators. For example, $A_M = \left(\begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix}\right)$ has the same resolvent traces as I_2, but e^{-tA_M} has arbitrarily large transient amplification as M grows. Our certificates apply to self-adjoint positive K.
7. The Wishart band is exact only for independent Gaussian sketches with known zero mean. It is not a generic measurement-noise theorem.
8. The OS Hamiltonian, exponential transfer clustering, and nuclear index are finite-dimensional. No thermodynamic limit, locality theorem, AQFT split inclusion, or Yang–Mills mass gap is claimed.

These restrictions make the proposal directly falsifiable. A physical retained-loop implementation outside the conservative window can be tested against the exact tetrahedral formula; a proposed gap can be refuted by a negative localizer; and every symbolic equality and numerical claim in the paper has a deterministic replay.

11 Conclusion

Forgotten quantum order does more than generate a double-commutator limit. At finite pulse, reversal-symmetric physical echoes form a self-adjoint random-unitary transfer operator. A quotient-action bound makes it strictly positive, so site and link reflection positivity, an OS Hamiltonian, exponential transfer clustering, and a finite nuclear index all follow before the continuum limit. The fourth-order physical expansion then connects its mass to the curvature-frame gap with an explicit $O(y^2)$ error.

The tetrahedral loop closes the finite-pulse problem exactly: all 24 physical orders average to one depolarizing eigenvalue, its positivity threshold is isolated, and its mass series converges to the sharp frame gap 64/3. The Hausdorff layer closes a different gap between a spectral statement and an auditable certificate. In finite dimension, the required depth is known and optimal; for rational frames, both acceptance and refutation can be checked in exact arithmetic. The Gaussian layer shows how a simultaneous covariance band preserves level under adaptive witness selection without disguising non-rejection as proof.

The resulting object is a complete chain from control architecture to positive transfer, from transfer to mass, and from mass to independently replayable data.

Reproducibility. The archive distributed with this manuscript contains both source scripts, frozen JSON outputs, the figure, a claim ledger, environment file, manifest, and a one-command replay. Exact and numerical outputs are regenerated from scratch and compared with the recorded hashes.

AI assistance disclosure. OpenAI Codex assisted with symbolic-code generation, numerical orchestration, repository and literature triage, adversarial proof checks, and manuscript editing. The author retains responsibility for every statement and the final submission.

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