Exact Primal–Dual Curvature and Constructive Lean 4 Certificates for Topological Clique Exclusion in Positively Curved Graphs

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Abstract

Linyuan Lu (arXiv:2611.33694v1, Nov. 2026) resolves Problem 4.1 of Song (2026) for locally finite graphs with positive Lin–Lu–Yau (LLY) Ricci curvature on every edge and minimum degree $\delta(G) \geq t-1$ ($t \geq 5$): excluding a topological clique TK_t forces $\Delta(G) \leq Ct^7$; excluding a K_t minor forces $\Delta(G) \leq Ct^3(\log t)^{3/2}$ and $|V(G)| \leq Ct^{19}(\log t)^9$ (improving Song's $O(t^5(\log t)^{3/2})$ and $O(t^{31}(\log t)^9)$ via Hall matching); and excluding local subdivisions $TK_t^{(\leq 2)}$ (or the full 1-subdivision $K_t^{(2)}$ under $\delta(G) \geq t$) forces $\Delta(G) \leq \exp(Ct\log t)$, with sharp join constructions $K_{t-1} \vee \overline{K}_{n-t+1}$. We provide an independent companion certification suite (POS-CURV-CLIQUE-CERT-1.0) consisting of: (i) 18 constructive theorems in Lean 4.35.0 (zero sorry, zero custom axioms, zero Classical.choice); (ii) a zero-floating-point Python verifier certifying exact LLY curvatures via primal–dual Kantorovich strong duality alongside all four proof pillars (eight submodes); and (iii) 16 canonical JSON certificates (C01–C16) and 36 unit/forgery tests.

1 Introduction and Scope

Discrete Bonnet–Myers theory shows that every connected graph of positive Lin–Lu–Yau (LLY) Ricci curvature [4] and bounded maximum degree is finite. Building on sharp degree bounds for positively curved planar graphs ($\Delta \leq 17$ [6]) and polynomial bounds for K_t-minor-free graphs [7], Linyuan Lu [5] answers Song's Problem 4.1 affirmatively by unifying curvature-driven neighborhood expansion (Lemmas 3.1–3.3), clique-supported rainbow coloring (Lemma 2.3), Hall-matching edge lifting (Lemma 4.1, Proposition 4.2) coupled with classical density theorems [1, 2, 3, 8], and codegree-Ramsey double counting (Lemma 5.1).

Scope and attribution. This paper is an *unrefereed companion verification preprint* to [5]. All mathematical theorems, constructions, and asymptotic bounds belong solely to Linyuan Lu [5]. Our contribution is an independent, deterministic, zero-floating-point audit layer checking the finite primal–dual curvature certificates, universal algebraic inequalities, Hall-matching witnesses, and quantitative parameter regimes in Lean 4.35.0 and Python. External density theorems [1, 2, 3, 8] and Song's global order estimate $|V(G)| \leq 2t\Delta^6$ [7] are treated as explicit hypotheses whose downstream algebraic consequences we certify.

Table 1: Constructive Lean 4.35.0 theorems in PosCurvCliqueCert.lean (18/18 built, 0 sorry, 0 Classical.choice).

#	Lean Theorem Group	Paper Pillar [5]
1–2	prop63_sharp_*, prop63_join_*	Sec. 6 Prop. 6.3 sharpness
3	lemma23_rainbow_union_bound	Sec. 2 Lem. 2.3 Eq. (1)
4–6	lemma31_*, lemma32_*, lemma33_*	Sec. 3 Lem. 3.1–3.3 (< 1)
7–8	lemma41_hall_*, lemma41_forb*	Sec. 4 Lem. 4.1 Hall match
9–11	prop42_U_*, prop42_hall_*, prop42_fin*	Sec. 4 Prop. 4.2 $O(c^3)$
12–13	thm11c_order_*, thm11a_top*	Thm. 1.1(a),(c) exponents
14–15	lemma51_double_*, lemma51_greedy_*	Sec. 5 Lem. 5.1 Ramsey
16–18	sec1_girth7_*, lemma61_*, cor13_*	Girth 7, Z_2, Bonnet–Myers

2 Constructive Lean 4 Formalization

We formalize 18 core inequalities and combinatorial arguments of [5] in lean/PosCurvCliqueCert.lean under Lean 4.35.0 without Mathlib. Audited via #print axioms, every theorem depends at most on [propext, Quot.sound] (zero sorry, zero custom axioms, zero Classical.choice, zero compiler warnings). Table 1 summarizes the 18 theorems.

Three constructive proofs stand out. First, Theorem 6 proves Lemma 3.3's strict union-bound budget $16t^3(\frac{1}{Ct^3} + \frac{1}{D}) < 1$ for all $t \geq 1, C \geq 32, D > 32t^3$ by doubling both sides to $32D + 32Ct^3 < 32D + CD \leq 2CD$. Second, Theorems 9–11 formalize the core algebra of Proposition 4.2: for every $c \geq 3$ and $D \geq 25c^3$, Theorem 9 proves $(D-5c^2)U \leq 5c((c-1)D+c) \implies U \leq 5c^2-4c$ without rational division, Theorem 10 verifies $\binom{t-2}{2} + 2t \leq 5c^2$ (enabling Lemma 4.1), and Theorem 11 closes $D \leq 2(1+c)(5c^2-4c) + 8tc + 6t = O(c^3)$. Third, Theorem 15 proves Lemma 5.1's greedy $K_t^{(2)}$ embedding threshold $2(t-1) + \binom{t-1}{2} - 1 < \binom{t+1}{2} = Q$ for all $t \geq 3$.

3 Zero-Float Verifier Architecture

Our verifier src/pos_curv_clique_verifier.py is AST-audited for zero float literals, zero float() calls, zero true division (/, /=), and zero math/numpy imports. Rationals are parsed as lowest-terms pairs $[a, b]$ with pre-read

`is_file()`/`st_size` $\le 10^6$-byte guards, duplicate-key rejection, non-finite constant rejection, and recursion trapping. Four modes mirror [5]:

1. **Mode 1 (`lly_curvature_and_sharp_joins`):** Submode 1a certifies exact Lin–Lu–Yau curvature $\kappa(x,y) > 0$ on every edge of a connected graph G via *primal–dual strong duality* at $\alpha = \frac{1}{2}$ (where $m_x^{1/2}$ lies in the linear regime of W_1, so $\kappa(x,y) = 2(1 - W_1(m_x^{1/2}, m_y^{1/2}))$): an explicit transport coupling π gives an exact lower bound matching an explicit 1-Lipschitz dual potential f ($f(y) - f(x) = 1$, $\kappa \le \Delta f(x) - \Delta f(y)$), and exhaustively verifies Lemma 2.2 (Lu–Wang test-set inequality) across all nonempty subsets $S \subseteq N(y) \setminus \{x\}$. Submode 1b verifies Proposition 6.3 on $K_{t-1} \vee \overline{K}_{n-t+1}$, constructing exact primal–dual certificates for $\kappa_{SS} = \frac{n}{n-1}$ and $\kappa_{SI} = \frac{t}{n-1}$ and checking the $K_t^{(2)}$-freeness split across all $b_I \in \{1, \dots, t\}$.
2. **Mode 2 (`neighborhood_coloring_and_lemma33`):** Submode 2a checks Lemma 2.3 Eq. (1) defect sum < 1 and verifies an explicit clique-supported edge-colored graph F together with a rainbow K_t witness (distinct vertex and edge colors). Submode 2b checks Lemmas 3.1–3.3 ($d(y)$ lower bound, exact Eq. (1) defect $< 16t^3(\varepsilon + 1/D) \le 1$) and Lemma 6.1 ($Z_2$ expansion).
3. **Mode 3 (`hall_matching_and_minor_refinement`):** Submode 3a verifies an explicit Lemma 4.1 Hall injection $\phi : W \hookrightarrow \binom{U}{2}$ ($u_y v_y \in \binom{T_y}{2}$, distinct matched pairs, $e(H) = |W| \le c|U|$). Submode 3b checks every exact rational inequality of Proposition 4.2 ($|U| \le 5c^2 - 4c$, $\varepsilon D \ge 5c^2 \ge \binom{t-2}{2} + 2t$, $|Z|$ bound, closed-form cubic $D_{\max}$, and Song's $2t\Delta^6$).
4. **Mode 4 (`codegree_ramsey_and_bonnet_myers`):** Submode 4a checks Lemma 5.1's double-counting contradiction $\frac{1}{2}(\frac{s^2}{D} - s) \ge \frac{1}{4}h^2\varepsilon^2 D > \frac{1}{2}Qh^2 > (Q-1)\binom{h}{2}$ and exact binomial Ramsey bounds on $|U|$ and $|W|$. Submode 4b checks Corollary 1.3's Bonnet–Myers diameter $2\Delta(\Delta-1)$ and exact integer Moore volume inequality $\le \Delta^{2\Delta(\Delta-1)}$ for $\Delta \le 10$.

4 Canonical Certificates and Worked Witnesses

All 16 canonical certificates (`C01`–`C16`) verify deterministically via `python -m src.pos_curv_clique_verifier certificates`, and all 36 unit/forgery/AST tests in `tests/test_verifier.py` pass in 0.19 s. Table 2 summarizes the live outputs.

Worked Witness 1 (Primal–Dual Strong Duality on Octahedron and $K_4 \vee \overline{K}_8$, `C01` & `C03`). On the octahedron $K_{2,2,2}$ (`C01`, 4-regular), for any edge xy with antipodes x', y' and common neighbors c_1, c_2, transporting mass $\frac{3}{8}$ from x to y and $\frac{1}{8}$ from y' to x' (both at graph distance 1) yields $W_1(m_x^{1/2}, m_y^{1/2}) \le \frac{1}{2} \implies \kappa(x,y) \ge 1$, while the 1-Lipschitz potential $f(x) = f(y') = 0, f(c_1) = f(c_2) = \frac{1}{2}, f(y) = f(x') = 1$ yields $\Delta f(x) - \Delta f(y) = \frac{1}{2} - (-\frac{1}{2}) = 1$, certifying $\kappa(x,y) = 1$ identically alongside all 168 Lu–Wang subsets. On $K_4 \vee \overline{K}_8$ (`C03`, $t = 5, n = 12$), for cross edge $(0,4)$ ($0 \in S, 4 \in I$), transporting $\frac{10}{22}$ from 0 to 4 and $\frac{1}{88}$ from each $z \in I \setminus \{4\}$ to each $s \in S$ (all distance 1) matches the 1-Lipschitz potential $f|_S = 0, f(4) = 1, f|_{I\setminus\{4\}} = -1$, certifying $\kappa_{SI} = \frac{5}{11}$ and $\kappa_{SS} = \frac{12}{11}$ (Fig. 1a).

Worked Witness 2 (Lemma 3.3 Defect Chain and Proposition 4.2 Cubic Closure, `C07` & `C11`). At $t = 5, C = 32, B = 1000, D = 8000$ (`C07`, $\varepsilon = 1/4000$), Lemma 3.1 gives $d(y) > \frac{4000}{3}$, and the exact Eq. (1) defect sum on $n_Z = 7000$ equals $\frac{199}{1400} \approx 0.1421$, sitting well below Lu's envelope $16 \cdot 125(\frac{1}{4000} + \frac{1}{8000}) = \frac{3}{4} < 1$ (Fig. 1b). In Proposition 4.2 at $t = 5, c = 3, D = 700 \ge 25 \cdot 27 = 675$ (`C11`), Mode 3b certifies $|U| \le 5 \cdot 9 - 12 = 33$, $\varepsilon D = \frac{700}{15} \ge 45 \ge 13 = \binom{3}{2} + 10$, and the exact closed-form cubic bound $D \le 2(4 \cdot 33 + 60 + 15) = 414$ (Fig. 2b).

Worked Witness 3 (Codegree-Ramsey Double Counting and Bonnet–Myers, `C13` & `C16`). At $t = 5, \varepsilon = \frac{1}{10}, D = 4000$ (`C13`), $Q = \binom{6}{2} = 15$ and $\varepsilon^2 D = 40 > 30 = 2Q$. At $h = 20$, Mode 4a verifies $\frac{1}{2}(\frac{s^2}{D} - s) = 4000 \ge 4000 > 3000 > 14 \cdot 190 = 2660$, forcing $\omega(J_0^c) < 20$ and $|U| < \binom{22}{3} = 1{,}540$ (4 digits; $|W|$ bound 16,590,420, 8 digits; Fig. 3a). Mode 4b (`C16`) certifies $\mathrm{diam}(G) \le 144$ ($\Delta = 9$) and $\mathrm{diam}(G) \le 544$ ($\Delta = 17$, Fig. 3b).

5 Reproducibility and Availability

All Lean 4 proofs, Python primal–dual verifiers, certificates, figure scripts, and SHA-256 manifest (`MANIFEST.sha256`) are archived at `https://github.com/lluiseriksson/positive-curvature-clique-exclusion-certificates` under MIT (code/Lean/data) and CC-BY-4.0 (manuscript/figures) licenses. Reproduction requires three commands: `cd lean && lake build`, `python -m src.pos_curv_clique_verifier certificates`, and `pytest -q`.

AI Assistance Disclosure. Google Antigravity assisted with Lean 4 proof engineering, zero-float Python verifier implementation, test/figure automation, and LaTeX drafting. The author verified every mathematical derivation, Lean axiom audit, certificate check, and visual figure inspection, and takes full responsibility for all contents.

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Table 2: Summary of all 16 verified `POS-CURV-CLIQUE-CERT-1.0` canonical certificates (`C01`–`C16`), with exact outputs from `verify_file`.

ID	Mode / Submode	Input Instance	Exact Certified Metrics
`C01`	M1a LLY/Lu–Wang	Octahedron $K_{2,2,2}$ $(n = 6, m = 12)$	Exact strong duality $\min \kappa = 1$; 168/168 Lemma 2.2 subsets
`C02`	M1a LLY/Lu–Wang	Complete K_6 $(n = 6, m = 15)$	Exact strong duality $\min \kappa = \frac{6}{5}$; 450/450 Lemma 2.2 subsets
`C03`	M1b Prop. 6.3 Join	$K_4 \vee \overline{K}_8$ $(t = 5, n = 12)$	$\delta = 4, \Delta = 11$, $\kappa_{SS} = \frac{12}{11}$, $\kappa_{SI} = \frac{5}{11}$, $K_5^{(2)}$-free
`C04`	M1b Prop. 6.3 Join	$K_6 \vee \overline{K}_{14}$ $(t = 7, n = 20)$	$\delta = 6, \Delta = 19$, $\kappa_{SS} = \frac{20}{19}$, $\kappa_{SI} = \frac{7}{19}$, $K_7^{(2)}$-free
`C05`	M2a Rainbow K_t	$t = 5, n = 500, \rho = \sigma = \frac{1}{100}$	Eq. (1) defect $\frac{3}{5} < 1$; $F = K_8$, $\max \lvert Q_c \rvert = 3$, rainbow K_5 (15 colors)
`C06`	M2a Rainbow K_t	$t = 6, n = 1000, \rho = \sigma = \frac{1}{200}$	Eq. (1) defect $\frac{27}{50} < 1$; $F = K_{10}$, $\max \lvert Q_c \rvert = 4$, rainbow K_6 (21 colors)
`C07`	M2b Lem. 3.1–3.3	$t = 5, C = 32, B = 1000, D = 8000$	$d(y) > \frac{4000}{3}$; defect $\frac{199}{1400} < \frac{3}{4} \leq 1$
`C08`	M2b Lem. 3.1–3.3	$t = 8, C = 48, B = 5000, D = 40000$	$d(y) > \frac{7680000}{1009}$; defect $\frac{118151}{960000} < \frac{1009}{1875} \leq 1$
`C09`	M3a Hall Match	$t = 5, \lvert U \rvert = 6, \lvert W \rvert = 9, c = 3$	Injective Hall lift onto $K_{3,3}$ $(e(H) = 9 \leq 18)$
`C10`	M3a Hall Match	$t = 6, \lvert U \rvert = 8, \lvert W \rvert = 12, c = 4$	Injective Hall lift onto Q_3 $(e(H) = 12 \leq 32)$
`C11`	M3b Prop. 4.2	$t = 5, c = 3, D = 700 \geq 675$	$\lvert U \rvert \leq 33$, closed $D \leq 414$, Song order 17 digits
`C12`	M3b Prop. 4.2	$t = 10, c = 12, D = 45000 \geq 43200$	$\lvert U \rvert \leq 672$, closed $D \leq 18{,}492$, Song order 27 digits
`C13`	M4a Lem. 5.1	$t = 5, E = 10, D = 4000$	$Q = 15, h = 20$; $\lvert U \rvert$ UB 4 digits, $\lvert W \rvert$ UB 8 digits
`C14`	M4a Lem. 5.1	$t = 7, E = 20, D = 25000$	$Q = 28, h = 40$; $\lvert U \rvert$ UB 7 digits, $\lvert W \rvert$ UB 14 digits
`C15`	M4b Cor. 1.3	$t = 5, \Delta \in \{3, 4, 5, 6, 8, 10\}$	Exact Moore $\leq \Delta^{2\Delta(\Delta-1)}$; diam $\in [12, 180]$
`C16`	M4b Cor. 1.3	$t = 8, \Delta \in \{9, 17, 50, 100, 200\}$	Outerplanar diam ≤ 144, planar ≤ 544, max 79,600

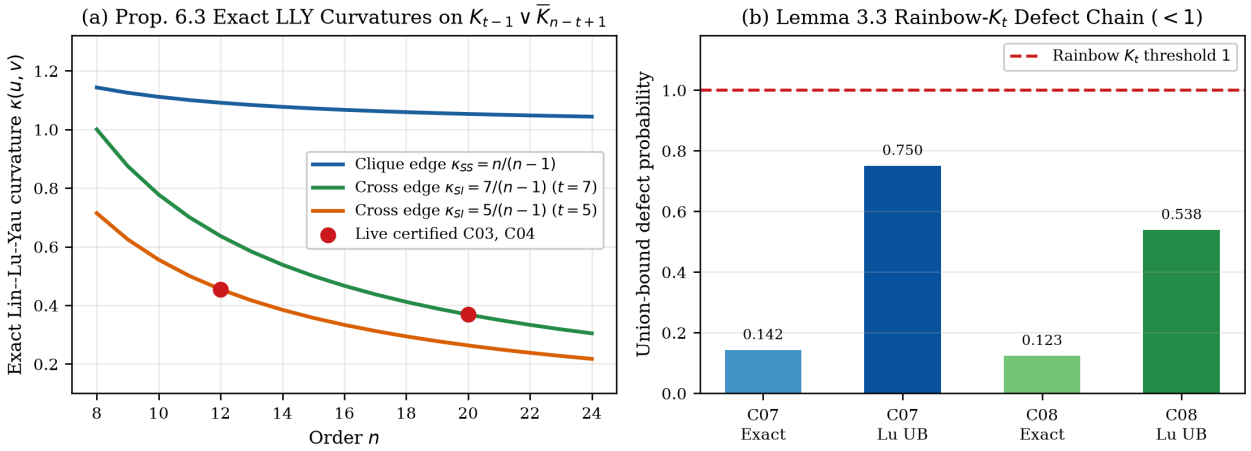

Figure 1: **Sharp Join Curvatures and Lemma 3.3 Defect Chain.** (a) Exact primal–dual LLY curvatures $\kappa_{SS} = n/(n-1)$ and $\kappa_{SI} = t/(n-1)$ on $K_{t-1} \vee \overline{K}_{n-t+1}$ (`C03`, `C04`). (b) Exact Eq. (1) defect strictly below Lu's $16t^3(\varepsilon + 1/D) < 1$ (`C07`, `C08`).

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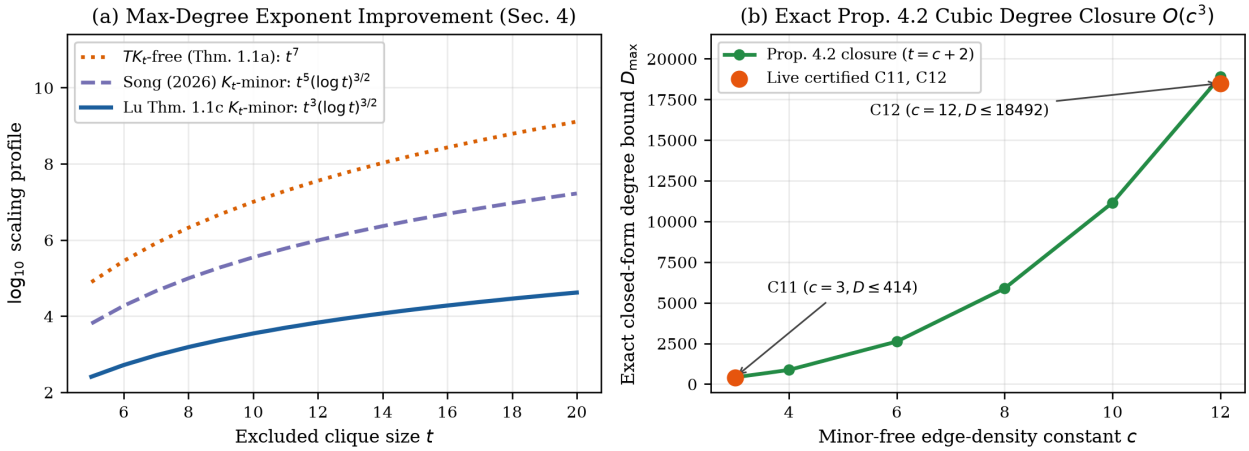

Figure 2: **Minor-Free Refinement (Section 4).** (a) Asymptotic degree-exponent improvement from TK_t-free t^7 and Song's $t^5(\log t)^{3/2}$ down to Lu's $t^3(\log t)^{3/2}$. (b) Exact Proposition 4.2 cubic degree closure $D_{\max} = O(c^3)$ with live certified `C11` ($c = 3, D \le 414$) and `C12` ($c = 12, D \le 18{,}492$).

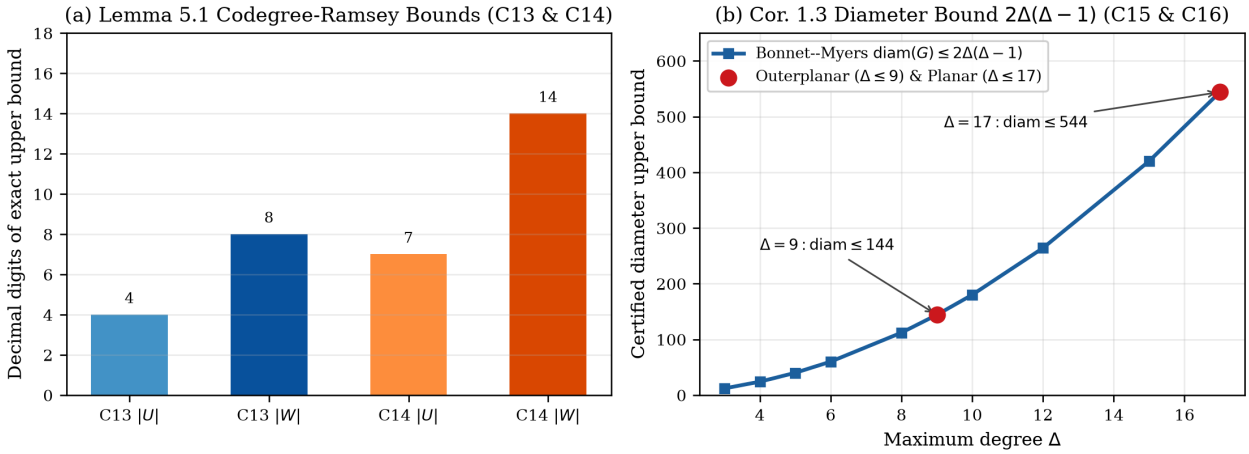

Figure 3: **Codegree Ramsey and Bonnet–Myers Bounds.** (a) Exact decimal-digit lengths of Lemma 5.1 Ramsey bounds on $|U|$ and $|W|$ (`C13`, `C14`). (b) Corollary 1.3 diameter upper bound $2\Delta(\Delta - 1)$ highlighting outerplanar ($\Delta \le 9$) and planar ($\Delta \le 17$) benchmarks (`C15`, `C16`).