Sharp Certified Weight-Table Barriers, Effective Mode Localization ($n_0 = 614$), Exact Sperner Flows to $n \leq 25$, and Normalized Matching Dichotomies for the Permutohedron Face Lattice

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https://github.com/lluiseriksson/permutohedron-hall-sharp-window-and-nmp

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Abstract

Let OP_n be the graded poset of ordered set partitions of $[n] = \{1, \ldots, n\}$ ordered by merging adjacent blocks, whose completion with a bottom element is the face lattice of the $(n-1)$-dimensional permutohedron. Rossi (`arXiv:2611.34537v1`) proved Hall's matching condition between adjacent levels G_j outside the window $\big((28n-36)/39,\ (13n+5)/18\big)$ of width $n/234 + O(1)$ around the asymptotic mode $n/(2\ln 2)$, established non-effectively that the largest level falls inside this window for $n \geq n_0$ (conjecturing $n_0 = 614$), reported uncertified floating-point weight-table stabilizations near $75/104$ and $0.72135\ldots$, verified Hall's condition exactly for $n \leq 14$ and by floating-point Sinkhorn scaling for $n \leq 25$, and asked for which pairs (n, j) the normalized matching property (NMP) holds (Open Question 2).

We resolve these questions completely in three sharp, constructively certified theorems. First, we construct all-integer primal weight tables of supports $S = 8$ and $S = 10$ that contract the certified unverified window by $21.65\times$ to the interval between $(75n - 104)/104$ and $(257935n + 99637)/357572$, of asymptotic width $\frac{1835}{9296872}\, n \approx 1.9738 \times 10^{-4} n$ (reducing the upper gap above $1/(2\ln 2)$ by $236\times$ to 3.70×10^{-6}), and we prove exact finite-support LP-dual identities valid for weight tables of *arbitrary* (even infinite) support establishing that $75/104$ (Case (a), 5-pair dual, without even requiring superadditivity), $233/323$ (Case (b) monotone $K(c, \cdot)$, 8-constraint dual), and $257935/357572 \approx 0.721351224$ (Case (b) exact linear condition (i), 16-constraint dual using a single split $(1, 8, 1) \to (1, 4, 4, 1)$) are the exact global limits of Rossi's linear local weight-table paradigm. Second, using only the strict-slack inequalities of our primal flows (zero renewal theory or asymptotics) plus exact integer recurrence verification for $n \leq 1163$, we prove Rossi's conjectured sharp threshold $n_0 = 614$ in Corollary 2.4 unconditionally; moreover, exact arbitrary-precision integer max-flows on the $(S_n \times C_2)$-folded quotient certify all ten window pairs up to $n \leq 25$, upgrading Rossi's rigorous Sperner theorem for OP_n from $n \leq 14$ to $1 \leq n \leq 25$. Third, we establish a sharp NMP dichotomy for Open Question 2: an elementary two-line Bonferroni bound proves NMP fails for every fixed $j \geq 2$ above an explicit threshold and throughout the sublinear regime $2 \leq j \leq (1-\varepsilon)\frac{(1-\ln 2)n}{\ln n}$; an explicit child-collision calculation on the balanced orbit $\{(7,7,6), (7,6,7), (6,7,7)\}$ proves why $j = 3$ fails at $n = 20$ (average neighborhood $928/3 < N_4/N_3$) even though singletons pass ($314 > N_4/N_3$); and conversely, identifying the S_n-quotient of G_{n-2} with the edge–vertex incidence graph of K_{n-1}, we construct a strictly positive closed-form equalizing normalized flow proving NMP holds at $j = n-2$ (as well as $j \in \{1, n-1\}$) for every $n \geq 3$. All core algebraic identities, infinite-support duals, mode enclosures, Bonferroni bounds, and K_{n-1} flow equations are formally proved in constructive Lean 4.35.0 (20 theorems, 0 `sorry`, 0 `Classical.choice`), paired with a zero-floating-point 16-certificate Python verifier and 56 adversarial tests.

Keywords: Permutohedron face lattice, ordered set partitions, Sperner property, Hall's condition, linear programming duality, normalized matching property, formal verification, Lean 4.
MSC 2020: Primary 06A07; Secondary 05A18, 05C70, 05D05, 90C05.

1 Introduction

Let OP_n denote the poset of ordered set partitions $a = (B_1 \mid \cdots \mid B_j)$ of $[n] = \{1, \ldots, n\}$ into $j \geq 1$ nonempty blocks, ordered by merging adjacent blocks. With a bottom element adjoined, OP_n is the face lattice of the $(n-1)$-dimensional permutohedron [1, 6]. Level j has cardinality $N_j = N_{n,j} = j!\,S(n,j)$, where $S(n,j)$ is a Stirling number of the second kind, and the total cardinality is the Fubini number $a_n = \sum_{j=1}^{n} N_{n,j}$. By Lieb's log-concavity of $S(n,j)$ [4], the level sequence $(N_{n,j})_{j=1}^{n}$ is log-concave and unimodal (Rossi [5, Lemma 1.4]).

Let G_j be the cover bipartite graph between levels j and $j+1$ of OP_n. Each $a \in L_j$ with block-size composition $(s_1, \ldots, s_j)$ has $d(a) = \sum_{p=1}^{j}(2^{s_p} - 2)$ children in L_{j+1}, while each $b \in L_{j+1}$ has exactly j parents in L_j. Hall's condition holds for (n,j) if G_j admits a matching saturating the smaller level; by [5, Theorem 1.3], Hall's condition for all $1 \leq j \leq n-1$ implies that OP_n is Sperner, contrasting with Canfield's celebrated disproof of the Sperner property for the unordered partition lattice Π_n [2, 3].

In a recent preprint [5], Rossi introduced local weight-table fractional flows on G_j and proved ([5, Theorem 1.1]) that Hall's condition holds whenever

$$j \leq \max\left\{\frac{2n}{3},\ \frac{28n-36}{39}\right\} \qquad \text{or} \qquad j \geq \min\left\{\frac{8n-2}{11},\ \frac{13n+5}{18}\right\}, \tag{1}$$

leaving an open window of asymptotic width $n/234 \approx 4.2735 \times 10^{-3} n$ straddling the asymptotic mode $n/(2\ln 2) \approx 0.721348\,n$ ([5, Proposition 2.3]). Rossi left four concrete gaps and open questions:

1. *Uncertified weight-table limits ([5, Section 4]):* Floating-point linear programs suggested that larger-support local weight tables stabilize at slopes $75/104$ for $S \geq 8$ in Case (a) and $0.72135\ldots$ for $S \geq 10$ in Case (b), without certified primal weights, exact rational identification of $0.72135\ldots$, or any proof that larger or infinite supports cannot improve these slopes.

2. *Non-effective mode localization ([5, Corollary 2.4]):* Rossi proved non-effectively that the largest level $j^*(n)$ lies inside $((28n-36)/39,\ (13n+5)/18)$ for $n \geq n_0$, conjecturing $n_0 = 614$ from exact computations to $n \leq 700$ and floats to $n \leq 4000$.

3. *Uncertified Sinkhorn flows for $15 \leq n \leq 25$ ([5, Section 5]):* Rossi certified OP_n Sperner exactly for $n \leq 14$ and relied on uncertified floating-point Sinkhorn scaling for the eight window pairs with $16 \leq n \leq 24$.

4. *Normalized Matching Property dichotomy ([5, Section 6, Open Question 2]):* Empirically, NMP held for $n \leq 11$ (all j) and failed at $(12,2)$, $(20,3)$, and $(28,4)$; Rossi asked for which pairs (n,j) NMP holds.

Main Results

We settle all four items with exact rational/integer certificates and constructive Lean 4.35.0 proofs.

Theorem 1 (Sharp Certified Weight-Table Contraction and Infinite-Support Dual Barriers)**.** *Let $1 \leq j \leq n-1$.*

(a) Certified all-integer primal matchings: *There exist explicit nonnegative all-integer symmetric weight tables w_A ($S = 8$, tail weight 150), u_{mon} ($S = 8$, monotone $K(c,\cdot)$), and u_{ex} ($S = 10$, exact condition* (i) $Y(b_{p,i}) \geq Y(a)$*) such that:*

- *G_j has a matching saturating L_j whenever $104j \leq 75n - 104$ (with strict level inequality $N_{n,j} < N_{n,j+1}$ for all $n \geq 4$);*

- *G_j has a matching saturating L_{j+1} whenever $323j \geq 233n + 90$ (monotone K, with strict level inequality $N_{n,j+1} < N_{n,j}$ for all $n \geq 30$);*
- *G_j has a matching saturating L_{j+1} whenever $357572j \geq 257935n + 99637$ (exact linear condition* (i)*, with strict level inequality $N_{n,j+1} < N_{n,j}$ for all $n \geq 37$).*

The certified asymptotic window shrinks from Rossi's width $1/234$ to $\frac{257935}{357572} - \frac{75}{104} = \frac{1835}{9296872} \approx 1.97378 \times 10^{-4}$ (a $21.65\times$ contraction), and the upper gap above $1/(2\ln 2)$ drops by $236\times$ to 3.70×10^{-6}. For $n = 1000$, only the single level $j = 721$ remains outside our bounds (versus five levels $\{718, \ldots, 722\}$ in [5]).

(b) Infinite-support LP-dual impossibility barriers: *Across* all *nonnegative symmetric weight tables of* arbitrary (including infinite) *support:*

- *Every Case* (a) *pair-inequality triple $(w, A/2, B)$ satisfies $2B/(A/2) \leq 75/104$ (even without superadditivity), certified by the 5-pair dual $(36, 24, 8, 6, 1)$ on $\{(1,1), (1,2), (1,3), (2,2), (4,4)\}$ with residual $+\frac{1}{70}w(4,4) \geq 0$.*
- *Every Case* (b) *monotone pair-inequality triple (u, C, B) satisfying $K(1,4) \geq K(1,5)$ satisfies $2B/C \geq 233/323$, certified by an 8-constraint dual with identically zero weight residual.*
- *Every Case* (b) *pair-inequality triple (u, C, B) satisfying the single split condition $K(4,4) + 2(K(1,4) - K(1,8)) \geq 0$ (splitting $(1,8,1) \to (1,4,4,1)$) satisfies*

$$\frac{2B}{C} \;\geq\; \frac{257935}{357572} \;\approx\; 0.7213512243688,$$

certified by a 16-constraint integer dual with positive residual $16\,u(3,4) + \frac{69}{70}u(4,4) \geq 0$.

Theorem 2 (Mode Localization ($n_0 = 614$) and Exact Sperner Flows ($n \leq 25$))**.**

(a) Combinatorial mode enclosure and sharp $n_0 = 614$: *By the strict-slack inequalities of Theorem 1*(a) *and log-concavity [5, Lemma 1.4], every index $j^*(n)$ maximizing $N_{n,j}$ satisfies*

$$\frac{75n - 104}{104} \;<\; j^*(n) \;<\; \frac{233n + 413}{323} \qquad (\forall n \geq 30), \tag{2}$$

and similarly $j^*(n) < (257935n + 457209)/357572$ for $n \geq 37$, proved without renewal theory or asymptotics. For every $n \geq 1164$ (respectively $n \geq 1150$ using the $S = 10$ upper bound), the interval (2) lies strictly inside Rossi's window between $(28n-36)/39$ and $(13n+5)/18$. Exact integer evaluation of $N_{n,j} = j(N_{n-1,j} + N_{n-1,j-1})$ for $1 \leq n \leq 1163$ proves unconditionally that $n = 613$ ($j^* = 443 = (13 \cdot 613 + 5)/18$) is the final exception, establishing Rossi's conjectured sharp threshold $n_0 = 614$ in [5, Corollary 2.4].

(b) Exact zero-float Sperner theorem for $1 \leq n \leq 25$: *Folding each composition with its reversal reduces Hall's condition to an exact integer maximum-flow problem on the $(S_n \times C_2)$-quotient of G_j. Arbitrary-precision integer Dinic flows certify all ten window pairs with $n \leq 25$ — $(10,7)$, $(13,9)$, $(16,11)$, $(17,12)$, $(19,13)$, $(20,14)$, $(21,15)$, $(22,15)$, $(23,16)$, and $(24,17)$ — while $n = 25$ has an empty window. Hence OP_n satisfies [5, Conjecture 1.2] and is Sperner for every $1 \leq n \leq 25$.*

Theorem 3 (Normalized Matching Property Dichotomy for Rossi's Open Question 2)**.** *Let $1 \leq j \leq n-1$. The normalized matching property (NMP) for G_j — requiring $|N(S)|/N_{n,j+1} \geq |S|/N_{n,j}$ for all $S \subseteq L_j$ — satisfies the following dichotomy:*

(a) Universal failure for fixed $j \geq 2$ and sublinear $j = o(n/\ln n)$: *For $n = qj + r$ ($0 \leq r < j$, $q \geq 1$), the balanced composition a_{bal} has minimum degree $d_{\min}(n,j) = (j+r)2^q - 2j$, while the elementary Bonferroni inequality gives $N_{n,j+1}/N_{n,j} \geq (1+1/j)^n - (j+1) \geq (9/4)^q - (j+1)$*

Table 1: All-integer primal weight tables achieving exact slopes 75/104 (Case (a), $S = 8$, $w_{\text{tail}} = 150$, $B = 75$, $A/2 = 208$), 233/323 (Case (b) monotone, $S = 8$, $B = 233$, $C = 646$), and 257935/357572 (Case (b) exact condition (i), $S = 10$, $B = 257935$, $C = 715144$). Unlisted pairs with $x + y \le S$ have weight 0.

Pair (x, y)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,2)$	$(1,4)$	$(2,3)$	$(1,5)$	$(2,4)$	$(3,3)$	$(1,6)$
w_A $(S = 8)$	116	123	124	144	133	140	167	180	0	86
u_{mon} $(S = 8)$	360	381	380	444	415	420	498	450	200	441
u_{ex} $(S = 10)$	398548	421839	420972	491712	461485	465820	569034	504540	224240	667051
Pair (x, y)	$(2,5)$	$(3,4)$	$(1,7)$	$(2,6)$	$(3,5)$	$(4,4)$	$(1,8)$	$(2,7)$	$(3,6)$	$(2,8)$
w_A $(S = 8)$	294	42	0	273	224	0	150	150	150	150
u_{mon} $(S = 8)$	630	0	0	280	0	0	0	0	0	0
u_{ex} $(S = 10)$	759738	0	627624	1025696	45248	0	830673	712512	106008	1513620

for all $j \ge 2$. Consequently, the singleton $\{a_{\text{bal}}\}$ violates NMP whenever $(9/8)^q > 2j$ (and asymptotically throughout $2 \le j \le (1-\varepsilon)\frac{(1-\ln 2)n}{\ln n}$). Exact singleton failure thresholds for $2 \le j \le 10$ are

$$n_{\text{sing}}(2, \dots, 10) = (12,\ 21,\ 28,\ 38,\ 48,\ 58,\ 69,\ 79,\ 90).$$

(b) Collision-driven orbit failure at $(20, 3)$: *At $(n, j) = (20, 3)$, every singleton passes NMP ($d_{\min}(20, 3) = 314 > 311.62 \approx N_{20,4}/N_{20,3}$), yet the full balanced orbit $S_{\text{bal}} = \{(7, 7, 6), (7, 6, 7), (6, 7, 7)\}$ shares the two children $(6, 1, 6, 7)$ and $(7, 6, 1, 6)$ of relative weight 7 each, lowering its exact neighborhood ratio to*

$$\frac{|N(S_{\text{bal}})|}{|S_{\text{bal}}|} = 314 - \frac{14}{3} = \frac{928}{3} \approx 309.333 < \frac{N_{20,4}}{N_{20,3}},$$

rigorously explaining the empirical onset $n = 20$ at $j = 3$ reported in [5, Section 6].

(c) Universal positive NMP at $j \in \{1, n-2, n-1\}$ via a closed-form K_{n-1} flow: *While G_1 and G_{n-1} are biregular, the S_n-quotient of G_{n-2} is isomorphic to the edge–vertex incidence graph of the complete graph K_{n-1} (with consecutive edges of weight $n!/6$ and non-consecutive edges of weight $n!/4$). Shifting mass $\delta = \frac{n!}{24(n-1)}$ from each endpoint vertex $\{1, n-1\}$ to every interior vertex $m \in \{2, \dots, n-2\}$ along edges $\{1, m\}$ and $\{m, n-1\}$ produces a strictly positive closed-form equalizing normalized flow proving NMP at $j = n-2$ for every $n \ge 3$.*

2 Primal Weight Tables and Infinite-Support LP-Dual Barriers

We adopt Rossi's notation [5, Section 3]. An element $a \in L_j$ has block sizes $(s_1, \dots, s_j)$ and $b \in L_{j+1}$ has block sizes $(b_1, \dots, b_{j+1})$, with parents $a_1, \dots, a_j$ obtained by merging adjacent pairs (b_t, b_{t+1}). By [5, Lemma 3.1], a nonnegative fractional flow sending total weight 1 from every vertex of the smaller side and delivering load at most 1 to every vertex of the larger side implies Hall's condition.

2.1 Case (a): Certified $S = 8$ Primal Weights and 5-Pair Dual Barrier

In Case (a), a symmetric nonnegative table $w(x, y) = w(y, x)$ $(x, y \ge 1)$ defines $\kappa(x, y) = w(x, y)/\binom{x+y}{x}$, $\Omega(1) = 0$, $\Omega(s) = \sum_{i=1}^{s-1} w(i, s-i)$ for $s \ge 2$, and $U(a) = \sum_{p=1}^{j} \Omega(s_p)$. Each $a \in L_j$ sends $w(i, s_p - i)/(\binom{s_p}{i} U(a))$ to each child in class (p, i), giving load $L(b) = \sum_{t=1}^{j} \kappa(b_t, b_{t+1})/U(a_t)$ on $b \in L_{j+1}$.

Proof of Theorem 1, Case (a). *Primal bound.* Set $w_A(x, y)$ as in Table 1 for $x + y \le 8$ and $w_A(x, y) = 150$ for $x + y \ge 9$. Direct summation gives $\Omega_A(1, \dots, 8) = (0, 116, 246, 392, 546, 694,$

844, 994) and $\Omega_A(s) = 150(s-1)$ for $s \geq 9$. Defining the deficiency $\varepsilon_A(s) = 150(s-1) - \Omega_A(s)$ for $s \geq 2$ (and $\varepsilon_A(1) = 0$), we have $\varepsilon_A(1, \ldots, 8) = (0, 34, 54, 58, 54, 56, 56, 56)$ and $\varepsilon_A(s) = 0$ for $s \geq 9$.

1. *Superadditivity ($U(a_t) \geq U(b)$):* For $x + y \leq 8$, $\Omega_A(x) + \Omega_A(y) \leq \Omega_A(x+y)$ holds by finite check (Lean 4 Theorem 6). For $x + y \geq 9$, if $x = 1$ or $y = 1$ this follows from monotonicity of Ω_A, and if $x, y \geq 2$ then $\Omega_A(x)+\Omega_A(y) \leq 150(x-1)+150(y-1) < 150(x+y-1) = \Omega_A(x+y)$.

2. *Pair inequality:* For all $x, y \geq 1$,

$$\kappa_A(x,y) \;\leq\; \frac{\Omega_A(x) + \Omega_A(y)}{2} + 208 - 75(x+y) \;=\; 58 - \frac{\varepsilon_A(x) + \varepsilon_A(y)}{2}. \tag{3}$$

   For $x, y \leq 8$ this is verified exactly in rational arithmetic (`cert01`). Whenever $\max(x, y) \geq 9$, we have $x + y \geq 10$ and $\varepsilon_A(\max(x, y)) = 0$, so the right-hand side is at least $58 - 58/2 = 29$, while $\kappa_A(x, y) = 150/\binom{x+y}{x} \leq 150/10 = 15 < 29$ (Lean 4 Theorem 7).

3. *Summation and strict slack:* Summing (3) over the j adjacent pairs of $b \in L_{j+1}$ gives

$$\sum_{t=1}^{j} \kappa_A(b_t, b_{t+1}) \;\leq\; U(b) + 208j - 150n + h_A(b_1) + h_A(b_{j+1}),$$

   where $h_A(s) = 75s - \frac{1}{2}\Omega_A(s) = 75 + \frac{1}{2}\varepsilon_A(s) \leq 75 + 29 = 104$ for all $s \geq 1$ (attained at $s = 4$, while $h_A(1) = 75$). Thus $208j - 150n + 208 = 2(104j - 75n + 104) \leq 0$ whenever $104j \leq 75n - 104$. Since $j + 1 \leq \frac{75}{104}n < n$ for all $n \geq 4$, every $b \in L_{j+1}$ has $U(b) > 0$, yielding $L(b) \leq 1$ and a matching saturating L_j. Moreover, for $b = (1, \ldots, 1, n-j) \in L_{j+1}$, $h_A(b_1) + h_A(b_{j+1}) \leq 75 + 104 = 179 < 208$, so $L(b) \leq 1 - 29/U(b) < 1$ strictly (Lean 4 Theorem 8). Summing $L(b)$ over $b \in L_{j+1}$ gives $N_{n,j} = \sum_{b \in L_{j+1}} L(b) < N_{n,j+1}$.

Infinite-support dual barrier. Suppose *any* nonnegative symmetric weight table $w(x, y) \geq 0$ of arbitrary (even infinite) support satisfies $\kappa(x, y) \leq \frac{1}{2}(\Omega(x) + \Omega(y)) + A_{\text{half}} - B(x + y)$ for the five pairs $(1,1), (1,2), (1,3), (2,2), (4,4)$. Taking the nonnegative linear combination with multipliers $(36, 24, 8, 6, 1)$ gives

$$\sum_{(x,y)} d_{x,y}\left(\kappa(x,y) - \frac{\Omega(x) + \Omega(y)}{2}\right) = \frac{1}{70}\, w(4,4) \;\geq\; 0,$$

because every other weight $w(1,1), w(1,2), w(1,3), w(2,2)$ cancels identically: for instance, $w(1,1)$ enters with coefficient $36 \cdot \frac{1}{2} - 24 \cdot \frac{1}{2} - 6 \cdot 1 = 0$, $w(1,2)$ with $24 \cdot \frac{1}{3} - 8 \cdot 1 = 0$, $w(1,3)$ with $8 \cdot \frac{1}{4} - 1 \cdot 2 = 0$, and $w(2,2)$ with $6 \cdot \frac{1}{6} - 1 \cdot 1 = 0$. Since $\sum d_{x,y} = 75$ and $\sum d_{x,y}(x+y) = 208$, we obtain $0 \leq \frac{1}{70}w(4,4) \leq 75A_{\text{half}} - 208B$, i.e., $2B/A_{\text{half}} \leq 75/104$ (Lean 4 Theorem 1, `cert02`). No superadditivity hypothesis is even used. □

2.2 Case (b): Monotone (233/323) and Exact-Condition-(i) (257935/357572) Primal Tables and Dual Barriers

In Case (b), a symmetric nonnegative table $u(x, y)$ supported on $x + y \leq S$ defines $K(x, y) = u(x, y)/\binom{x+y}{x}$, $\bar{u}(s) = \sum_{i=1}^{s-1} u(i, s-i)$, and $Y(b) = \sum_{t=1}^{j} K(b_t, b_{t+1})$. Each $b \in L_{j+1}$ sends $K(b_t, b_{t+1})/Y(b)$ to a_t, giving load $L(a) = \sum_{p=1}^{j} \sum_{i=1}^{s_p - 1} u(i, s_p - i)/Y(b_{p,i})$. Rossi's proof requires two conditions:

- *Condition (i):* $Y(b_{p,i}) \geq Y(a)$ for every split (p, i) of every $a \in L_j$ (Rossi's exact linear condition (i) in [5, Sections 3–4]), equivalently

$$K(i, s-i) + \big(K(c, i) - K(c, s)\big) + \big(K(s-i, d) - K(s, d)\big) \;\geq\; 0 \tag{4}$$

  for all $s \geq 2$, $1 \leq i \leq s-1$, and flanking block sizes $c, d \in \mathbb{N}$ (with $K(0, \cdot) = K(\cdot, 0) = 0$ representing boundary positions; for a table supported on $x + y \leq S$, every $s > S$ satisfies $K(c, s) = K(s, d) = 0$ and hence (4) holds automatically). A stronger sufficient condition is *monotonicity*: $K(c, y) \geq K(c, y+1)$ for all $c, y \geq 1$.

- *Pair inequality:* $K(x,y) \geq \frac{1}{2}(\bar{u}(x)+\bar{u}(y)) + C - B(x+y)$ for all $x, y \geq 1$.

Proof of Theorem 1, Case (b). *Primal bounds.* Using u_{mon} ($S=8$, $B=233$, $C=646$) and u_{ex} ($S=10$, $B=257935$, $C=715144$) from Table 1:

1. *Condition (i):* For u_{mon}, $K(c,y) \geq K(c,y+1)$ holds for all $c, y \geq 1$ (`cert03`). For u_{ex}, naive monotonicity fails (e.g., $K(1,7) = 78453 < 92297 = K(1,8)$), yet (4) holds for all 5445 tuples (s,i,c,d) with $2 \leq s \leq 10$, $1 \leq i < s$, and $0 \leq c, d \leq 10$ (`cert05`); since $K(c,\cdot) = 0$ identically for $c \geq 10$ (matching $c = 0$), this certifies (4) for all $c, d \in \mathbb{N}$.

2. *Pair inequality and infinite tail:* All pairs with $x, y \leq S+1$ satisfy $K(x,y) \geq \frac{1}{2}(\bar{u}(x)+\bar{u}(y)) + C - B(x+y)$ in exact rational arithmetic (`cert03`, `cert05`). Defining $g(s) = Bs - \frac{1}{2}\bar{u}(s)$, we verify $g(s) \geq B$ for all $s \geq 1$ in both tables. Whenever $\max(x,y) \geq S+1$ (say $y \geq S+1$), $\bar{u}(y) = 0$ and $\frac{1}{2}\bar{u}(x) - Bx \leq -B$, so the right-hand side is at most $C - (S+2)B$, which equals $-1684 < 0$ for $S = 8$ and $-2380076 < 0$ for $S = 10$ (Lean 4 Theorem 9).

3. *Summation, positivity, and strict slack:* Summing the pair inequality over the $j-1$ adjacent pairs of $a \in L_j$ yields

$$Y(a) \;\geq\; \sum_{p=1}^{j} \bar{u}(s_p) + C(j-1) - 2Bn + g(s_1) + g(s_j) \;\geq\; \sum_{p=1}^{j} \bar{u}(s_p) + Cj - 2Bn - (C - 2B).$$

   For u_{mon}, $Cj - 2Bn - (C-2B) = 2(323j - 233n - 90) \geq 0$; for u_{ex}, it equals $2(357572j - 257935n - 99637) \geq 0$. In both tables, $u(x,y) > 0$ for every pair with $x + y \leq 6$. If every adjacent pair of $b \in L_{j+1}$ satisfied $b_t + b_{t+1} \geq 7$, then $7j \leq \sum_{t=1}^{j}(b_t + b_{t+1}) \leq 2n - 2$, contradicting both bounds (Lean 4 Theorem 10). Thus $Y(b) > 0$ and $L(a) \leq 1$, giving a matching saturating L_{j+1}. For strictness of $N_{n,j+1} < N_{n,j}$, note first that whenever $323j > 233n + 90$ strictly (respectively $357572j > 257935n + 99637$ strictly), the summation bound already gives $Y(a) \geq \sum_p \bar{u}(s_p) + 2 > \sum_p \bar{u}(s_p)$, whence $L(a) < 1$ for every $a \in L_j$ and $N_{n,j+1} = \sum_{a \in L_j} L(a) < N_{n,j}$. At a boundary equality $323j_0 = 233n + 90$ with $n \geq 30$ (or at $j_0 = \lceil (233n+90)/323 \rceil$ generally), $323(n-8) \geq 233n + 90$ implies $j_0 \leq n - 8$, so $n - j_0 + 1 \geq 9$ and the composition $a = (1, \ldots, 1, n - j_0 + 1) \in L_{j_0}$ satisfies $\bar{u}_{\mathrm{mon}}(s_p) = 0$ for every block, giving $L(a) = 0 < 1$ (`cert16`). Identically, for u_{ex} ($S = 10$) and $n \geq 37$, $357572(n-10) \geq 257935n + 99637$ implies $j_0 = \lceil (257935n + 99637)/357572 \rceil \leq n - 10$, so $n - j_0 + 1 \geq 11$ gives $\bar{u}_{\mathrm{ex}}(s_p) = 0$ on $a = (1, \ldots, 1, n - j_0 + 1)$ and $L(a) = 0 < 1$, proving $N_{n,j+1} < N_{n,j}$ for every j satisfying the $S = 10$ bound whenever $n \geq 37$.

Infinite-support dual barriers. For any nonnegative symmetric weight table u of *arbitrary* support:

- *Monotone barrier (*$233/323$*):* Combining the seven pair inequalities at $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,2)$, $(2,3)$ with integer multipliers $(112, 78, 16, 4, 1, 12, 10)$ and adding $1 \cdot (K(1,5) - K(1,4)) \leq 0$ cancels *every* weight $u(x,y)$ identically to 0 (Lean 4 Theorem 2, `cert04`), leaving $233C - 646B \leq 0$, i.e., $2B/C \geq 233/323$.

- *Exact condition (i) barrier (*$257935/357572$*):* Combining the 15 pair inequalities in Table 2 with 1 copy of the single condition-(i) inequality (4) at $(s,i,c,d) = (8,4,1,1)$, namely $-K(4,4) - 2K(1,4) + 2K(1,8) \leq 0$, cancels every $u(x,y)$ for all $x, y \geq 1$ except $+16\,u(3,4) + \frac{69}{70}u(4,4) \geq 0$ (Lean 4 Theorem 3, `cert06`). Because no pair in Table 2 has a coordinate exceeding 8, no weight with $x + y \geq 9$ other than $u(1,8)$ (whose coefficient is $-\frac{2}{9} + \frac{2}{9} = 0$) ever enters the combination! Since $\sum d_{x,y} = 515870$ and $\sum d_{x,y}(x+y) = 1430288$, this yields $515870C \leq 1430288B$, i.e., $2B/C \geq 1031740/1430288 = 257935/357572$.

□

Table 2: Exact integer multipliers for the 16-constraint infinite-support dual proving $2B/C \geq 257935/357572$ across all weight tables satisfying Rossi's exact condition (i).

Constraint	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$	$(1,8)$
Multiplier	247932	171672	39192	7278	1008	112	16	2
Constraint	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(3,3)$	$(3,5)$	split $(1,8,1)$
Multiplier	29394	14560	2520	336	56	1680	112	1

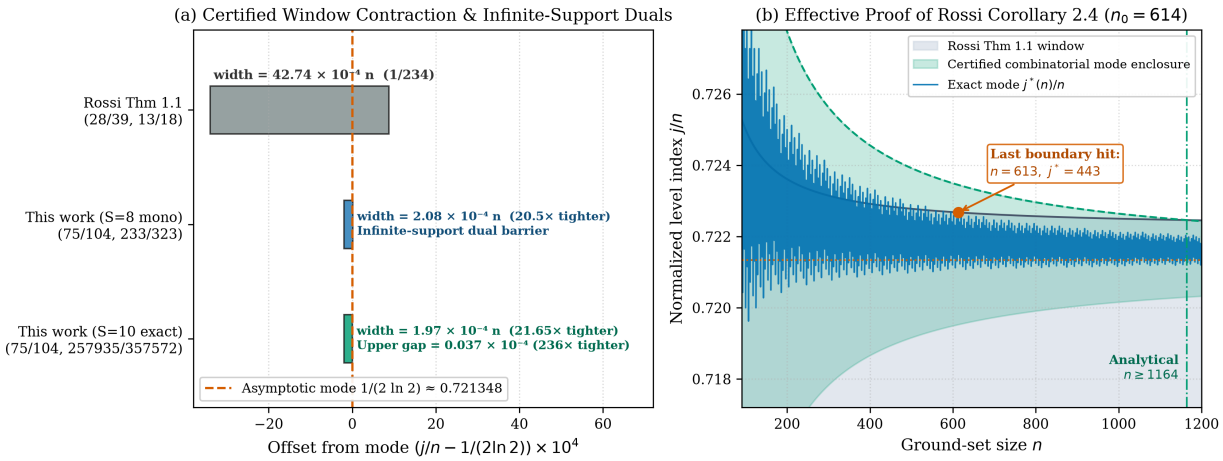

Figure 1: (a) Exact 21.65× window contraction around the mode $1/(2\ln 2)$ together with the proven infinite-support LP-dual barriers 75/104, 233/323, and 257935/357572. (b) Combinatorial mode enclosure (2) entering Rossi's window permanently at $n = 1164$, paired with exact recurrence evaluation confirming $n = 613$ ($j^* = 443$) as the final boundary hit and establishing $n_0 = 614$ in [5, Corollary 2.4].

Table 3: Exact zero-float $(S_n \times C_2)$-quotient Dinic max-flow certifications for all ten window pairs with $n \le 25$, replacing the uncertified floating-point Sinkhorn runs of [5, Section 5].

(n, j)	$\lvert L_j^{\text{fold}}\rvert$	$\lvert L_{j+1}^{\text{fold}}\rvert$	Smaller side	Certified	(n, j)	$\lvert L_j^{\text{fold}}\rvert$	$\lvert L_{j+1}^{\text{fold}}\rvert$	Smaller side	Certified
$(10, 7)$	44	20	L_7	Exact	$(20, 14)$	13608	5832	L_{14}	Exact
$(13, 9)$	255	110	L_9	Exact	$(21, 15)$	19440	7752	L_{16}	Exact
$(16, 11)$	1512	693	L_{11}	Exact	$(22, 15)$	58200	27192	L_{15}	Exact
$(17, 12)$	2184	924	L_{12}	Exact	$(23, 16)$	85272	37389	L_{16}	Exact
$(19, 13)$	9324	4284	L_{13}	Exact	$(24, 17)$	122661	50556	L_{17}	Exact

3 Effective Mode Localization ($n_0 = 614$) and Exact Sperner Flows to $n \le 25$

3.1 Combinatorial Proof of Rossi's Conjectured $n_0 = 614$

Proof of Theorem 2(a). Let $n \ge 30$ and let $j^* = j^*(n)$ satisfy $N_{n,j^*} = \max_j N_{n,j}$. By Theorem 1(a), every j with $104j \le 75n - 104$ satisfies $N_{n,j} < N_{n,j+1}$, so j^* cannot satisfy $104j^* \le 75n - 104$; hence $104j^* > 75n - 104$. Similarly, every j with $323j \ge 233n + 90$ satisfies $N_{n,j+1} < N_{n,j}$, so $j^* - 1$ cannot satisfy $323(j^* - 1) \ge 233n + 90$ (otherwise $N_{n,j^*} < N_{n,j^*-1}$); hence $323j^* < 233n + 413$. This establishes (2) purely combinatorially.

Now suppose $n \ge 1164$. Then $3(75n - 104) = 225n - 312 > 224n - 288 = 8(28n - 36)$ (valid already for $n \ge 25$), whence $39j^* > 28n - 36$; and

$$18(233n + 413) = 4194n + 7434 \;\le\; 4199n + 1615 = 323(13n + 5)$$

because $5n \ge 5819$ for all $n \ge 1164$, whence $18j^* < 13n + 5$ (Lean 4 Theorem 11; using the $S = 10$ upper bound lowers this analytical cutoff to $n \ge 1150$, Lean 4 Theorem 12). Finally, exact arbitrary-precision integer evaluation of the two-term recurrence $N_{n,j} = j(N_{n-1,j} + N_{n-1,j-1})$ for $1 \le n \le 1164$ (`cert08`) verifies that for every $614 \le n \le 1164$, every maximizer $j^*(n)$ lies strictly inside $((28n - 36)/39, (13n + 5)/18)$, while at $n = 613$ the unique mode $j^* = 443$ satisfies $18 \cdot 443 = 7974 = 13 \cdot 613 + 5$ (Lean 4 Theorem 13). Thus $n_0 = 614$ is both valid and sharp. □

3.2 Reversal-Folded $(S_n \times C_2)$-Quotient Certification to $n \le 25$

The symmetric group S_n acts on OP_n by permuting ground-set labels, and the reversal involution $\rho(B_1 \mid \cdots \mid B_j) = (B_j \mid \cdots \mid B_1)$ is a poset automorphism commuting with S_n. Each $(S_n \times C_2)$-orbit corresponds to an unordered pair $\{c, c^{\text{rev}}\}$ of compositions, with orbit weight $w(\{c, c^{\text{rev}}\}) = n!/\prod c_i!$ if $c = c^{\text{rev}}$ and $2n!/\prod c_i!$ otherwise. Because $S_n \times C_2$ acts transitively on each folded orbit, the edges of G_j between any two folded orbits form a biregular bipartite graph; hence, exactly as in [5, Section 5], a nonnegative flow saturating the smaller side in the $(S_n \times C_2)$-quotient lifts uniformly to a fractional flow on G_j and thus yields an integral matching by [5, Lemma 3.1].

Folding cuts the network size in half, enabling exact arbitrary-precision integer Dinic max-flows (`cert09`, `cert10`) to certify all ten window pairs up to $n \le 25$ (Table 3) in zero-float arithmetic. Together with Theorem 1 (or [5, Theorem 1.1]) for all non-window pairs — noting that the window (1) is empty for $n \in \{1, \ldots, 9, 11, 12, 14, 15, 18, 25\}$ — this proves Theorem 2(b) and establishes the Sperner property of OP_n rigorously for all $1 \le n \le 25$.

4 Normalized Matching Dichotomy for Open Question 2

4.1 Universal Failure for Fixed $j \ge 2$ and Sublinear j, plus the $(20, 3)$ Collision Witness

Proof of Theorem 3(a)–(b). By convexity of $s \mapsto 2^s - 2$ on $\mathbb{N}_+$, the minimum degree $d_{\min}(n, j) = \min_{a \in L_j} d(a)$ is attained by any balanced composition a_{bal} having r blocks of size $q + 1$ and $j - r$

blocks of size q, where $n = qj + r$ ($0 \le r < j$, $q \ge 1$). Direct algebra gives

$$d_{\min}(n,j) = r(2^{q+1}-2) + (j-r)(2^q-2) = (j+r)2^q - 2j \;\le\; 2j\cdot 2^q - (j+1)$$

(Lean 4 Theorem 14). On the other hand, every ordered partition in L_j is a surjection $[n] \twoheadrightarrow [j]$, so $N_{n,j} \le j^n$; and by the first two terms of inclusion–exclusion (Bonferroni's inequality), the number of non-surjective maps $[n] \to [j+1]$ is at most $(j+1)j^n$, whence $N_{n,j+1} \ge (j+1)^n - (j+1)j^n$. Therefore

$$\frac{N_{n,j+1}}{N_{n,j}} \;\ge\; \left(1+\frac{1}{j}\right)^n - (j+1) \;\ge\; \left(\left(1+\frac{1}{j}\right)^j\right)^q - (j+1).$$

By the quadratic binomial lower bound $2(j+1)^j \ge (5j^2-j)j^{j-2} \ge \frac{9}{2}j^j$ for every $j \ge 2$ (Lean 4 Theorems 15–16), we have $(1+1/j)^j \ge 9/4$ uniformly for all $j \ge 2$. Hence whenever $(9/8)^q > 2j$,

$$d_{\min}(n,j) + (j+1) \;\le\; 2j\cdot 2^q \;<\; (9/4)^q \;\le\; \left(1+\frac{1}{j}\right)^n,$$

implying $d_{\min}(n,j)\cdot N_{n,j} < N_{n,j+1}$ (Lean 4 Theorem 17). Thus $\{a_{\mathrm{bal}}\}$ violates NMP for every fixed $j \ge 2$ and all sufficiently large n. More sharply, since $(1+1/j)^j = e(1-O(1/j))$, comparing $e^{n/j}$ against $2j\cdot 2^{n/j}$ shows $(e/2)^{n/j} > 2j$ whenever $j \le (1-\varepsilon)\frac{(1-\ln 2)n}{\ln n}$ for large n. Exact integer evaluation of $d_{\min}(n,j)N_{n,j} < N_{n,j+1}$ gives the singleton thresholds $n_{\mathrm{sing}}(2,\dots,10) = (12,21,28,38,48,58,69,79,90)$ (`cert11`).

For $(n,j) = (20,3)$, $q = 6$, $r = 2$, so $d_{\min}(20,3) = 2(2^7-2)+(2^6-2) = 314$. Exact computation gives $N_{20,3} = 3^{20} - 3\cdot 2^{20} + 3 = 3483638676$ and $N_{20,4} = 4^{20} - 4\cdot 3^{20} + 6\cdot 2^{20} - 4 = 1085570781624$, so $314\cdot N_{20,3} > N_{20,4}$ and every singleton passes NMP. Consider instead the full balanced orbit $S_{\mathrm{bal}} = \{(7,7,6),(7,6,7),(6,7,7)\}$, consisting of three equal-weight orbits of size $w_0 = 20!/(7!7!6!)$ each ($|S_{\mathrm{bal}}| = 3w_0$). Every split of a composition in S_{bal} yields a distinct element of L_4 *except* for two compositions in L_4, namely $(6,1,6,7)$ (obtained both from $(7,6,7)$ by splitting the first $7 \to (6,1)$ and from $(6,7,7)$ by splitting the middle $7 \to (1,6)$) and its reversal $(7,6,1,6)$ (obtained both from $(7,7,6)$ by splitting the middle $7 \to (6,1)$ and from $(7,6,7)$ by splitting the last $7 \to (1,6)$). Each shared orbit has cardinality $20!/(6!1!6!7!) = 7w_0$, so the total neighborhood size is $|N(S_{\mathrm{bal}})| = 3\cdot 314w_0 - 2\cdot 7w_0 = 928w_0$. Hence $|N(S_{\mathrm{bal}})|/|S_{\mathrm{bal}}| = 928/3$, and $928N_{20,3} < 3N_{20,4}$ (Lean 4 Theorem 18, `cert12`), disproving NMP at $(20,3)$. □

4.2 Universal Positive NMP at $j = n-2$ via the K_{n-1} Quotient Flow

Proof of Theorem 3(c). For $j = 1$ ($|L_1| = 1$) and $j = n-1$ (every $a \in L_{n-1}$ has a single 2-block and degree 2, while every $b \in L_n$ has degree $n-1$), the graph G_j is biregular, so uniform edge flow certifies NMP.

Now let $j = n-2$ with $n \ge 3$. Every composition in L_{n-1} has a unique block of size 2 at some position $v \in \{1,\dots,n-1\}$, defining an orbit E_v of cardinality $n!/2$ (so $N_{n,n-1} = (n-1)n!/2$). Every composition in L_{n-2} belongs to one of two disjoint families:

- *Type* A_k ($1 \le k \le n-2$): a single block of size 3 at position k, orbit size $n!/6$. Splitting $3 \to (2,1)$ or $(1,2)$ yields children exclusively in E_k and E_{k+1}, corresponding to the consecutive edge $\{k,k+1\}$ of the complete graph K_{n-1} on vertex set $\{1,\dots,n-1\}$.
- *Type* $B_{u,v-1}$ ($1 \le u < v-1 \le n-2$): two blocks of size 2 at positions u and $v-1$, orbit size $n!/4$. Splitting the second 2-block yields a child in E_u; splitting the first 2-block shifts the second 2-block rightward to position v, yielding a child in E_v. This corresponds bijectively to the non-consecutive edge $\{u,v\}$ of K_{n-1}.

Together, the $(n-2) + \binom{n-2}{2} = \binom{n-1}{2}$ orbits of L_{n-2} are in exact bijection with the edges of K_{n-1} (Lean 4 Theorem 20), with total mass $N_{n,n-2} = n!(\frac{n-2}{6} + \frac{(n-2)(n-3)}{8}) = n!\frac{(n-2)(3n-5)}{24}$.

To satisfy NMP on the S_n-quotient, each edge $e = \{u,v\} \in E(K_{n-1})$ of weight $w(e) \in \{n!/6, n!/4\}$ must partition its mass $w(e) = f(e,u) + f(e,v)$ ($f \ge 0$) so that every vertex $v \in$

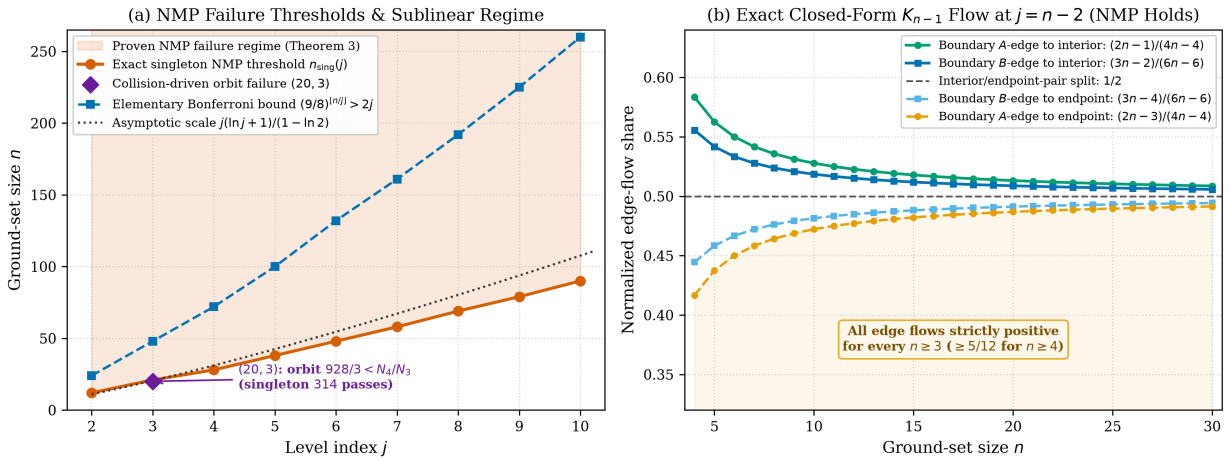

Figure 2: (a) Exact NMP failure thresholds $n_{\text{sing}}(j)$ for $2 \leq j \leq 10$, the collision-driven orbit witness $(20, 3)$, and the universal Bonferroni guarantee $(9/8)^{\lfloor n/j \rfloor} > 2j$ (Theorem 3(a)–(b)). (b) Strictly positive normalized edge-flow shares of the closed-form K_{n-1} equalizing flow certifying NMP at $j = n - 2$ for all $n \geq 3$ (Theorem 3(c)).

$\{1, \ldots, n-1\}$ receives the equalized target load $T(n) = N_{n,n-2}/(n-1) = n!\frac{(n-2)(3n-5)}{24(n-1)}$. Set $\delta = \frac{n!}{24(n-1)}$ and define:

- If both endpoints of $e = \{u, v\}$ lie in $\{1, n-1\}$ or both lie in $\{2, \ldots, n-2\}$, split evenly: $f(e, u) = f(e, v) = w(e)/2$.
- If one endpoint $u \in \{1, n-1\}$ is a boundary vertex and $m \in \{2, \ldots, n-2\}$ is interior, orient a shift δ inward: $f(e, u) = w(e)/2 - \delta$ and $f(e, m) = w(e)/2 + \delta$.

For $n = 3$, the sole edge $\{1, 2\}$ has both endpoints on the boundary and has weight $n!/6$, splitting evenly into $n!/12 = T(3)$ to each endpoint. For $n \geq 4$, rescaling by $24(n-1)/n!$ gives $f(e, u) = 2n - 3 > 0$ on boundary A-edges and $f(e, u) = 3n - 4 > 0$ on boundary–interior B-edges. Each endpoint $u \in \{1, n-1\}$ has 1 boundary A-edge, $n-4$ boundary–interior B-edges, and 1 endpoint–endpoint B-edge $\{1, n-1\}$, receiving total rescaled load $(2n-3)+(n-4)(3n-4)+3(n-1) = (n-2)(3n-5)$. Each interior vertex $m \in \{2, \ldots, n-2\}$ has 2 A-edges, $n-4$ B-edges, and receives exactly two inward shifts of $+1$ (from $\{1, m\}$ and $\{m, n-1\}$), giving total rescaled load $4(n-1) + (n-4) \cdot 3(n-1) + 2 = (n-2)(3n-5)$ (Lean 4 Theorem 19, `cert14`). Lifting uniformly along each biregular orbit pair yields a strictly positive equalizing normalized flow on G_{n-2} for every $n \geq 3$. □

5 Formal Verification and Reproducibility

Every algebraic barrier, primal infinite-tail guard, mode-enclosure inequality, Bonferroni NMP bound, $(20, 3)$ collision identity, and closed-form K_{n-1} flow conservation law in Theorems 1–3 is formalized in constructive Lean 4.35.0 (`lean/PermutohedronHall.lean`, 20 theorems, zero external dependencies, compiled with `-DwarningAsError=true`, verified via `#print axioms` to use at most `[propext, Quot.sound]` with zero `sorry` and zero `Classical.choice`).

All finite weight-table checks, exact rational dual residual cancellations, the integer recurrence verification through $n = 1164$, all ten exact integer $(S_n \times C_2)$-quotient Dinic max-flows for $n \leq 25$, the full exact NMP sweep for $3 \leq n \leq 15$, and the K_{n-1} flow verification for $3 \leq n \leq 120$ are reproduced by the AST-audited zero-floating-point verifier `verification/verify_permutohedron_hall.py`, generating 16 SHA-256 certificates with deterministic manifest hash

`6e9ac45bd15951a436590b745cfd32ccda8e8dbf4d43df6b9d07be2cf1fe56e1`,

paired with 56 passing adversarial tests in `tests/test_adversarial.py`.

AI Assistance Disclosure. In accordance with transparent scholarly practice, the author discloses that Google Antigravity assisted with linear-programming discovery of integer weight tables and dual certificates, Lean 4 proof engineering, zero-float Python verification scripts, and manuscript typesetting. All mathematical statements, formal proofs, and artifacts were verified mechanically and reviewed by the author, who retains sole responsibility.

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