The Reconstructed Theory Has One Mass

A machine-checked volume-uniform transfer gap for a finite Ising strip

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Abstract

We give a finite-dimensional Osterwalder–Schrader (OS) reconstruction for an anisotropic Ising model on open rectangular strips and prove a spectral gap whose rate is uniform in the spatial extent. For temporal coupling β, spatial coupling γ, and

$$0 < \alpha < 1, \qquad 2\tanh|\beta| + 2\tanh|\gamma| \le \alpha,$$

one number $m > 0$ works for every spatial length. Reflection positivity is proved directly for the Gibbs measure; the null space of the reflected form is identified with the kernel of an explicit boundary-collapse map; the OS quotient is linearly equivalent to the finite boundary-vector space; and the transfer operator forced by the site and bond forms is intertwined with a symmetrised transfer matrix. Dobrushin comparison supplies positive Perron data and the bound

$$\|T_L - |\Omega_L\rangle\langle\Omega_L|\| \le e^{-m} \quad \text{for every } L.$$

Consequently, connected reconstructed correlations decay at the same common rate. Lean 4 checks the composition without assuming reflection positivity, Perron data, a vacuum, a gap, or clustering at the public endpoint. A conservative priority survey found earlier machine-checked OS axioms and abstract OS reconstruction, and earlier volume-uniform measure-side clustering, but no earlier artifact combining a concrete reconstructed transfer operator with a proved volume-uniform spectral gap.

1 Why the composition matters

Reflection positivity and high-temperature mixing normally live in different mathematical descriptions. The first is a positivity statement about a reflected Euclidean measure and produces a physical Hilbert space and a transfer operator. The second is a contraction statement about conditional distributions and produces decay of measure-level correlations. A paper that merely places those facts side by side has not shown that the reconstructed operator inherits the quantitative rate. Conversely, an abstract structure containing a field named "gap" does not prove a gap for a model.

The result here is the missing composition in a finite system. All spaces are finite, but the spatial dimension of the slice grows with L, and the conclusion uses one m before quantifying over L. Thus the statement is not the tautology that every fixed positive matrix has some finite-volume gap. It is also not a thermodynamic-limit construction or a continuum Yang–Mills mass-gap theorem; the precise boundary of the claim is discussed in Section 11.

This manuscript supersedes—rather than supplements—the three internal stepping-stone manuscripts on the reconstructed gap, correlator decay, and OS-forced transfer operator.

This is the promised *commutant plus bulk dissipation* candidate in its classical reduction. On the reconstructed boundary space, multiplication by functions on Σ_L is a maximal

abelian algebra, hence is its own commutant. The vacuum line is the invariant scalar sector, while the orthogonal complement is the bulk sector. The estimate proved below is exactly uniform dissipation of that complement. Nothing here promotes this classical statement to a noncommutative commutant theorem; doing so would be a separate paper and is deliberately not claimed.

The formal development is in `YangMills/OS/OSReconstructionUniform.lean`. It composes earlier, independently checked modules rather than replacing them with a monolithic proof. This makes the logical inputs visible and permits a direct non-vacuity audit.

2 The finite anisotropic Ising strip

Let

$$\Sigma_L = \{0,1\}^{\{0,\dots,L\}}$$

be the spatial slice space, with the two spin values interpreted as ± 1. The open-chain spatial weight is

$$w_{\gamma,L}(\sigma) = \prod_{j=0}^{L-1} \exp\bigl(\gamma s(\sigma_j) s(\sigma_{j+1})\bigr) > 0,$$

and the bond between consecutive time slices is

$$K_{\beta,L}(\sigma,\tau) = \prod_{j=0}^{L} \exp\bigl(\beta s(\sigma_j) s(\tau_j)\bigr).$$

For a path $X = (X_0, \dots, X_N)$ of spatial slices, the unnormalised Gibbs weight is

$$W_{L,N}(X) = \prod_{t=0}^{N} w_{\gamma,L}(X_t) \prod_{t=0}^{N-1} K_{\beta,L}(X_t, X_{t+1}).$$

These definitions contain no operator. The model-to-operator dictionary is therefore a theorem rather than a definitional identification.

The symmetrised kernel is

$$S_L(\sigma,\tau) = \sqrt{w_{\gamma,L}(\sigma)}\, K_{\beta,L}(\sigma,\tau) \sqrt{w_{\gamma,L}(\tau)}.$$

If $\lambda_L > 0$ is its Perron value, define the normalised kernel $M_L = S_L/\lambda_L$. Its Euclidean continuous linear operator is denoted T_L. For a strictly positive Perron vector Ω_L, its Euclidean normalisation is denoted $\widehat{\Omega}_L$.

3 The three load-bearing inputs

3.1 Reflection positivity on the measure

For half-path observables F_i and coefficients $c_i \in \mathbb{C}$, the formal theorem `os_ising_measure_gram_nonneg` proves that the full Gibbs sum

$$\sum_{i,j} \overline{c_i} c_j \sum_{X} \overline{F_i(X_{\mathrm{past}})} F_j(X^{\mathrm{rev}}_{\mathrm{future}}) W_{L,N}(X)$$

is a nonnegative real number when $\beta \geq 0$. This is a statement about the measure itself, not only about an auxiliary quadratic form.

3.2 The OS quotient is identified

The boundary-collapse map sums a half-path observable against its Gibbs weight while retaining the terminal slice. The theorem `os_ising_null_space_iff` states that its kernel is exactly the null space of the reflected site form. Surjectivity then gives the bundled linear equivalence

$$\frac{\{\text{half-path observables}\}}{\ker(\text{collapse})} \simeq_{\mathbb{C}} \mathbb{C}^{\Sigma_L},$$

implemented as `osIsingPhysicalEquiv`. Hence no unidentified quotient is hidden in the reconstruction.

The transfer operator $\mathcal{T}_L$ on boundary vectors is forced by

$$\langle u, \mathcal{T}_L v\rangle_{\text{site}} = \langle u, v\rangle_{\text{bond}}.$$

The theorem `os_ising_transfer_is_measure_sum` identifies this matrix element with the reflected Gibbs two-point sum one time step farther apart. Thus the operator conclusion has a measure-level reading.

3.3 A uniform Dobrushin feed

The rectangular interaction has row sum bounded by $2\tanh|\beta| + 2\tanh|\gamma|$. The previously proved theorem `dobrushin_ising_uniform_gap` converts the common Dobrushin window into one $m > 0$ and, for every L, positive Perron data satisfying

$$M_L\Omega_L = \Omega_L, \qquad \|T_L - |\widehat{\Omega}_L\rangle\langle\widehat{\Omega}_L|\| \le e^{-m}.$$

The order of quantifiers is essential: m is selected before L.

4 Main theorem

Theorem 4.1 (Machine-checked OS reconstruction with one mass)**.** *Let $0 < \alpha < 1$, $\beta \ge 0$, and*

$$2\tanh|\beta| + 2\tanh|\gamma| \le \alpha.$$

Then the finite Gibbs measure is reflection positive at every reflection depth. Its OS null space is the kernel of the explicit collapse map and its physical quotient is linearly equivalent to $\mathbb{C}^{\Sigma_L}$. Moreover, there exists $m > 0$ such that for every L there are $\lambda_L > 0$ and $\Omega_L > 0$ for which:

1. *$M_L\Omega_L = \Omega_L$;*
2. *$(T_L, \widehat{\Omega}_L)$ is a symmetric unit-vacuum transfer pair;*
3. *$\|T_L - |\widehat{\Omega}_L\rangle\langle\widehat{\Omega}_L|\| \le e^{-m}$;*
4. *for every real boundary vector v and $n \in \mathbb{N}$,*

$$\left|\langle v, T_L^n v\rangle - \langle\widehat{\Omega}_L, v\rangle^2\right| \le \|v\|^2 e^{-mn};$$

5. *multiplication by $\sqrt{w_{\gamma,L}}$ intertwines the transfer operator forced by the OS forms with the unnormalised symmetrised kernel S_L.*

In Lean, the quantitative portion is the single declaration `os_reconstruction_uniform_gap`. Its hypotheses are exactly the three scalar inequalities $0 < \alpha$, $\alpha < 1$, and the Dobrushin window. The condition $\beta \geq 0$ belongs only to the separate reflection-positivity theorem. In particular, the endpoint does not take any of items (1)–(4) as an argument.

Remark 4.2 (Meaning of "one mass")**.** *This means one $m > 0$ chosen before L and valid for every finite L; it asserts neither a unique particle excitation nor an infinite-volume Hamiltonian or isolated mass shell.*

5 Proof architecture

The formal proof is short because the interfaces are strong; its content is the exact compatibility of those interfaces.

First, the Dobrushin corollary produces m, and for each L produces λ_L, Ω_L, positivity, the fixed-point equation, and the projected norm bound. Symmetry of M_L, strict positivity of Ω_L, and the fixed-point equation are passed to `vacuumTransfer_opOf`. That theorem proves, entrywise, symmetry of the operator, unit norm of $\widehat{\Omega}_L$, and $T_L\widehat{\Omega}_L = \widehat{\Omega}_L$.

Second, `clustering_of_gap` applies the projected norm estimate to the identity

$$\langle v, T_L^n v\rangle - \langle \widehat{\Omega}_L, v\rangle^2 = \langle v, (T_L - P_{\widehat{\Omega}_L})^n v\rangle \quad (n \geq 1),$$

with the $n = 0$ case treated by Cauchy–Schwarz. This yields the same m, not a second rate selected after L.

Third, the existing similarity theorem `transferOp_sqrtWeightEquiv` gives

$$\mathcal{T}_L Q_L = Q_L S_L, \qquad (Q_L u)(\sigma) = \sqrt{w_{\gamma,L}(\sigma)}\, u(\sigma).$$

Strict positivity of $w_{\gamma,L}$ makes Q_L invertible. The operator whose projected norm is controlled is therefore the normalised version of the same symmetrised transfer matrix that is intertwined with the OS-forced operator.

Finally, the two measure-identification theorems attach that operator back to the reflected Gibbs sums. This last step is what rules out an isolated finite-matrix result with no reconstructed interpretation.

6 Adversarial non-vacuity audit

Two failure modes were tested directly.

Empty antecedent. The window is inhabited: for example $\beta = \gamma = 0.1$ and $\alpha = 0.5$ satisfy it. More generally, continuity of tanh gives a nonempty open high-temperature neighbourhood of the origin. Every Boltzmann weight is a strictly positive exponential, so partition functions and the normalising Perron mass are nonzero.

Conclusion smuggled into the hypotheses. The public endpoint contains no argument of type `VacuumTransfer`, no Perron vector, no spectral-norm inequality, and no clustering bound. Its only hypotheses are scalar parameter inequalities. Reflection positivity is also not postulated: the Gram theorem consumes only $\beta \geq 0$ and constructs the nonnegative real witness for the actual Gibbs sum.

The quotient is not circular either. Its null subspace is proved equal to the kernel of a concrete surjective linear map before the quotient equivalence is formed. Likewise, the transfer operator is not selected to have a convenient spectrum; it is uniquely forced by the site/bond equation and only afterwards intertwined with S_L.

7 Why the general Birkhoff route was rejected

A tempting route is to seek volume-uniform contraction in the standard positive cone using the projective diameter of $S_L = \sqrt{w}\,K\sqrt{w}$. A pre-registered numerical falsifier evaluated all pairwise Hilbert-metric ratios for $L = 2, \ldots, 8$, both inside and outside the Dobrushin window. It passed 88 of 88 internal consistency checks in both ordinary and optimised Python modes and found

$$\Delta(S_L) = \Delta(K_L) = 4|\beta|(L+1), \qquad \kappa_L = \tanh\big(|\beta|(L+1)\big) \longrightarrow 1.$$

The equality of projective diameters is structural: positive diagonal congruence does not change the relevant column cross-ratios. Hence the slice weight cannot rescue standard-cone Birkhoff contraction, and tensor additivity makes the obstruction grow with L. This does not exclude a genuinely different cone, but it kills the proposed general Birkhoff proof. The paper therefore uses the surviving Dobrushin reduction and makes no beyond-window claim.

8 Formal artefacts and reuse map

Table 1 records the load-bearing in-repository chain. All paths are relative to the repository root.

File	Reused declarations	Role here
`YangMills/OS/SpatialOS.lean`	`gibbsSum_reflected_gram_nonneg`	Gibbs-measure RP
`YangMills/OS/SpatialReconstruction.lean`	`mem_ker_collapseL_iff`, `physicalEquiv`, `osPairing_transfer_gibbsSum`, similarity theorem	quotient, forced operator, measure dictionary
`YangMills/OS/DobrushinCorollary.lean`	`dobrushin_ising_uniform_gap`, `sliceW`	one mass and model family
`YangMills/OS/DobrushinTransport.lean`	`vacuumTransfer_opOf`	vacuum packaging
`YangMills/OS/TransferGap.lean`	`clustering_of_gap`	operator gap to decay
`YangMills/OS/OSReconstructionUniform.lean`	five public declarations	new composition endpoint

Table 1: Load-bearing formal reuse.

Five satellite repositories were inspected at their public heads on 4 August 2026. Their role is negative or terminological, not load-bearing:

Repository	inspected HEAD	reusable verdict
`lean-os-positivity`	`2ea7692f`	RP vocabulary; mother tree stronger
`lean-transfer-matrix`	`4e2366a4`	1D Ising gap only
`lean-gaussian-field`	`53af447b`	finite covariance algebra only
`aqft-split-inclusion-series`	`078c39ae`	TeX/numerics, no Lean
`physmath-lean-lemmas`	`b26ce4c4`	unrelated summability closure

No theorem was vendored from these repositories. This is deliberate: source presence is not treated as a materialised object file or as verification.

9 Related machine-checked work and priority boundary

Douglas, Hoback, Mei, and Nissim formalised the Glimm–Jaffe/OS axioms for the four-dimensional Gaussian free field in Lean 4 [5]. This already rules out any claim of first machine-checked OS axioms or reflection positivity. The repository `mrdouglasny/reflection-positivity` constructs an abstract physical Hilbert space and a self-adjoint contractive transfer operator from reflection-positive data [6]; its `GappedTransfer` interface carries the gap as a hypothesis rather than proving it for a concrete family. The repository `mrdouglasny/lgt` proves Dobrushin-style, volume-uniform measure-side correlation decay for lattice gauge theory [7], but has no OS reconstruction or transfer operator. The active $P(\phi)_2$ project contains transfer-matrix spectral infrastructure, with its current headline uniform estimates still represented by named project assumptions [8].

Accordingly, the narrow priority statement is: the public survey found no earlier machine-checked result combining (i) a concrete interacting Gibbs family, (ii) an OS-reconstructed transfer operator, (iii) a gap proved rather than assumed, and (iv) one rate uniform in spatial volume. This is an evidence-qualified statement, not a claim that abstract OS reconstruction, reflection positivity, Dobrushin uniqueness, or finite-matrix spectral gaps are new.

The classical foundations are the OS reconstruction papers [1, 2], the lattice reflection-positive construction of Osterwalder and Seiler [3], and Dobrushin's comparison method [4]. The present work is a finite, machine-checked quantitative composition of these themes.

10 Reproducibility and audit status

The repository is pinned to Lean `v4.29.0-rc6` and Mathlib commit

`07642720480157414db592fa85b626dafb71355b`.

The OS-R numerical gate contains 110 registered checks and is run with Python assertions enabled and disabled. Lean compilation, full core compilation, and the repository-wide axiom oracle are executed only on a Colab Pro+ CPU high-RAM runtime. The pre-measurement predictions are 8481 core jobs and 3015 oracle reports. On the exact frozen merged tree the realised totals are 8482 core jobs and 3015 oracle reports, with zero `sorryAx`; the incorrect core prediction is retained in the record rather than repaired after measurement. The final fresh-clone run ended with every child sentinel and the atomic parent sentinel equal to zero.

The final frozen commit, byte counts, LF and CRLF hashes, realised build counts, and Colab opening/closing times are recorded in the accompanying verification ledger and release manifest. This paper does not self-certify the terminal verdict: it supplies the frozen object and the attacks attempted; an independent audit task must issue the final assessment.

11 Limitations and next targets

The theorem concerns finite open rectangles and uniformity in the spatial extent. It does not construct an infinite-volume measure, a continuum limit, a relativistic Wightman theory, or a Hamiltonian by functional calculus. The correlation theorem is stated for real Euclidean boundary vectors. Complex observables can be split into real and imaginary parts, but that corollary is not needed for the headline endpoint.

Uniform finite-volume bounds alone do not produce an infinite-volume transfer operator: one still needs compatible maps between boundary spaces, convergence of states and operators, and independence of finite-volume boundary conditions. These inputs are not present here.

The Dobrushin window is sufficient and deliberately non-sharp. The failed standard-cone Birkhoff test explains why a naive tensor-product improvement cannot enlarge it uniformly.

A credible beyond-window project would need a new cone, a block dynamics with a proved uniform contraction, or a comparison principle that survives the tensorial wall. None is claimed here.

Conclusion

The main formal fact is compact: one scalar high-temperature window produces one mass before the spatial volume is chosen, and the resulting operator is the one reconstructed from the reflected Gibbs measure. Keeping positivity, quotient identification, similarity, Perron normalisation, and decay in the same theorem suite closes the two most dangerous loopholes: a vacuous implication and a gap smuggled into the assumptions. The result is modest in physical scope but, within that scope, fully quantitative and mechanically checkable.

References

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