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Constant-Curvature Embeddability Spectra of Finite Metric Spaces: The Leaf-Tree Dichotomy, Sharp Gromov Curvature Ceilings, Unified Non-Euclidean Descartes Laws, and Canonical Phase Spectra

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Abstract

Given a finite metric space (X, d) on $n \geq 3$ points, we define its *constant-curvature embeddability spectrum* $\mathcal{K}(X, d) \subset \mathbb{R}$ as the set of all sectional curvatures $K \in \mathbb{R}$ for which (X, d) embeds isometrically into the complete constant-curvature space form $\mathbb{M}^N_K$ ($\mathbb{H}^N_K$ for $K < 0$, $\mathbb{R}^N$ for $K = 0$, $\mathbb{S}^N_K$ for $K > 0$) for some $N \geq 1$. We develop a unified spectral, metric-geometric, and algebraic theory characterizing the structure, extremal bounds, and phase transitions of $\mathcal{K}(X, d)$.

First, we establish the closedness of $\mathcal{K}(X, d) \subset (-\infty, \pi^2/\operatorname{diam}(X)^2]$ and give a self-contained elementary proof—via an exact four-dimensional Euclidean vector decomposition—of the *Exponential Ptolemy Inequality* $e^{\frac{\kappa}{2}(d_{12}+d_{34})} \leq e^{\frac{\kappa}{2}(d_{13}+d_{24})} + e^{\frac{\kappa}{2}(d_{14}+d_{23})}$ in $\mathbb{H}^N_K$ ($\kappa = \sqrt{-K}$). As an immediate consequence, every non-tree metric space with Gromov 4-point hyperbolicity defect $\delta(X, d) > 0$ obeys the sharp universal negative-curvature ceiling $\sqrt{-K} \leq \frac{\log 2}{\delta(X,d)}$ for all $K \in \mathcal{K}(X, d) \cap (-\infty, 0)$, with sharp constant $\log 2$.

Second, we prove the *Complete Asymptotic Hyperbolic Embeddability Dichotomy*: $\mathcal{K}(X, d)$ is unbounded below if and only if $(-\infty, -K_0] \subset \mathcal{K}(X, d)$ for some finite $K_0 \geq 0$, if and only if (X, d) is either collinear ($X \subset \mathbb{R}^1$, with $K_0 = 0$) or a *strict leaf-tree metric* (the restriction of a weighted tree strictly to its leaves, with positive pendant edge weights $w_i > 0$). Conversely, every non-collinear tree metric containing even a single non-leaf point of X satisfies $\mathcal{K}(X, d) \cap (-\infty, 0] = \emptyset$. Using a least-common-ancestor clade decomposition, diagonal congruence, and Gershgorin's circle theorem, we prove the explicit quantitative threshold bound $\sqrt{K_0(X, d_T)} \leq \frac{\log(\sqrt{n}+1)}{2w_{\min}}$, and show $K_0 = 0$ for every finite ultrametric.

Third, for every n-leaf star metric $S(w_1, \ldots, w_n)$ ($d_{ij} = w_i + w_j$ for $i \neq j$, $w_i > 0$), rank-2 Gram reduction yields a single unified *Non-Euclidean Descartes–Soddy–Gosset Master Law* across all three geometries: $K \in \mathcal{K}(S(w))$ if and only if $f_S(K) := (n-2)\sum_{i=1}^n \cot^2_K(w_i) - (\sum_{i=1}^n \cot_K(w_i))^2 + 2(n-2)K \leq 0$. Combining Chebyshev's sum inequality with an exact large-angle separation argument, we prove global strict monotonicity $f'_S(K) \geq 2(n-2) > 0$ on $(-\infty, (\pi/\operatorname{diam})^2)$, establishing $\mathcal{K}(S(w)) = (-\infty, K_*(w)]$, the exact sign trichotomy governed by the Euclidean Descartes defect $\mathcal{D}_0(w) = (n-2)\sum w_i^{-2} - (\sum w_i^{-1})^2$, and the explicit universal curvature bound $|K_*(w)| \leq |\mathcal{D}_0(w)|/(2(n-2))$.

Fourth, we determine exact curvature spectra across canonical graph and metric families—proving $\mathcal{K}(K_{1,3}) = \mathcal{K}(Q_d) = \emptyset$ ($d \geq 3$), $\mathcal{K}(C_m) = \{(2\pi/m)^2\}$ ($m \geq 4$), an exact five-regime phase diagram for all 4-point Klein-4 metrics (d_1, d_2, d_3), and isolated singletons $\mathcal{K}(X_5(K_0)) =$

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$\{K_0\}$ for every $K_0 \in \mathbb{R}$ via 5-point centered orthogonal crosses—realizing ten distinct sign-stratified interval types of $\mathcal{K}(X,d)$ and motivating the Curvature Spectrum Connectedness Conjecture, supported by 1,200 random metric tests with zero counterexamples.

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1 Introduction

The characterization of finite metric spaces that embed isometrically into Euclidean space $\mathbb{R}^N$, spheres $\mathbb{S}^N_K$ of sectional curvature $K > 0$, or hyperbolic spaces $\mathbb{H}^N_K$ of sectional curvature $K < 0$ is a classical pillar of distance geometry dating back to Cayley, Menger [9], Schoenberg [11, 12, 13], and Blumenthal [1]. Modern metric geometry, phylogenetic tree reconstruction, and machine learning in non-Euclidean spaces [5, 2, 6, 10] have renewed interest in a fundamental global question: rather than testing a fixed target space, *across what set of sectional curvatures $K \in \mathbb{R}$ does a given finite metric space (X,d) admit an isometric embedding into the space form $\mathbb{M}^N_K$?*

Definition 1.1 (Constant-Curvature Embeddability Spectrum)**.** For $K \in \mathbb{R}$ and $N \geq 1$, let $\mathbb{M}^N_K$ denote the complete N-dimensional Riemannian space form of constant sectional curvature K (simply connected for $N \geq 2$): hyperbolic space $\mathbb{H}^N_K$ for $K < 0$, Euclidean space $\mathbb{R}^N$ for $K = 0$, and the sphere $\mathbb{S}^N_K$ of radius $1/\sqrt{K}$ (equipped with its intrinsic geodesic metric; a circle of radius $1/\sqrt{K}$ when $N = 1$) for $K > 0$. For a finite metric space (X,d) with $|X| = n \geq 3$, the *constant-curvature embeddability spectrum* of (X,d) is

$$\mathcal{K}(X,d) \;:=\; \{K \in \mathbb{R} : \exists\, N \geq 1 \text{ and an isometric embedding } f\colon (X,d) \hookrightarrow \mathbb{M}^N_K\}. \tag{1}$$

Although the pointwise linear-algebraic criteria for $K \in \mathcal{K}(X,d)$ are classical [11, 1, 6], the *global* structure of $\mathcal{K}(X,d)$ as a subset of $\mathbb{R}$ exhibits rich and previously uncharacterized phenomena: some spaces admit embeddings at no curvature at all ($\mathcal{K} = \emptyset$), some at a single isolated positive curvature ($\mathcal{K} = \{K_0\}$), some only on a purely hyperbolic half-line $(-\infty, K_*]$ with $K_* < 0$, some on a half-line crossing Euclidean space into spherical space, and some on a compact interval $[K_-, K_+]$ that may straddle 0, anchor at 0, or lie strictly inside $(0,\infty)$.

1.1 Summary of Main Results

This paper establishes four main theorems with completely self-contained proofs and formulates one open connectedness conjecture:

1. **Exponential Ptolemy Inequality and Sharp Gromov Ceiling (Theorem 2.1).** After verifying the unified Gram/Schoenberg criterion and proving that $\mathcal{K}(X,d)$ is a closed subset of $(-\infty, \pi^2/\operatorname{diam}(X)^2]$ (including across the singular limit $K \to 0$), we give an elementary six-line proof via a four-dimensional Euclidean vector split of the *Exponential Ptolemy Inequality*

$$e^{\frac{\kappa}{2}(d_{12}+d_{34})} \;\le\; e^{\frac{\kappa}{2}(d_{13}+d_{24})} + e^{\frac{\kappa}{2}(d_{14}+d_{23})} \qquad (\kappa = \sqrt{-K})$$

in $\mathbb{H}^N_K$ (closely related to Bourdon's visual metric on CAT(-1) spaces [4]). This immediately yields the sharp universal negative-curvature ceiling $\sqrt{-K} \le \frac{\log 2}{\delta(X,d)}$ whenever the Gromov 4-point hyperbolicity defect $\delta(X,d) > 0$, with exact sharp constant $\log 2$.

2. **Complete Asymptotic Hyperbolic Embeddability Dichotomy (Theorem 3.2).** We prove that $\mathcal{K}(X,d)$ is unbounded below if and only if $(-\infty, -K_0] \subset \mathcal{K}(X,d)$ for some finite $K_0 \ge 0$, if and only if (X,d) is either collinear ($X \subset \mathbb{R}^1$, $K_0 = 0$) or a *strict leaf-tree metric* (the restriction of a weighted tree metric exclusively to its leaves, with positive pendant edge weights $w_i > 0$). In sharp contrast, every non-collinear tree metric containing a non-leaf point of X (whether of degree 2 or degree ≥ 3) satisfies $\mathcal{K}(X,d) \cap (-\infty, 0] = \emptyset$. For every strict leaf-tree metric, an explicit least-common-ancestor clade decomposition and Gershgorin congruence argument prove the quantitative bound $\sqrt{K_0(X,d_T)} \le \frac{\log(\sqrt{n}+1)}{2w_{\min}}$, while every finite ultrametric satisfies $K_0 = 0$.

3. **Unified Non-Euclidean Descartes Law and Global Strict Monotonicity (Theorem 4.1).** For every n-leaf star metric $S(w_1, \dots, w_n)$ ($d_{ij} = w_i + w_j$ for $i \ne j$, $w_i > 0$, representing the centers of n mutually tangent balls), rank-2 perturbation reduction yields a single unified analytic criterion across all $K \in (-\infty, (\pi/\operatorname{diam})^2]$:

$$K \in \mathcal{K}(S(w)) \iff f_S(K) \;:=\; (n-2)\sum_{i=1}^{n} \cot_K^2(w_i) - \left(\sum_{i=1}^{n} \cot_K(w_i)\right)^2 + 2(n-2)K \;\le\; 0,$$

extending the classical Descartes–Soddy–Gosset kissing-sphere law [8, 7] to all curvatures. We prove the global derivative bound $f_S'(K) \ge 2(n-2) > 0$ on the entire physical domain $(-\infty, (\pi/\operatorname{diam})^2)$ (combining Chebyshev's sum inequality with an exact separation of the unique angle exceeding $\pi/2$), establishing $\mathcal{K}(S(w)) = (-\infty, K_*(w)]$, the exact sign trichotomy of $K_*(w)$ via $\mathcal{D}_0(w) = f_S(0)$, and the universal bound $|K_*(w)| \le |\mathcal{D}_0(w)|/(2(n-2))$.

4. **Exact Curvature Spectra of Canonical Families and Connectedness Conjecture (Theorem 5.1 and Conjecture 5.2).** We derive exact closed-form curvature spectra for $K_{1,3}$, all hypercubes Q_d ($d \ge 3$), all cycle graphs C_m ($m \ge 4$), all 4-point Klein-4 metrics ($d_1 \le d_2 \le d_3$), and 5-point centered orthogonal crosses $X_5(K_0)$ (realizing every $K_0 \in \mathbb{R}$ as an isolated singleton $\{K_0\}$), exhibiting ten distinct sign-stratified interval types of $\mathcal{K}(X,d)$ and showing that the compressed Klein-4 rhombus $(1,1,c)$ saturates the sharp Gromov ceiling $\kappa\delta \to \log 2$ as $c \downarrow 1$. All results are verified numerically in Section 6, including 1,200 random 4-, 5-, and 6-point metrics supporting the Curvature Spectrum Connectedness Conjecture.

2 Unified Spectral Criterion, Exponential Ptolemy Inequality, and Sharp Gromov Ceiling

Throughout, (X, d) denotes a finite metric space with $X = \{1, \dots, n\}$ ($n \geq 3$), distance matrix $D = (d_{ij})_{i,j=1}^n$, diameter $\operatorname{diam}(X) := \max_{i,j} d_{ij} > 0$, centering matrix $J := I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$, and entrywise square $D^{\circ 2} := (d_{ij}^2)$. For $n \geq 4$, the Gromov 4-point hyperbolicity defect of (X, d) is

$$\delta(X, d) \;:=\; \max_{\{i,j,k,\ell\} \subset X} \frac{S_3(i,j,k,\ell) - S_2(i,j,k,\ell)}{2} \;\geq\; 0, \tag{2}$$

where $S_1 \leq S_2 \leq S_3$ denote the three opposite-pair sums $\{d_{ij} + d_{k\ell},\ d_{ik} + d_{j\ell},\ d_{i\ell} + d_{jk}\}$ sorted in nondecreasing order (for $n = 3$, set $\delta(X, d) := 0$). A space with $\delta(X, d) = 0$ is called 0-*hyperbolic* (or a *tree metric* [3]). For symmetric $M \in \mathbb{R}^{n\times n}$, we write its eigenvalues in nonincreasing order as $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M) = \lambda_{\min}(M)$.

Theorem 2.1 (Spectral Criterion, Exponential Ptolemy Inequality, and Sharp Gromov Ceiling)**.** *Let (X, d) be a finite metric space on $n \geq 3$ points.*

(a) ***Unified spectral criterion and closedness.*** *We have $K \in \mathcal{K}(X, d)$ if and only if (X, d) embeds isometrically into $\mathbb{M}_K^{n-1}$, which holds if and only if:*

- *($K < 0$, $\kappa = \sqrt{-K}$) the matrix $H(\kappa) := (\cosh(\kappa d_{ij}))_{i,j=1}^n$ has at most one positive eigenvalue, i.e., $\lambda_2(H(\kappa)) \leq 0$;*
- *($K = 0$) the Euclidean Gram matrix $G_0 := -\frac{1}{2}JD^{\circ 2}J$ is positive semidefinite, i.e., $\lambda_{\min}(G_0) \geq 0$;*
- *($K > 0$, $\kappa = \sqrt{K}$) $\kappa \operatorname{diam}(X) \leq \pi$ and $S(\kappa) := (\cos(\kappa d_{ij}))_{i,j=1}^n \succeq 0$, i.e., $\lambda_{\min}(S(\kappa)) \geq 0$.*

In particular, $\mathcal{K}(X, d)$ is a closed subset of $(-\infty, \pi^2/\operatorname{diam}(X)^2]$.

(b) ***Exponential Ptolemy Inequality in $\mathbb{H}_K^N$.*** *For any $K = -\kappa^2 < 0$, any $N \geq 1$, and any four points $x_1, x_2, x_3, x_4 \in \mathbb{H}_K^N$ with distances $d_{ij} = d_{\mathbb{H}_K^N}(x_i, x_j)$,*

$$e^{\frac{\kappa}{2}(d_{12}+d_{34})} \;\leq\; e^{\frac{\kappa}{2}(d_{13}+d_{24})} + e^{\frac{\kappa}{2}(d_{14}+d_{23})}. \tag{3}$$

(c) ***Sharp universal Gromov curvature ceiling.*** *If $\delta(X, d) > 0$, then every negative curvature $K \in \mathcal{K}(X, d) \cap (-\infty, 0)$ obeys*

$$\sqrt{-K} \;\leq\; \frac{\log 2}{\delta(X, d)}. \tag{4}$$

The constant $\log 2$ is sharp: ideal regular quadrilaterals in $\mathbb{H}_{-1}^2$ satisfy $\sup_{X \subset \mathbb{H}_{-1}^2, |X|=4} \delta(X) = \log 2$.

Proof. Part (a). The Euclidean ($K = 0$) criterion is Schoenberg's theorem [11]. For $K > 0$, $\mathbb{S}_K^N$ is the sphere of radius κ^{-1} in $\mathbb{R}^{N+1}$ with geodesic distance $d_{\mathbb{S}_K^N}(u, v) = \kappa^{-1}\arccos(\kappa^2\langle u, v\rangle) \in [0, \pi/\kappa]$, so $f\colon X \to \mathbb{S}_K^N$ is an isometry iff $\kappa d_{ij} \leq \pi$ and the Gram matrix $(\kappa^2\langle f(i), f(j)\rangle) = (\cos(\kappa d_{ij})) = S(\kappa) \succeq 0$ (of rank $r \leq n$, yielding an embedding into $\mathbb{S}_K^{r-1} \subset \mathbb{S}_K^{n-1}$). For $K < 0$, realizing $\mathbb{H}_K^N$ as the upper sheet $\{x \in \mathbb{R}^{N,1} : \langle x, x\rangle_L = -1,\ x_0 > 0\}$ of Minkowski space $\mathbb{R}^{N,1}$ equipped with the rescaled Lorentzian metric $\kappa^{-2}\,ds_L^2$ (where $\langle x, y\rangle_L = -x_0y_0 + \sum_{a=1}^N x_a y_a$, so that $\cosh(\kappa d_{\mathbb{H}_K^N}(x, y)) = -\langle x, y\rangle_L$), an isometric embedding into $\mathbb{H}_K^N$ for some $N \geq 1$ exists iff $-H(\kappa) = X^\top J_{N,1} X$ for some $N \geq 1$ and $X \in \mathbb{R}^{(N+1)\times n}$, where $J_{N,1} := \operatorname{diag}(-1, 1, \dots, 1) \in \mathbb{R}^{(N+1)\times(N+1)}$ is the Minkowski signature matrix; by Sylvester's law of inertia, this holds iff $-H(\kappa)$

has at most one negative eigenvalue, equivalently $H(\kappa)$ has at most one positive eigenvalue. Since $H(\kappa)_{11} = 1 > 0$, $H(\kappa)$ has exactly one positive eigenvalue and $r \le n-1$ negative eigenvalues; choosing X gives $\langle x_i, x_i\rangle_L = -1$ and $\langle x_i, x_j\rangle_L = -\cosh(\kappa d_{ij}) \le -1 < 0$, so all x_i lie in the same sheet (flipping $X \mapsto -X$ if needed places all $x_i \in \mathbb{H}^r_K \subset \mathbb{H}^{n-1}_K$).

Closedness of $\mathcal{K}(X,d)$ on $(-\infty, 0)$ and on $(0, \pi^2/\operatorname{diam}(X)^2]$ follows from continuity of eigenvalues of $H(\kappa)$ and $S(\kappa)$. To prove closedness at $K = 0$, suppose $K_m \to 0$ with $K_m \in \mathcal{K}(X,d)$. If $K_m = +\kappa_m^2 > 0$, then for every $v \in \mathbf{1}_n^\perp$, $0 \le v^\top S(\kappa_m) v = -\frac{\kappa_m^2}{2} v^\top D^{\circ 2} v + O(\kappa_m^4)$; dividing by κ_m^2 and letting $m \to \infty$ gives $v^\top G_0 v \ge 0$. If $K_m = -\kappa_m^2 < 0$, then $H(\kappa_m) = \mathbf{1}_n \mathbf{1}_n^\top + \frac{\kappa_m^2}{2} D^{\circ 2} + O(\kappa_m^4)$, so the simple top eigenvalue n of $\mathbf{1}_n\mathbf{1}_n^\top$ perturbs smoothly with top unit eigenvector $u_+(\kappa_m) = n^{-1/2}\mathbf{1}_n + O(\kappa_m^2)$. For any $v \in \mathbf{1}_n^\perp$, choosing $\alpha_m := -\frac{\langle v, u_+(\kappa_m)\rangle}{\langle \mathbf{1}_n, u_+(\kappa_m)\rangle} = O(\kappa_m^2)$ makes $v_m := v + \alpha_m \mathbf{1}_n \perp u_+(\kappa_m)$. Since $\lambda_2(H(\kappa_m)) \le 0$,

$$0 \;\ge\; v_m^\top H(\kappa_m) v_m \;=\; n^2\alpha_m^2 + \frac{\kappa_m^2}{2} v^\top D^{\circ 2} v + O(\kappa_m^4) \;\ge\; \frac{\kappa_m^2}{2} v^\top D^{\circ 2} v + O(\kappa_m^4).$$

Dividing by κ_m^2 and letting $m \to \infty$ yields $v^\top G_0 v = -\frac{1}{2} v^\top D^{\circ 2} v \ge 0$ for all $v \in \mathbf{1}_n^\perp$, hence $0 \in \mathcal{K}(X,d)$.

Part (b). By rescaling the metric by κ, assume without loss of generality that $K = -1$ ($\kappa = 1$). Relabel $o := x_4$, $x := x_1$, $y := x_2$, $z := x_3$, and recall the Gromov product $(u|w)_o := \frac{1}{2}(d(u,o) + d(w,o) - d(u,w)) \ge 0$. Dividing (3) by $e^{\frac{1}{2}(d(x,o)+d(y,o)+d(z,o))}$, we must prove the visual triangle inequality

$$e^{-(x|y)_o} \;\le\; e^{-(x|z)_o} + e^{-(z|y)_o}. \tag{5}$$

If any of x, y, z coincides with o, (5) is immediate (since $(u|w)_o \ge 0$ implies $e^{-(u|w)_o} \le 1$). Assume $a := d(o,x) > 0$, $b := d(o,y) > 0$, $c := d(o,z) > 0$, and let $\alpha := \angle_o(x,z)$, $\beta := \angle_o(z,y)$, $\gamma := \angle_o(x,y) \in [0,\pi]$ be the geodesic angles at o in $T_o\mathbb{H}^N_{-1}$. By the spherical triangle inequality on the unit tangent sphere $\mathbb{S}^{N-1} \subset T_o\mathbb{H}^N_{-1}$, we have $\gamma \le \alpha + \beta$.

Extend the geodesic ray from o through z to the point $z(C)$ at distance $C \ge c$ from o. For $C' > C$, the triangle inequality gives $d(x, z(C')) - d(x, z(C)) \le C' - C$, so $(x|z(C))_o$ is nondecreasing in C. By the hyperbolic law of cosines,

$$\cosh d(x, z(C)) \;=\; \cosh a \cosh C - \sinh a \sinh C \cos\alpha \;=\; \tfrac{1}{2} e^C (\cosh a - \sinh a \cos\alpha) + O(e^{-C}),$$

so $d(x, z(C)) - C \to \log(\cosh a - \sinh a \cos\alpha)$ as $C \to \infty$. Putting $u := e^{-a} \in (0,1)$, $v := e^{-b} \in (0,1)$, $\tilde\alpha := \alpha/2$, $\tilde\beta := \beta/2$, $\tilde\gamma := \gamma/2$, it follows that

$$e^{-2(x|z)_o} \;\ge\; \lim_{C\to\infty} e^{d(x,z(C))-C-a} \;=\; e^{-a}(\cosh a - \sinh a\cos\alpha) \;=\; \sin^2\tilde\alpha + u^2\cos^2\tilde\alpha \;=:\; p(\tilde\alpha)^2,$$

and similarly $e^{-2(z|y)_o} \ge \sin^2\tilde\beta + v^2\cos^2\tilde\beta =: q(\tilde\beta)^2$. On the other hand, since $e^d = \cosh d + \sinh d < 2\cosh d$ for all $d \ge 0$, the hyperbolic law of cosines for (o,x,y) gives

$$\begin{aligned} e^{-2(x|y)_o} &= e^{d(x,y)-a-b} \;<\; 2e^{-a-b}\cosh d(x,y) \\ &= \sin^2\tilde\gamma\,(1+u^2v^2) + \cos^2\tilde\gamma\,(u^2+v^2) \;=:\; \Phi(\tilde\gamma). \end{aligned}$$

Note that $\Phi(\tilde\gamma) = (u^2+v^2) + (1-u^2)(1-v^2)\sin^2\tilde\gamma$ is nondecreasing in $\tilde\gamma \in [0, \frac{\pi}{2}]$. Setting $\tilde\beta' := \min(\tilde\beta, \frac{\pi}{2} - \tilde\alpha) \in [0, \tilde\beta]$, we have $\tilde\gamma \le \tilde\alpha + \tilde\beta' \le \frac{\pi}{2}$ and $q(\tilde\beta') \le q(\tilde\beta)$ (since $q(\theta) = \sqrt{v^2 + (1-v^2)\sin^2\theta}$ is nondecreasing on $[0, \frac{\pi}{2}]$). Write $s_\alpha := \sin\tilde\alpha$, $c_\alpha := \cos\tilde\alpha$, $s_\beta := \sin\tilde\beta'$, $c_\beta := \cos\tilde\beta'$, and define two vectors in $\mathbb{R}^4$:

$$X := (s_\alpha c_\beta,\; u c_\alpha c_\beta,\; -v s_\alpha s_\beta,\; uv c_\alpha s_\beta), \qquad Y := (c_\alpha s_\beta,\; -u s_\alpha s_\beta,\; v c_\alpha c_\beta,\; uv s_\alpha c_\beta).$$

Direct computation yields

$$\|X\|_2^2 = (s_\alpha^2 + u^2 c_\alpha^2)(c_\beta^2 + v^2 s_\beta^2) = p(\tilde{\alpha})^2 (c_\beta^2 + v^2 s_\beta^2) \le p(\tilde{\alpha})^2,$$
$$\|Y\|_2^2 = (s_\beta^2 + v^2 c_\beta^2)(c_\alpha^2 + u^2 s_\alpha^2) = q(\tilde{\beta}')^2 (c_\alpha^2 + u^2 s_\alpha^2) \le q(\tilde{\beta}')^2,$$
$$X + Y = \big(\sin(\tilde{\alpha} + \tilde{\beta}'),\ u\cos(\tilde{\alpha} + \tilde{\beta}'),\ v\cos(\tilde{\alpha} + \tilde{\beta}'),\ uv\sin(\tilde{\alpha} + \tilde{\beta}')\big),$$

so $\|X + Y\|_2^2 = \Phi(\tilde{\alpha} + \tilde{\beta}')$. By the Euclidean triangle inequality in $\mathbb{R}^4$,

$$\begin{aligned} e^{-(x|y)_o} < \sqrt{\Phi(\tilde{\gamma})} \ &\le \ \sqrt{\Phi(\tilde{\alpha} + \tilde{\beta}')} \ = \ \|X + Y\|_2 \\ &\le \ \|X\|_2 + \|Y\|_2 \ \le \ p(\tilde{\alpha}) + q(\tilde{\beta}') \ \le \ e^{-(x|z)_o} + e^{-(z|y)_o}, \end{aligned}$$

proving (5) and hence (3).

Part (c). Let $K = -\kappa^2 \in \mathcal{K}(X,d) \cap (-\infty, 0)$, and take any 4-point subset $\{x_1, x_2, x_3, x_4\} \subset X$ with opposite-pair sums $S_1 \le S_2 \le S_3$. Applying (3) to the pairing realizing S_3 gives

$$e^{\frac{\kappa}{2}S_3} \ \le \ e^{\frac{\kappa}{2}S_1} + e^{\frac{\kappa}{2}S_2} \ \le \ 2\, e^{\frac{\kappa}{2}S_2} \ \Longrightarrow \ \kappa\, \frac{S_3 - S_2}{2} \ \le \ \log 2.$$

Taking the maximum over all 4-point subsets yields $\kappa\, \delta(X,d) \le \log 2$, proving (4). For sharpness, consider the four vertices $z_k = r\, e^{ik\pi/2}$ $(k = 0, 1, 2, 3,\ 0 < r < 1)$ in the Poincaré disk model $\mathbb{B}^2$ of $\mathbb{H}^2_{-1}$, where $\cosh d(z,w) = 1 + \frac{2|z-w|^2}{(1-|z|^2)(1-|w|^2)}$. Then $d_{\text{adj}} = \operatorname{arcosh}(1 + \frac{4r^2}{(1-r^2)^2})$ and $d_{\text{diag}} = \operatorname{arcosh}(1 + \frac{8r^2}{(1-r^2)^2})$, so $S_1 = S_2 = 2d_{\text{adj}}$, $S_3 = 2d_{\text{diag}}$, and

$$\delta(\{z_0, z_1, z_2, z_3\}) \ = \ d_{\text{diag}} - d_{\text{adj}} \ = \ \log\!\left(\frac{8r^2 + O((1-r^2)^2)}{4r^2 + O((1-r^2)^2)}\right) \ \longrightarrow \ \log 2 \qquad (r \uparrow 1). \qquad \square$$

3 Asymptotic Hyperbolic Embeddability Dichotomy

We now determine exactly which finite metric spaces embed isometrically into $\mathbb{H}^N_K$ for arbitrarily large negative curvature $K \to -\infty$. Recall by Buneman's theorem [3] that a finite metric space (X,d) is 0-hyperbolic ($\delta(X,d) = 0$) if and only if there exists a finite weighted tree (T,w) (with strictly positive edge weights $w_e > 0$) and a map $\phi\colon X \to V(T)$ such that $d(x,y) = d_T(\phi(x), \phi(y))$ for all $x, y \in X$; moreover, the *minimal* tree realization (T,w)—obtained by contracting zero-weight edges, deleting unlabelled leaves, and suppressing unlabelled degree-2 vertices—is unique up to isomorphism.

Definition 3.1 (Strict Leaf-Tree Metric)**.** A finite metric space (X,d) ($|X| = n \ge 3$) is a *strict leaf-tree metric* if it is 0-hyperbolic and, in its unique minimal weighted tree realization (T,w), every point of X maps bijectively onto a leaf (degree-1 vertex) of T, with strictly positive pendant edge weights $w_1, \dots, w_n > 0$ and $w_{\min} := \min_{1 \le i \le n} w_i > 0$.

Note that every 3-point metric space is either collinear (one triangle inequality is tight) or a strict 3-leaf star tree (all triangle inequalities strict, with $w_1 = \frac{d_{12} + d_{13} - d_{23}}{2} > 0$, etc.). For $n \ge 4$, however, tree metrics split cleanly into three disjoint classes: collinear metrics ($X \subset \mathbb{R}^1$), strict leaf-tree metrics, and non-collinear tree metrics with at least one point of X at an internal vertex of T (degree 2 or degree ≥ 3).

Theorem 3.2 (Complete Asymptotic Hyperbolic Embeddability Dichotomy)**.** *Let (X,d) be a finite metric space on $n \ge 3$ points.*

(a) ***Exact characterization.*** *The following three conditions are equivalent:*

(i) $\mathcal{K}(X,d)$ is unbounded below (there exists a sequence $K_m \in \mathcal{K}(X,d)$ with $K_m \to -\infty$);

(ii) the infimal hyperbolic threshold $K_0(X,d) := \inf\{K_0 \geq 0 : (-\infty, -K_0] \subset \mathcal{K}(X,d)\}$ *is finite (in which case* $(-\infty, -K_0(X,d)] \subset \mathcal{K}(X,d)$ *by closedness of* $\mathcal{K}(X,d)$*);*

(iii) (X,d) *is either* collinear *(isometric to a subset of* $\mathbb{R}^1$*, in which case* $K_0 = 0$ *and* $(-\infty, 0] \subset \mathcal{K}(X,d)$*) or a* strict leaf-tree metric*.*

(b) ***Non-leaf obstruction.*** *If* (X,d) *is a non-collinear* 0*-hyperbolic metric space whose minimal tree realization* T *places at least one point* $x \in X$ *at a non-leaf vertex (*$\deg_T(x) \geq 2$*), then* (X,d) *admits no isometric embedding into* $\mathbb{M}_K^N$ *for any* $K \leq 0$*:*

$$\mathcal{K}(X,d) \cap (-\infty, 0] \;=\; \emptyset.$$

(c) ***Explicit Gershgorin threshold bound and ultrametric universality.*** *For every strict leaf-tree metric* (X, d_T) *with minimal pendant weight* $w_{\min} > 0$*,* (X, d_T) *embeds isometrically into* $\mathbb{H}_K^{n-1}$ *for every* $K = -\kappa^2 < 0$ *satisfying*

$$\kappa \;\geq\; \frac{\log(\sqrt{n}+1)}{2\,w_{\min}}, \qquad \textit{i.e.,} \qquad K_0(X, d_T) \;\leq\; \left(\frac{\log(\sqrt{n}+1)}{2\,w_{\min}}\right)^2. \tag{6}$$

Furthermore, every finite ultrametric (X,d) *(*$d(x,z) \leq \max\{d(x,y), d(y,z)\}$ *for all* x, y, z*) satisfies* $K_0(X,d) = 0$*, i.e.,* $(-\infty, 0] \subset \mathcal{K}(X,d)$*.*

Proof. Part (b) and (i) $\Rightarrow$ *(iii).* Suppose $K_m \in \mathcal{K}(X,d)$ with $K_m \to -\infty$. By Theorem 2.1(c), $\delta(X,d) \leq \frac{\log 2}{\sqrt{-K_m}} \to 0$, so $\delta(X,d) = 0$ and (X,d) has a unique minimal weighted tree realization (T, w). If T is a path (no vertex of degree ≥ 3), then (X,d) is isometric to a subset of $\mathbb{R}^1$, which embeds isometrically as a geodesic line in $\mathbb{H}_K^1$ for all $K \leq 0$.

Now suppose T is not a path (so T has at least one branch vertex of degree ≥ 3), and suppose for contradiction that some $x \in X$ is not a leaf of T, so $\deg_T(x) \geq 2$. Deleting x disconnects T into $\deg_T(x) \geq 2$ components, and since T is minimal, every component contains at least one leaf in X. Because T contains a branch vertex v of degree ≥ 3:

- if $\deg_T(x) \geq 3$, pick three distinct leaves $a, b, c \in X$ in three different components of $T \setminus \{x\}$;
- if $\deg_T(x) = 2$, let C_1 be a component of $T \setminus \{x\}$ containing a branch vertex v of degree ≥ 3, pick a leaf $a \in X$ in the other component C_0, and pick two leaves $b, c \in X$ lying in two different components of $T \setminus \{v\}$ not containing x (both necessarily contained in C_1).

In both cases, $x \notin \{a, b, c\}$ lies strictly on the unique T-path $[a, b]$ and strictly on the unique T-path $[a, c]$, so $d(a,b) = d(a,x) + d(x,b)$ and $d(a,c) = d(a,x) + d(x,c)$, whereas both T-paths $[x, b]$ and $[x, c]$ pass through v (with $v = x$ in the first case), giving $d(x,b) = d(x,v) + d(v,b)$ and $d(x,c) = d(x,v) + d(v,c)$, while $[b, c]$ also passes through v with $d(b,v), d(v,c) > 0$, yielding $d(b,c) = d(b,v) + d(v,c) > |d(v,b) - d(v,c)| = |d(x,b) - d(x,c)|$.

Now suppose $f\colon X \to \mathbb{M}_K^N$ were an isometric embedding for any $K \leq 0$. In $\mathbb{R}^N$ and $\mathbb{H}_K^N$, geodesics between any two points are unique and non-branching: $d(a,b) = d(a,x) + d(x,b)$ forces $f(b)$ to lie on the unique geodesic ray from $f(a)$ through $f(x)$ beyond $f(x)$, and $d(a,c) = d(a,x) + d(x,c)$ forces $f(c)$ to lie on the *exact same* geodesic ray beyond $f(x)$. Hence $f(x), f(b), f(c)$ are collinear on that ray, forcing $d(b,c) = |d(x,b) - d(x,c)|$, a contradiction! This simultaneously proves Part (b) and (i) $\Rightarrow$ (iii).

Part (c) and (iii) $\Rightarrow$ *(ii)* $\Rightarrow$ *(i).* Obviously (ii) $\Rightarrow$ (i), and the collinear case of (iii) gives $K_0 = 0$. Suppose (X, d_T) is a strict leaf-tree metric with minimal tree realization (T, w) and pendant weights $w_1, \ldots, w_n > 0$. Pick any internal vertex $r_0 \in V(T) \setminus X$ as root (note T has at least one internal vertex since $n \geq 3$), let $h_i := d_T(r_0, i) \geq w_i > 0$ be the root-to-leaf distance, and let

$\ell_{ij} := \frac{1}{2}(h_i + h_j - d_T(i,j)) \geq 0$ be the depth of $\mathrm{lca}(i,j)$ from r_0. Because the pendant edge of leaf i is not shared with any leaf $j \neq i$, we have $\ell_{ii} = h_i$ and $\ell_{ij} \leq h_i - w_i$ for all $j \neq i$.

Set $D_h := \mathrm{diag}(e^{\kappa h_1}, \ldots, e^{\kappa h_n})$. Since $d_T(i,j) = h_i + h_j - 2\ell_{ij}$, we have $e^{\kappa d_T(i,j)} = e^{\kappa h_i} e^{\kappa h_j} e^{-2\kappa \ell_{ij}}$, so

$$H(\kappa) \;=\; \frac{1}{2}\Big(e^{\kappa D_T} + e^{-\kappa D_T}\Big) \;=\; \frac{1}{2}\, D_h\Big(\mathbf{1}_n\mathbf{1}_n^\top - \big(W(\kappa) - E(\kappa)\big)\Big) D_h, \tag{7}$$

where $W(\kappa)_{ij} := 1 - e^{-2\kappa\ell_{ij}}$ and $E(\kappa)_{ij} := e^{-\kappa(h_i+h_j+d_T(i,j))}$. Define the truncated internal depth $\tilde{\ell}_{ij} := \ell_{ij}$ for $i \neq j$ and $\tilde{\ell}_{ii} := h_i - w_i$ (the depth of the unique internal neighbor of leaf i). Then

$$W(\kappa) \;=\; W_{\mathrm{int}}(\kappa) + \mathrm{diag}\Big(e^{-2\kappa(h_i - w_i)}\big(1 - e^{-2\kappa w_i}\big)\Big), \qquad W_{\mathrm{int}}(\kappa)_{ij} := 1 - e^{-2\kappa\tilde{\ell}_{ij}}.$$

Let $0 = t_0 < t_1 < \cdots < t_m$ be the distinct positive values taken by $\tilde{\ell}_{ij}$. For each $k \in \{1, \ldots, m\}$, the relation $\tilde{\ell}_{ij} \geq t_k$ is an equivalence relation on the leaves of depth $h_i - w_i \geq t_k$ (partitioning them into clades at distance t_k from r_0), so the indicator matrix $P_k := (\mathbf{1}_{\{\tilde{\ell}_{ij} \geq t_k\}})_{i,j=1}^n$ is a sum of rank-1 clade matrices $\mathbf{1}_C\mathbf{1}_C^\top \succeq 0$. By Abel summation,

$$W_{\mathrm{int}}(\kappa) \;=\; \sum_{k=1}^{m} \underbrace{\big(e^{-2\kappa t_{k-1}} - e^{-2\kappa t_k}\big)}_{>0} P_k \;\succeq\; 0.$$

Now conjugate $M(\kappa) := W(\kappa) - E(\kappa)$ by the positive diagonal matrix $S := \mathrm{diag}(e^{\kappa(h_i - w_i)})$:

$$S\,M(\kappa)\,S \;=\; S\,W_{\mathrm{int}}(\kappa)\,S + R(\kappa), \qquad R(\kappa) \;:=\; \mathrm{diag}\big(1 - e^{-2\kappa w_i}\big) - S\,E(\kappa)\,S.$$

The diagonal entries of $R(\kappa)$ are $R(\kappa)_{ii} = 1 - e^{-2\kappa w_i} - e^{2\kappa(h_i - w_i)} e^{-2\kappa h_i} = 1 - 2e^{-2\kappa w_i}$, while for $i \neq j$, using $d_T(i,j) \geq w_i + w_j$,

$$|R(\kappa)_{ij}| \;=\; e^{\kappa(h_i - w_i) + \kappa(h_j - w_j) - \kappa(h_i + h_j + d_T(i,j))} \;=\; e^{-\kappa(w_i + w_j + d_T(i,j))} \;\leq\; e^{-2\kappa(w_i + w_j)} \;\leq\; e^{-4\kappa w_{\min}}.$$

By Gershgorin's circle theorem, $R(\kappa) \succeq 0$ whenever $x := e^{-2\kappa w_{\min}}$ satisfies $1 - 2x - (n-1)x^2 \geq 0$, i.e., $x \leq \frac{-1+\sqrt{n}}{n-1} = \frac{1}{\sqrt{n}+1}$, which is equivalent to (6). Whenever (6) holds, $SM(\kappa)S \succeq 0$, hence $M(\kappa) \succeq 0$, so by (7), Sylvester's law of inertia, and Cauchy interlacing, $H(\kappa)$ has at most one positive eigenvalue. By Theorem 2.1(a), $(X, d_T) \hookrightarrow \mathbb{H}_K^{n-1}$.

Finally, let (X, d) be any finite ultrametric with diameter $D_{\max} = \mathrm{diam}(X)$ and distinct nonzero distance levels $0 = r_0 < r_1 < \cdots < r_m = D_{\max}$. For each $s \in \{0, \ldots, m-1\}$, the relation $d(i,j) \leq r_s$ is an equivalence relation on X, so its indicator matrix $Q_s := (\mathbf{1}_{\{d(i,j) \leq r_s\}})_{i,j=1}^n$ is block-diagonal all-ones, hence $Q_s \succeq 0$. For every $\kappa > 0$, writing $\psi(r) := \cosh(\kappa D_{\max}) - \cosh(\kappa r)$ (decreasing with $\psi(D_{\max}) = 0$) gives

$$\cosh(\kappa D_{\max}) - \cosh(\kappa d_{ij}) \;=\; \sum_{s=0}^{m-1} \underbrace{\big(\psi(r_s) - \psi(r_{s+1})\big)}_{>0} (Q_s)_{ij} \;\succeq\; 0.$$

Hence $H(\kappa) = \cosh(\kappa D_{\max})\mathbf{1}_n\mathbf{1}_n^\top - P(\kappa)$ with $P(\kappa) \succeq 0$, so $\lambda_2(H(\kappa)) \leq 0$ for all $\kappa > 0$; closedness at $K = 0$ (Theorem 2.1(a)) gives $(-\infty, 0] \subset \mathcal{K}(X, d)$. □

4 Unified Non-Euclidean Descartes Law and Strict Monotonicity for n-Leaf Stars

A distinguished subclass of strict leaf-tree metrics is the n-leaf weighted star $S(w) = S(w_1, \ldots, w_n)$ ($n \geq 3$, $w_i > 0$), defined by $d_{ij} := w_i + w_j$ ($i \neq j$) and $d_{ii} := 0$. Geometrically, $S(w)$ is the

distance matrix of the centers of n mutually externally tangent geodesic balls of radii $w_1, \dots, w_n$ in $\mathbb{M}_K^{n-1}$. Define the *generalized cotangent* of curvature $K \in \mathbb{R}$ by

$$\cot_K(w) \;:=\; \begin{cases} \sqrt{-K}\coth(\sqrt{-K}\,w), & K<0,\\ 1/w, & K=0,\\ \sqrt{K}\cot(\sqrt{K}\,w), & K>0, \end{cases} \tag{8}$$

which is real-analytic in K on $(-\infty,(\pi/w)^2)$. Ordering $0<w_1\le w_2\le\cdots\le w_n$, we have $\operatorname{diam}(S(w))=w_{n-1}+w_n>w_n$ (since $w_{n-1}>0$), so every w_i satisfies $\sqrt{K}w_i<\pi$ on the entire physical curvature domain $K\in(-\infty,K_{\mathrm{phys}}(w)]$, where $K_{\mathrm{phys}}(w):=\big(\frac{\pi}{w_{n-1}+w_n}\big)^2$.

Theorem 4.1 (Unified Non-Euclidean Descartes Law and Strict Monotonicity)**.** *Let $n\ge 3$ and $w=(w_1,\dots,w_n)\in(0,\infty)^n$. Define the Non-Euclidean Descartes functional on $(-\infty,K_{\mathrm{phys}}(w)]$ by*

$$f_S(K) \;:=\; (n-2)\sum_{i=1}^{n}\cot_K^2(w_i)-\left(\sum_{i=1}^{n}\cot_K(w_i)\right)^2+2(n-2)\,K, \tag{9}$$

and let $\mathcal{D}_0(w):=f_S(0)=(n-2)\sum_{i=1}^n w_i^{-2}-\big(\sum_{i=1}^n w_i^{-1}\big)^2$ be the Euclidean Descartes–Soddy–Gosset defect. Then:

(a) ***Unified master criterion.*** *For every $K\in(-\infty,K_{\mathrm{phys}}(w)]$,*

$$K\in\mathcal{K}(S(w)) \iff f_S(K)\;\le\;0, \tag{10}$$

with equality $f_S(K)=0$ if and only if $S(w)$ embeds isometrically into the codimension-1 space form $\mathbb{M}_K^{n-2}$.

(b) ***Global strict monotonicity.*** *On the entire physical curvature domain $(-\infty,K_{\mathrm{phys}}(w))$,*

$$\begin{aligned} &f_S'(K)\ge\;2(n-2)\;>\;0,\\ &f_S'(0)=\frac{2}{3}\left[\left(\sum_{i=1}^{n}\frac{1}{w_i}\right)\left(\sum_{i=1}^{n}w_i\right)-(n-2)(n-3)\right]\;\ge\;\frac{2}{3}(5n-6). \end{aligned} \tag{11}$$

(c) ***Exact half-line spectrum, Descartes trichotomy, and explicit curvature bound.*** *There exists a unique root $K_*(w)\in(-\infty,K_{\mathrm{phys}}(w))$ of $f_S(K)=0$, and the exact curvature embeddability spectrum of $S(w)$ is the half-infinite closed interval*

$$\mathcal{K}(S(w))\;=\;(-\infty,\;K_*(w)]. \tag{12}$$

Its sign obeys the exact trichotomy:

$$\begin{aligned} &\mathcal{D}_0(w)<0 \iff K_*(w)>0, \qquad \mathcal{D}_0(w)=0 \iff K_*(w)=0,\\ &\mathcal{D}_0(w)>0 \iff K_*(w)<0 \;\;(\textit{purely hyperbolic}). \end{aligned}$$

Moreover, $K_*(w)$ obeys the universal explicit bound

$$|K_*(w)|\;\le\;\frac{|\mathcal{D}_0(w)|}{2(n-2)}. \tag{13}$$

Proof. Part (a). First let $K=-\kappa^2<0$, and put $c_i:=\cosh(\kappa w_i)>1$, $s_i:=\sinh(\kappa w_i)>0$, $t_i:=\coth(\kappa w_i)=c_i/s_i>1$. For $i\ne j$, $\cosh(\kappa(w_i+w_j))=c_ic_j+s_is_j$, whereas on the diagonal $H(\kappa)_{ii}=1=c_i^2-s_i^2=(c_i^2+s_i^2)-2s_i^2$. Hence

$$H(\kappa)\;=\;cc^{\top}+ss^{\top}-2\operatorname{diag}(s_1^2,\dots,s_n^2)\;=\;D_s(tt^{\top}+\mathbf{1}_n\mathbf{1}_n^{\top}-2I_n)D_s,$$

where $D_s := \mathrm{diag}(s_1, \dots, s_n) \succ 0$. By Sylvester's law of inertia, $H(\kappa)$ has at most one positive eigenvalue iff the rank-2 matrix $M := tt^\top + \mathbf{1}_n\mathbf{1}_n^\top$ has $\lambda_2(M) \le 2$. On $\mathrm{span}(t, \mathbf{1}_n)$ (and noting $\lambda_2(M) = 0 < 2$ if $t \propto \mathbf{1}_n$), the two top eigenvalues $\mu_1 \ge \mu_2 \ge 0$ of M are the eigenvalues of the 2×2 Gram matrix $G_M := \left(\begin{smallmatrix} \|t\|_2^2 & \langle t, \mathbf{1}_n\rangle \\ \langle t, \mathbf{1}_n\rangle & n \end{smallmatrix}\right)$. Since $\|t\|_2^2 > n \ge 3 > 2$, we always have $\mu_1 \ge \max(\|t\|_2^2, n) > 2$, so $\mu_2 \le 2 \iff (\mu_1 - 2)(\mu_2 - 2) = \det(G_M - 2I_2) \le 0$:

$$\det\begin{pmatrix} \|t\|_2^2 - 2 & \langle t, \mathbf{1}_n\rangle \\ \langle t, \mathbf{1}_n\rangle & n - 2 \end{pmatrix} \;=\; (n-2)\|t\|_2^2 - \langle t, \mathbf{1}_n\rangle^2 - 2(n-2) \;\le\; 0.$$

Multiplying by $\kappa^2 = -K > 0$ yields $f_S(K) \le 0$ (and $\mu_2 = 2$ drops the rank of $H(\kappa)$ by 1, yielding an embedding into $\mathbb{H}^{n-2}_K$).

Next let $K = +\kappa^2 \in (0, K_{\mathrm{phys}}(w)]$, and put $\tilde{c}_i := \cos(\kappa w_i)$, $\tilde{s}_i := \sin(\kappa w_i) > 0$, $u_i := \cot(\kappa w_i) = \tilde{c}_i/\tilde{s}_i$. Since $\cos(\kappa(w_i + w_j)) = \tilde{c}_i\tilde{c}_j - \tilde{s}_i\tilde{s}_j$ $(i \ne j)$ and $1 = \tilde{c}_i^2 + \tilde{s}_i^2 = (\tilde{c}_i^2 - \tilde{s}_i^2) + 2\tilde{s}_i^2$, we obtain

$$S(\kappa) \;=\; D_{\tilde{s}}(uu^\top - \mathbf{1}_n\mathbf{1}_n^\top + 2I_n)D_{\tilde{s}}, \qquad D_{\tilde{s}} := \mathrm{diag}(\tilde{s}_1, \dots, \tilde{s}_n) \succ 0.$$

The rank-at-most-2 symmetric matrix $N := uu^\top - \mathbf{1}_n\mathbf{1}_n^\top$ has two eigenvalues $\nu_+ \ge 0 \ge \nu_-$ on $\mathrm{span}(u, \mathbf{1}_n)$ (allowing one or both to vanish when $u \propto \mathbf{1}_n$, while all remaining $n-2$ eigenvalues are 0) satisfying $\nu_+ + \nu_- = \|u\|_2^2 - n$ and $\nu_+\nu_- = -(n\|u\|_2^2 - \langle u, \mathbf{1}_n\rangle^2) \le 0$. Since $2 + \nu_+ \ge 2 > 0$, we have $S(\kappa) \succeq 0 \iff \nu_- \ge -2 \iff (2 + \nu_+)(2 + \nu_-) \ge 0$:

$$4 + 2(\|u\|_2^2 - n) - (n\|u\|_2^2 - \langle u, \mathbf{1}_n\rangle^2) \;\ge\; 0 \iff (n-2)\|u\|_2^2 - \langle u, \mathbf{1}_n\rangle^2 + 2(n-2) \;\le\; 0.$$

Multiplying by $\kappa^2 = K > 0$ again yields $f_S(K) \le 0$ (with equality iff $S(\kappa)$ is singular, giving an embedding into $\mathbb{S}^{n-2}_K$). Finally, for $K = 0$, observe that $D_{ij}^{\circ 2} = w_i^2 + w_j^2 + 2w_iw_j$ for $i \ne j$ and $D_{ii}^{\circ 2} = 0 = 4w_i^2 - 4w_i^2$, so $D^{\circ 2} = w^{\circ 2}\mathbf{1}_n^\top + \mathbf{1}_n(w^{\circ 2})^\top + 2ww^\top - 4D_w^2$, where $D_w := \mathrm{diag}(w_1, \dots, w_n) \succ 0$. Since $J\mathbf{1}_n = 0$,

$$G_0 \;=\; -\frac{1}{2}JD^{\circ 2}J \;=\; J\,D_w(2I_n - \mathbf{1}_n\mathbf{1}_n^\top)D_w\,J.$$

Setting $b := D_w^{-1}\mathbf{1}_n = (w_1^{-1}, \dots, w_n^{-1})^\top$ and changing variables $y = D_w x$ maps $\mathbf{1}_n^\perp$ isomorphically onto $b^\perp$. The orthogonal projection of $\mathbf{1}_n$ onto $b^\perp$ is $\mathbf{1}_n^\perp := \mathbf{1}_n - \frac{\langle \mathbf{1}_n, b\rangle}{\|b\|_2^2} b$, with squared norm $\|\mathbf{1}_n^\perp\|_2^2 = n - \frac{(\sum w_i^{-1})^2}{\sum w_i^{-2}}$. Hence by Cauchy–Schwarz,

$$\min_{y \in b^\perp,\, \|y\|_2 = 1} y^\top(2I_n - \mathbf{1}_n\mathbf{1}_n^\top)y \;=\; 2 - \|\mathbf{1}_n^\perp\|_2^2 \;=\; \frac{\left(\sum w_i^{-1}\right)^2 - (n-2)\sum w_i^{-2}}{\sum w_i^{-2}} \;=\; -\frac{\mathcal{D}_0(w)}{\sum w_i^{-2}},$$

showing directly that $G_0 \succeq 0 \iff \mathcal{D}_0(w) \le 0$, and that $\mathrm{rank}(G_0) \le n - 2$ (embedding into $\mathbb{R}^{n-2}$) holds iff G_0 has a nonzero nullvector in $\mathbf{1}_n^\perp$, i.e., iff $\mathcal{D}_0(w) = 0$.

Part (b). Assume $0 < w_1 \le w_2 \le \cdots \le w_n$. Set $c_i := \cot_K(w_i)$ and $g_i := -\frac{d}{dK}\cot_K(w_i)$. Direct differentiation using $\frac{d}{dK} = \frac{1}{2\kappa}\frac{d}{d\kappa}$ (for $K = \kappa^2 > 0$) and $\frac{d}{dK} = -\frac{1}{2\kappa}\frac{d}{d\kappa}$ (for $K = -\kappa^2 < 0$) gives

$$g_i \;=\; \begin{cases} \dfrac{\sinh(2\kappa w_i) - 2\kappa w_i}{4\kappa\sinh^2(\kappa w_i)}, & K = -\kappa^2 < 0, \\[2ex] \dfrac{w_i}{3}, & K = 0, \\[2ex] \dfrac{2\kappa w_i - \sin(2\kappa w_i)}{4\kappa\sin^2(\kappa w_i)}, & K = \kappa^2 > 0. \end{cases}$$

We verify two monotonicity properties in w for every fixed $K < K_{\mathrm{phys}}(w)$:

- $c(w) = \cot_K(w)$ is strictly decreasing in $w \in (0, \pi/\sqrt{K_+})$ (since $(\coth u)' = -\operatorname{csch}^2 u < 0$, $(1/w)' < 0$, $(\cot u)' = -\csc^2 u < 0$), so $c_1 \geq c_2 \geq \cdots \geq c_n$.
- $g(w) > 0$ is strictly increasing in w: for $K < 0$, $\frac{d}{du}\frac{\sinh(2u)-2u}{\sinh^2 u} = \frac{4\sinh u(u\cosh u - \sinh u)}{\sinh^4 u} > 0$ for $u > 0$ (as $\tanh u < u$); for $K > 0$, $\frac{d}{du}\frac{2u-\sin(2u)}{\sin^2 u} = \frac{4\sin u(\sin u - u\cos u)}{\sin^4 u} > 0$ for $u \in (0, \pi)$ (as $\frac{d}{du}(\sin u - u\cos u) = u\sin u > 0$). Thus $0 < g_1 \leq g_2 \leq \cdots \leq g_n$.

Differentiating (9) yields

$$\frac{1}{2}f_S'(K) \;=\; (n-2) - (n-2)\sum_{i=1}^{n} c_i g_i + \left(\sum_{i=1}^{n} c_i\right)\left(\sum_{i=1}^{n} g_i\right). \tag{14}$$

At $K = 0$, $c_i g_i = \frac{1}{3}$ for all i, so (14) immediately gives the exact formula in (11), and Cauchy–Schwarz $(\sum w_i^{-1})(\sum w_i) \geq n^2$ gives $f_S'(0) \geq \frac{2}{3}(n^2 - (n-2)(n-3)) = \frac{2}{3}(5n-6)$.

For general $K < K_{\text{phys}}(w)$, we consider two cases:

- *Case 1:* $c_n \geq 0$ (which covers all $K \leq 0$, and all $K > 0$ with $\sqrt{K}w_n \leq \frac{\pi}{2}$). Because $\{c_i\}$ is nonincreasing and $\{g_i\}$ is nondecreasing, Chebyshev's sum inequality gives $(\sum c_i)(\sum g_i) \geq n\sum c_i g_i$. Substituting into (14) yields

$$\frac{1}{2}f_S'(K) \;\geq\; (n-2) + 2\sum_{i=1}^{n} c_i g_i \;\geq\; n-2.$$

- *Case 2:* $c_n < 0$ (so $K = \kappa^2 > 0$ and $u_n := \kappa w_n \in (\frac{\pi}{2}, \pi)$). Set $\theta := \pi - u_n \in (0, \frac{\pi}{2})$. Since $K < K_{\text{phys}}(w)$, every $i \in \{1, \ldots, n-1\}$ satisfies $u_i := \kappa w_i < \pi - u_n = \theta < \frac{\pi}{2}$. Consequently:

$$c_i \;=\; \kappa\cot u_i \;>\; \kappa\cot\theta \;=\; -\kappa\cot(\pi-\theta) \;=\; |c_n| \;>\; 0,$$

and since $2\pi - 4\theta + 2\sin(2\theta) > 0$ on $(0, \frac{\pi}{2})$,

$$g_n \;=\; \frac{2(\pi-\theta)+\sin(2\theta)}{4\kappa\sin^2\theta} \;>\; \frac{2\theta-\sin(2\theta)}{4\kappa\sin^2\theta} \;=\; g(\theta/\kappa) \;>\; g(w_i) \;=\; g_i \;>\; 0.$$

Therefore $c_i g_n + c_n g_i = c_i g_n - |c_n| g_i > |c_n|(g_n - g_i) > 0$ for every $i \leq n-1$. Separating the n-th index in (14) with $S_c := \sum_{i=1}^{n-1} c_i > 0$ and $S_g := \sum_{i=1}^{n-1} g_i > 0$, and applying Chebyshev's sum inequality to the $n-1$ pairs $\{(c_i, g_i)\}_{i=1}^{n-1}$, gives

$$\frac{1}{2}f_S'(K) \;=\; (n-2) + \underbrace{\Big[S_c S_g - (n-2)\sum_{i=1}^{n-1} c_i g_i\Big]}_{\geq \sum_{i=1}^{n-1} c_i g_i > 0} + \underbrace{(n-3)|c_n| g_n}_{\geq 0} + \sum_{i=1}^{n-1} \underbrace{(c_i g_n - |c_n| g_i)}_{>0} \;>\; n-2.$$

This proves (11) on all of $(-\infty, K_{\text{phys}}(w))$.

Part (c). As $K = -\kappa^2 \to -\infty$, $\coth(\kappa w_i) \to 1$, so $f_S(K)/|K| \to (n-2)n - n^2 - 2(n-2) = -4n + 4 < 0$, whence $\lim_{K\to-\infty} f_S(K) = -\infty$. At the upper physical endpoint $K_{\text{phys}} = (\pi/(w_{n-1} + w_n))^2$, we have $\kappa(w_{n-1} + w_n) = \pi$, so $c_n = -c_{n-1}$; setting $S_0 := \sum_{i=1}^{n-2} c_i = \sum_{i=1}^{n} c_i$, Cauchy–Schwarz gives $(n-2)\sum_{i=1}^{n-2} c_i^2 \geq S_0^2$, and therefore

$$f_S(K_{\text{phys}}) \;=\; \Big[(n-2)\sum_{i=1}^{n-2} c_i^2 - S_0^2\Big] + 2(n-2)c_{n-1}^2 + 2(n-2)K_{\text{phys}} \;\geq\; 2(n-2)K_{\text{phys}} \;>\; 0.$$

By strict monotonicity $f_S'(K) \geq 2(n-2) > 0$, f_S vanishes at a unique point $K_*(w) \in (-\infty, K_{\text{phys}}(w))$, establishing (12) and the exact sign trichotomy. Finally, by the Mean Value Theorem between 0 and $K_*(w)$, $|\mathcal{D}_0(w)| = |f_S(0) - f_S(K_*(w))| \geq 2(n-2)|K_*(w)|$, proving (13). □

5 Exact Curvature Spectra of Canonical Families and the Connectedness Conjecture

We now compute exact curvature spectra across canonical finite graphs and the complete 3-parameter family of 4-point Klein-4 metric spaces (spaces with $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ and translation-invariant distances $d(x, y) \in \{d_1, d_2, d_3\}$ for $x \neq y$, where $0 < d_1 \leq d_2 \leq d_3 \leq d_1 + d_2$).

Theorem 5.1 (Exact Curvature Spectra of Canonical Metric Families)**.** *The constant-curvature embeddability spectra of the following canonical finite metric families are given explicitly as follows (collectively realizing ten distinct sign-stratified closed-interval types relative to $K = 0$: $\emptyset$; the singletons $\{K_0\}$ for $K_0 < 0$, $K_0 = 0$, and $K_0 > 0$; the half-lines $(-\infty, K_*]$ for $K_* < 0$, $K_* = 0$, and $K_* > 0$; and the compact intervals $[K_-, K_+]$ with $K_- < 0 < K_+$, $0 = K_- < K_+$, and $0 < K_- < K_+$):*

(a) ***Empty spectrum ($\mathcal{K} = \emptyset$).*** *Both the 4-vertex star graph $K_{1,3}$ (with center vertex included) and every d-dimensional hypercube graph Q_d ($d \geq 3$) satisfy $\mathcal{K}(K_{1,3}) = \mathcal{K}(Q_d) = \emptyset$.*

(b) ***Isolated positive singleton ($\mathcal{K} = \{K_0\}$, $K_0 > 0$).*** *For every integer $m \geq 4$, the unit-edge cycle graph C_m satisfies*

$$\mathcal{K}(C_m) \;=\; \left\{ \left(\frac{2\pi}{m} \right)^2 \right\}. \tag{15}$$

(c) ***Purely hyperbolic ($K_* < 0$), hyperbolic-to-Euclidean ($K_* = 0$), and hyperbolic-to-spherical ($K_* > 0$) half-lines ($\mathcal{K} = (-\infty, K_*]$).*** *By Theorem 4.1, every n-leaf star $S(w)$ with $\mathcal{D}_0(w) > 0$ realizes a purely hyperbolic half-line ($K_* < 0$), while every $S(w)$ with $\mathcal{D}_0(w) \leq 0$—in particular, the regular n-simplex K_n of unit edge length, which has $\mathcal{K}(K_n) = (-\infty, (\arccos(-\frac{1}{n-1}))^2]$—realizes a half-line with $K_* \geq 0$.*

(d) ***Exact five-regime Klein-4 classification (realizing purely spherical and two-sided bounded intervals).*** *Let $X_{\mathrm{K4}}(d_1, d_2, d_3)$ be the 4-point Klein-4 metric space with sorted nonzero distances $0 < d_1 \leq d_2 \leq d_3 \leq d_1 + d_2$. Let $\kappa_+ \in (0, \pi/d_3]$ be the unique root of $1 + \cos(\kappa d_1) + \cos(\kappa d_2) + \cos(\kappa d_3) = 0$, and put $K_+ := \kappa_+^2 > 0$. Then:*

(i) If $d_3 = d_2$ (ultrametric): $\mathcal{K} = (-\infty, K_+]$.

(ii) If $d_2 < d_3 < \sqrt{d_1^2 + d_2^2}$ (compressed rectangle): $\mathcal{K} = [K_-, K_+]$ with $-\infty < K_- < 0 < K_+$, where $K_- = -\kappa_-^2$ and $\kappa_- > 0$ is the unique positive root of $\sinh^2 \frac{\kappa d_3}{2} = \sinh^2 \frac{\kappa d_1}{2} + \sinh^2 \frac{\kappa d_2}{2}$. Moreover, for unit-side rhombi $(1, 1, c)$ ($1 < c < \sqrt{2}$), $\delta = c - 1$ and $\sqrt{-K_-}\,\delta \to \log 2$ as $c \downarrow 1$, asymptotically saturating the sharp Gromov ceiling (4).

(iii) If $d_3 = \sqrt{d_1^2 + d_2^2}$ (Euclidean rectangle): $\mathcal{K} = [0, K_+]$ (no negative curvature embeds).

(iv) If $\sqrt{d_1^2 + d_2^2} < d_3 < d_1 + d_2$ (stretched rectangle): $\mathcal{K} = [K_-, K_+]$ with $0 < K_- < K_+$ (purely spherical compact interval), where $K_- = \kappa_-^2$ and $\kappa_- \in (0, \kappa_+)$ is the unique root of $\sin^2 \frac{\kappa d_3}{2} = \sin^2 \frac{\kappa d_1}{2} + \sin^2 \frac{\kappa d_2}{2}$.

(v) If $d_3 = d_1 + d_2$ (degenerate C_4): $\mathcal{K} = \{(\pi/d_3)^2\}$ (isolated singleton).

(e) ***Arbitrary isolated singletons ($\mathcal{K} = \{K_0\}$ for every $K_0 \in \mathbb{R}$).*** *For any $K_0 \in \mathbb{R}$, choose $r > 0$ with $\sqrt{K_0}\, r < \pi/2$ whenever $K_0 > 0$, and let $X_5(K_0) = \{o, p_1, p_2, p_3, p_4\} \subset \mathbb{M}^2_{K_0}$ be the 5-point centered orthogonal cross formed by an origin o and four points at geodesic distance r along orthogonal rays at angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (so $d(o, p_i) = r$, $d(p_1, p_3) = d(p_2, p_4) = 2r$, and $d(p_i, p_{i+1}) = a(K_0, r)$ with $\cosh(\sqrt{-K_0}\, a) = \cosh^2(\sqrt{-K_0}\, r)$ for $K_0 < 0$, $a = \sqrt{2}\, r$ for $K_0 = 0$, and $\cos(\sqrt{K_0}\, a) = \cos^2(\sqrt{K_0}\, r)$ for $K_0 > 0$). Then*

$$\mathcal{K}(X_5(K_0)) \;=\; \{K_0\}. \tag{16}$$

Proof. Part (a). For $K_{1,3}$, Theorem 3.2(b) gives $\mathcal{K}(K_{1,3}) \cap (-\infty, 0] = \emptyset$. For $K = \kappa^2 > 0$ with $\kappa \in (0, \pi/2]$ (as $\operatorname{diam}(K_{1,3}) = 2$), set $c := \cos\kappa \in [0, 1)$. On the 2-dimensional subspace of vectors $(x_0, y, y, y)^\top$, the matrix $S(\kappa)$ restricts (in the basis $\mathbf{e}_0$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$) to the Gram matrix $\left(\begin{smallmatrix} 1 & 3c \\ 3c & 3(1+2(2c^2-1)) \end{smallmatrix}\right)$, whose determinant is $3(4c^2 - 1) - 9c^2 = 3(c^2 - 1) = -3\sin^2\kappa < 0$ for all $\kappa \in (0, \pi/2]$. Thus $S(\kappa) \not\succeq 0$ and $\mathcal{K}(K_{1,3}) = \emptyset$.

For the hypercube Q_d ($d \geq 3$), the vertices 0^d and 1^d at distance d are joined by $d \geq 3$ distinct shortest paths of length d, which is impossible in $\mathbb{M}_K^N$ unless $K > 0$ and $0^d, 1^d$ are antipodal, forcing $\sqrt{K} = \pi/d$. However, $(0, 0, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$ lie at distance $2 < d = \pi/\sqrt{K}$ and yet are joined by two distinct shortest paths (through $(1, 0, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$), contradicting geodesic uniqueness for non-antipodal points on $\mathbb{S}_K^N$. Hence $\mathcal{K}(Q_d) = \emptyset$.

Part (b). For $m = 2r \geq 4$ even, opposite vertices 0 and r of C_{2r} have two distinct shortest paths of length r, forcing $K > 0$ and $\sqrt{K} = \pi/r = 2\pi/m$; at $K = (2\pi/m)^2$, the regular m-gon on $\mathbb{S}^1_{(2\pi/m)^2}$ (the circle of radius $m/(2\pi)$) is an isometric embedding. For $m = 2r + 1 \geq 5$ odd, $\operatorname{diam}(C_m) = r \geq 2$, so any $K \in \mathcal{K}(C_m)$ with $K > 0$ satisfies $\kappa = \sqrt{K} \leq \pi/r$. Note $2\kappa < \pi$ strictly: for $r \geq 3$ this is immediate from $2\pi/r < \pi$, while for $r = 2$ (C_5) having $\kappa = \pi/2$ would force both v_2 and v_3 to be antipodal to v_0, contradicting $d(v_2, v_3) = 1$. Thus for any $K \in \mathcal{K}(C_m)$ (whether $K \leq 0$ or $K > 0$), points at distance 2 have a unique geodesic in $\mathbb{M}_K^N$. Since $d(v_{i-1}, v_{i+1}) = 2 = d(v_{i-1}, v_i) + d(v_i, v_{i+1})$ for all $i \pmod m$, every triple (v_{i-1}, v_i, v_{i+1}) lies monotonically on a common geodesic without backtracking, forcing all m vertices to advance by unit steps around a single complete geodesic γ. If $K \leq 0$, $\gamma \cong \mathbb{R}^1$ cannot close up, a contradiction. If $K > 0$, γ is a great circle of circumference $2\pi/\kappa \geq 2r$, and returning to v_0 after m monotonic unit steps requires $m = q\,(2\pi/\kappa)$ for some positive integer q; since $r \geq 2$, we have $1 \leq q = m/(2\pi/\kappa) \leq (2r + 1)/(2r) < 2$, forcing $q = 1$ and $\kappa = 2\pi/m$. Conversely, the regular m-gon on $\mathbb{S}^1_{(2\pi/m)^2}$ embeds C_m isometrically.

Part (c). The star statement is Theorem 4.1(c). For the unit regular simplex $K_n = S(\frac{1}{2}, \ldots, \frac{1}{2})$, $f_S(K) = -2nK\cot^2(\frac{\sqrt{K}}{2}) + 2(n-2)K \leq 0 \iff \cot^2(\frac{\sqrt{K}}{2}) \geq \frac{n-2}{n} \iff \cos\sqrt{K} \geq -\frac{1}{n-1}$, yielding $K_* = (\arccos(-\frac{1}{n-1}))^2$.

Part (d). Because D is a Cayley matrix on $\mathbb{Z}_2 \times \mathbb{Z}_2$, the four Walsh–Hadamard characters $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(1,-1,-1,1)$ simultaneously diagonalize $H(\kappa)$, G_0, and $S(\kappa)$ for all κ. At $K = 0$, $G_0 = -\frac{1}{2}JD^{\circ 2}J$ has eigenvalue 0 on $(1,1,1,1)$ and eigenvalues

$$\gamma_1 = \frac{d_2^2 + d_3^2 - d_1^2}{2} > 0, \qquad \gamma_2 = \frac{d_1^2 + d_3^2 - d_2^2}{2} > 0, \qquad \gamma_3 = \frac{d_1^2 + d_2^2 - d_3^2}{2}$$

on the three nontrivial characters; thus $0 \in \mathcal{K}(X_{\mathrm{K4}}) \iff d_3 \leq \sqrt{d_1^2 + d_2^2}$, which immediately includes $K = 0$ in regimes (i)–(iii) and excludes $K = 0$ in regimes (iv)–(v).

For $K = +\kappa^2 \in (0, (\pi/d_3)^2]$, the four eigenvalues of $S(\kappa)$ are

$$\mu_0(\kappa) = 1 + \cos(\kappa d_1) + \cos(\kappa d_2) + \cos(\kappa d_3),$$
$$\mu_1(\kappa) = 2\Big(\sin^2\tfrac{\kappa d_2}{2} + \sin^2\tfrac{\kappa d_3}{2} - \sin^2\tfrac{\kappa d_1}{2}\Big) > 0, \qquad \mu_2(\kappa) = 2\Big(\sin^2\tfrac{\kappa d_1}{2} + \sin^2\tfrac{\kappa d_3}{2} - \sin^2\tfrac{\kappa d_2}{2}\Big) > 0,$$
$$\mu_3(\kappa) = 2\Big(\sin^2\tfrac{\kappa d_1}{2} + \sin^2\tfrac{\kappa d_2}{2} - \sin^2\tfrac{\kappa d_3}{2}\Big),$$

where $\mu_1, \mu_2 > 0$ because $\frac{\kappa d_1}{2} \leq \frac{\kappa d_2}{2} \leq \frac{\kappa d_3}{2} \leq \frac{\pi}{2}$ and sin is increasing on $[0, \frac{\pi}{2}]$. On $(0, \pi/d_3)$, $\mu_0'(\kappa) = -\sum_{j=1}^3 d_j \sin(\kappa d_j) < 0$ strictly, with $\mu_0(0) = 4 > 0$ and

$$\mu_0(\pi/d_3) \;=\; \cos\frac{\pi d_1}{d_3} + \cos\frac{\pi d_2}{d_3} \;=\; 2\cos\left(\frac{\pi(d_1 + d_2)}{2d_3}\right)\cos\left(\frac{\pi(d_2 - d_1)}{2d_3}\right) \;\leq\; 0$$

(since $\frac{d_1+d_2}{2d_3} \in [\frac{1}{2}, 1]$ and $\frac{d_2-d_1}{2d_3} \in [0, \frac{1}{2})$), with equality iff $d_3 = d_1 + d_2$. Thus $\mu_0(\kappa) \geq 0 \iff \kappa \leq \kappa_+$ for a unique $\kappa_+ \in (0, \pi/d_3]$.

Next, for any $\alpha \in (0,1)$, the function $u \mapsto \frac{\sin(\alpha u)}{\sin u}$ is strictly increasing on $(0, \frac{\pi}{2}]$ (its derivative has the sign of $\alpha \tan u - \tan(\alpha u) > 0$ since $\frac{\tan x}{x}$ is strictly increasing on $(0, \frac{\pi}{2})$). Consequently, the ratio $R_+(\kappa) := (\sin^2 \frac{\kappa d_1}{2} + \sin^2 \frac{\kappa d_2}{2}) / \sin^2 \frac{\kappa d_3}{2}$ satisfies $\lim_{\kappa \downarrow 0} R_+(\kappa) = \frac{d_1^2+d_2^2}{d_3^2}$ and is strictly increasing on $(0, \pi/d_3]$ whenever $d_3 > d_2$ (and strictly > 1 always when $d_3 = d_2$). Moreover, at $\kappa = \kappa_+$, $\mu_0(\kappa_+) = 0$ gives $\mu_3(\kappa_+) = 2(1 + \cos(\kappa_+ d_3)) \geq 0$, strictly > 0 whenever $d_3 < d_1 + d_2$ (as $\kappa_+ < \pi/d_3$). Hence $\mu_3(\kappa) \geq 0$ on all of $(0, \kappa_+]$ if $d_3 \leq \sqrt{d_1^2 + d_2^2}$, on $[\kappa_-, \kappa_+]$ for a unique $\kappa_- \in (0, \kappa_+)$ if $\sqrt{d_1^2 + d_2^2} < d_3 < d_1 + d_2$, and only at $\kappa_+ = \pi/d_3$ if $d_3 = d_1 + d_2$.

For $K = -\kappa^2 < 0$, the non-Perron eigenvalues of $H(\kappa)$ are

$$\lambda_1 = 2\Big(\sinh^2 \tfrac{\kappa d_1}{2} - \sinh^2 \tfrac{\kappa d_2}{2} - \sinh^2 \tfrac{\kappa d_3}{2}\Big) < 0, \qquad \lambda_2 = 2\Big(\sinh^2 \tfrac{\kappa d_2}{2} - \sinh^2 \tfrac{\kappa d_1}{2} - \sinh^2 \tfrac{\kappa d_3}{2}\Big) < 0,$$

and $\lambda_3(\kappa) = 2(\sinh^2 \frac{\kappa d_3}{2} - \sinh^2 \frac{\kappa d_1}{2} - \sinh^2 \frac{\kappa d_2}{2})$. Since $\frac{\tanh x}{x}$ is strictly decreasing on $(0, \infty)$, $u \mapsto \frac{\sinh(\alpha u)}{\sinh u}$ is strictly decreasing on $(0, \infty)$ for every $\alpha \in (0,1)$, so $R_-(\kappa) := (\sinh^2 \frac{\kappa d_1}{2} + \sinh^2 \frac{\kappa d_2}{2}) / \sinh^2 \frac{\kappa d_3}{2}$ decreases strictly from $\frac{d_1^2+d_2^2}{d_3^2}$ to 0 when $d_3 > d_2$ (and stays strictly > 1 for all $\kappa > 0$ when $d_3 = d_2$). Thus $\lambda_3(\kappa) \leq 0$ for all $\kappa > 0$ when $d_3 = d_2$, on $(0, \kappa_-]$ for a unique $\kappa_- > 0$ when $d_2 < d_3 < \sqrt{d_1^2 + d_2^2}$, and for no $\kappa > 0$ when $d_3 \geq \sqrt{d_1^2 + d_2^2}$. Finally, for $(1, 1, c)$ with $c \in (1, \sqrt{2})$, we have $\delta = c - 1$ and $\sinh^2 \frac{\kappa_- c}{2} = 2 \sinh^2 \frac{\kappa_-}{2}$; as $c \downarrow 1$, $\kappa_- \to \infty$, so $\frac{1}{4} e^{\kappa_- c}(1 + o(1)) = \frac{1}{2} e^{\kappa_-}(1 + o(1))$, giving $e^{\kappa_-(c-1)} \to 2$ and $\kappa_- \delta \to \log 2$.

Part (e). Clearly $K_0 \in \mathcal{K}(X_5(K_0))$ by construction. Note that $a = a(K_0, r) \in (r, 2r)$ strictly for all $K_0 \in \mathbb{R}$: for $K_0 < 0$, $\cosh(\kappa_0 a) = \cosh^2(\kappa_0 r) \in (\cosh(\sqrt{2}\kappa_0 r), \cosh(2\kappa_0 r))$ gives $a \in (\sqrt{2}\, r, 2r)$; for $K_0 = 0$, $a = \sqrt{2}\, r$; and for $K_0 > 0$ with $\kappa_0 r < \frac{\pi}{2}$, $0 < \cos^2(\kappa_0 r) < \cos(\kappa_0 r) < 1$ gives $a \in (r, \sqrt{2}\, r)$ (since $\sin(\kappa_0 a/2) / \sin(\kappa_0 r) = 1/\sqrt{2}$ and $\sin(\alpha u)/\sin u > \alpha$ forces $\alpha = a/(2r) < 1/\sqrt{2}$). Suppose $X_5(K_0)$ embeds isometrically into $\mathbb{M}_K^N$ for some $K \in \mathbb{R}$. If $K = \kappa^2 > 0$, then $\operatorname{diam}(X_5(K_0)) = 2r$ gives $\kappa r \leq \frac{\pi}{2}$; equality $\kappa r = \frac{\pi}{2}$ is impossible, because it would make $p_3 = -p_1$ and $p_4 = -p_2$ on the equator orthogonal to o, forcing $\langle p_1, p_2 \rangle = \langle p_2, p_3 \rangle = -\langle p_1, p_2 \rangle = 0$ and hence $a = \frac{\pi}{2\kappa} = r$, contradicting $a > r$. Thus for every $K \in \mathcal{K}(X_5(K_0))$, geodesics of length $2r$ are unique in $\mathbb{M}_K^N$, so $d(p_1, o) + d(o, p_3) = 2r = d(p_2, o) + d(o, p_4)$ forces o to be the unique geodesic midpoint of both $[p_1, p_3]$ and $[p_2, p_4]$. Letting $\theta := \angle_o(p_1, p_2) \in [0, \pi]$, we have $\angle_o(p_2, p_3) = \pi - \theta$, and equating the law of cosines for $d(p_1, p_2) = d(p_2, p_3) = a$ forces $\cos\theta = 0$ in every geometry, yielding $\sinh(\frac{\kappa a}{2}) / \sinh(\kappa r) = \frac{1}{\sqrt{2}}$ if $K = -\kappa^2 < 0$, $a = \sqrt{2}\, r$ if $K = 0$, and $\sin(\frac{\kappa a}{2}) / \sin(\kappa r) = \frac{1}{\sqrt{2}}$ if $K = \kappa^2 > 0$. Setting $\alpha := \frac{a}{2r} \in (\frac{1}{2}, 1)$ and $u := \kappa r$, the ratio monotonicity established in Part (d) implies: for $K_0 < 0$ ($\alpha > \frac{1}{\sqrt{2}}$), only $K < 0$ can attain $\frac{1}{\sqrt{2}}$, and $u \mapsto \sinh(\alpha u)/\sinh u$ is strictly decreasing on $(0, \infty)$, so $K = K_0$ is unique; for $K_0 = 0$ ($\alpha = \frac{1}{\sqrt{2}}$), $\sinh(\alpha u)/\sinh u < \alpha$ for $K < 0$ and $\sin(\alpha u)/\sin u > \alpha$ for $K > 0$, forcing $K = 0$; and for $K_0 > 0$ ($\alpha < \frac{1}{\sqrt{2}}$), only $K > 0$ can attain $\frac{1}{\sqrt{2}}$, and $u \mapsto \sin(\alpha u)/\sin u$ is strictly increasing on $(0, \frac{\pi}{2})$, so $K = K_0$ is unique. □

Notice that across every family analyzed in Theorems 3.2, 4.1, and 5.1—every n-leaf star, every Klein-4 metric, every cycle graph, every hypercube, and every ultrametric—as well as 1,200 random 4-, 5-, and 6-point metrics tested in Section 6, $\mathcal{K}(X, d)$ is always a single connected interval (possibly empty, a singleton, a half-line, or a compact interval). This motivates our central open problem:

Conjecture 5.2 (Curvature Spectrum Connectedness Conjecture)**.** *For every finite metric space (X, d), the constant-curvature embeddability spectrum $\mathcal{K}(X, d)$ is a connected (closed) interval in $\mathbb{R}$—that is, if (X, d) embeds isometrically into both $\mathbb{M}_{K_1}^{N_1}$ and $\mathbb{M}_{K_2}^{N_2}$ ($K_1 < K_2$), then (X, d) embeds isometrically into $\mathbb{M}_K^{|X|-1}$ for every $K \in [K_1, K_2]$.*

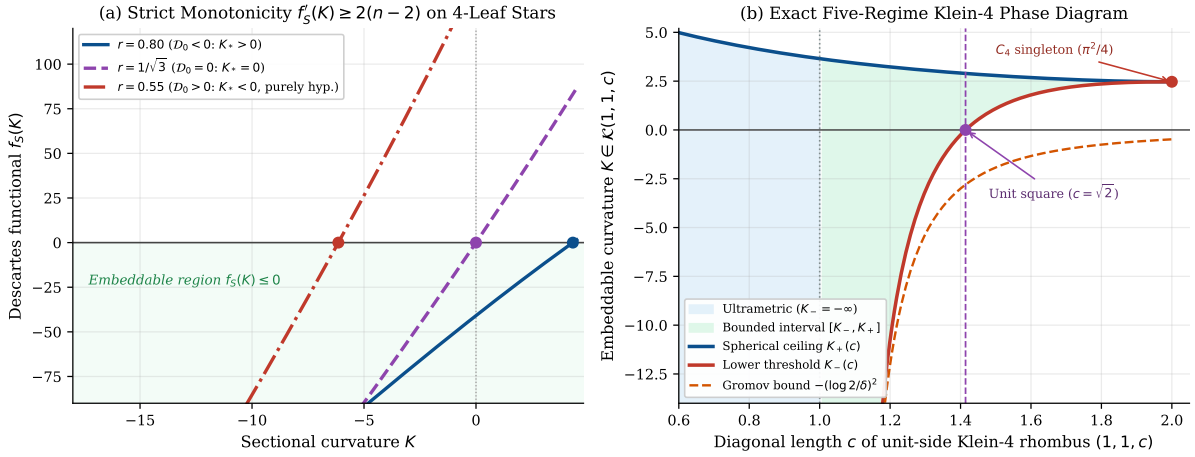


Figure 1: **(a)** Non-Euclidean Descartes functional $f_S(K)$ for 4-leaf stars $(0.5, 0.5, 0.5, r - 0.5)$ across all three defect regimes ($\mathcal{D}_0 < 0$, $\mathcal{D}_0 = 0$, $\mathcal{D}_0 > 0$), illustrating strict global monotonicity $f_S'(K) \geq 2(n-2) = 4$ and unique root $K_*(w)$ (Theorem 4.1). **(b)** Exact five-regime curvature phase diagram $\mathcal{K}(1,1,c)$ for unit-side Klein-4 rhombi as c sweeps from 0.6 to 2.0 (Theorem 5.1(d)), showing the ultrametric half-line ($c \leq 1$), two-sided bounded interval ($1 < c < \sqrt{2}$), unit square ($c = \sqrt{2}$), purely spherical interval ($\sqrt{2} < c < 2$), and isolated C_4 singleton ($c = 2$, $K = \pi^2/4$), together with the sharp Gromov ceiling $-(\log 2/\delta)^2$.

6 Computational Verification and Quantitative Benchmarks

All four theorems, exact threshold formulas, and numerical tests for Conjecture 5.2 are independently verified by the open-source Python suite in `https://github.com/lluiseriksson/curvature-embeddability-spectra` (tag `v1.0.5`), comprising `scripts/verify_curvature_embeddability.py`, `scripts/make_figures.py`, and the `pytest` suite in `tests/test_curvature_embeddability.py` (4/4 test modules passing in 2.33 s).

6.1 Summary of Verification Results

- **Theorem 2.1:** Across 10,000 random 4-point configurations in $\mathbb{H}^4_{-1}$, the maximum Exponential Ptolemy ratio $e^{S_3/2}/(e^{S_1/2} + e^{S_2/2})$ was $0.99493689 \leq 1$ and $\max \delta = 0.64122830 < \log 2 \approx 0.69314718$. In 40-digit `mpmath` precision, ideal regular quadrilaterals at $r = 0.999999$ reached $\log 2 - \delta = 5.000 \times 10^{-13}$.

- **Theorem 3.2:** All 300/300 random ultrametric checks ($n \in \{4,5,6,8\}$, $K \in [-100, 0]$) passed with $K_0 = 0$; all 100/100 random strict leaf-tree metrics ($n \in \{4,5,6,7\}$, including 14 purely hyperbolic non-Euclidean leaf-trees) satisfied $\kappa_0 \leq \frac{\log(\sqrt{n}+1)}{2w_{\min}}$ with maximum ratio 0.318983; and 40-digit eigensolver checks confirmed $\lambda_2(H(\kappa)) > 0$ across $K \in [-25, 0)$ (and $\lambda_{\min}(G_0) < 0$ at $K = 0$) for both $K_{1,3}$ and a degree-2 subdivided star.

- **Theorem 4.1:** Across 75 random stars (1,275 star-curvature pairs, $n \in \{3,4,5,6,8\}$), the Descartes master criterion (10) matched direct matrix eigensolving (with 40-digit `mpmath` fallback when $\kappa \operatorname{diam} > 12$) with 1,275/1,275 exact agreement, $\min f_S'(K)/(2(n-2)) = 2.277941 \geq 1$, and $\max |K_*(w)|/(|\mathcal{D}_0(w)|/(2(n-2))) = 0.396608 \leq 1$ on the random ensemble; deterministic 4-leaf benchmarks (where the regular simplex attains ratio $3.650519/8 = 0.456315 \leq 1$ and the apex $r = 0.8$ attains $4.324597/10.2222 = 0.423059 \leq 1$) are reported in Table 1.

- **Theorem 5.1 and Conjecture 5.2:** Exact singleton spectra were verified for C_4–C_8 and for 5-point centered orthogonal crosses $X_5(K_0)$ ($K_0 \in \{-2, -1, 0, 0.5, 1\}$), empty spectra

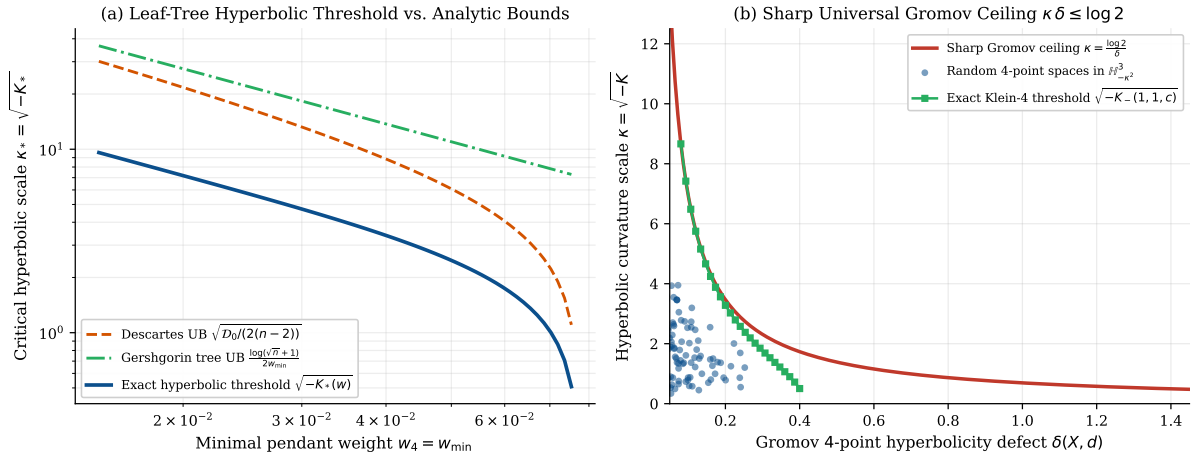


Figure 2: **(a)** Exact purely hyperbolic threshold $\kappa_* = \sqrt{-K_*(w)}$ for 4-leaf stars $(0.5, 0.5, 0.5, w_4)$ $(w_4 \in [0.015, 1/\sqrt{3} - 0.5))$ compared with the universal Gershgorin tree bound $\frac{\log(\sqrt{n}+1)}{2w_{\min}}$ (Theorem 3.2(c)) and the Descartes bound $\sqrt{\mathcal{D}_0/(2(n-2))}$ (Theorem 4.1(c)). **(b)** Scatter of $(\delta(X, d), \kappa)$ across 180 random 4-point spaces in $\mathbb{H}^3_{-\kappa^2}$ and the exact Klein-4 boundary $\sqrt{-K_-(1, 1, c)}$, demonstrating saturation of the sharp universal Gromov ceiling $\kappa\delta \le \log 2$ (Theorem 2.1(c)) as $\delta \downarrow 0$.

Table 1: Non-Euclidean Descartes benchmarks for 4-leaf stars $S(w)$ (Theorem 4.1), displaying the Euclidean defect $\mathcal{D}_0(w)$, exact critical curvature $K_*(w)$, explicit upper bound $|\mathcal{D}_0(w)|/(2(n-2))$, and slope $f'_S(0)$ at $K = 0$.

4-leaf star $S(w_1, w_2, w_3, w_4)$	$\mathcal{D}_0(w)$	Exact $K_*(w)$	$\lvert\mathcal{D}_0\rvert/(2(n-2))$	$f'_S(0)$
Regular simplex $(0.5, 0.5, 0.5, 0.5)$	-32.0000	$+3.650519$	8.0000	9.3333
Apex $r = 0.8$ $(0.5, 0.5, 0.5, 0.3)$	-40.8889	$+4.324597$	10.2222	9.8667
Descartes critical $r = 1/\sqrt{3}$	0.0000	0.000000	0.0000	18.5709
Purely hyperbolic $r = 0.55$ $(0.5, 0.5, 0.5, 0.05)$	$+148.0000$	-6.145569	37.0000	25.5333
Purely hyperbolic $(0.1, 1.0, 1.0, 1.0)$	$+37.0000$	-1.536392	9.2500	25.5333
Asymmetric $(0.1, 0.1, 0.1, 2.0)$	-329.7500	$+2.010559$	82.4375	45.4333

for Q_3, Q_4, all five Klein-4 regimes in Table 2, and 1,200/1,200 random 4-, 5-, and 6-point metrics yielded connected curvature spectra (0 counterexamples).

7 Concluding Remarks and Open Directions

We have established a unified quantitative framework for the constant-curvature embeddability spectrum $\mathcal{K}(X, d)$ of finite metric spaces, proving the sharp $\log 2$ Gromov curvature ceiling via an elementary $\mathbb{R}^4$ Exponential Ptolemy proof, completely characterizing asymptotic hyperbolic embeddability via the leaf-tree dichotomy and Gershgorin clade bound, solving the unified Non-Euclidean Descartes law with global strict monotonicity for n-leaf stars, and determining exact canonical phase spectra realizing ten distinct sign-stratified interval types (including arbitrary isolated singletons $\{K_0\}$ for all $K_0 \in \mathbb{R}$). Two natural open problems stand out:

1. **Connectedness of $\mathcal{K}(X, d)$ (Conjecture 5.2).** Proving (or finding a high-cardinality counterexample to) the connectedness of $\mathcal{K}(X, d)$ for general $n \ge 4$—even for general 4-point metrics beyond Klein-4 and star families—would establish a discrete analog of Alexandrov–Toponogov comparison intervals for exact isometric embeddability.

Table 2: Exact five-regime curvature spectra $[K_-, K_+]$ for unit-side Klein-4 rhombi $X_{\mathrm{K4}}(1, 1, c)$ (Theorem 5.1(d)), showing Gromov defect $\delta = \max(0, c-1)$ and the Gromov negative-curvature ceiling $(\log 2/\delta)^2 \geq -K_-$.

Diagonal c	Phase	$K_-(c)$	$K_+(c)$	δ	$(\log 2/\delta)^2$
0.8000	Ultrametric half-line	$-\infty$	+4.21639	0.0000	∞
1.0000	Regular simplex K_4	$-\infty$	+3.65052	0.0000	∞
1.2000	Two-sided bounded	−10.75364	+3.22767	0.2000	12.0113
1.3000	Two-sided bounded	−3.12813	+3.05787	0.3000	5.3384
$\sqrt{2}$	Euclidean unit square	0.00000	+2.89245	0.4142	2.8003
1.6000	Purely spherical	+1.79485	+2.68101	0.6000	1.3346
1.8000	Purely spherical	+2.35798	+2.52864	0.8000	0.7507
1.9500	Purely spherical	+2.46215	+2.47191	0.9500	0.5324
2.0000	C_4 singleton ($\pi^2/4$)	+2.46740	+2.46740	1.0000	0.4805

2. **Sharp Thresholds on General Strict Leaf-Trees.** For general strict leaf-trees with multiple internal vertices (beyond single-center stars), determining the exact critical curvature $K_0(X, d_T)$ in terms of the tree's internal edge lengths and degree sequence, and characterizing precisely when $K_0(X, d_T) = 0$ (Euclidean embeddability of leaf-trees), remains an attractive open direction.

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