Endpoint Parity Loss in a Bessel Wronskian:
an exact obstruction to kernel-and-anchor proofs of global ratio monotonicity

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August 2026

Abstract

For $\beta > 0$ write $I_m = I_m(\beta)$ and define

$$a_m = I_m^2\big((m-1)I_{m-1}^2 + (m+1)I_{m+1}^2\big), \qquad b_m = mI_m^4,$$

with sine series $F_A(t) = \sum_{m\geq 1} a_m \sin(mt)$ and $F_B(t) = \sum_{m\geq 1} b_m \sin(mt)$. The associated Wronskian $\mathcal{W} = 2(F_A'F_B - F_AF_B')$ is negative exactly when F_A/F_B is decreasing. We isolate what can and cannot be proved from two natural inputs: the Neumann convolution kernel $I_0(2\beta\sin(\phi/2))$ and the small-coupling anchor $(-4096/\beta^{10})\mathcal{W} \to 4\sin^3 t$.

First, the convolution does prove the denominator statement $F_B(t) > 0$ for $0 < t < \pi$. Second, the endpoint is governed by two alternating quantities, c_3 and B_π, through the exact law $\mathcal{W}(\pi - d) = -4c_3B_\pi d^3 + O(d^5)$. We prove $B_\pi > 0$, derive integral representations, and establish the logarithmic cancellation law

$$\lim_{\beta\to\infty} -\frac{1}{\beta}\log\frac{|c_3B_\pi|}{c_3^{\mathrm{abs}}B_\pi^{\mathrm{abs}}} = 8 - 4\sqrt{2}.$$

Thus a positive-term majorant loses an exponentially decisive factor. Finally, we give a smooth one-parameter perturbation theorem: one may keep F_B and its kernel unchanged, preserve positivity and strict coefficient-ratio ordering, and preserve every jet at $\beta = 0$, while choosing either sign of the endpoint cubic coefficient. Consequently those structural data, even taken together, do not imply global Wronskian negativity. This is a no-go theorem for a proof architecture, not a counterexample to the Bessel conjecture. A short high-precision kill-test accompanies the paper; no computer-assisted inequality is used in the proofs.

1 The question and the status of the result

The following ratio-monotonicity statement arises after an exact Wronskian reduction of a surface-positivity problem.

Conjecture 1 (Bessel Wronskian conjecture)**.** For every $\beta > 0$,

$$F_B(t) > 0, \qquad \left(\frac{F_A}{F_B}\right)'(t) < 0 \quad (0 < t < \pi).$$

Equivalently, $F_B > 0$ and $\mathcal{W} < 0$ on $(0, \pi)$.

The first assertion will be proved exactly below. The second assertion is *not* proved here. Our main result instead separates a valid part of the proposed kernel route from a fatal loss at $t = \pi$. The distinction is important: the perturbations in Theorem 7 do not obey all modified-Bessel recurrences, and hence do not refute Conjecture 1.

The Wronskian identity itself is elementary. If $F_x(t) = \sum_{m\geq 1} x_m \sin(mt)$, then

$$2(F_a'F_b - F_aF_b') = \sum_{m<n} (a_m b_n - a_n b_m)\big[(m-n)\sin((m+n)t) + (m+n)\sin((n-m)t)\big]. \quad (1)$$

It follows by expanding the products and antisymmetrizing. Throughout, all Bessel series and their differentiated series converge locally uniformly; the rearrangements below are therefore legitimate.

2 The convolution kernel closes $F_B > 0$

The Neumann addition theorem gives the two parity forms

$$K_+(\phi) = I_0\left(2\beta\cos\frac{\phi}{2}\right) = \sum_{m\in\mathbb{Z}} I_m(\beta)^2 e^{im\phi}, \quad (2)$$

$$K_-(\phi) = I_0\left(2\beta\sin\frac{\phi}{2}\right) = \sum_{m\in\mathbb{Z}} (-1)^m I_m(\beta)^2 e^{im\phi}. \quad (3)$$

For periodic convolution normalized by $(f * g)(t) = (2\pi)^{-1}\int_0^{2\pi} f(s)g(t-s)\,ds$, both squares give

$$P_4 := K_+ * K_+ = K_- * K_- = \sum_{m\in\mathbb{Z}} I_m(\beta)^4 e^{imt}, \qquad F_B = -\frac{1}{2}P_4'. \quad (4)$$

Lemma 2 (strict circular convolution)**.** *If q is a nonconstant, even, continuous function on the circle that is strictly decreasing with geodesic distance from 0, then $q * q$ is strictly decreasing on $(0,\pi)$ and $(q*q)' < 0$ there whenever q is smooth.*

Proof. Use the layer-cake representation of $q - \min q$ by centered circular arcs. The convolution is a positive superposition of overlap lengths of two such arcs whose centers are separated by t. Each overlap length is nonincreasing for $0 < t < \pi$. At any fixed t there is an open set of pairs of level arcs whose endpoints meet in the unsaturated regime; on this set the derivative of the overlap is strictly negative. Strict decrease of q makes this set have positive level measure. Differentiation is allowed for smooth q and yields the pointwise derivative claim. □

Theorem 3. *For every $\beta > 0$, $F_B(t) > 0$ for $0 < t < \pi$.*

Proof. On $[0,\pi]$, $K_+(t) = I_0(2\beta\cos(t/2))$ is strictly decreasing, because $I_0'(x) = I_1(x) > 0$ for $x > 0$. Apply Lemma 2 to (4): $P_4'(t) < 0$, and hence $F_B(t) = -P_4'(t)/2 > 0$. □

This proof uses the kernel at full strength and contains no alternating majorization. The obstruction begins with the numerator and becomes sharp at the opposite endpoint.

3 The exact small-coupling anchor

The fixed-order Taylor expansions of I_m give

$$a_1 = \frac{\beta^6}{128} + O(\beta^8), \qquad b_1 = \frac{\beta^4}{16} + O(\beta^6),$$

$$a_2 = \frac{\beta^6}{256} + O(\beta^8), \qquad b_2 = \frac{\beta^8}{2048} + O(\beta^{10}).$$

All remaining modes enter later. Substitution in the Wronskian yields, uniformly for $t \in [0, \pi]$,

$$\mathcal{W}(t;\beta) = -\frac{\beta^{10}}{1024}\sin^3 t + O(\beta^{12}), \qquad -\frac{4096}{\beta^{10}}\mathcal{W}(t;\beta) \longrightarrow 4\sin^3 t. \tag{5}$$

Thus the parabolic anchor has the desired sign for every fixed interior t when β is small. Uniformity of the absolute error in (5), however, does not supply a relative estimate near $t = \pi$, where the leading term has a triple zero.

4 The endpoint coefficient

Put $d = \pi - t$. The linear coefficient of $F_A(\pi - d)$ vanishes by an edgewise telescoping:

$$\sum_{m\geq 1}(-1)^{m+1} m a_m = 0. \tag{6}$$

Indeed, the two appearances of $I_m^2 I_{m+1}^2$ have coefficients

$$(-1)^{m+1} m(m+1) \quad \text{and} \quad (-1)^{m+2}(m+1)m,$$

which cancel. Define

$$c_3(\beta) = \frac{1}{6}\sum_{m\geq 1}(-1)^{m+1} m(m+1)(2m+1) I_m^2 I_{m+1}^2, \tag{7}$$

$$B_\pi(\beta) = \sum_{m\geq 1}(-1)^{m+1} m^2 I_m^4. \tag{8}$$

Taylor expansion, followed by the same edge collection used in (6), gives

$$F_A(\pi - d) = c_3 d^3 + O(d^5), \qquad F_B(\pi - d) = B_\pi d + O(d^3), \tag{9}$$

and consequently

$$\mathcal{W}(\pi - d) = -4c_3 B_\pi d^3 + O(d^5), \qquad \lim_{t\uparrow\pi}\frac{-\mathcal{W}(t)}{\sin^3 t} = 4c_3 B_\pi. \tag{10}$$

The denominator coefficient has a positive integral. From $P_4''(\pi) = 2B_\pi$, differentiating the convolution and integrating by parts once gives

$$B_\pi = \frac{\beta^2}{\pi}\int_0^{\pi/2} \sin u \cos u \, I_1(2\beta\cos u) I_1(2\beta \sin u)\, du > 0. \tag{11}$$

Therefore the desired endpoint sign is exactly the nontrivial statement $c_3(\beta) > 0$.

For later use, set $f(x) = I_1(x)/2$, $y = 2\beta \sin u$ and $z = 2\beta\cos u$. Applying (3), differentiating the correlation, and integrating by parts gives the half-range identity

$$c_3 = \frac{1}{6\pi}\int_0^{\pi/2} J(u,\beta)\, du, \tag{12}$$

where

$$J = -z^2 f'(y) f'(z) + zy^2 f'(y) f''(z) + y f(y) f'(z). \tag{13}$$

The integrand is not pointwise positive. This is not a numerical accident: for every sufficiently large β, its negative boundary region and positive saddle region are both nonempty. Hence (12) is a signed cancellation identity rather than a positive-kernel proof of $c_3 > 0$.

5 The exponential parity loss

Introduce the positive envelopes

$$c_3^{\mathrm{abs}} = \frac{1}{6}\sum_{m\ge 1} m(m+1)(2m+1) I_m^2 I_{m+1}^2, \tag{14}$$

$$B_\pi^{\mathrm{abs}} = \sum_{m\ge 1} m^2 I_m^4. \tag{15}$$

We record two elementary estimates used to keep the asymptotic proof uniform. They also make explicit the tail control that is hidden if one invokes Laplace's method only formally.

Lemma 4 (fixed-order and moment bounds)**.** *For fixed integer $n \ge 0$ and $j = 0, 1, 2$, as $w \to \infty$,*

$$I_n^{(j)}(w) = \frac{e^w}{\sqrt{2\pi w}}\left(1 + O_{n,j}(w^{-1})\right). \tag{16}$$

The remainder is uniform for $w \ge w_0 > 0$, after increasing its constant, and $|I_n^{(j)}(w)| \le e^w$ for $w \ge 0$. Moreover, for each integer $k \ge 0$ there is C_k such that

$$S_k(\beta) := \sum_{m\ge 0} (m+1)^k I_m(\beta) \le C_k (1+\beta)^k e^\beta \qquad (\beta > 0). \tag{17}$$

Proof. For integer order,

$$I_n(w) = \frac{1}{\pi}\int_0^\pi e^{w\cos\theta}\cos(n\theta)\, d\theta.$$

Differentiating under the integral inserts $(\cos\theta)^j$. The standard one-saddle expansion at $\theta = 0$, with the complement bounded by $e^{w\cos\delta}$, proves (16) with a differentiable remainder. Taking absolute values in the same integral gives the global bound.

For (17), put $s = (1+\beta)^{-1}$. The elementary inequality $(m+1)^k \le C_k s^{-k} e^{sm}$ and positivity give

$$S_k(\beta) \le C_k s^{-k} \sum_{m\ge 0} e^{sm} I_m(\beta) \le C_k s^{-k} e^{\beta\cosh s}.$$

Since $\beta(\cosh s - 1)$ is uniformly bounded, this is (17). □

Theorem 5 (endpoint cancellation exponent)**.** *As $\beta \to \infty$,*

$$c_3(\beta) = \frac{\beta^{3/2} e^{2\sqrt{2}\beta}}{24\, 2^{1/4} \pi^{3/2}}\left(1 + O(\beta^{-1})\right), \tag{18}$$

$$B_\pi(\beta) = \frac{\beta^{1/2} e^{2\sqrt{2}\beta}}{4\, 2^{3/4} \pi^{3/2}}\left(1 + O(\beta^{-1})\right). \tag{19}$$

Moreover

$$\lim_{\beta\to\infty} \frac{1}{\beta}\log c_3^{\mathrm{abs}} = \lim_{\beta\to\infty}\frac{1}{\beta}\log B_\pi^{\mathrm{abs}} = 4, \tag{20}$$

and therefore

$$\boxed{\lim_{\beta\to\infty} -\frac{1}{\beta}\log\frac{|c_3 B_\pi|}{c_3^{\mathrm{abs}} B_\pi^{\mathrm{abs}}} = 8 - 4\sqrt{2}.} \tag{21}$$

Proof. We separate the saddle, its tails, and the positive envelopes.

The signed c_3 saddle. In (12), the phase is $y+z=2\beta(\sin u+\cos u)$. It has a unique nondegenerate maximum at $u_0=\pi/4$. Write

$$x=\sqrt{2}\beta, \qquad u=u_0+v, \qquad y=x(\cos v+\sin v), \quad z=x(\cos v-\sin v).$$

On the central window $|v|\le x^{-2/5}$, both y and z are comparable to x. Lemma 4, applied to $f=I_1/2$, gives uniformly there

$$J(u,\beta)=\frac{e^{y+z}}{8\pi}y^{3/2}z^{1/2}(1+O(x^{-1})). \tag{22}$$

Indeed, $zy^2f'(y)f''(z)$ is the displayed term, while the ratios of $z^2f'(y)f'(z)$ and $yf(y)f'(z)$ to it are respectively $O(x^{-1})$ and $O(x^{-2})$.

The phase is exactly $y+z=2x\cos v$. Set $v=s/\sqrt{x}$. Its expansion is

$$2x\cos(s/\sqrt{x})=2x-s^2+O(s^4/x).$$

The amplitude $y^{3/2}z^{1/2}/x^2$ is smooth near zero. Its odd part integrates to zero on the symmetric central window because the phase is even; its even part is $1+O(v^2)$. Splitting further at $|s|=x^{1/10}$, using a Gaussian majorant inside and $e^{-cx^{1/5}}$ outside, gives the one-term expansion with a relative $O(x^{-1})$ remainder:

$$\int_{|v|\le x^{-2/5}} J(u_0+v,\beta)\,dv=\frac{x^2e^{2x}}{8\pi}\sqrt{\frac{\pi}{x}}(1+O(x^{-1})). \tag{23}$$

For completeness, the omitted part is exponentially smaller than (23). Lemma 4 and the polynomial factors in (13) imply on the full half interval

$$|J(u,\beta)|\le C(1+\beta)^3e^{y+z}. \tag{24}$$

Choose a fixed $0<\delta<\pi/4$. When $x^{-2/5}\le|v|\le\delta$, $2x-(y+z)=2x(1-\cos v)\ge cx^{1/5}$; when $\delta\le|v|\le\pi/4$, the deficit is at least $c_\delta x$. Thus the two tail integrals are, respectively, a polynomial times $e^{2x-cx^{1/5}}$ and $e^{2x-c_\delta x}$. Combining this with (12) and (23) proves (18), including its positive sign and differentiated remainder.

The signed B_π saddle. The same decomposition applies to (11); its integrand is positive, so no additional cancellation is hidden. At u_0 the prefactor $\sin u\cos u$ equals $1/2$, and $I_1(x)^2=e^{2x}(2\pi x)^{-1}(1+O(x^{-1}))$. After $v=s/\sqrt{x}$ the same even Gaussian and the same two deficit estimates give

$$B_\pi=\frac{\beta^2}{\pi}\frac{1}{2}\frac{e^{2x}}{2\pi x}\sqrt{\frac{\pi}{x}}(1+O(x^{-1})),$$

which is (19).

The absolute envelopes. Any fixed positive summand gives the lower exponential rate 4, since $I_m(\beta)=e^\beta(2\pi\beta)^{-1/2}(1+O(\beta^{-1}))$ for fixed m. For the upper rate, positivity and $m(m+1)(2m+1)\le 6(m+1)^3$ give the explicit product bounds

$$B_\pi^{\mathrm{abs}}\le S_2S_0^3, \qquad c_3^{\mathrm{abs}}\le S_3S_0^3. \tag{25}$$

To see these inequalities, expand the products on the right: for each m, the required diagonal monomial occurs among their nonnegative terms. Lemma 4 now bounds both expressions by a fixed polynomial in β times $e^{4\beta}$. This proves (20) without a tail truncation or an interchange of asymptotic series. Combining the two signed rates $2\sqrt{2}$ with the two absolute rates 4 proves (21). ∎

The rate $8-4\sqrt{2}=2.3431457505\ldots$ is the kill-test for any positive-term or absolute-value barrier. A barrier whose slack decays only algebraically cannot resolve the endpoint sign at large β.

6 A structural no-go theorem

We now show that the loss is logical, not merely a weakness of a particular estimate. Let $p=(2,1,0,\ldots)$. It satisfies

$$\sum_{m\geq 1}(-1)^{m+1}mp_m=0, \qquad -\frac{1}{6}\sum_{m\geq 1}(-1)^{m+1}m^3p_m=1. \tag{26}$$

The next radius packages every inequality that can change under this two-mode perturbation. Write $r_m=A_m/B_m$ and $D_{12}=r_2-r_1$, $D_{23}=r_3-r_2$, and define

$$\rho(\beta)=\min\left\{\frac{A_1}{4},\ \frac{A_2}{2},\ \frac{D_{12}}{2(2/B_1+1/B_2)},\ \frac{B_2D_{23}}{2}\right\}. \tag{27}$$

Under the strict ratio-order hypothesis, ρ is continuous and strictly positive on $(0,\infty)$.

Lemma 6 (explicit perturbation radius)**.** *If $|h(\beta)|<\rho(\beta)$ and $\widetilde{A}=A+h(2,1,0,\ldots)$, then all $\widetilde{A}_m$ are positive and $\widetilde{A}_m/B_m$ is strictly increasing in m at that β. For large β there is a constant $c>0$ such that*

$$\rho(\beta)\geq ce^{4\beta}\beta^{-4}. \tag{28}$$

Proof. Only modes 1 and 2 change. The first two terms in (27) give $A_1+2h>A_1/2$ and $A_2+h>A_2/2$. Moreover

$$\begin{aligned}\widetilde{r}_2-\widetilde{r}_1&=D_{12}+h(1/B_2-2/B_1)>D_{12}/2,\\ \widetilde{r}_3-\widetilde{r}_2&=D_{23}-h/B_2>D_{23}/2.\end{aligned}$$

All later gaps are unchanged.

For fixed m, the standard expansion of I_m gives

$$B_m=\frac{me^{4\beta}}{4\pi^2\beta^2}(1+O(\beta^{-1})), \qquad r_{m+1}-r_m=\frac{4(2m+1)}{\beta^2}+O(\beta^{-3}). \tag{29}$$

Also A_1,A_2 have scale $e^{4\beta}\beta^{-2}$. Substitution in the four entries of (27) proves (28). □

Theorem 7 (structural non-identifiability)**.** *Let $(A_m(\beta),B_m(\beta))=(a_m(\beta),b_m(\beta))$ be the Bessel arrays above, and suppose the known strict ordering $A_m/B_m<A_{m+1}/B_{m+1}$ is included among the structural inputs. There are two families $\widetilde{A}^+$ and $\widetilde{A}^-$ such that:*

1. *B is unchanged; hence the convolution representation and $F_B>0$ are unchanged;*
2. *every $\widetilde{A}^\pm_m$ is positive and $\widetilde{A}^\pm_m/B_m$ remains strictly increasing in m for every $\beta>0$;*
3. *$\widetilde{A}^\pm-A$ is flat at $\beta=0$, so the two families have the same complete small-β asymptotic expansion and in particular the same anchor (5);*
4. *at one common $\beta=\beta_*$, the endpoint cubic coefficients have opposite signs. Thus their Wronskians have opposite signs in a punctured left neighborhood of $t=\pi$.*

Consequently items 1–3 cannot imply global Wronskian negativity.

Proof. For a scalar function h, put

$$\widetilde{A}_m = A_m + h(\beta)p_m, \qquad \widetilde{B}_m = B_m.$$

By (26), the endpoint linear cancellation is preserved and the cubic coefficient becomes $\widetilde{c}_3 = c_3 + h$.

Choose a large center $\beta_* = R$. Theorem 5 and Lemma 6 give

$$\frac{|c_3(R)|}{\rho(R)} = O\Big(R^{11/2}e^{-(4-2\sqrt{2})R}\Big) \longrightarrow 0.$$

Increase R until $|c_3(R)| < \rho(R)/4$, and fix $0 < \varepsilon < \rho(R)/4$.

For $\sigma > 0$, define

$$\psi_{R,\sigma}(\beta) = \exp\left(-\frac{1}{\beta^2} + \frac{1}{R^2} - \frac{2(\beta - R)}{R^3} - \frac{(\beta - R)^2}{\sigma^2}\right), \tag{30}$$

$$h_\pm(\beta) = (-c_3(R) \pm \varepsilon)\psi_{R,\sigma}(\beta). \tag{31}$$

The exponent in (30) has derivative zero at R and second derivative $-6\beta^{-4} - 2\sigma^{-2} < 0$. Hence its unique global maximum is zero at R:

$$0 < \psi_{R,\sigma}(\beta) \le 1, \qquad \psi_{R,\sigma}(R) = 1. \tag{32}$$

It is real analytic on $(0,\infty)$ and extends by 0 to a C^∞ function flat at 0. At R, $\widetilde{c}_3^{\pm}(R) = c_3(R) + h_\pm(R) = \pm\varepsilon$.

We now choose one σ that works on the entire half-line. Put $q = |c_3(R)| + \varepsilon < \rho(R)/2$. By continuity there is a neighborhood U of R on which $q < \rho(\beta)$; by (32), $|h_\pm| \le q < \rho$ on U.

Near zero, each entry in (27) is a positive quotient or product of fixed-order Bessel functions and strict ratio gaps. Their convergent Taylor–Laurent expansions have a first nonzero positive coefficient. Consequently there are $C_0 > 0$ and an integer N such that $\rho(\beta) \ge C_0\beta^N$ for all sufficiently small β. Equation (30) is $O(e^{-1/(2\beta^2)})$ there, uniformly for σ in a bounded interval, so $q\psi < \rho$ near zero.

Near infinity, (28) applies, whereas

$$\psi_{R,\sigma}(\beta) = \exp(-\beta^2/\sigma^2 + O_{R,\sigma}(\beta)).$$

Thus $q\psi < \rho$ for all sufficiently large β. The portion left after removing these two tails and U is compact and has positive distance from R. On it ρ has a positive minimum while $\psi_{R,\sigma} \to 0$ uniformly as $\sigma \downarrow 0$. Taking σ sufficiently small therefore gives $|h_\pm(\beta)| < \rho(\beta)$ for every $\beta > 0$. Lemma 6 proves items 1–2, flatness proves item 3, and (10) proves item 4. □

Remark 8 (exact scope). The theorem rules out any argument whose hypotheses see only the B-kernel, positivity of the coefficients, strict ordering of A_m/B_m, and the full small-coupling jet. A successful proof of Conjecture 1 must use additional Bessel structure that rejects the perturbation, for example an exact signed identity, recurrence-compatible differential inequality, or a cancellation-preserving certificate.

7 Reproducible kill-test

The companion script `scripts/wronskian_endpoint_kill_test.py` is a diagnostic, not an interval proof. It uses 180-decimal arithmetic, truncates well beyond the active Bessel modes, and checks the exact normalizations before measuring the cancellation. A representative run is:

β	endpoint ratio	log(ratio)	next fitted rate
8	$3.86868117\,10^{-5}$	-10.1600118	2.1068310
16	$1.85217583\,10^{-12}$	-27.0146600	2.2301610
32	$5.90114568\,10^{-28}$	-62.6972361	2.2878524
64	$9.45545116\,10^{-60}$	-135.9085142	—

Here "endpoint ratio" is $|c_3 B_\pi|/(c_3^{\mathrm{abs}} B_\pi^{\mathrm{abs}})$. The fitted rates approach $8 - 4\sqrt{2}$. At $\beta = 32$ and $d = 10^{-4}$, the quotient of $\mathcal{W}(\pi - d)/d^3$ by $-4c_3 B_\pi$ is 1.0000009109773. The finite perturbation that flips c_3 has relative size $c_3/A_1 = 1.270694641\,10^{-12}$ and preserves positivity and the checked ratio order. The explicit preservation radius (27) satisfies $\rho/c_3 = 9.65917182921\,10^8$ at the same parameter, so the tested sign-flip lies well inside the proved local admissible radius. This number is a normalization check, not an ingredient of Theorem 7.

The acceptance checks use explicit exceptions rather than Python `assert`; the same run is required under ordinary Python and `python -O`. The option `-self-test-mutations` injects a false predicate and requires the harness to reject it. None of these floating-point observations is used to establish Theorems 3, 5, or 7.

8 Implications for proof design

The two proposed legs now have sharply different outcomes.

1. The convolution kernel is strong enough to prove $F_B > 0$ globally. It should be retained as the denominator lemma.

2. The parabola $4\sin^3 t$ is an exact small-β anchor but is not a global barrier by itself. Any continuation in (t, β) must carry an endpoint variable normalized by $c_3 B_\pi$, not merely absolute Fourier moments.

3. The large-β endpoint window requires signed resolution on the scale $e^{-(8-4\sqrt{2})\beta}$ relative to the naive positive envelope. Polynomially accurate structural bounds necessarily miss it.

4. Coefficient-ratio order is useful but logically insufficient for sine-series ratio monotonicity. Theorem 7 remains true even after adjoining all jets at $\beta = 0$.

A plausible surviving architecture is therefore modular: use the exact convolution for F_B; isolate $c_3 > 0$ by a cancellation-preserving signed identity; prove a normalized endpoint window after factoring $\sin^3 t$; and bridge that window to a compact interior region. The present paper does not claim that this program has been completed.

9 Priority and limitations

The addition formulas (2)–(3) are classical Neumann–Graf identities; standard sources are Watson's treatise and the NIST Digital Library of Mathematical Functions [3, 6]. The large-argument expansions with differentiable error control are also classical [4, 5]. Ratio criteria for ordinary power series go back at least to Biernacki and Krzyż [2]; they do not directly apply to sine series. Positivity and monotonicity phenomena for Bessel sums have a substantial earlier literature, including Askey–Steinig [1].

The contribution claimed here is narrower: the exact endpoint reduction, the cancellation exponent (21), and the perturbative non-identifiability theorem for the specified proof architecture. We do not claim novelty for the classical identities or Laplace method. We also do not claim a proof or disproof of Conjecture 1. Related programme manuscripts record the Wronskian reduction and a separate certificate-based route to the global inequality [7, 8]; the present result is useful precisely because it explains why a much shorter kernel-and-anchor proof cannot close without a new signed lemma.

Data and verification statement

The source, diagnostic script, invocation commands, output transcript, byte counts, and SHA-256 hashes are distributed with the manuscript. The paper proofs are ordinary analytic arguments. No Lean build, interval oracle, or sustained remote computation was used for this paper. The numerical kill-test is labelled diagnostic and is reproducible on a single CPU process.

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