

An exact relaxation optimum in a one-parameter family of four quantum measurement bases

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Abstract

Four orthonormal bases define a quantum measurement channel by averaging their nonselective rank-one projective measurements. We determine the exact Hilbert–Schmidt relaxation rate of the Poisson semigroup generated by this channel minus the identity, for the one-parameter subfamily introduced by Raynal, Lü and Englert in their study of approximately mutually unbiased bases in dimension six. A block decomposition gives the complete spectrum as explicit quadratic polynomials in the parameter. The largest relaxation rate within this subfamily, with equal measurement probabilities and unit total event rate, is exactly $73/100$, attained uniquely at $u = 7/10$. This parameter does not maximize the mean squared chordal distance between the bases. An exact rational comparison exhibits a larger mean distance with a smaller relaxation rate. We also prove the $2/3$ ceiling for arbitrary mixtures of product-vector measurement bases on $\mathbb{C}^2 \otimes \mathbb{C}^3$, which the construction exceeds. The optimality statement is restricted to the specified subfamily; no global four-basis optimality or new mutually unbiased basis construction is claimed.

1 Introduction and scope

Mutually unbiased bases are useful in quantum state determination and related measurement problems [1, 4]. In dimension six, Raynal, Lü and Englert (RLE) constructed a two-parameter family of four bases and identified a one-parameter subfamily that contains their maximizer of the mean squared chordal distance [3]. Their family provides an explicit setting in which to compare geometric and dynamical objectives.

We use those existing bases, with their attribution retained throughout. The quantity optimized here is the slowest exponential decay rate of a continuous-time, randomized measurement process. The resulting exact spectrum yields a different optimum from the mean-distance objective. The general possibility that distinct objectives have distinct optimizers is not the contribution; the result is the explicit spectral formula, its sharp optimization on the RLE subfamily, and an exact comparison of the two objectives there.

Measurement-induced mixing is related in motivation to noncommuting block-measurement diffusion [6], but the processes differ. The present generator is a fixed average of complete rank-one measurements minus the identity. It is not a small-angle product limit and contains no additional Hamiltonian. Measurement-channel spectra also occur in quantum verification [4] and classical shadows [5]. We use the standard operator and Gram-matrix framework rather than claim that framework as new.

Every optimality assertion below specifies the admissible family and fixes both the measurement probabilities and the total event rate. In particular, the theorem does not optimize over the full two-parameter RLE family, over all four bases in $\mathbb{C}^6$, or over physical implementation

costs. The proof is analytic; exact symbolic calculations supplied with the manuscript provide reproducibility checks rather than a substitute for the proof. Version 1 received two advisory AI assessments. This revision addresses their corrections; those reports remain specific to version 1 and are not treated as assessments of the present PDF.

2 Measurement dynamics and its Gram matrix

Let $B_j = (b_{j1}, \dots, b_{j6})$, $j = 0, 1, 2, 3$, be orthonormal bases of $\mathbb{C}^6$. Put $\Pi_{jk} = |b_{jk}\rangle\langle b_{jk}|$ and

$$\mathcal{E}_j(X) = \sum_{k=1}^6 \Pi_{jk} X \Pi_{jk}, \quad \mathcal{T} = \frac{1}{4} \sum_{j=0}^3 \mathcal{E}_j, \quad \mathcal{L} = \mathcal{T} - \text{id}. \quad (1)$$

At each event of a rate-one Poisson process, one chooses a basis uniformly, measures, and discards the outcome. The averaged evolution is $e^{t\mathcal{L}}$. Indeed,

$$e^{t(\mathcal{T}-\text{id})} = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{T}^n,$$

which is completely positive, trace preserving and unital. Each $\mathcal{E}_j$ is an orthogonal projection in the Hilbert–Schmidt inner product. Thus $\mathcal{T}$ is self-adjoint and positive semidefinite as a superoperator, with spectrum in $[0, 1]$.

Let $\mathcal{H}_0 = \{X \in M_6(\mathbb{C}) : \text{Tr } X = 0\}$. Define

$$\gamma = 1 - \lambda_{\max}(\mathcal{T}|_{\mathcal{H}_0}). \quad (2)$$

This definition gives $\gamma = 0$ if a nonscalar stationary operator exists. For every state ρ ,

$$\|e^{t\mathcal{L}}(\rho) - I_6/6\|_2 \leq e^{-\gamma t} \|\rho - I_6/6\|_2. \quad (3)$$

The exponent is sharp: the largest eigenvalue on $\mathcal{H}_0$ has a nonzero Hermitian eigenoperator, and a sufficiently small multiple can be added to $I_6/6$ to obtain a density matrix.

Write $v_{jk} = \Pi_{jk} - I_6/6$ and form the real symmetric 24×24 matrix

$$G_{(j,k),(\ell,r)} = \text{Tr}(v_{jk}v_{\ell r}) = |\langle b_{jk}, b_{\ell r} \rangle|^2 - \frac{1}{6}. \quad (4)$$

For $X \in \mathcal{H}_0$, the identity term cancels because

$$\sum_{k=1}^6 v_{jk} \text{Tr}(v_{jk}X) = \sum_{k=1}^6 \Pi_{jk} \text{Tr}(\Pi_{jk}X) - \frac{I_6}{6} \text{Tr } X = \mathcal{E}_j(X).$$

Thus the frame operator associated with these vectors is $4\mathcal{T}|_{\mathcal{H}_0}$; see also the operator-frame formulation in [2]. With column vectorization, $|\Pi_{jk}\rangle = \bar{b}_{jk} \otimes b_{jk}$, so the full channel is $\frac{1}{4} \sum_{jk} |\Pi_{jk}\rangle\langle \Pi_{jk}|$. This makes explicit the standard connection to verification operators discussed in [4]. The centered frame operator $4\mathcal{T}|_{\mathcal{H}_0}$ and G have the same nonzero eigenvalues, including multiplicity. Therefore

$$\gamma = 1 - \frac{1}{4} \lambda_{\max}(G). \quad (5)$$

The distinction between the dimension-24 Gram matrix and the dimension-35 traceless operator space matters for zero eigenvalues.

3 The RLE subfamily and its complete spectrum

We give all bases explicitly to fix phases, ordering and the parameter convention. These matrices are the RLE construction [3, Eqs. (5)–(7)], expressed in notation adapted to the channel. Set

$$\omega = e^{2\pi i/3}, \quad F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Z = \text{diag}(1, -1), \quad X = \text{diag}(\bar{x}, x), \quad H = \begin{pmatrix} 1 & \omega t^2 \\ 1 & -\omega t^2 \end{pmatrix},$$

where $|x| = |t| = 1$. A bar denotes entrywise complex conjugation. Define $U_0 = I_6$ and

$$U_1 = \frac{1}{\sqrt{6}} \text{diag}(X, i\bar{\omega}tZ\bar{X}^2, X) \begin{pmatrix} F & F & F \\ F & \omega F & \bar{\omega}F \\ H & \bar{\omega}H & \omega H \end{pmatrix}, \quad (6)$$

$$U_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} F & F & F \\ H & \omega H & \bar{\omega}H \\ H & \bar{\omega}H & \omega H \end{pmatrix}, \quad (7)$$

$$U_3 = \frac{1}{\sqrt{6}} \text{diag}(\bar{X}, \bar{\omega}\bar{X}, -itZX^2) \begin{pmatrix} F & F & F \\ H & \omega H & \bar{\omega}H \\ F & \bar{\omega}F & \omega F \end{pmatrix}. \quad (8)$$

The columns of U_j are B_j . Every displayed matrix is unitary; the unitarity of the central block matrices follows from $FF^* = HH^* = 2I_2$ and $1 + \omega + \bar{\omega} = 0$.

For the subfamily studied here, choose

$$\frac{1}{4} \leq u \leq 1, \quad x = \sqrt{1-u} + i\sqrt{u}, \quad q = \frac{1-2u}{\sqrt{u}}, \quad t = e^{-i\pi/3}(q + i\sqrt{1-q^2}). \quad (9)$$

The condition $|q| \leq 1$ is equivalent to $1/4 \leq u \leq 1$. Equation (9) fixes a branch of the RLE relation $\cos(\theta_t + \pi/3) = \cos(2\theta_x)/\sin \theta_x$, with $u = \sin^2 \theta_x$. Equivalent phase branches are not needed for the theorem.

3.1 Overlap blocks

For $j \neq \ell$, let $P_{j\ell}$ have entries $|(U_j^* U_\ell)_{kr}|^2$. The canonical basis is unbiased with each of the other three, so $P_{0j} = J_6/6$, where J_n is the all-ones matrix. The remaining probabilities depend on just three real numbers:

$$A = \frac{4}{3}(1-u)^2, \quad B = (1-2u)^2, \quad C = \frac{(1-u)(4u-1)}{3}, \quad A + B + 4C = 1. \quad (10)$$

With

$$D = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} B & A \\ A & B \end{pmatrix}, \quad O = CJ_2,$$

direct multiplication gives

$$P_{12} = \begin{pmatrix} D & O & O \\ O & D & O \\ O & O & D \end{pmatrix}, \quad P_{23} = \begin{pmatrix} O & D & O \\ O & O & D \\ D & O & O \end{pmatrix}, \quad P_{31} = \begin{pmatrix} O & O & \tilde{D} \\ \tilde{D} & O & O \\ O & \tilde{D} & O \end{pmatrix}. \quad (11)$$

For completeness, one may obtain the first entry in (10) from the RLE overlap amplitude $\alpha = 4\sqrt{1-u}(1 - \omega\bar{t}\sqrt{u})$. Since $\omega\bar{t} = -q + i\sqrt{1-q^2}$,

$$\frac{|\alpha|^2}{36} = \frac{4}{9}(1-u)((1 + \sqrt{u}q)^2 + u(1-q^2)) = \frac{4}{3}(1-u)^2.$$

Here is the remaining algebra, rather than an inference from row normalization. Put $p = \sqrt{u}$, $c = \sqrt{1-u}$, $r = \sqrt{1-q^2}$, and write $t = e^{i\theta_t}$. The branch in (9) gives

$$\Re t = \frac{q + \sqrt{3}r}{2}, \quad \cos(\theta_t - 2\pi/3) = -q, \quad \cos(\theta_t + 2\pi/3) = \frac{q - \sqrt{3}r}{2}.$$

In RLE's overlap formulas [3, Eqs. (8)–(12)], denote their γ amplitude by η to distinguish it from the relaxation rate. The remaining amplitudes are

$$\begin{aligned} \beta &= -2i\bar{\omega}t(1 - 2u - 2p\cos(\theta_t - 2\pi/3)) = -6i\bar{\omega}t(1 - 2u), \\ \eta &= -2\bar{\omega}c(\bar{\omega} + 2\bar{t}p), \\ \delta &= -2it(1 - 2u - 2p\cos\theta_t) = 2i\sqrt{3}prt, \\ \epsilon &= -2i\bar{\omega}\bar{t}(1 - 2u - 2p\cos(\theta_t + 2\pi/3)) = -2i\sqrt{3}pr\bar{\omega}\bar{t}. \end{aligned}$$

Consequently,

$$\begin{aligned} |\beta|^2 &= 36(1 - 2u)^2, \\ |\eta|^2 &= 4(1 - u)(1 + 4u - 4pq) = 12(1 - u)(4u - 1), \\ |\delta|^2 &= |\epsilon|^2 = 12u(1 - q^2) = 12(1 - u)(4u - 1). \end{aligned}$$

RLE's cyclic placement of these amplitudes, including the interchange of α and β in the 31 block, therefore gives (11) after division by 36. This proves equality of all four C entries before invoking normalization. The supplement also verifies every entry directly from (6)–(9) for symbolic u , not just at the optimum.

Theorem 3.1 (Complete Gram spectrum). *Set*

$$s = A - B = \frac{1 + 4u - 8u^2}{3}, \quad k = 1 - 6C = 8u^2 - 10u + 3. \quad (12)$$

For the bases (6)–(9), the Gram spectrum, with multiplicities, is

$$\text{spec}(G) = \{0^{(4)}, 1^{(5)}, (1 - 2s)^{(3)}, (1 + s)^{(6)}, (1 + 2k)^{(2)}, (1 - k)^{(4)}\}. \quad (13)$$

When eigenvalues coincide, their multiplicities add. In particular,

$$\gamma(u) = 1 - \frac{1 + \max\{0, -2s, s, 2k, -k\}}{4}. \quad (14)$$

Proof. Each diagonal block of G is $Q = I_6 - J_6/6$. The canonical block has no off-diagonal couplings and contributes $0^{(1)}$ and $1^{(5)}$.

For the other three blocks, identify the six coordinates with $\mathbb{C}^3 \otimes \mathbb{C}^2$ and split each two-coordinate block into $e_+ = (1, 1)/\sqrt{2}$ and $e_- = (1, -1)/\sqrt{2}$. Let

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R^3 = I_3.$$

Every 2×2 block above, and its transpose, preserves the e_+ and e_- lines. Hence the splitting is invariant for the full symmetric Gram matrix. On the antisymmetric space $\mathbb{C}^3 \otimes e_-$, Q is the identity, and the blocks P_{12}, P_{23}, P_{31} act respectively as $sI_3, sR, -sR^2$. Conjugating the three-basis block by $\text{diag}(I_3, I_3, R^2)$ reduces it to

$$\begin{pmatrix} 1 & s & -s \\ s & 1 & s \\ -s & s & 1 \end{pmatrix} \otimes I_3.$$

The scalar 3×3 factor has eigenvalues $1 - 2s$ once and $1 + s$ twice, giving multiplicities three and six.

On the symmetric space, the constant vector in $\mathbb{C}^3$ gives a null vector for each basis block, contributing three more zeros. Its orthogonal complement has dimension two. There the three overlap blocks, after subtracting $J_6/6$, act as kI_3, kR, kR^2 restricted to that complement: this uses $A + B - 2C = 1 - 6C = k$. The same conjugation gives the scalar matrix

$$\begin{pmatrix} 1 & k & k \\ k & 1 & k \\ k & k & 1 \end{pmatrix}$$

on each of two copies. Its eigenvalues are $1 + 2k$ once and $1 - k$ twice. This accounts for all 24 dimensions and proves (13). Equation (14) follows from (5). $\square$

4 The exact optimum and the distance comparison

Theorem 4.1 (Sharp subfamily optimum). *For equal probabilities and unit total measurement rate, the largest relaxation rate within (9) is $73/100$. It is attained uniquely at $u = 7/10$.*

Proof. Put $h(u) = \max\{0, -2s, s, 2k, -k\}$. We prove $h(u) \geq 2/25$ over the entire parameter interval. For $1/4 \leq u \leq 3/5$, the function s is nonincreasing and

$$h(u) \geq s(u) \geq s(3/5) = 13/75 > 2/25.$$

For $3/5 \leq u \leq 7/10$,

$$-k - \frac{2}{25} = -8 \left(u - \frac{11}{20}\right) \left(u - \frac{7}{10}\right) \geq 0,$$

with equality in that interval only at $u = 7/10$. For $7/10 \leq u \leq 1$,

$$-2s - \frac{2}{25} = \frac{16}{3} \left(u - \frac{7}{10}\right) \left(u + \frac{1}{5}\right) \geq 0,$$

again with equality only at $7/10$. At that parameter, $s = -1/25$, $k = -2/25$ and $h = 2/25$. Substitution into (14) gives $1 - (27/25)/4 = 73/100$. $\square$

Corollary 4.2 (An explicit algebraic construction). *Taking*

$$x = \frac{\sqrt{30} + i\sqrt{70}}{10}, \quad q = -\frac{2\sqrt{70}}{35}, \quad t = \frac{1 - i\sqrt{3}}{2}(q + i\sqrt{1 - q^2})$$

in (6)–(8) gives four bases whose channel on $\mathcal{H}_0$ has spectrum

$$\left\{ 0^{(15)}, \left(\frac{21}{100}\right)^{(2)}, \left(\frac{24}{100}\right)^{(6)}, \left(\frac{1}{4}\right)^{(5)}, \left(\frac{27}{100}\right)^{(7)} \right\}. \quad (15)$$

Proof. Here $(A, B, C) = (3/25, 4/25, 9/50)$. Use Theorem 3.1 and the nonzero-spectrum correspondence after (4). The Gram rank is 20, so the traceless channel has nullity $35 - 20 = 15$. $\square$

For comparison with the original geometric objective, define the mean squared chordal distance as in [3]:

$$\overline{D^2} = 1 - \frac{1}{30} \sum_{0 \leq j < \ell \leq 3} \|P_{j\ell} - J_6/6\|_F^2. \quad (16)$$

Each noncanonical pair has squared Frobenius defect $6(A^2 + B^2 + 4C^2) - 1$, while the three canonical pairs have zero defect. Hence

$$\overline{D^2}(u) = -\frac{448u^4 - 1024u^3 + 888u^2 - 352u + 25}{30}. \quad (17)$$

For any four orthonormal bases in dimension six, the four diagonal Gram blocks have squared Frobenius norm five each, while each off-diagonal pair is counted twice. Thus

$$\text{Tr } G = 20, \quad \overline{D^2} = 1 - \frac{\|G\|_F^2 - 20}{60}.$$

The two objectives therefore depend on different parts of the same spectrum: the distance depends on its second moment, whereas the relaxation rate depends on its largest eigenvalue.

At the spectral optimum,

$$\overline{D^2}(7/10) = \frac{6239}{6250}, \quad \left. \frac{d\overline{D^2}}{du} \right|_{u=7/10} = -\frac{12}{625} \neq 0. \quad (18)$$

Thus this interior parameter cannot maximize the distance objective. Table 1 gives a strict comparison using only rational values. The increase in distance in its first row over its second is exactly $38513/750000000$, whereas its relaxation rate is smaller by $29/20000$.

Parameter u	Mean distance $\overline{D^2}$	Relaxation rate γ
139/200	748718513/750000000	14571/20000
7/10	6239/6250	73/100

Table 1: A strict reversal between the geometric and dynamical objectives. Both columns are to be maximized. The first row has larger distance; the second has faster relaxation.

The distance-maximizing parameter identified in [3] has $u \simeq 0.694552035120092$, the real root of $112u^3 - 192u^2 + 111u - 22 = 0$. Substitution into our rate formula gives approximately 0.728425. The rate gain over this geometric optimizer is approximately 0.2162%, distinct from the larger product-class comparison below. Neither the proof of Theorem 4.1 nor the exact comparison in Table 1 depends on this decimal evaluation.

5 Product measurements and unrestricted four-basis bounds

Proposition 5.1 (Product-vector ceiling). *Let $\mathcal{T}$ be any probability-weighted mixture of complete orthonormal bases consisting of product vectors on $\mathbb{C}^2 \otimes \mathbb{C}^3$. For the rate-one generator $\mathcal{T} - \text{id}$, $\gamma \leq 2/3$. This includes orthonormal product bases that are not tensor products of two fixed local bases.*

Proof. Consider the three-dimensional operator subspace $V = \{A \otimes I_3 : \text{Tr } A = 0\}$. The orthogonal projection of a rank-one product projector $\Pi = \rho_A \otimes \rho_B$ onto V is

$$Q_V \Pi = (\rho_A - I_2/2) \otimes I_3/3, \quad \|Q_V \Pi\|_2^2 = \frac{1}{6}.$$

Every basis has six projectors, and its dephasing superoperator is $\sum_k |\Pi_k\rangle\langle\Pi_k|$. Its compressed superoperator trace on V is therefore one, and the same holds for every probability mixture. The largest Rayleigh quotient on V is at least this compressed trace divided by $\dim V$, namely $1/3$. Maximizing over all of $\mathcal{H}_0$ can only increase it; invariance of V under $\mathcal{T}$ is not assumed. Equation (2) proves the claim. $\square$

The ceiling is sharp. For example, take the uniform mixture of the $3 \times 4 = 12$ tensor-product bases obtained from the three Pauli measurement bases on the qubit and four mutually unbiased measurement bases on the qutrit. The local traceless eigenvalues are $1/3$ and $1/4$, and the correlation-sector eigenvalue is $1/12$, giving $\gamma = 2/3$. These finite local constructions are standard [1]. The rate $73/100$ is therefore 9.5% larger than the product-vector ceiling, with the total event rate held fixed. This is a rate comparison, not a statement about gate counts, laboratory time per measurement or energetic cost.

Let $\gamma_4^*(6)$ denote the supremum over all four bases with uniform probabilities in dimension six. The construction and the elementary projector bound give

$$\frac{73}{100} \leq \gamma_4^*(6) \leq \frac{3}{4}. \quad (19)$$

To see the upper bound, restrict each $\mathcal{E}_j$ to $\mathcal{H}_0$, where it is a nonzero orthogonal projection. On a unit vector in the range of any one of them, the quadratic form of their sum is at least one. Thus $\lambda_{\max}(4\mathcal{T}|_{\mathcal{H}_0}) \geq 1$. If equality holds, every unit vector in any one projection range must be annihilated by the other three positive projections. The four traceless projection ranges are therefore mutually orthogonal, equivalently the bases are mutually unbiased. Conversely, four mutually unbiased bases attain equality. The maximum is attained because $U(6)^4$ is compact and the largest eigenvalue is continuous. Hence $\gamma_4^*(6) = 3/4$ if and only if four mutually unbiased bases exist in dimension six; this standard projection connection is also present in the verification formulation of [4]. We neither establish equality nor sharpen the unrestricted upper bound. In particular, $73/100$ is not asserted to be the global optimum.

6 Reproducibility, attribution and limitations

The version-2 source bundle contains `verify.py`, `verify_generic.py`, `requirements.txt`, and the dated run records `verification.json` and `generic-verification.json`. Run the two Python scripts with SymPy 1.14.0. The first constructs the four bases at the algebraic optimum, checks unitarity, forms the Gram matrix directly, and checks its exact eigenvalues and multiplicities. It also verifies the characteristic polynomial of the polynomial block matrix over $\mathbb{Q}[u]$, the distance comparison, and the factorized optimization inequalities.

The second uses a multiquadratic algebra over $\mathbb{Q}(u)$ to construct the original matrices for symbolic u . It verifies phase normalization, every unitarity and canonical-unbiasedness entry, and all three overlap-probability blocks. The radical squares are $1 - u$, u , $4u - 1$, 3 and -1 ; only nonzero powers of u occur in denominators, so the identities extend throughout $[1/4, 1]$, including both endpoints. Neither script relies on floating-point tolerances. These are reproducibility checks, not proof-assistant certificates.

For release provenance, version 1 described an accompanying executable supplement, but its public package did not contain those files. They were supplied privately to its AI referees. Version 2 corrects that discrepancy by including the source, environment requirements and actual run outputs in its own versioned bundle; no file is retroactively added to the version-1 release. The mathematical argument remains self-contained.

The RLE bases, their parameter relation and their original distance objective are prior work. The projector-frame interpretation is also standard. A focused literature search did not identify the particular sharp subfamily relaxation optimum proved here, but that search is not an exhaustive priority certificate. The paper makes a narrowly stated contribution and leaves its novelty and significance open to independent assessment. No formal proof-assistant verification or independent human review is claimed.

AI assistance. This manuscript was prepared with substantial assistance from OpenAI Codex during a research session on 11 October 2026. The assistant performed literature

searches, proposed the spectral optimization problem, derived the formulas, wrote the draft and verification code, and executed symbolic and numerical checks. The author commissioned the work and requested preparation for AIRR publication. Version 1 subsequently received separate advisory reviews from Claude Opus 5.5 (max, Claude Code) and GPT-6 Astra (xhigh). Their comments informed this revision and the expanded symbolic verifier. The generic verifier adapts the exact-algebra approach used in Astra’s version-1 review. Their reports and ratings remain attached to version 1; this PDF requires its own exact-version assessment. The AI assistance and computational checks should not be read as independent peer review.

Conflict of interest. The author operates AIRR.science. Hosting or release there is distinct from independent scientific assessment. This revision is a working manuscript, without human scientific acceptance. AI reports are advisory and version-specific; no independent human peer review is claimed.

Revision record. Version 2 supplies the missing B, C overlap derivation, corrects the decimal of the geometric optimizer, explicitly counts the 12 product bases in the sharpness example, and publishes the reproducibility bundle. It also clarifies the frame identity, invariant sectors, compression argument, spectral-moment comparison and unrestricted-bound equality. The principal theorem, parameter domain and scope of the novelty claim are unchanged. The two-parameter family and unrestricted optimum remain open directions, not conclusions of this revision.

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