AIRR.SCIENCE · ARR-2026-3M3D0JXMHQ8TCRPK v1 [Mathematics] 28 Nov 2026

# THE VORTICITY-IMMUNE SUBSPACE, KIRCHHOFF CIRCULATION RANK, AND EXACT TOPOLOGICAL OBSTRUCTIONS TO NON-REVERSIBLE ACCELERATION OF FINITE MARKOV CHAINS

LLUIS ERIKSSON

Abstract. Adding a divergence-free skew-symmetric vorticity $J \in \mathcal{Z}_1(G)$ to a reversible finite Markov generator $L$ on a weighted graph $(G, w, \pi)$ preserves the stationary measure $\pi$ and never increases the asymptotic variance $\sigma^2(f, L+D_\pi^{-1}J)$ of any observable $f \in L_0^2(\pi)$. Which observables resist *all* non-reversible perturbations on a given graph topology $G$, and which graphs admit a *single* circulation driving the worst-case asymptotic variance $V^*(L+\gamma D_\pi^{-1}J)$ to zero as $\gamma \to \infty$?

We establish a complete graph-topological and algebraic characterization resolving both questions sharply. First, via the Schur-complement variance identity $\Sigma(L+D_\pi^{-1}J) = 2(S+\Omega^\top S^{-1}\Omega)^{-1}$ and monotone shorted-operator convergence, we prove that as $|\gamma| \to \infty$ the limiting variance $\lim_{|\gamma|\to\infty} \sigma^2(f, L+\gamma D_\pi^{-1}J)$ equals twice the squared Dirichlet norm of the $\mathcal{E}$-orthogonal projection of $(-L)^{-1}f$ onto $\ker(J)$. Second, we determine the exact *universal vorticity-immune potential space* $\mathcal{I}(G) := \bigcap_{J\in\mathcal{Z}_1(G)} \ker(J)$ for every finite graph $G$: a potential $v \in \mathbb{R}^V$ lies in $\mathcal{I}(G)$ if and only if $v$ is constant on every non-bipartite 2-connected block of $G$ and constant on each bipartition color class of every bipartite 2-connected block ($|B| \geq 3$). Consequently, $\dim \mathcal{I}(G) = c(G) + b_{\text{bridge}}(G) + b_{\text{bip}}(G)$, and every non-zero observable $f \in L_0^2(\pi)$ admits strict variance reduction within the Markov-admissible polytope $\mathcal{M}(L, \pi)$ if and only if $G$ is connected, bridgeless, and has no bipartite 2-connected block. Third, we uncover a sharp gap between per-observable acceleration and simultaneous global collapse governed by the *Kirchhoff circulation rank* $\rho_{\text{circ}}(G) := \max_{J\in\mathcal{Z}_1(G)} \operatorname{rank}(J)$: using row-sum deletion, Tutte's theorem, and Lovász's odd ear decomposition, we prove that simultaneous global variance collapse $\lim_{|\gamma|\to\infty} V^*(L+\gamma D_\pi^{-1}J) = 0$ is achievable by a single circulation if and only if $\rho_{\text{circ}}(G) = n-1$, if and only if $G$ is *factor-critical*. For every non-factor-critical graph (including all even-order graphs, all bipartite graphs, and odd non-factor-critical graphs such as $K_{2,5}+e$), the circulation nullity $\nu_{\text{circ}}(G) := n-1-\rho_{\text{circ}}(G) \geq 1$ enforces a strictly positive Courant–Fischer variance floor $V^*(L+D_\pi^{-1}J) \geq 2/\lambda_{n+1-\nu_{\text{circ}}(G)}(-L)$. Fourth, we prove the Block–Tutte upper bound $\rho_{\text{circ}}(G) \leq \sum_{|B|\geq 3} 2\min_{v\in B} \nu(B-v)$, prove strict separation already at $n = 8$ for both 2-connected bipartite and 2-connected non-bipartite graphs via independent-set and bipartite doubly-zero-sum obstructions (together with larger 2-connected examples at $n = 10, 12, 14$), and verify computationally across all 1,252 vertex-non-empty graphs on $n \leq 7$ vertices (1,245 edge-non-empty) that $n = 8$ is the minimal order of strict separation. Finally, within $\mathcal{M}(L, \pi)$, we establish the exact continuous-time holding-time decomposition, an exact Chebyshev closed form for the total asymptotic variance on $C_n$, and sharp spectral-gap versus variance decoupling on bottleneck cycles $C_n(\delta)$.

## 1. Introduction

Breaking detailed balance is a foundational strategy for accelerating convergence and reducing Monte Carlo variance in Markov chain sampling and non-equilibrium statistical physics [6, 7, 13, 12, 1, 9]. Let $G = (V, E)$ be a finite simple connected graph with $|V| = n \geq 2$, positive symmetric edge conductances $w_{ij} = w_{ji} > 0$ for $\{i, j\} \in E$ ($w_{ij} = 0$ otherwise), and a strictly positive target

2020 *Mathematics Subject Classification.* Primary 60J27, 05C50; Secondary 05C70, 15A03, 60J22.

*Key words and phrases.* non-reversible Markov chains, asymptotic variance, vorticity, Kirchhoff circulation, factor-critical graphs, odd ear decomposition, Tutte rank, block-cut tree.

ORCID: 0009-0006-7494-9040. Reproducible verification suite, exhaustive Graph Atlas tests, 50-digit tables, and figures archived at `https://github.com/lluiseriksson/kirchhoff-vorticity-markov` (release tag `v1.0.3`).

distribution $\pi \in \mathbb{R}^V_{>0}$ ($\sum_i \pi_i = 1$). The associated reversible continuous-time Markov generator $L \in \mathbb{R}^{V\times V}$ has off-diagonal rates $L_{ij} = w_{ij}/\pi_i$ ($i \neq j$) and exit rates $q_i := -L_{ii} = \sum_{j\neq i} L_{ij} > 0$, satisfying detailed balance $\pi_i L_{ij} = \pi_j L_{ji} = w_{ij}$ and stationarity $\pi^\top L = 0$.

Any non-reversible generator $Q$ supported on $G$ sharing both the stationary measure $\pi$ and the reversible skeleton (additive symmetrization) $\frac{1}{2}(Q + D_\pi^{-1}Q^\top D_\pi) = L$ takes the unique form

$$Q \;=\; L + D_\pi^{-1}J, \qquad J \in \mathcal{Z}_1(G) \;:=\; \big\{J \in \mathbb{R}^{V\times V} : J^\top = -J,\ \mathrm{supp}(J) \subseteq E,\ J\mathbf{1} = 0\big\}, \tag{1}$$

where $D_\pi := \mathrm{diag}(\pi)$, $\mathrm{supp}(J) := \{\{i,j\} : J_{ij} \neq 0\}$, and $\mathcal{Z}_1(G)$ is the real *Kirchhoff circulation space* (cycle space) of $G$. Requiring non-negative off-diagonal transition rates $Q_{ij} \geq 0$ ($i \neq j$) restricts $J$ to the symmetric convex *Markov-admissible vorticity polytope*

$$\mathcal{M}(L,\pi) \;:=\; \big\{Q = L + D_\pi^{-1}J : J \in \mathcal{Z}_1(G),\ |J_{ij}| \leq w_{ij}\ \forall\{i,j\} \in E\big\}, \tag{2}$$

while the unbounded ray $Q_\gamma := L + \gamma D_\pi^{-1}J$ ($\gamma \in \mathbb{R}$) governs large-vorticity / small-diffusivity acceleration limits (equivalently scaling the reversible skeleton $L$ by $|\gamma|^{-1}$ at fixed $J$, as in [3, 12, 9]).

For any zero-mean observable $f \in L^2_0(\pi) := \{f \in \mathbb{R}^V : \langle f, \mathbf{1}\rangle_\pi = 0\}$ (with $\langle u, v\rangle_\pi := \sum_i \pi_i u_i v_i$), the asymptotic variance of the ergodic average $T^{-1}\int_0^T f(X_t)\,dt$ is

$$\sigma^2(f, Q) \;:=\; 2\big\langle f,\, (-Q)^{-1}f\big\rangle_\pi, \tag{3}$$

where $(-Q)^{-1}$ denotes the unique inverse of $-Q$ on $L^2_0(\pi)$ (which exists for every $Q = L + D_\pi^{-1}J$, $J \in \mathcal{Z}_1(G)$, even outside $\mathcal{M}(L,\pi)$, since $\langle u, (-Q)u\rangle_\pi = \langle u, (-L)u\rangle_\pi > 0$ for all non-constant $u$). It is well known [8, 12, 1, 9] that adding vorticity never increases $\sigma^2(f, L + D_\pi^{-1}J) \leq \sigma^2(f, L)$ for any $f \in L^2_0(\pi)$.

Yet three fundamental structural questions concerning the *topological limits* of non-reversible acceleration on a finite graph $G$ have remained open:

(1) *Universal Vorticity Immunity:* Which observables $f \in L^2_0(\pi)$ satisfy $\sigma^2(f, L + D_\pi^{-1}J) = \sigma^2(f, L)$ for *every* circulation $J \in \mathcal{Z}_1(G)$, and how does the dimension of this immune subspace depend on the block structure of $G$?

(2) *Global Single-Circulation Collapse vs. Parity and Matching Obstructions:* When does a *single* circulation $J \in \mathcal{Z}_1(G)$ drive the worst-case asymptotic variance $V^*(L + \gamma D_\pi^{-1}J) := \sup_{\|f\|_\pi = 1,\, f\in L^2_0(\pi)} \sigma^2(f, L + \gamma D_\pi^{-1}J)$ to 0 as $|\gamma| \to \infty$, and what universal lower bound governs $V^*$ when global collapse fails?

(3) *Kirchhoff Circulation Rank vs. Tutte Matching Rank:* For unconstrained skew-symmetric matrices supported on $G$, Tutte's classical theorem [14] yields $\max \mathrm{rank}(A) = 2\nu(G)$, where $\nu(G)$ is the matching number. How does imposing Kirchhoff's conservation law $J\mathbf{1} = 0$ constrain the maximum achievable rank $\rho_{\mathrm{circ}}(G) := \max_{J\in\mathcal{Z}_1(G)} \mathrm{rank}(J)$?

**Overview of Main Results.**

- **Shorted-Operator Infinite-Vorticity Limit and Holding-Time Floor (Theorem 2.1).** In $L^2_0(\pi)$-orthonormal coordinates, $\Sigma(L + D_\pi^{-1}J) = 2(S + \Omega^\top S^{-1}\Omega)^{-1}$. Sending $|\gamma| \to \infty$ converges monotonically to the Dirichlet shorted operator $\lim_{|\gamma|\to\infty} \sigma^2(f, L + \gamma D_\pi^{-1}J) = 2\|P^{\mathcal{E}}_{\ker(J)}(-L)^{-1}f\|^2_{\mathcal{E}}$. Within $\mathcal{M}(L,\pi)$, every generator obeys $\sigma^2(f,Q) = \sum_i (\pi_i/q_i) f_i^2 + \sum_i \pi_i q_i \,\mathrm{Var}_{P(i,\cdot)}(v) \geq \langle f, D_q^{-1}f\rangle_\pi$, isolating the irreducible exponential holding-time variance floor.
- **Block–Bipartite Formula for the Universal Vorticity-Immune Subspace (Theorem 3.1).** For every finite graph $G$, a potential $v \in \mathbb{R}^V$ belongs to $\mathcal{I}(G) := \bigcap_{J\in\mathcal{Z}_1(G)} \ker(J)$ if and only if $v$ is constant on every non-bipartite 2-connected block of $G$ and constant on

each color class of every bipartite 2-connected block ($|B| \geq 3$). Hence

$$\dim \mathcal{I}(G) \;=\; c(G) + b_{\text{bridge}}(G) + b_{\text{bip}}(G).$$

On every connected bipartite graph (including hypercubes $Q_d$, even cycles $C_{2k}$, and complete bipartite graphs $K_{m,n}$), the bipartition parity observable $(-L)(\mathbf{1}_A - \mathbf{1}_B)$ is completely immune to every non-reversible perturbation. Moreover, every non-zero $f \in L^2_0(\pi)$ admits strict variance reduction within $\mathcal{M}(L,\pi)$ if and only if $G$ is connected, bridgeless, and has no bipartite 2-connected block ($b_{\text{bridge}}(G) = b_{\text{bip}}(G) = 0$).

- **Factor-Criticality Equivalence for Global Collapse and Courant–Fischer Floors (Theorem 4.2).** Simultaneous global variance collapse $\lim_{|\gamma|\to\infty} V^*(L + \gamma D_\pi^{-1} J) = 0$ is achievable by a single circulation $J \in \mathcal{Z}_1(G)$ if and only if $\rho_{\text{circ}}(G) = n - 1$, if and only if $G$ is *factor-critical* (every $G - v$ has a perfect matching; equivalently, $G$ admits an odd ear decomposition [11]). Whenever $G$ is not factor-critical—including *every* graph of even order (Pfaffian parity obstruction), every bipartite graph, and odd non-factor-critical graphs such as $K_{2,5} + e$—the circulation nullity $\nu_{\text{circ}}(G) := n - 1 - \rho_{\text{circ}}(G) \geq 1$ enforces the universal variance floor

$$\inf_{J \in \mathcal{Z}_1(G)} V^*(L + D_\pi^{-1} J) \;\geq\; \frac{2}{\lambda_{n+1-\nu_{\text{circ}}(G)}(-L)} \;\geq\; \frac{2}{\lambda_{\max}(-L)} \;>\; 0.$$

In particular, on every even complete graph $K_{2k}$ ($k \geq 2$), every individual observable can be driven to zero variance by an observable-dependent circulation, yet *no* single circulation reduces $V^*$ at all ($V^*(L + D_\pi^{-1} J) = V^*(L)$ for all $J \in \mathcal{Z}_1(K_{2k})$).

- **Kirchhoff Circulation Rank Bounds and Strict Separation from Tutte Rank (Theorem 4.3).** By row/column deletion under $J\mathbf{1} = 0$ and block decomposition, $\rho_{\text{circ}}(G) \leq \sum_{|B|\geq 3} 2 \min_{v \in B} \nu(B - v)$. Independent-set and bipartite doubly-zero-sum obstructions strictly beat this Block–Tutte bound at $n = 8$ for both 2-connected bipartite ($G_8$, $\rho_{\text{circ}} = 4 < 6$) and 2-connected non-bipartite ($G_8'$, $\rho_{\text{circ}} = 4 < 6$) graphs, as well as larger 2-connected examples at $n = 10, 12, 14$; exhaustive Graph Atlas verification in Section 6 confirms equality across all 1,252 vertex-non-empty graphs on $n \leq 7$ vertices, showing computationally that $n = 8$ is minimal.
- **Exact Chebyshev Trace Collapse and Bottleneck Decoupling (Theorem 5.1).** On the uniform cycle $C_n$ with Markov-admissible vorticity $\gamma \in (0, 1]$, the total asymptotic variance satisfies the exact Chebyshev formula $\text{Tr}_{L^2_0(\pi)}(\Sigma(Q_\gamma)) = \frac{n}{2\gamma} \frac{1+((1-\gamma)/(1+\gamma))^n}{1-((1-\gamma)/(1+\gamma))^n} - \frac{1}{2\gamma^2}$, interpolating from $(n^2-1)/6$ to the holding-time floor $(n-1)/2$ with zero spectral-gap increase; introducing a single bottleneck edge of conductance $\delta \in (0, 1]$ decouples spectral-gap enhancement (peaking at $+66\%$ near $\delta \approx 0.11$ on $C_{10}(\delta)$) from variance reduction (choked quadratically as $O(\delta^2)$ as $\delta \to 0^+$, and observed numerically to grow monotonically from 1 to $10.47\times$ on $C_{10}(\delta)$).

## 2. Schur-Complement Variance Operator, Shorted Limit, and Holding-Time Floor

Let $(G, w, \pi)$ be a finite connected weighted graph ($|V| = n \geq 2$) with reversible generator $L = D_\pi^{-1}(W - D_w)$, where $W = (w_{ij})$, $D_w = \text{diag}(d_i)$ ($d_i = \sum_j w_{ij} > 0$), and exit rates $q_i = d_i/\pi_i > 0$. Define the Dirichlet form $\mathcal{E}(u, v) := \langle u, (-L)v\rangle_\pi = \frac{1}{2}\sum_{i,j} w_{ij}(u_i - u_j)(v_i - v_j)$ and its induced seminorm $\|u\|_{\mathcal{E}} := \mathcal{E}(u,u)^{1/2}$. Passing to $L^2(\pi)$-isometric coordinates $\tilde{f} = D_\pi^{1/2} f$, the symmetric operator $S_{\text{full}} := D_\pi^{-1/2}(D_w - W)D_\pi^{-1/2} \succeq 0$ has simple kernel $\text{span}(\sqrt{\pi})$ and eigenvalues $0 = \lambda_1(-L) < \lambda_2(-L) \leq \cdots \leq \lambda_n(-L)$. Fix an orthonormal basis matrix $U \in \mathbb{R}^{n\times(n-1)}$ for $\sqrt{\pi}^\perp$, and set

$$S \;:=\; U^\top S_{\text{full}} U \;\succ\; 0, \qquad \Omega(J) \;:=\; U^\top D_\pi^{-1/2} J D_\pi^{-1/2} U \;=\; -\Omega(J)^\top \in \mathbb{R}^{(n-1)\times(n-1)}.$$

Because $J\mathbf{1} = 0$ and $J^\top = -J$, the skew-symmetric matrix $D_\pi^{-1/2} J D_\pi^{-1/2}$ annihilates $\sqrt{\pi}$ and leaves $\sqrt{\pi}^\perp$ invariant. Therefore, on $L_0^2(\pi)$ identified isometrically with $\mathbb{R}^{n-1}$ via $f \mapsto \tilde{f}_0 := U^\top D_\pi^{1/2} f$, the operator $-Q_\gamma = -L - \gamma D_\pi^{-1} J$ acts as $S - \gamma\Omega(J)$.

**Theorem 2.1** (Schur-complement variance, shorted infinite-vorticity limit, and holding-time floor)**.** *Let $J \in \mathcal{Z}_1(G)$ and $Q_\gamma := L + \gamma D_\pi^{-1} J$ for $\gamma \in \mathbb{R}$.*

(a) Schur-complement identity. *For every $\gamma \in \mathbb{R}$, $S - \gamma\Omega(J)$ is invertible and the asymptotic variance operator $\Sigma(Q_\gamma) := 2\,\mathrm{Sym}((S - \gamma\Omega(J))^{-1})$ on $\mathbb{R}^{n-1} \cong L_0^2(\pi)$ satisfies*

$$\Sigma(Q_\gamma) \;=\; 2\big(S + \gamma^2 \Omega(J)^\top S^{-1} \Omega(J)\big)^{-1} \;\preceq\; 2S^{-1} \;=\; \Sigma(L), \tag{4}$$

*with $\sigma^2(f, Q_\gamma) = \tilde{f}_0^\top \Sigma(Q_\gamma) \tilde{f}_0$ for all $f \in L_0^2(\pi)$.*

(b) Exact shorted-operator limit as $|\gamma| \to \infty$. *As $|\gamma| \to \infty$, $\sigma^2(f, Q_\gamma)$ decreases monotonically to*

$$\lim_{|\gamma|\to\infty} \sigma^2(f, Q_\gamma) \;=\; 2\,\big\|P^{\mathcal{E}}_{\ker(J)}(-L)^{-1} f\big\|_{\mathcal{E}}^2 \;=\; 2 \sup_{v \in \ker(J)} \big\{2\langle f, v\rangle_\pi - \mathcal{E}(v, v)\big\}, \tag{5}$$

*where $P^{\mathcal{E}}_{\ker(J)}$ denotes the $\mathcal{E}$-orthogonal projection on $\mathbb{R}^V / \operatorname{span}(\mathbf{1})$ onto $\ker(J)/\operatorname{span}(\mathbf{1})$. In particular, for any $f \in L_0^2(\pi)$:*

- *$\sigma^2(f, Q_\gamma) = \sigma^2(f, L)$ for all $\gamma \in \mathbb{R}$ (equivalently, for some $\gamma \neq 0$) $\iff$ $(-L)^{-1} f \in \ker(J)$;*
- *$\lim_{|\gamma|\to\infty} \sigma^2(f, Q_\gamma) = 0 \iff \langle f, v\rangle_\pi = 0$ for every $v \in \ker(J)$.*

(c) Continuous-time holding-time decomposition. *Whenever $Q \in \mathcal{M}(L, \pi)$ is a valid Markov generator, write $-Q = D_q(I - P)$ where $q_i = -Q_{ii} = -L_{ii} > 0$ and $P$ is the embedded jump stochastic matrix. For every $f \in L_0^2(\pi)$ with Poisson solution $v = (-Q)^{-1} f$,*

$$\sigma^2(f, Q) \;=\; \sum_{i=1}^n \frac{\pi_i}{q_i} f_i^2 \;+\; \sum_{i=1}^n \pi_i q_i \,\mathrm{Var}_{P(i,\cdot)}(v) \;\geq\; \langle f, D_q^{-1} f\rangle_\pi. \tag{6}$$

*Equality holds in $\sigma^2(f, Q) = \langle f, D_q^{-1} f\rangle_\pi$ for all $f \in L_0^2(\pi)$ if and only if $P$ is a single cyclic permutation (a directed Hamiltonian cycle); within $\mathcal{M}(L, \pi)$, this is attainable if and only if either $G = K_2$ (with $J = 0$ and arbitrary $w_{12} > 0$), or $(G, w)$ is a single cycle $C_n$ ($n \geq 3$) with uniform edge conductances $w_{ij} = c > 0$ and maximal circulation $|J_{ij}| = c$.*

*Proof. Proof of (a).* Since $S \succ 0$ and $\Omega := -\Omega^\top = \Omega(J)$, for any non-zero $x \in \mathbb{R}^{n-1}$ we have $x^\top(S - \gamma\Omega)x = x^\top S x > 0$, so $S - \gamma\Omega$ and $S + \gamma\Omega = (S - \gamma\Omega)^\top$ are invertible. Using the algebraic identity

$$\begin{aligned}\frac{1}{2}\Big((S - \gamma\Omega)^{-1} + (S + \gamma\Omega)^{-1}\Big) &= (S - \gamma\Omega)^{-1} S (S + \gamma\Omega)^{-1} \\ &= \big((S + \gamma\Omega) S^{-1} (S - \gamma\Omega)\big)^{-1} \;=\; \big(S + \gamma^2 \Omega^\top S^{-1} \Omega\big)^{-1}\end{aligned}$$

(where $\pm\gamma\Omega$ cancels and $-\Omega S^{-1}\Omega = \Omega^\top S^{-1}\Omega$) establishes (4) (cf. [8, 1, 9]). Since $\gamma^2 \Omega^\top S^{-1} \Omega \succeq 0$, operator monotonicity of inversion on positive definite matrices gives $\Sigma(Q_\gamma) \preceq 2S^{-1} = \Sigma(L)$.

*Proof of (b).* By quadratic Legendre duality for the positive definite matrix $S + \gamma^2 \Omega^\top S^{-1} \Omega$,

$$\frac{1}{2}\sigma^2(f, Q_\gamma) \;=\; \tilde{f}_0^\top (S + \gamma^2 \Omega^\top S^{-1} \Omega)^{-1} \tilde{f}_0 \;=\; \sup_{y \in \mathbb{R}^{n-1}} \Big\{2\tilde{f}_0^\top y - y^\top S y - \gamma^2 \|\Omega y\|_{S^{-1}}^2\Big\}. \tag{7}$$

Let $K := \ker(\Omega) \subseteq \mathbb{R}^{n-1}$ and let $K^\perp$ be its Euclidean orthogonal complement. Restricting the supremum in (7) to $y \in K$ eliminates the $\gamma^2$ penalty and yields the uniform lower bound

$$\frac{1}{2}\sigma^2(f, Q_\gamma) \;\geq\; \sup_{y \in K} \big\{2\tilde{f}_0^\top y - y^\top S y\big\}.$$

Conversely, since $|\gamma| \mapsto \sigma^2(f, Q_\gamma)$ is non-increasing, write the unique maximizer $y_\gamma = (S + \gamma^2 \Omega^\top S^{-1} \Omega)^{-1} \tilde{f}_0$ in (7) as $y_\gamma = y_\gamma^K + y_\gamma^\perp$ ($y_\gamma^K \in K$, $y_\gamma^\perp \in K^\perp$). Evaluating at the maximizer gives

$$y_\gamma^\top S y_\gamma + \gamma^2 \|\Omega y_\gamma^\perp\|_{S^{-1}}^2 \;=\; \frac{1}{2}\sigma^2(f, Q_\gamma) \;\le\; \frac{1}{2}\sigma^2(f, L).$$

Since $S \succ 0$ and $\Omega|_{K^\perp}$ is injective, $\|y_\gamma\|_2$ is uniformly bounded while $\|y_\gamma^\perp\|_2 = O(|\gamma|^{-1}) \to 0$. Extracting a subsequence with $y_{\gamma_m}^K \to y_\infty \in K$ gives

$$\limsup_{|\gamma|\to\infty} \frac{1}{2}\sigma^2(f, Q_\gamma) \;\le\; \limsup_{m\to\infty} \big(2\tilde{f}_0^\top y_{\gamma_m} - y_{\gamma_m}^\top S y_{\gamma_m}\big) \;=\; 2\tilde{f}_0^\top y_\infty - y_\infty^\top S y_\infty \;\le\; \sup_{y\in K}\big\{2\tilde{f}_0^\top y - y^\top S y\big\}.$$

Now note that $v \mapsto y := U^\top D_\pi^{1/2} v$ maps $\mathbb{R}^V/\operatorname{span}(\mathbf{1})$ isomorphically onto $\mathbb{R}^{n-1}$, satisfies $Uy = D_\pi^{1/2}(v - \langle v, \mathbf{1}\rangle_\pi \mathbf{1})$, and maps $\ker(J)/\operatorname{span}(\mathbf{1})$ onto $K = \ker(\Omega)$ because $J\mathbf{1} = 0$ implies

$$\Omega(U^\top D_\pi^{1/2} v) = 0 \iff D_\pi^{-1/2} J D_\pi^{-1/2}(U U^\top D_\pi^{1/2} v) = 0 \iff J(v - \langle v, \mathbf{1}\rangle_\pi \mathbf{1}) = 0 \iff Jv = 0.$$

Moreover, $2\tilde{f}_0^\top y - y^\top S y = 2\langle f, v\rangle_\pi - \mathcal{E}(v, v) = 2\mathcal{E}(v_0, v) - \mathcal{E}(v, v)$ where $v_0 := (-L)^{-1} f$, whose supremum over $v \in \ker(J)$ equals $\|P^{\mathcal{E}}_{\ker(J)} v_0\|_{\mathcal{E}}^2$, establishing (5). For the two equivalences: the unique maximizer of (7) at $\gamma = 0$ is $y_0 = S^{-1}\tilde{f}_0$ (corresponding to $v_0 = (-L)^{-1} f$), so $\sigma^2(f, Q_\gamma) = \sigma^2(f, L)$ for some (hence all) $\gamma \neq 0$ iff $\Omega y_0 = 0$, i.e., $v_0 \in \ker(J)$; and (5) vanishes iff $v_0 \perp_{\mathcal{E}} \ker(J)$, i.e., $\mathcal{E}(v_0, v) = \langle f, v\rangle_\pi = 0$ for all $v \in \ker(J)$.

*Proof of (c).* Note that $\tilde{\pi}_i := \pi_i q_i / \bar{q}$ ($\bar{q} := \sum_i \pi_i q_i$) is stationary for the jump chain $P$ because $\sum_i \pi_i q_i P_{ij} = \sum_{i\neq j} \pi_i Q_{ij} = \pi_j q_j$ (using $P_{ii} = 0$ and $\pi^\top Q = 0$). Since $(-Q)v = D_q(I - P)v = f$, we obtain the continuous-time counterpart of Gordin's martingale decomposition [5, 10]:

$$\begin{aligned}
\sigma^2(f, Q) &= 2\langle D_q(I-P)v,\, v\rangle_\pi \;=\; 2\bar{q}\langle (I-P)v,\, v\rangle_{\tilde{\pi}} \\
&= \bar{q}\|(I-P)v\|_{\tilde{\pi}}^2 + \bar{q}\big(\|v\|_{\tilde{\pi}}^2 - \|Pv\|_{\tilde{\pi}}^2\big) \\
&= \sum_{i=1}^n \pi_i q_i \left(\frac{f_i}{q_i}\right)^2 + \sum_{i=1}^n \pi_i q_i \Big(\sum_{j=1}^n P_{ij} v_j^2 - \big((Pv)_i\big)^2\Big),
\end{aligned}$$

where the last step uses $\sum_i \tilde{\pi}_i (P(v^2))_i = \sum_i \tilde{\pi}_i v_i^2 = \|v\|_{\tilde{\pi}}^2$. This proves (6). Equality $\sigma^2(f, Q) = \langle f, D_q^{-1} f\rangle_\pi$ holds for all $f \in L_0^2(\pi)$ iff $\operatorname{Var}_{P(i,\cdot)}(v) = 0$ for every $i \in V$ and every $v \in \mathbb{R}^V$, which holds iff each row $P(i, \cdot)$ is a point mass at some state $\tau(i) \in V \setminus \{i\}$. Because $\tilde{\pi}_j = \sum_i \tilde{\pi}_i P_{ij} > 0$ for every $j \in V$, every column of $P$ must contain at least one 1, so $\tau : V \to V$ is surjective and hence a permutation; connectivity of $G$ (irreducibility of $Q$) forces $\tau$ to be a single $n$-cycle. For attainability within $\mathcal{M}(L, \pi)$: if $n = 2$ ($G = K_2$), then $\mathcal{Z}_1(K_2) = \{0\}$, $Q = L \in \mathcal{M}(L, \pi)$ already has jump matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (a 2-cycle permutation), and equality holds for every $w_{12} > 0$ and $\pi \in \mathbb{R}^2_{>0}$. Now suppose $n \ge 3$ and $Q = L + D_\pi^{-1} J \in \mathcal{M}(L, \pi)$ has jump matrix $P$ equal to a single directed $n$-cycle. For every edge $\{i, j\} \in E$, we have $w_{ij} > 0$ and $P_{ij} = (w_{ij} + J_{ij})/d_i$, $P_{ji} = (w_{ij} - J_{ij})/d_j$; in particular, $P_{ij}$ and $P_{ji}$ cannot both vanish (otherwise $J_{ij} = -w_{ij}$ and $J_{ij} = +w_{ij}$, contradicting $w_{ij} > 0$). Since a directed $n$-cycle with $n \ge 3$ contains no bidirected 2-cycle, $P_{ij}$ and $P_{ji}$ cannot both equal 1 either. Therefore, on every edge $\{i, j\} \in E$, exactly one of $P_{ij}, P_{ji}$ equals 1 and the other equals 0, which forces $J_{ij} = \pm w_{ij}$ on every edge, identifies the undirected support $G$ with the single cycle $C_n$, and implies via $J\mathbf{1} = 0$ that all edge conductances $w_{ij} = |J_{ij}|$ are equal. Conversely, on $C_n$ ($n \ge 3$) with uniform $w_{ij} = c > 0$, choosing $J_{i,i+1} = c$ yields the clockwise $n$-cycle permutation $P$ in $\mathcal{M}(L, \pi)$. □

## 3. The Universal Vorticity-Immune Subspace $\mathcal{I}(G)$

Theorem 2.1(b) shows that an observable $f \in L_0^2(\pi)$ is unaffected by a circulation $J \in \mathcal{Z}_1(G)$ if and only if its reversible Poisson potential $(-L)^{-1} f$ lies in $\ker(J)$. We therefore define the

*universal vorticity-immune potential space* and *universal vorticity-immune observable space* of $G$:

$$\mathcal{I}(G) \;:=\; \bigcap_{J\in\mathcal{Z}_1(G)} \ker(J) \;\subseteq\; \mathbb{R}^V, \qquad \mathcal{F}_{\mathrm{imm}}(G,L) \;:=\; (-L)\mathcal{I}(G) \;\subseteq\; L^2_0(\pi). \tag{8}$$

Remarkably, $\mathcal{I}(G)$ depends *solely on the unweighted graph topology* $G$. Recall that every finite simple graph $G$ decomposes uniquely into its maximal 2-vertex-connected subgraphs (blocks) $\mathcal{B}(G)$: each block $B \in \mathcal{B}(G)$ is either an isolated vertex ($|B| = 1$), a bridge ($|B| = 2$, i.e., a $K_2$ cut-edge), or a 2-connected subgraph with $|B| \geq 3$. Let $c(G)$ denote the number of connected components of $G$, $b_{\mathrm{bridge}}(G)$ the number of bridges, and $b_{\mathrm{bip}}(G)$ the number of bipartite blocks with $|B| \geq 3$.

**Theorem 3.1** (Block–bipartite characterization of the universal vorticity-immune subspace)**.** *Let $G = (V, E)$ be any finite simple graph.*

(a) Exact structural description. *A potential $v \in \mathbb{R}^V$ belongs to $\mathcal{I}(G)$ if and only if:*
- *$v$ is constant on every non-bipartite 2-connected block $B \in \mathcal{B}(G)$ ($|B| \geq 3$); and*
- *$v$ is constant on each of the two bipartition color classes $A_B, B_B$ of every bipartite 2-connected block $B \in \mathcal{B}(G)$ ($|B| \geq 3$).*

*(Bridges and isolated vertices impose no constraint on $v$.)*

(b) Exact dimension formula.

$$\dim \mathcal{I}(G) \;=\; c(G) + b_{\mathrm{bridge}}(G) + b_{\mathrm{bip}}(G). \tag{9}$$

*When $G$ is connected with reversible generator $L$, $\dim \mathcal{F}_{\mathrm{imm}}(G,L) = b_{\mathrm{bridge}}(G) + b_{\mathrm{bip}}(G)$.*

(c) Universal lower bound and strict-reduction dichotomy. *Suppose $G$ is connected ($n \geq 2$). For every $f \in L^2_0(\pi)$ and every $J \in \mathcal{Z}_1(G)$,*

$$\sigma^2(f, L + D_\pi^{-1}J) \;\geq\; 2\left\|P^{\mathcal{E}}_{\mathcal{I}(G)}(-L)^{-1}f\right\|^2_{\mathcal{E}}, \tag{10}$$

*and $\sigma^2(f, L + D_\pi^{-1}J) = \sigma^2(f, L)$ holds for all $J \in \mathcal{Z}_1(G)$ if and only if $f \in \mathcal{F}_{\mathrm{imm}}(G,L)$. Consequently, the following are equivalent:*

(i) *every non-zero $f \in L^2_0(\pi)$ admits strict variance reduction within the Markov-admissible polytope: $\inf_{Q\in\mathcal{M}(L,\pi)} \sigma^2(f,Q) < \sigma^2(f,L)$;*

(ii) *every non-zero $f \in L^2_0(\pi)$ admits strict variance reduction over $\mathcal{Z}_1(G)$: $\inf_{J\in\mathcal{Z}_1(G)} \sigma^2(f, L + D_\pi^{-1}J) < \sigma^2(f,L)$;*

(iii) *$\mathcal{I}(G) = \mathrm{span}(\mathbf{1})$, i.e., $G$ is bridgeless and every 2-connected block of $G$ is non-bipartite ($b_{\mathrm{bridge}}(G) = b_{\mathrm{bip}}(G) = 0$).*

*Proof. Proof of (a).* Because the simple cycles of $G$ span $\mathcal{Z}_1(G)$ as a real vector space, $v \in \mathcal{I}(G)$ if and only if $J_C v = 0$ for every simple cycle $C = (u_0, u_1, \ldots, u_{k-1}, u_0)$ of $G$, where $J_C \in \mathcal{Z}_1(G)$ assigns unit flow $u_r \to u_{r+1}$ around $C$ (indices modulo $k$). Evaluating $J_C v$ at $u_r$ gives

$$(J_C v)_{u_r} \;=\; v(u_{r+1}) - v(u_{r-1}) \qquad (r = 0, 1, \ldots, k-1), \tag{11}$$

and $(J_C v)_x = 0$ for $x \notin V(C)$. Because every simple cycle lies inside a single 2-connected block $B \in \mathcal{B}(G)$ ($|B| \geq 3$), the condition $v \in \mathcal{I}(G)$ decouples block by block, with bridges and isolated vertices imposing no constraints.

Fix a 2-connected block $B$ with $|B| \geq 3$. *Sufficiency:* If $B$ is non-bipartite and $v|_B$ is constant, (11) vanishes for all cycles $C \subseteq B$. If $B$ is bipartite with color classes $A_B \sqcup B_B$ and $v$ is constant on $A_B$ and constant on $B_B$, then along any cycle $C = (u_0, \ldots, u_{k-1}) \subseteq B$, both $u_{r-1}$ and $u_{r+1}$ are neighbors of $u_r$, hence lie in the same color class; thus $v(u_{r+1}) = v(u_{r-1})$ and $J_C v = 0$.

*Necessity:* Suppose $J_C v = 0$ for every simple cycle $C \subseteq B$. Let $x, y \in V(B)$ be any two distinct vertices sharing a common neighbor $z \in V(B)$ (a 2-edge path $x - z - y$ in $B$). Because $B$ is 2-vertex-connected, $B - z$ is connected, so there exists a simple path $P$ in $B - z$ joining $x$ to $y$. Adjoining the edges $xz$ and $zy$ yields a simple cycle $C = P \cup \{xz, zy\} \subseteq B$ in which $x$ and $y$ are

the two neighbors of $z$; evaluating (11) at $z$ forces $v(x) = v(y)$. Hence $v$ is constant on every connected component of the 2-step graph $B^{[2]}$ (vertex set $V(B)$, with an edge between distinct $x, y \in V(B)$ whenever $B$ contains a 2-edge path $x - z - y$). Since $B$ is connected:

- if $B$ is bipartite with color classes $A_B \sqcup B_B$, any two vertices in the same color class are joined by an even-length walk in $B$, hence lie in the same component of $B^{[2]}$, forcing $v$ to be constant on $A_B$ and constant on $B_B$;
- if $B$ is non-bipartite, $B$ contains an odd cycle, so any pair of adjacent vertices—and hence every pair of vertices in $B$—is joined by an even-length walk in $B$, making $B^{[2]}$ connected and forcing $v$ to be constant on all of $B$.

*Proof of (b).* It suffices to consider a single connected component ($c(G) = 1$). If $|V| = 1$, $b_{\text{bridge}} = b_{\text{bip}} = 0$ and $\dim \mathcal{I}(G) = 1$, matching (9). For $|V| \geq 2$, every block has $|B| \geq 2$, and the *block-cut tree* $T(G)$ (whose vertices are the blocks and cut-vertices of $G$, with edges joining each cut-vertex to its incident blocks) is a tree [2, Lemma 3.1.4]. Root $T(G)$ at an arbitrary block $B_0$ and traverse $T(G)$ outward. Pinning the value of $v$ at a single chosen vertex in $B_0$ gives 1 initial degree of freedom. Whenever a block $B \in \mathcal{B}(G)$ is visited, it shares exactly one vertex (its parent cut-vertex in $T(G)$, or the root vertex for $B_0$) with the previously visited union of blocks; by Part (a), extending $v$ across $B$ adds:

- 0 degrees of freedom if $B$ is non-bipartite ($|B| \geq 3$, $v|_B$ constant);
- 1 degree of freedom if $B$ is a bridge ($|B| = 2$, value at the opposite endpoint);
- 1 degree of freedom if $B$ is bipartite ($|B| \geq 3$, constant value on the opposite color class).

Because $T(G)$ is acyclic, these choices are independent, giving $\dim \mathcal{I}(G) = 1 + b_{\text{bridge}}(G) + b_{\text{bip}}(G)$. Summing over connected components yields (9). For connected $G$, $\ker(-L) = \operatorname{span}(\mathbf{1}) \subseteq \mathcal{I}(G)$, so $\dim \mathcal{F}_{\text{imm}}(G, L) = \dim \mathcal{I}(G) - 1 = b_{\text{bridge}}(G) + b_{\text{bip}}(G)$.

*Proof of (c).* Since $\mathcal{I}(G) \subseteq \ker(J)$ for every $J \in \mathcal{Z}_1(G)$, Theorem 2.1(a)–(b) gives

$$\sigma^2(f, L + D_\pi^{-1}J) \;\geq\; 2\big\|P^{\mathcal{E}}_{\ker(J)}(-L)^{-1}f\big\|^2_{\mathcal{E}} \;\geq\; 2\big\|P^{\mathcal{E}}_{\mathcal{I}(G)}(-L)^{-1}f\big\|^2_{\mathcal{E}},$$

proving (10). By Theorem 2.1(b), $\sigma^2(f, L + D_\pi^{-1}J) = \sigma^2(f, L)$ for all $J \in \mathcal{Z}_1(G)$ iff $(-L)^{-1}f \in \bigcap_{J \in \mathcal{Z}_1(G)} \ker(J) = \mathcal{I}(G)$, i.e., $f \in \mathcal{F}_{\text{imm}}(G, L)$. For the equivalence (i)⇔(ii)⇔(iii): (i)⇒(ii) is immediate since $\mathcal{M}(L, \pi) \subseteq \{L + D_\pi^{-1}J : J \in \mathcal{Z}_1(G)\}$; (ii)⇒(iii) follows because any non-zero $f \in \mathcal{F}_{\text{imm}}(G, L)$ cannot be strictly reduced; and (iii)⇒(i) holds because if $\mathcal{I}(G) = \operatorname{span}(\mathbf{1})$ and $f \in L^2_0(\pi) \setminus \{0\}$, then $v_0 := (-L)^{-1}f \notin \mathcal{I}(G)$, so there exists $J \in \mathcal{Z}_1(G)$ with $Jv_0 \neq 0$, and scaling $J$ by $\varepsilon > 0$ small enough that $|\varepsilon J_{ij}| \leq w_{ij}$ for all $\{i, j\} \in E$ produces a valid Markov generator $Q = L + \varepsilon D_\pi^{-1}J \in \mathcal{M}(L, \pi)$ with $\sigma^2(f, Q) < \sigma^2(f, L)$ strictly by Theorem 2.1(b). □

**Corollary 3.2** (Universal immunity of bipartition parity on bipartite graphs)**.** *Let $G = (A \sqcup B, E)$ be any connected bipartite graph ($|A|, |B| \geq 1$) with arbitrary conductances $w$ and stationary distribution $\pi$. Let $v_{\text{bip}} := \mathbf{1}_A - \mathbf{1}_B \in \mathcal{I}(G)$ and $f_{\text{bip}} := (-L)v_{\text{bip}} \in L^2_0(\pi)$ (for a $d$-regular walk with unit edge rates $L_{ij} = 1$, $f_{\text{bip}} = 2d(\mathbf{1}_A - \mathbf{1}_B)$ is the top Laplacian eigenfunction). Then $\sigma^2(f_{\text{bip}}, L + D_\pi^{-1}J) = \sigma^2(f_{\text{bip}}, L)$ identically for every circulation $J \in \mathcal{Z}_1(G)$ of arbitrary magnitude.*

*Proof.* Every block of $G$ is either a bridge or a bipartite 2-connected block whose color classes are restrictions of $A$ and $B$, so $v_{\text{bip}} \in \mathcal{I}(G)$ by Theorem 3.1(a), and Theorem 3.1(c) applies. □

## 4. Kirchhoff Circulation Rank, Factor-Criticality, and Exact Obstructions

Theorem 3.1 characterizes when *each individual observable* can be accelerated. We now turn to the much sharper global problem: when can a *single* circulation $J \in \mathcal{Z}_1(G)$ simultaneously

suppress the worst-case asymptotic variance

$$V^*(Q) \;:=\; \sup_{f\in L_0^2(\pi),\,\|f\|_\pi=1} \sigma^2(f,Q) \;=\; \lambda_{\max}\big(\Sigma(Q)\big)$$

to 0 along the ray $Q_\gamma = L + \gamma D_\pi^{-1} J$ as $|\gamma| \to \infty$?

**Definition 4.1** (Kirchhoff circulation rank and single-circulation nullity)**.** For any finite simple graph $G = (V, E)$ with $|V| = n$, define its *Kirchhoff circulation rank* by

$$\rho_{\mathrm{circ}}(G) \;:=\; \max_{J\in\mathcal{Z}_1(G)} \operatorname{rank}(J).$$

When $G$ is connected ($n \geq 1$), define its *single-circulation nullity* by $\nu_{\mathrm{circ}}(G) := (n-1) - \rho_{\mathrm{circ}}(G) \geq 0$. Recall that a graph $G$ is *factor-critical* (Gallai [4], Lovász [11]) if $G - v$ has a perfect matching for every vertex $v \in V(G)$ (equivalently, $\nu(G - v) = (n - 1)/2$ for all $v \in V$, where $\nu(H)$ denotes the maximum matching cardinality of $H$).

**Theorem 4.2** (Factor-criticality equivalence for global variance collapse and Courant–Fischer floors)**.** *Let $(G, w, \pi)$ be a finite connected weighted graph with $|V| = n \geq 2$ and reversible generator $L$ having eigenvalues $0 = \lambda_1(-L) < \lambda_2(-L) \leq \cdots \leq \lambda_n(-L)$.*

(a) Exact five-way equivalence. *The following five statements are equivalent:*
   (i) *there exists a single circulation $J \in \mathcal{Z}_1(G)$ such that $\lim_{|\gamma|\to\infty} V^*(L + \gamma D_\pi^{-1} J) = 0$;*
   (ii) *there exists $J \in \mathcal{Z}_1(G)$ with $\ker(J) = \operatorname{span}(\mathbf{1})$;*
   (iii) *$\rho_{\mathrm{circ}}(G) = n - 1$ (i.e., $\nu_{\mathrm{circ}}(G) = 0$);*
   (iv) *$G$ is factor-critical;*
   (v) *$G$ admits an odd ear decomposition starting from $K_1$.*

   *Moreover, whenever (i)–(v) hold, a generic $J \in \mathcal{Z}_1(G)$ achieves $V^*(L+\gamma D_\pi^{-1} J) = O(\gamma^{-2})$ as $|\gamma| \to \infty$.*

(b) Courant–Fischer universal variance floor. *Whenever $G$ is* not *factor-critical (so $d := \nu_{\mathrm{circ}}(G) \geq 1$), for* every *single circulation $J \in \mathcal{Z}_1(G)$ (of arbitrary magnitude),*

$$V^*(L + D_\pi^{-1} J) \;\geq\; \frac{2}{\lambda_{n+1-d}(-L)} \;\geq\; \frac{2}{\lambda_{\max}(-L)} \;>\; 0. \tag{12}$$

   *In particular, global single-circulation collapse is impossible on every graph of even order $n$ (Pfaffian parity obstruction: $\operatorname{rank}(J)$ is always even, so $d \geq 1$), every bipartite graph, every graph with a bridge, and every odd non-factor-critical graph.*

(c) Sharp separation on even complete graphs $K_{2k}$ ($k \geq 2$). *On $K_{2k}$ with unit off-diagonal transition rates $L_{ij} = 1$ ($i \neq j$, conductances $w_{ij} = \frac{1}{2k}$, uniform $\pi_i = \frac{1}{2k}$), every* individual *observable $f \in L_0^2(\pi)$ satisfies $\inf_{J\in\mathcal{Z}_1(K_{2k})} \sigma^2(f, L + D_\pi^{-1} J) = 0$, yet for* every *single circulation $J \in \mathcal{Z}_1(K_{2k})$, $V^*(L + D_\pi^{-1} J) = V^*(L) = \frac{1}{k}$ (zero worst-case reduction).*

*Proof. Proof of (a). (i)⇔(ii)⇔(iii):* Since $\mathbf{1} \in \ker(J)$ for every $J \in \mathcal{Z}_1(G)$, $\operatorname{rank}(J) = n - 1 \iff \ker(J) = \operatorname{span}(\mathbf{1})$, which gives (ii)⇔(iii). If $\ker(J) = \operatorname{span}(\mathbf{1})$, then $\Omega(J)$ is invertible on $\mathbb{R}^{n-1}$, so $\Omega(J)^\top S^{-1}\Omega(J) \succeq c_0 I$ for some $c_0 > 0$, whence $\Sigma(L + \gamma D_\pi^{-1} J) = 2(S + \gamma^2 \Omega^\top S^{-1}\Omega)^{-1} \preceq \frac{2}{c_0\gamma^2} I$ and $V^*(L + \gamma D_\pi^{-1} J) = O(\gamma^{-2}) \to 0$. Conversely, if $\dim\ker(J) \geq 2$, pick any $v \in \ker(J) \setminus \operatorname{span}(\mathbf{1})$ and set $f := (-L)v \in L_0^2(\pi) \setminus \{0\}$; by Theorem 2.1(b), $\sigma^2(f, L + \gamma D_\pi^{-1} J) = \sigma^2(f, L) > 0$ for all $\gamma \in \mathbb{R}$, so $V^*(L + \gamma D_\pi^{-1} J) \geq \sigma^2(f, L)/\|f\|_\pi^2 > 0$ cannot tend to 0. Because $\det(\Omega(J))$ is a polynomial in the basis coordinates of $J \in \mathcal{Z}_1(G)$, if it does not vanish identically it is non-zero on a Zariski-open dense subset of $\mathcal{Z}_1(G)$.

*(iii)⇒(iv):* Let $J \in \mathcal{Z}_1(G)$ have $\operatorname{rank}(J) = n - 1$. Fix any vertex $v \in V(G)$ and let $J_{G-v}$ be the $(n - 1) \times (n - 1)$ principal submatrix obtained by deleting row $v$ and column $v$. Because $J\mathbf{1} = 0$

and $\mathbf{1}^\top J = 0$, row $v$ is minus the sum of the other $n-1$ rows and column $v$ is minus the sum of the other $n-1$ columns. Deleting row $v$ and column $v$ therefore leaves the rank unchanged:

$$\operatorname{rank}(J) \;=\; \operatorname{rank}(J_{G-v}) \qquad \forall\, v \in V(G). \tag{13}$$

Since $J_{G-v}$ is a skew-symmetric matrix supported on $G-v$, Tutte's theorem [14] yields $\operatorname{rank}(J_{G-v}) \le 2\nu(G-v)$ (indeed, every non-zero term in the Pfaffian of a $2r \times 2r$ principal submatrix corresponds to a $r$-edge matching in $G-v$). Thus $n-1 = \operatorname{rank}(J) \le 2\nu(G-v) \le n-1$, forcing $2\nu(G-v) = n-1$ for every $v \in V(G)$, i.e., $G$ is factor-critical.

*(iv)⇔(v):* This is Lovász's classical ear-decomposition theorem [11]: a graph $G$ is factor-critical if and only if there exists a sequence of subgraphs $K_1 = H_0 \subset H_1 \subset \cdots \subset H_t = G$ where each $H_j$ is obtained from $H_{j-1}$ by adding an *odd ear* $P_j$—a path of odd length $2m_j+1$ ($m_j \ge 0$) whose two endpoints $x_j, y_j$ lie in $V(H_{j-1})$ (with $x_j = y_j$ allowed when $m_j \ge 1$, forming a closed odd cycle attached at $x_j$) and whose $2m_j$ internal vertices are disjoint from $V(H_{j-1})$.

*(v)⇒(iii):* We prove by induction on $j = 0, 1, \ldots, t$ that every $H_j$ in the odd ear decomposition is connected and admits $J_j \in \mathcal{Z}_1(H_j)$ with $\operatorname{rank}(J_j) = |V(H_j)|-1$. For $j=0$, $H_0 = K_1$ has $|V(H_0)|-1 = 0$ and $J_0 = 0$ works trivially. Assume the claim holds for $H_{j-1}$ with circulation $J_{j-1} \in \mathcal{Z}_1(H_{j-1})$ of rank $|V(H_{j-1})|-1$, and let $H_j = H_{j-1} \cup P_j$ where $P_j = (x_j = u_0, u_1, \ldots, u_{2m}, u_{2m+1} = y_j)$ has $2m$ new internal vertices ($m \ge 0$).

- If $m = 0$, then $P_j$ is a single chord edge $x_j y_j$ with no new vertices, so $J_j := J_{j-1} \in \mathcal{Z}_1(H_{j-1}) \subseteq \mathcal{Z}_1(H_j)$ already satisfies $\operatorname{rank}(J_j) = |V(H_{j-1})| - 1 = |V(H_j)| - 1$.
- If $m \ge 1$, since $H_{j-1}$ is connected there exists a simple path $Q_j$ in $H_{j-1}$ from $y_j$ to $x_j$ (empty if $x_j = y_j$), so $C_j := P_j \cup Q_j$ is a simple cycle in $H_j$. Let $J_{C_j} \in \mathcal{Z}_1(H_j)$ be the unit circulation along $C_j$ oriented $u_0 \to u_1 \to \cdots \to u_{2m+1} \to Q_j \to u_0$, and define $J(\varepsilon) := J_{j-1} + \varepsilon J_{C_j} \in \mathcal{Z}_1(H_j)$ for $\varepsilon > 0$. Partitioning $V(H_j) = V(H_{j-1}) \sqcup \{u_1, \ldots, u_{2m}\}$ gives the block form

$$J(\varepsilon) \;=\; \begin{pmatrix} J_{j-1} + \varepsilon (J_{C_j})_{HH} & \varepsilon B \\ -\varepsilon B^\top & \varepsilon A_{P_{2m}} \end{pmatrix},$$

  where $A_{P_{2m}} \in \mathbb{R}^{2m \times 2m}$ is the skew-symmetric adjacency matrix of the directed path $u_1 \to u_2 \to \cdots \to u_{2m}$ (entries $(A_{P_{2m}})_{r,r+1} = +1$, $(A_{P_{2m}})_{r+1,r} = -1$, 0 elsewhere). Expanding $\det(A_{P_{2m}})$ along row 1 gives $\det(A_{P_{2m}}) = \det(A_{P_{2m-2}}) = \cdots = \det(A_{P_2}) = 1 \ne 0$, so $\varepsilon A_{P_{2m}}$ is invertible for every $\varepsilon > 0$. By Schur-complement rank additivity,

$$\operatorname{rank}(J(\varepsilon)) \;=\; \operatorname{rank}(\varepsilon A_{P_{2m}}) + \operatorname{rank}\Big( J_{j-1} + \varepsilon\big((J_{C_j})_{HH} + B A_{P_{2m}}^{-1} B^\top\big)\Big).$$

  Because $\operatorname{rank}(J_{j-1}) = |V(H_{j-1})| - 1$, some $(|V(H_{j-1})|-1) \times (|V(H_{j-1})|-1)$ minor of $J_{j-1}$ is non-zero; by continuity of the determinant, this minor remains non-zero for all sufficiently small $\varepsilon > 0$. Hence $\operatorname{rank}(J(\varepsilon)) \ge 2m + (|V(H_{j-1})|-1) = |V(H_j)| - 1$. Since $J(\varepsilon)\mathbf{1}_{H_j} = 0$, $\operatorname{rank}(J(\varepsilon)) \le |V(H_j)| - 1$, proving $\operatorname{rank}(J(\varepsilon)) = |V(H_j)| - 1$.

At $j = t$, this yields $J_t \in \mathcal{Z}_1(G)$ with $\operatorname{rank}(J_t) = n-1$.

*Proof of (b).* Let $J \in \mathcal{Z}_1(G)$ be arbitrary. Then $\operatorname{rank}(\Omega(J)) = \operatorname{rank}(J) \le \rho_{\mathrm{circ}}(G)$, so $W := \ker(\Omega(J)) \subseteq \mathbb{R}^{n-1}$ has dimension $\dim(W) \ge (n-1) - \rho_{\mathrm{circ}}(G) = d \ge 1$. Since $S \succ 0$, the subspace $E := SW \subseteq \mathbb{R}^{n-1}$ also has $\dim(E) \ge d$. For every non-zero $\tilde{f}_0 = Sy \in E$ ($y \in W$), $\Omega(J)y = 0$ implies $(S + \Omega(J)^\top S^{-1}\Omega(J))y = Sy = \tilde{f}_0$, whence $(S + \Omega(J)^\top S^{-1}\Omega(J))^{-1}\tilde{f}_0 = y = S^{-1}\tilde{f}_0$ and

$$\frac{\tilde{f}_0^\top \Sigma (L + D_\pi^{-1} J)\tilde{f}_0}{\|\tilde{f}_0\|_2^2} \;=\; \frac{2\tilde{f}_0^\top S^{-1}\tilde{f}_0}{\|\tilde{f}_0\|_2^2}.$$

Ordering the eigenvalues of $S^{-1}$ in non-decreasing order as $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$ where $\mu_k = 1/\lambda_{n+1-k}(-L)$, the Courant–Fischer min-max theorem over the subspace $E$ of dimension $\ge d$

gives

$$V^*(L+D_\pi^{-1}J) \;\geq\; \max_{\tilde{f}_0\in E\setminus\{0\}} \frac{2\tilde{f}_0^\top S^{-1}\tilde{f}_0}{\|\tilde{f}_0\|_2^2} \;\geq\; 2\mu_d \;=\; \frac{2}{\lambda_{n+1-d}(-L)} \;\geq\; \frac{2}{\lambda_{\max}(-L)}.$$

*Proof of (c).* On $K_{2k}$ ($k\geq 2$) with unit off-diagonal rates $L_{ij}=1$ ($i\neq j$, conductances $w_{ij}=\frac{1}{2k}$, $\pi_i=\frac{1}{2k}$), we have $-L=2kI-\mathbf{1}\mathbf{1}^\top$, so every non-zero eigenvalue of $-L$ equals $2k$ and $V^*(L)=\frac{2}{2k}=\frac{1}{k}$. Since $2k$ is even, $\nu_{\mathrm{circ}}(K_{2k})\geq 1$, so (12) yields $V^*(L+D_\pi^{-1}J)\geq 2/\lambda_{\max}(-L)=\frac{1}{k}=V^*(L)$ for every $J\in\mathcal{Z}_1(K_{2k})$ (and $V^*(L+D_\pi^{-1}J)\leq V^*(L)$ by (4), giving exact equality). On the other hand, because $K_{2k}$ is complete, $\mathcal{Z}_1(K_{2k})$ consists of *all* skew-symmetric matrices on $\mathbb{R}^{2k}$ annihilating $\mathbf{1}$. Given any $f\in L_0^2(\pi)\setminus\{0\}$, choose a non-zero vector $u\in\mathbf{1}^\perp$ orthogonal to $f$ (which exists since $\dim\mathbf{1}^\perp=2k-1\geq 3$) and let $J_f\in\mathcal{Z}_1(K_{2k})$ be any skew-symmetric matrix with $\ker(J_f)=\mathrm{span}(\mathbf{1},u)$ (which exists since $\dim(\mathrm{span}(\mathbf{1},u)^\perp)=2k-2$ is even). Then $\langle f,v\rangle_\pi=0$ for all $v\in\ker(J_f)$, so Theorem 2.1(b) gives $\lim_{|\gamma|\to\infty}\sigma^2(f,L+\gamma D_\pi^{-1}J_f)=0$. □

We now analyze the Kirchhoff circulation rank $\rho_{\mathrm{circ}}(G)$ for general (non-factor-critical) graphs and establish how Kirchhoff's conservation law strictly separates $\rho_{\mathrm{circ}}(G)$ from Tutte's unconstrained skew-symmetric rank $2\nu(G)$.

**Theorem 4.3** (Block–Tutte upper bound and Kirchhoff obstruction separations)**.** *Let $G=(V,E)$ be any finite simple graph.*

(a) Block–Tutte upper bound.

$$\rho_{\mathrm{circ}}(G) \;\leq\; \sum_{\substack{B\in\mathcal{B}(G)\\ |B|\geq 3}} 2\min_{v\in V(B)}\nu(B-v). \tag{14}$$

(b) Independent-set Kirchhoff obstruction. *Let $I\subsetneq V$ be any non-empty independent set containing no isolated vertices, let $S:=N_G(I)$, and let $c(I,S)$ denote the number of connected components of the bipartite edge-induced subgraph $G[I,S]$ consisting of all edges between $I$ and $S$. Then*

$$\rho_{\mathrm{circ}}(G) \;\leq\; 2\left\lfloor\frac{|V\setminus I|+|S|-c(I,S)-1}{2}\right\rfloor. \tag{15}$$

(c) Bipartite doubly-zero-sum cover obstruction. *If $G=(A\sqcup B,E)$ is bipartite and $X\subsetneq A$ is a non-empty subset such that $A\setminus X$ has no isolated vertices, let $Y':=N_G(A\setminus X)\subseteq B$ and let $c(A\setminus X,Y')$ be the number of connected components of $G[A\setminus X,Y']$. Then*

$$\rho_{\mathrm{circ}}(G) \;\leq\; 2\big(|X|-1+|Y'|-c(A\setminus X,Y')\big), \tag{16}$$

*and symmetrically with $A$ and $B$ interchanged.*

(d) Strict separation at $n=8$ and larger 2-connected examples. *Equality in (14) fails strictly already at $n=8$ for both 2-connected bipartite and 2-connected non-bipartite graphs:*

- *($n=8$, bipartite $G_8$): on $A=\{1,2,3,4\}$ and $B=\{5,6,7,8\}$ with edge set $(\{1,2\}\times\{5,6,7,8\})\cup(\{3,4\}\times\{5,6\})$, $G_8$ is 2-connected with $2\min_v\nu(G_8-v)=6$ yet $\rho_{\mathrm{circ}}(G_8)=4$;*
- *($n=8$, non-bipartite $G_8'$): adjoining edge $\{1,2\}$ to $G_8$ yields a 2-connected non-bipartite graph with $2\min_v\nu(G_8'-v)=6$ yet $\rho_{\mathrm{circ}}(G_8')=4$.*

*Strict separation also holds in the larger 2-connected examples* $\mathrm{ring}(2K_{2,3})$ *($n=10$, bipartite, $\rho_{\mathrm{circ}}=4<6$),* $\mathrm{ring}(3C_4^{\mathrm{opp}})$ *($n=12$, non-bipartite, $\rho_{\mathrm{circ}}=8<10$), and* $3C_4^{\mathrm{opp}}+2$ *($n=14$, bipartite, $\rho_{\mathrm{circ}}=8<10$).*

*Proof. Proof of (a).* Since every simple cycle of $G$ lies inside a single block $B\in\mathcal{B}(G)$ with $|B|\geq 3$, we have $\mathcal{Z}_1(G)=\bigoplus_{|B|\geq 3}\mathcal{Z}_1(B)$. Writing $J=\sum_{|B|\geq 3}J_B$ ($J_B\in\mathcal{Z}_1(B)$ extended

by zeros) yields $\operatorname{rank}(J) \le \sum_{|B|\ge 3} \operatorname{rank}(J_B)$. On each block $B$, (13) and Tutte's theorem give $\operatorname{rank}(J_B) = \operatorname{rank}((J_B)_{B-v}) \le 2\nu(B-v)$ for every $v \in V(B)$; taking the minimum over $v \in V(B)$ proves (14).

*Proof of (b).* Fix $J \in \mathcal{Z}_1(G)$. Because $I$ is an independent set, $J_{II} = 0$ and every row $i \in I$ has support contained in $S = N_G(I)$. Partition $G[I,S]$ into its $c := c(I,S)$ connected components $(I_1 \sqcup S_1), \dots, (I_c \sqcup S_c)$. Since $I$ has no isolated vertices and $S = N_G(I)$, the sets $S_1, \dots, S_c$ partition $S$ into non-empty disjoint subsets. For each $r \in \{1, \dots, c\}$ and each $i \in I_r$, the condition $(J\mathbf{1})_i = 0$ reads $\sum_{s \in S_r} J_{is} = 0$, so row $i$ lies in $\mathbf{1}_{S_r}^{\perp} \subset \mathbb{R}^{S_r}$ (dimension $|S_r| - 1$). Hence

$$\operatorname{rank}(J_{I,S}) \;\le\; \sum_{r=1}^{c} (|S_r| - 1) \;=\; |S| - c(I,S).$$

By skew-symmetry $J_{S,I} = -J_{I,S}^{\top}$, $\dim\ker(J_{S,I}) = |I| - \operatorname{rank}(J_{I,S}) \ge |I| - |S| + c(I,S)$. Every vector $w \in \ker(J_{S,I})$ extends by zeros on $V \setminus I$ to a vector $(0_{V\setminus I}, w) \in \ker(J)$ (since $J_{II} = 0$ and $J_{V\setminus(I\cup S),I} = 0$). Because $V \setminus I \neq \emptyset$, the all-ones vector $\mathbf{1} \in \ker(J)$ is linearly independent of $\{0_{V\setminus I}\} \times \ker(J_{S,I})$, whence

$$\dim\ker(J) \;\ge\; |I| - |S| + c(I,S) + 1.$$

Thus $\operatorname{rank}(J) = |V| - \dim\ker(J) \le |V \setminus I| + |S| - c(I,S) - 1$, and because $\operatorname{rank}(J)$ is even, (15) follows.

*Proof of (c).* On a bipartite graph $G = (A \sqcup B, E)$, every $J \in \mathcal{Z}_1(G)$ has block form $J = \left(\begin{smallmatrix} 0 & F \\ -F^{\top} & 0 \end{smallmatrix}\right)$ with $\operatorname{rank}(J) = 2\operatorname{rank}(F)$, $F\mathbf{1}_B = 0$, and $F^{\top}\mathbf{1}_A = 0$. Partition the rows of $F$ into $X$ and $A \setminus X$. By the same component-wise row-sum bound as in Part (b), $\operatorname{rank}(F_{A\setminus X,\cdot}) = \operatorname{rank}(F_{A\setminus X,Y'}) \le |Y'| - c(A \setminus X, Y')$. Moreover, $F^{\top}\mathbf{1}_A = 0$ gives $\sum_{x\in X} r_x = -\sum_{a\in A\setminus X} r_a \in \operatorname{rowspan}(F_{A\setminus X,\cdot})$, where $r_i$ denotes row $i$ of $F$. Thus the $|X|$ rows of $F_{X,\cdot}$ contribute at most $|X|-1$ linearly independent rows beyond $\operatorname{rowspan}(F_{A\setminus X,\cdot})$, proving $\operatorname{rank}(F) \le (|X|-1) + |Y'| - c(A\setminus X, Y')$ and hence (16).

*Proof of (d).*

- *Minimal bipartite separation $G_8$ ($n = 8$):* $G_8$ is 2-connected because deleting any vertex leaves a connected graph. For every $v \in V(G_8)$, $|V(G_8 - v)| = 7$ implies $\nu(G_8 - v) \le 3$, and explicit 3-edge matchings across the four automorphism orbits achieve $\nu(G_8 - v) = 3$: for $v \in \{1,2\}$ (say $v = 1$), $\{(2,7),(3,5),(4,6)\}$; for $v = 3$, $\{(1,7),(2,8),(4,5)\}$; for $v = 5$, $\{(1,7),(2,8),(3,6)\}$; for $v = 7$, $\{(1,8),(2,5),(3,6)\}$. Thus $2\min_v \nu(G_8 - v) = 6$. By (16) with $X = \{1,2\}$ ($A \setminus X = \{3,4\}$, $Y' = \{5,6\}$, $c(A\setminus X, Y') = 1$), $\rho_{\mathrm{circ}}(G_8) \le 2(2 - 1 + 2 - 1) = 4$. Conversely, $G_8$ contains the 6-cycle $C = (1,5,3,6,2,7)$, whose unit circulation $J_C \in \mathcal{Z}_1(G_8)$ has $\operatorname{rank}(J_C) = 6 - 2 = 4$ (since on an even cycle $C_{2k}$, $\ker(J_{C_{2k}}) = \operatorname{span}(\mathbf{1}_C, v_{\mathrm{bip}})$ has dimension 2 on $V(C)$); hence $\rho_{\mathrm{circ}}(G_8) = 4 < 6$.
- *Minimal non-bipartite separation $G_8' = G_8 + \{1,2\}$ ($n = 8$):* $G_8'$ is 2-connected and non-bipartite (triangle $1 - 2 - 5$), with $3 = \nu(G_8 - v) \le \nu(G_8' - v) \le 3$ for every $v$, so $2\min_v \nu(G_8' - v) = 6$. Applying (15) to the independent set $I = \{3,4,7,8\}$ ($S = N_{G_8'}(I) = \{1,2,5,6\}$, where $G_8'[I,S]$ splits into two connected components $\{3,4,5,6\}$ and $\{7,8,1,2\}$, so $c(I,S) = 2$) yields $\rho_{\mathrm{circ}}(G_8') \le 2\lfloor(4 + 4 - 2 - 1)/2\rfloor = 4$. Since $\mathcal{Z}_1(G_8) \subseteq \mathcal{Z}_1(G_8')$, $\rho_{\mathrm{circ}}(G_8') \ge \rho_{\mathrm{circ}}(G_8) = 4$, proving $\rho_{\mathrm{circ}}(G_8') = 4 < 6$.
- *Larger 2-connected examples ($n = 10, 12, 14$):* (i) On $\mathrm{ring}(2K_{2,3})$ ($n = 10$, two copies of $K_{2,3}$ on $\{0,1\} \times \{2,3,4\}$ and $\{5,6\} \times \{7,8,9\}$ linked by $\{0,5\}, \{1,6\}$, bipartite with $A = \{0,1,7,8,9\}$, $B = \{5,6,2,3,4\}$), for $v = 0$ the set $\{1,5,6\}$ is a 3-vertex cover of $G - 0$, so $\nu(G - 0) \le 3$, and matching $\{(1,2),(5,7),(6,8)\}$ attains $\nu(G - 0) = 3$ (while deleting any $v \in \{2,3,4,7,8,9\}$, say $v = 2$, admits the 4-matching $\{(0,3),(1,4),(5,7),(6,8)\}$); hence $2\min_v \nu(G - v) = 6$. By (16) with $X = \{0,1\}$ ($A \setminus X = \{7,8,9\}$, $Y' = \{5,6\}$, $c = 1$),

$\rho_{\mathrm{circ}} \le 4$, attained by two vertex-disjoint 4-cycles $(0,2,1,3)$ and $(5,7,6,8)$ $(2+2=4)$. (ii) On ring$(3C_4^{\mathrm{opp}})$ ($n=12$, three 4-cycles $(a_k,b_k,c_k,d_k)$, $k=1,2,3$, linked by $\{c_1,a_2\}$, $\{c_2,a_3\}$, $\{c_3,a_1\}$), every $G-v$ has 11 vertices ($\nu \le 5$) and admits a 5-matching: for $v=a_1$, $\{(b_1,c_1),(a_2,b_2),(c_2,d_2),(a_3,b_3),(c_3,d_3)\}$; for $v=b_1$, $\{(a_1,d_1),(c_1,a_2),(b_2,c_2),(a_3,b_3),(c_3,d_3)\}$; and symmetrically for $c_1,d_1$. Thus $2\min_v \nu(G-v)=10$. By (15) with $I=\{b_1,d_1,b_2,d_2,b_3,d_3\}$ ($S=\{a_1,c_1,a_2,c_2,a_3,c_3\}$, $c(I,S)=3$), $\rho_{\mathrm{circ}} \le 2\lfloor(6+6-3-1)/2\rfloor = 8$. Conversely, the single odd 9-cycle $(a_1,b_1,c_1,a_2,b_2,c_2,a_3,b_3,c_3)$ carries a unit circulation of exact rank $9-1=8$, proving $\rho_{\mathrm{circ}}=8<10$. (iii) On $3C_4^{\mathrm{opp}}+2$ ($n=14$, three 4-cycles $(a_k,b_k,c_k,d_k)$, $k=1,2,3$, attached to two hubs $0,1$ via $\{0,a_k\},\{1,c_k\}$, bipartite with $A=\{0,1,b_1,d_1,b_2,d_2,b_3,d_3\}$ and $B=\{a_1,c_1,a_2,c_2,a_3,c_3\}$), deleting any $b\in B$ leaves 5 vertices in $B$, so $\nu(G-b)\le 5$, while every $G-v$ admits a 5-matching: for $v=a_1$, $\{(b_1,c_1),(a_2,b_2),(c_2,d_2),(a_3,b_3),(c_3,d_3)\}$; for $v=0$, $\{(a_1,b_1),(c_1,d_1),(a_2,b_2),(c_2,d_2),(a_3,b_3)\}$; and for $v=b_1$, $\{(0,a_1),(1,c_1),(a_2,b_2),(c_2,d_2),(a_3,b_3)\}$. Hence $2\min_v \nu(G-v)=10$. By (16) with $X=\{0,1\}$ ($Y'=B$, $|Y'|=6$, $c(A\setminus X,B)=3$), $\rho_{\mathrm{circ}} \le 2(1+6-3)=8$. Conversely, the vertex-disjoint union of the 8-cycle $(0,a_1,b_1,c_1,1,c_2,b_2,a_2)$ (rank 6) and the 4-cycle $(a_3,b_3,c_3,d_3)$ (rank 2) gives $\rho_{\mathrm{circ}}=8<10$.

□

*Remark* 4.4 (Computational minimality of $n=8$). Exhaustive Graph Atlas verification in Section 6 confirms that equality holds in the Block–Tutte bound (14) for all 1,252 vertex-non-empty simple graphs on $1\le|V|\le 7$ vertices (comprising 1,245 edge-non-empty graphs and 7 edgeless graphs). Together with Theorem 4.3(d), this shows computationally that $n=8$ is the exact minimal order of strict separation for both bipartite and non-bipartite graphs.

## 5. Markov-Admissible Acceleration on Cycles: Exact Chebyshev Trace and Bottleneck Decoupling

Within the physical Markov-admissible polytope $\mathcal{M}(L,\pi)=\{|J_{ij}|\le w_{ij}\}$, Kirchhoff's conservation law couples edge capacities globally across cycles. We analyze both the uniform cycle $C_n$ (which attains the holding-time floor of Theorem 2.1(c)) and the bottleneck cycle $C_n(\delta)$, revealing a stark decoupling between spectral-gap acceleration and asymptotic-variance reduction.

**Theorem 5.1** (Exact Chebyshev trace on $C_n$ and spectral-gap vs. variance decoupling)**.** *Consider the uniform continuous-time random walk on $C_n$ ($n\ge 3$) with $\pi_i=\frac{1}{n}$, conductances $w_{i,i+1}=\frac{1}{n}$ (so reversible rates are $L_{i,i\pm1}=1$, exit rate $q=2$), and Markov-admissible vorticity $Q_\gamma = L+\gamma D_\pi^{-1}J_{\mathrm{cyc}} \in \mathcal{M}(L,\pi)$ ($\gamma\in[-1,1]$, clockwise rate $1+\gamma$, counterclockwise rate $1-\gamma$).*

(a) Zero spectral-gap gain with quadratic variance collapse. *For every $\gamma\in[-1,1]$, the spectral gap $\gamma_{\mathrm{spec}}(Q_\gamma):=\min_{\lambda\in\sigma(-Q_\gamma)\setminus\{0\}}\mathrm{Re}\,\lambda$ is identically constant:*

$$\gamma_{\mathrm{spec}}(Q_\gamma) \;=\; 2\big(1-\cos\tfrac{2\pi}{n}\big) \;=\; \gamma_{\mathrm{spec}}(L).$$

*In contrast, the worst-case asymptotic variance is*

$$V^*(Q_\gamma) \;=\; \frac{1}{1+\gamma^2-(1-\gamma^2)\cos(2\pi/n)}, \tag{17}$$

*which drops from $V^*(L)=\frac{1}{1-\cos(2\pi/n)}\sim\frac{n^2}{2\pi^2}$ at $\gamma=0$ down to the exact holding-time Loewner floor $V^*(Q_{\pm1})=\frac{1}{2}$ at $|\gamma|=1$ (a speedup of $\frac{2}{1-\cos(2\pi/n)}\sim\frac{n^2}{\pi^2}$).*

(b) Exact Chebyshev closed form for the total asymptotic variance. *For every $n\ge 3$ and every $\gamma\in(0,1)$, setting $r:=\frac{1-\gamma}{1+\gamma}\in(0,1)$,*

$$\mathrm{Tr}_{L_0^2(\pi)}\big(\Sigma(Q_\gamma)\big) \;=\; \sum_{k=1}^{n-1}\frac{1}{1+\gamma^2-(1-\gamma^2)\cos(2\pi k/n)} \;=\; \frac{n}{2\gamma}\,\frac{1+r^n}{1-r^n} \;-\; \frac{1}{2\gamma^2}, \tag{18}$$

*which interpolates smoothly between* $\mathrm{Tr}_{L_0^2(\pi)}(\Sigma(L)) = \frac{n^2-1}{6}$ *as* $\gamma \to 0^+$ *and* $\mathrm{Tr}_{L_0^2(\pi)}(\Sigma(Q_1)) = \frac{n-1}{2}$ *as* $\gamma \to 1^-$ *(an asymptotic reduction factor of* $\frac{n+1}{3}$*).*

(c) Kirchhoff bottleneck choking and spectral-gap enhancement on $C_n(\delta)$. *Let* $C_n(\delta)$ *(*$n \geq 4$*,* $\delta \in (0,1)$*) have uniform stationary measure* $\pi_i = \frac{1}{n}$*, conductances* $w_{0,n-1} = \frac{\delta}{n}$ *on edge* $\{0, n-1\}$ *and* $w_{j,j+1} = \frac{1}{n}$ *on the remaining* $n-1$ *edges, and let* $J_{\mathrm{cyc}} \in \mathcal{Z}_1(C_n)$ *be the clockwise circulation with* $(J_{\mathrm{cyc}})_{j,j+1} = \frac{1}{n}$*,* $(J_{\mathrm{cyc}})_{j+1,j} = -\frac{1}{n}$ *(indices modulo* $n$*), so* $\widetilde{J}_0 := D_\pi^{-1} J_{\mathrm{cyc}}$ *has entries* $\pm 1$ *along* $C_n$*. Then Kirchhoff's law* $J\mathbf{1} = 0$ *forces* $|J_{ij}| \leq \delta/n$ *on* every *edge simultaneously, so* $\max_{Q \in \mathcal{M}(L,\pi)}(V^*(L) - V^*(Q)) = O(\delta^2)$ *as* $\delta \to 0^+$*. Conversely, for every fixed* $\delta \in (0,1)$ *the Fiedler eigenvalue* $\lambda_2(-L(\delta))$ *is simple with unit eigenvector* $v_2 \in \mathbf{1}^\perp$ *satisfying* $\widetilde{J}_0 v_2 \neq 0$*, and for all sufficiently small* $|\gamma|$ *the spectral gap* $\gamma_{\mathrm{spec}}(L(\delta) + \gamma\widetilde{J}_0)$ *is attained by the perturbed real eigenvalue* $\lambda_2(\gamma)$ *with strictly positive curvature*

$$\frac{d^2}{d\gamma^2}\bigg|_{\gamma=0} \gamma_{\mathrm{spec}}\big(L(\delta) + \gamma\widetilde{J}_0\big) \;=\; 2\sum_{k=3}^{n} \frac{\big(v_k^\top \widetilde{J}_0 v_2\big)^2}{\lambda_k(-L(\delta)) - \lambda_2(-L(\delta))} \;>\; 0,$$

*where* $\{v_2, \dots, v_n\}$ *is an orthonormal eigenbasis of* $-L(\delta)$ *on* $\mathbf{1}^\perp$*.*

*Proof. Proof of (a).* The operators $-L$ and $D_\pi^{-1}J_{\mathrm{cyc}}$ are circulant matrices simultaneously diagonalized by the discrete Fourier modes $e_k(j) = e^{2\pi ijk/n}$ $(k = 0, 1, \dots, n-1)$, with eigenvalues $\mu_k(\gamma) = 2(1 - \cos\theta_k) - 2i\gamma \sin\theta_k$ for $-Q_\gamma = -L - \gamma D_\pi^{-1} J_{\mathrm{cyc}}$ $(\theta_k = 2\pi k/n)$. Hence $\mathrm{Re}\,\mu_k(\gamma) = 2(1 - \cos\theta_k)$ is independent of $\gamma$, giving $\gamma_{\mathrm{spec}}(Q_\gamma) = 2(1 - \cos\frac{2\pi}{n})$. For $k \geq 1$, the eigenvalue of $\Sigma(Q_\gamma) = 2\,\mathrm{Sym}((-Q_\gamma)^{-1})$ is

$$\begin{aligned} 2\,\mathrm{Re}\,\frac{1}{\mu_k(\gamma)} &= \frac{4(1-\cos\theta_k)}{4(1-\cos\theta_k)^2 + 4\gamma^2\sin^2\theta_k} \\ &= \frac{1}{(1-\cos\theta_k) + \gamma^2(1+\cos\theta_k)} \;=\; \frac{1}{1+\gamma^2 - (1-\gamma^2)\cos\theta_k}, \end{aligned}$$

whose maximum over $k = 1, \dots, n-1$ is attained at $k = 1$ (since $1 - \gamma^2 \geq 0$), yielding (17).

*Proof of (b).* Fix $\gamma \in (0,1)$ and $r = \frac{1-\gamma}{1+\gamma} \in (0,1)$. Observe that for every $\theta \in \mathbb{R}$,

$$|e^{i\theta} - r|^2 \;=\; 1 + r^2 - 2r\cos\theta \;=\; \frac{2}{(1+\gamma)^2}\Big(1 + \gamma^2 - (1-\gamma^2)\cos\theta\Big),$$

while $1 - r^2 = \frac{4\gamma}{(1+\gamma)^2}$. Therefore,

$$\frac{1}{1+\gamma^2-(1-\gamma^2)\cos\theta} \;=\; \frac{1}{2\gamma}\,\frac{1-r^2}{|e^{i\theta}-r|^2} \;=\; \frac{1}{2\gamma}\,\mathrm{Re}\left(\frac{e^{i\theta}+r}{e^{i\theta}-r}\right).$$

Logarithmic differentiation of $z^n - 1 = \prod_{k=0}^{n-1}(z - e^{2\pi ik/n})$ at $z = r \in (0,1)$ gives $\sum_{k=0}^{n-1} \frac{r}{r - e^{2\pi ik/n}} = \frac{nr^n}{r^n - 1}$, whence

$$\sum_{k=0}^{n-1} \frac{e^{2\pi ik/n} + r}{e^{2\pi ik/n} - r} \;=\; \sum_{k=0}^{n-1}\left(1 - \frac{2r}{r - e^{2\pi ik/n}}\right) \;=\; n - \frac{2nr^n}{r^n - 1} \;=\; n\,\frac{1+r^n}{1-r^n}.$$

Multiplying by $\frac{1}{2\gamma}$ and subtracting the $k = 0$ summand $\frac{1}{1+\gamma^2-(1-\gamma^2)} = \frac{1}{2\gamma^2}$ yields (18). Taking $\gamma \to 1^-$ $(r \to 0^+)$ gives $\frac{n-1}{2}$; expanding $r = e^{-2\gamma - \frac{2}{3}\gamma^3 + O(\gamma^5)}$ as $\gamma \to 0^+$ recovers $\frac{n^2-1}{6}$.

*Proof of (c).* On $C_n(\delta)$ $(n \geq 4,\ \delta \in (0,1))$, $\dim \mathcal{Z}_1(C_n) = 1$, so every $J \in \mathcal{Z}_1(C_n)$ has the form $J = \gamma J_{\mathrm{cyc}}$, and $|J_{0,n-1}| \leq w_{0,n-1} = \delta/n$ forces $|\gamma| \leq \delta$. Letting $\Omega_0 := U^\top \widetilde{J}_0 U \in \mathbb{R}^{(n-1)\times(n-1)}$ (since $\pi_i = \frac{1}{n}$, $D_\pi^{-1/2} J_{\mathrm{cyc}} D_\pi^{-1/2} = \widetilde{J}_0$), Theorem 2.1(a) gives $\Sigma(Q_\gamma) = 2(S(\delta) + \gamma^2 \Omega_0^\top S(\delta)^{-1}\Omega_0)^{-1}$,

where $\lambda_2(S(\delta)) \geq \lambda_2(S_{\text{path}}) > 0$ uniformly as $\delta \to 0^+$ (removing $\{0, n-1\}$ leaves the connected path $P_n$); hence $V^*(L) - V^*(Q_\gamma) = O(\delta^2)$.

For the spectral gap: by reflection symmetry $j \mapsto n-1-j$ of $C_n(\delta)$, $-L(\delta) = S_{\text{path}} + \delta(e_0 - e_{n-1})(e_0 - e_{n-1})^\top$ decomposes into even (symmetric) and odd (antisymmetric) subspaces on $\mathbf{1}^\perp$. On the even subspace, $(e_0 - e_{n-1})^\top v = 0$, so every even eigenvalue is independent of $\delta$ and coincides with the even eigenvalues of $P_n$ (smallest non-zero even eigenvalue $2(1-\cos\frac{2\pi}{n})$). On the odd subspace, let $\lambda_{\text{odd}}(\delta)$ be the lowest eigenvalue and $v(\delta)$ a unit odd eigenvector. By oddness, $v(\delta)_{n-1} = -v(\delta)_0$, so $(e_0 - e_{n-1})^\top v(\delta) = 2v(\delta)_0$. If $v(\delta)_0 = 0$ for some $\delta \in [0,1]$, then $v(\delta)_{n-1} = 0$ as well, and the row-0 eigenvalue equation $(-L(\delta)v(\delta))_0 = (1+\delta)v(\delta)_0 - v(\delta)_1 - \delta v(\delta)_{n-1} = \lambda_{\text{odd}}(\delta)v(\delta)_0$ forces $v(\delta)_1 = 0$; inductively, the row-$j$ equation $2v(\delta)_j - v(\delta)_{j-1} - v(\delta)_{j+1} = \lambda_{\text{odd}}(\delta)v(\delta)_j$ forces $v(\delta)_{j+1} = 0$ for all $j$, contradicting $v(\delta) \neq 0$. Thus $(e_0 - e_{n-1})^\top v(\delta) \neq 0$ for every $\delta \in [0,1]$. By the variational characterization of the minimum eigenvalue, $\lambda_{\text{odd}}(\delta)$ is strictly increasing on $[0,1]$ (for any $0 \leq \delta_1 < \delta_2 \leq 1$, $\lambda_{\text{odd}}(\delta_1) \leq v(\delta_2)^\top(-L(\delta_1))v(\delta_2) = \lambda_{\text{odd}}(\delta_2) - (\delta_2 - \delta_1)((e_0 - e_{n-1})^\top v(\delta_2))^2 < \lambda_{\text{odd}}(\delta_2)$). Since $\lambda_{\text{odd}}(1) = 2(1-\cos\frac{2\pi}{n})$ (where the lowest odd and even modes meet on the uniform cycle $C_n$), we have $\lambda_{\text{odd}}(\delta) < 2(1-\cos\frac{2\pi}{n})$ for every $\delta \in (0,1)$. Moreover, because $-L(\delta) = S_{\text{path}} + \delta(e_0 - e_{n-1})(e_0 - e_{n-1})^\top \succeq S_{\text{path}}$ on the odd subspace and the odd eigenvalues of the path Laplacian $S_{\text{path}}$ are $2(1-\cos\frac{(2m-1)\pi}{n})$ ($m = 1, \ldots, \lfloor n/2 \rfloor$), Weyl's monotonicity theorem implies that the second odd eigenvalue satisfies $\lambda_{\text{odd},2}(\delta) \geq 2(1-\cos\frac{3\pi}{n}) > 2(1-\cos\frac{2\pi}{n})$ for every $n \geq 4$. Combining this with the lowest non-zero even eigenvalue $2(1-\cos\frac{2\pi}{n})$ yields $\lambda_2(\delta) := \lambda_{\text{odd}}(\delta) < 2(1-\cos\frac{2\pi}{n}) \leq \lambda_3(\delta)$ for every $\delta \in (0,1)$—proving $\lambda_2(\delta)$ is *simple* with unit eigenvector $v_2 = v(\delta) \in \mathbf{1}^\perp$. Moreover, $\lambda_2(\delta) < 2(1-\cos\frac{2\pi}{n}) \leq 2$ for all $n \geq 4$, whereas for even $n \geq 4$ the unique direction in $\ker(\widetilde{J}_0) \cap \mathbf{1}^\perp$ is $v_{\text{bip}}$ ($(v_{\text{bip}})_j = (-1)^j$), whose Rayleigh quotient is $\frac{v_{\text{bip}}^\top(-L(\delta))v_{\text{bip}}}{\|v_{\text{bip}}\|_2^2} = \frac{4(n-1)+4\delta}{n} > 3 > 2 > \lambda_2(\delta)$ (and for odd $n$, $\ker(\widetilde{J}_0) \cap \mathbf{1}^\perp = \{0\}$). Hence $\widetilde{J}_0 v_2 \neq 0$. In $\mathbb{R}^{n-1}$ with $\tilde{v}_k := U^\top v_k$ ($k = 2, \ldots, n$), $\lambda_2(\delta)$ is a simple real eigenvalue of $S(\delta) - \gamma\Omega_0$ at $\gamma = 0$, strictly smaller than $\lambda_3(\delta) \leq \cdots \leq \lambda_n(\delta)$; by continuity of eigenvalues in $\gamma$, for all sufficiently small $|\gamma|$ the perturbed real analytic eigenvalue $\lambda_2(\gamma)$ strictly minimizes $\operatorname{Re}\lambda$ over $\sigma(S(\delta) - \gamma\Omega_0)$ (so $\gamma_{\text{spec}}(L(\delta) + \gamma\widetilde{J}_0) = \lambda_2(\gamma)$), and second-order Rayleigh–Schrödinger perturbation theory yields $\lambda_2'(0) = -\tilde{v}_2^\top\Omega_0\tilde{v}_2 = 0$ and $\lambda_2''(0) = 2\sum_{k=3}^n \frac{(\tilde{v}_k^\top\Omega_0\tilde{v}_2)^2}{\lambda_k - \lambda_2} = 2\sum_{k=3}^n \frac{(v_k^\top\widetilde{J}_0 v_2)^2}{\lambda_k - \lambda_2} > 0$. □

## 6. High-Precision Numerical and Exhaustive Verification

All five theorems have been verified via our reproducible Python suite (the verification script `verify_kirchhoff_vorticity.py` and pytest suite `test_kirchhoff_vorticity.py` in the archived repository):

- **Theorem 2.1:** Across 25 random connected weighted graphs ($n \in [5, 10]$) with random Dirichlet stationary measures $\pi$ and random Markov-admissible vorticities $J \in \mathcal{M}(L, \pi)$, the Schur-complement formula (4) and holding-time decomposition (6) match direct Poisson solutions to maximum relative error $2.78 \times 10^{-15}$ and $2.95 \times 10^{-15}$, respectively, and the shorted limit (5) matches $\gamma = 10^4$ to $1.05 \times 10^{-8}$.
- **Theorems 3.1, 4.2, and 4.3(a) (Remark 4.4):** Exhaustive testing over all 1,252 vertex-non-empty non-isomorphic graphs in the NetworkX Graph Atlas ($1 \leq n \leq 7$, comprising 1,245 edge-non-empty graphs and 7 edgeless graphs, including 994 connected graphs on $3 \leq n \leq 7$ vertices, of which 431 are factor-critical) plus 213 connected Erdős–Rényi graphs ($5 \leq n \leq 13$, 75 factor-critical) yields **zero mismatches** for $\dim \mathcal{I}(G) = c(G) + b_{\text{bridge}}(G) + b_{\text{bip}}(G)$, **zero mismatches** for the factor-criticality equivalence $\rho_{\text{circ}}(G) = n - 1 \iff G$ factor-critical, and **zero mismatches** for exactness of the Block–Tutte bound (14) across all $n \leq 7$.

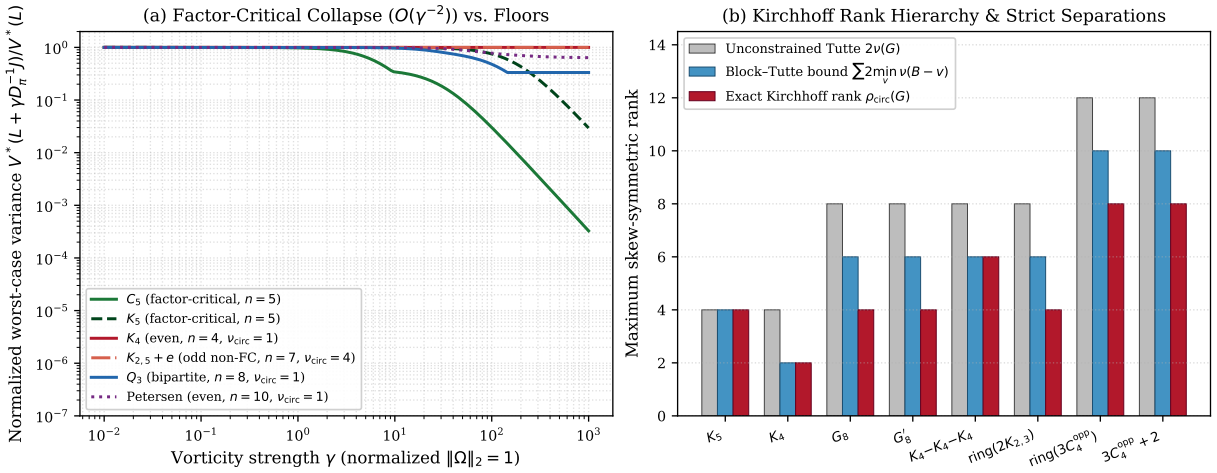



FIGURE 1. (a) Normalized worst-case asymptotic variance $V^*(L+\gamma D_\pi^{-1}J)/V^*(L)$ along a generic circulation $J \in \mathcal{Z}_1(G)$ (normalized to $\|\Omega\|_2 = 1$): factor-critical graphs ($C_5, K_5$) collapse as $O(\gamma^{-2})$, whereas non-factor-critical graphs ($K_4$, odd non-FC $K_{2,5}+e$, bipartite $Q_3$, Petersen) saturate at strictly positive floors governed by Theorem 4.2(b). (b) Hierarchy comparing unconstrained Tutte rank $2\nu(G)$, the Block–Tutte bound (14), and exact Kirchhoff circulation rank $\rho_{\text{circ}}(G)$, highlighting the minimal strict separations at $n = 8$ ($G_8, G_8'$) and larger 2-connected examples at $n = 10, 12, 14$ (Theorem 4.3(d)).

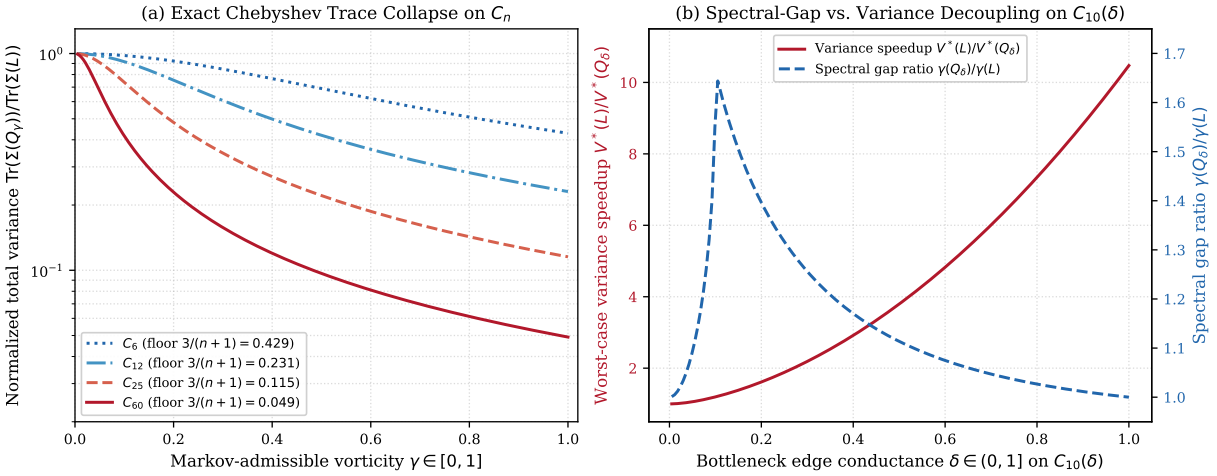



FIGURE 2. (a) Exact Chebyshev trace collapse (18) on uniform cycles $C_n$ across $\gamma \in (0, 1]$, approaching the holding-time floor $3/(n+1)$ with zero spectral-gap gain. (b) Spectral-gap vs. variance decoupling on the bottleneck cycle $C_{10}(\delta)$ (Theorem 5.1(c)): maximal Markov-admissible vorticity $|\gamma| = \delta$ boosts the spectral gap by up to +66% near $\delta \approx 0.11$ while variance speedup grows monotonically from 1 to 10.47× as $\delta \to 1$.

- **Theorem 5.1:** The Chebyshev trace formula (18) matches direct summation across $n \in \{6, 10, 20, 50\}$ and $\gamma \in \{0.1, 0.5, 0.9, 1.0\}$ in 50-digit `mpmath` arithmetic to maximum absolute error $4.11 \times 10^{-48}$.

TABLE 1. Exact invariants and large-vorticity ($\gamma = 10^4$, generic $J \in \mathcal{Z}_1(G)$ normalized to $\|\Omega(J)\|_2 = 1$, matching Figure 1(a)) worst-case asymptotic variance $V^*$ across canonical topologies (unit conductances $w_{ij} = 1$, uniform $\pi_i = 1/n$, reversible generator $L = n(W - D_w)$). Here $K_{2,5}+e$ adds an edge inside the 2-vertex part of $K_{2,5}$, $K_4$-$K_4$-$K_4$ is the linear chain of three $K_4$ cliques sharing two distinct cut-vertices, and $G_8, G_8', \mathrm{ring}(2K_{2,3}), \mathrm{ring}(3C_4^{\mathrm{opp}}), 3C_4^{\mathrm{opp}} + 2$ are defined in Theorem 4.3(d).

| Graph $G$ | $n$ | FC | $\dim \mathcal{I}$ | $\rho_{\mathrm{circ}}$ | BT UB | $\nu_{\mathrm{circ}}$ | $V^*(L)$ | CF Floor | $V^*(\gamma = 10^4)$ |
|---|---|---|---|---|---|---|---|---|---|
| $C_5$ | 5 | Yes | 1 | 4 | 4 | 0 | 0.289443 | 0.000000 | $9.47 \times 10^{-7}$ |
| $K_5$ | 5 | Yes | 1 | 4 | 4 | 0 | 0.080000 | 0.000000 | $4.45 \times 10^{-5}$ |
| $C_6$ | 6 | No | 2 | 4 | 4 | 1 | 0.333333 | 0.083333 | 0.083333 |
| $K_4$ | 4 | No | 1 | 2 | 2 | 1 | 0.125000 | 0.125000 | 0.125000 |
| $K_{2,3}$ | 5 | No | 2 | 2 | 2 | 2 | 0.200000 | 0.133333 | 0.200000 |
| $K_{2,5} + e$ | 7 | No | 1 | 2 | 2 | 4 | 0.142857 | 0.142857 | 0.142857 |
| $Q_3$ | 8 | No | 2 | 6 | 6 | 1 | 0.125000 | 0.041667 | 0.041667 |
| $G_8$ | 8 | No | 2 | 4 | 6 | 3 | 0.213388 | 0.062500 | 0.158608 |
| $G_8'$ | 8 | No | 1 | 4 | 6 | 3 | 0.213388 | 0.062500 | 0.166032 |
| Petersen | 10 | No | 1 | 8 | 8 | 1 | 0.100000 | 0.040000 | 0.048569 |
| $K_4$-$K_4$-$K_4$ | 10 | No | 1 | 6 | 6 | 3 | 0.373205 | 0.050000 | 0.088183 |
| $\mathrm{ring}(2K_{2,3})$ | 10 | No | 2 | 4 | 6 | 5 | 0.318614 | 0.100000 | 0.111702 |
| $\mathrm{ring}(3C_4^{\mathrm{opp}})$ | 12 | No | 1 | 8 | 10 | 3 | 0.343032 | 0.041667 | 0.083341 |
| $3C_4^{\mathrm{opp}} + 2$ | 14 | No | 2 | 8 | 10 | 5 | 0.325825 | 0.047619 | 0.107373 |

TABLE 2. Spectral-gap vs. asymptotic-variance decoupling on the bottleneck cycle $C_{10}(\delta)$ under maximal Markov-admissible vorticity $|\gamma| = \delta$ (rate 1 on 9 edges, rate $\delta$ on edge $\{0, 9\}$).

| $\delta$ | $V^*(L)$ | $V^*(Q_\delta)$ | Speedup $V^*(L)/V^*(Q_\delta)$ | $\gamma_{\mathrm{spec}}(L)$ | $\gamma_{\mathrm{spec}}(Q_\delta)$ | Gap gain |
|---|---|---|---|---|---|---|
| 0.01 | 18.939000 | 18.882866 | 1.002973 | 0.105602 | 0.106023 | +0.399% |
| 0.05 | 14.846865 | 14.013378 | 1.059478 | 0.134709 | 0.147239 | +9.302% |
| 0.10 | 11.955287 | 9.997396 | 1.195840 | 0.167290 | 0.271946 | +62.559% |
| 0.25 | 8.233516 | 4.374473 | 1.882173 | 0.242910 | 0.319848 | +31.674% |
| 0.50 | 6.299353 | 1.659680 | 3.795522 | 0.317493 | 0.353573 | +11.364% |
| 0.75 | 5.589060 | 0.839760 | 6.655547 | 0.357842 | 0.370757 | +3.609% |
| 1.00 | 5.236068 | 0.500000 | 10.472136 | 0.381966 | 0.381966 | +0.000% |

## 7. Open Questions

Two natural algebraic-combinatorial questions emerge from our results:

(1) *Exact min-max formula for $\rho_{\mathrm{circ}}(G)$:* Theorem 4.2 characterizes exactly when $\rho_{\mathrm{circ}}(G) = n - 1$, and Theorem 4.3 together with Remark 4.4 shows that the Block–Tutte bound is tight across all $n \le 7$ graphs but fails strictly starting at $n = 8$. Is there a polynomial-time combinatorial min-max formula (generalizing Tutte–Berge and Edmonds–Lovász matroid parity) determining $\rho_{\mathrm{circ}}(G)$ for every finite graph $G$?

(2) *Per-observable attainability of the universal immune floor:* Theorem 3.1(c) proves $\sigma^2(f, L + D_\pi^{-1} J) \ge 2\|P^{\mathcal{E}}_{\mathcal{I}(G)}(-L)^{-1} f\|^2_{\mathcal{E}}$ for all $f$ and $J$, with exact per-observable attainability proved on factor-critical graphs and even complete graphs $K_{2k}$ (and verified numerically on

all tested graphs including $\mathrm{ring}(2K_{2,3})$ and $\mathrm{ring}(3C_4^{\mathrm{opp}})$). Does $\bigcup_{J\in\mathcal{Z}_1(G)} \mathrm{im}(J) = \mathcal{I}(G)^\perp$ hold for every finite graph $G$?

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Independent Researcher
*Email address*: editor@airr.science