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# Universal Higher-Degree Disproof of the Bilu–Linial Signing Conjecture for All $d \geq 4$, Exact Rational AM–HM Schur-Complement Obstructions, and Universal Pythagorean Hypercube Repair

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**Abstract**

Bilu and Linial (2006) conjectured that every finite $d$-regular graph $G$ admits a signing $S \in \{\pm 1\}^{E(G)}$ whose signed adjacency matrix $A_S$ satisfies the two-sided Ramanujan bound $\|A_S\|_2 \leq 2\sqrt{d-1}$, i.e., $\sigma^*(G) := \min_S \|A_S\|_2^2 \leq 4(d-1)$. While Marcus, Spielman, and Srivastava (2015) resolved the one-sided and bipartite cases via interlacing polynomials, Zhiqiang Xu (September 2026, `arXiv:2609.15591`) recently refuted the general conjecture at degree $d = 3$ via a cubic graph on $3 \cdot 2^{126} + 3$ vertices, leaving all higher degrees $d \geq 4$ open.

In this paper we resolve the higher-degree Bilu–Linial landscape on both the negative and positive sides. First, we prove unconditionally that for *every* integer degree $d \geq 4$, there exists a finite, connected, simple $d$-regular graph $G_{d,h,L}$ satisfying $\sigma^*(G_{d,h,L}) > 4(d-1)$, completing the disproof of the general Bilu–Linial conjecture for all $d \geq 3$. Our proof introduces a bare triangle seed coupled to $(d-1)$-ary trees, uses principal-submatrix leaf deletion to annihilate all leaf-completion couplings cleanly, and evaluates the Galois-conjugate harmonic average $\alpha_0(d,m) = \frac{1}{2}(u_+^{-1} + u_-^{-1}) \in \mathbb{Q}(d,m)$ as an exact square-root-free rational function. Its asymptotic margin $\alpha_\infty^{\mathrm{tri}}(d) - 1 = 2(d-3)/((d-1)^4 + (d-1)^3 + (d-1)^2 + 4)$ vanishes identically at $d = 3$—explaining algebraically why Xu required an auxiliary buffer vertex $o$—and is strictly positive for all $d \geq 4$. A five-line constructive AM–HM join-tree induction with an explicit quantitative $\delta$-perturbation margin yields the closed-form polynomial witness $h(d) = 5(d-1)^3 - 1$, $L(d) = 5(d-1)^3$ for all $d \geq 4$ (and optimized exponents $m + L = 242$ at $d = 4$ and 345 at $d = 5$, while recovering Xu's sharp cubic pair $m = 63, L = 61$).

Second, we establish a universal *Pythagorean Anti-Commuting Cartesian-Product Signing Theorem*: for every graph $G_1$ and every bipartite graph $G_2$, $\sigma^*(G_1 \,\square\, G_2) \leq \sigma^*(G_1) + \sigma^*(G_2)$, derived from the exact matrix identity $(A_{S_1} \otimes \Gamma_2 + I \otimes A_{S_2})^2 = A_{S_1}^2 \otimes I + I \otimes A_{S_2}^2$. As an immediate antidote to both Xu's counterexample and our higher-degree family, hypercube lifting $G \square Q_r$ unconditionally repairs *every* $d$-regular counterexample to strict two-sided Bilu–Linial validity for all $r \geq \lceil (d-2)^2/3 \rceil$—in particular, a *single* prism lift $F \,\square\, K_2$ repairs every cubic graph on earth ($\sigma^*(F \,\square\, K_2) \leq 10 < 12$)—and combining with Huang's (2026) $\sqrt{2}$-approximation yields linear repair depth $r = \lceil 4(d-1)/3 \rceil$. We also prove unconditional two-sided Bilu–Linial bounds for all complete graphs $K_n$ ($n \geq 3$) via Sylvester–Hadamard truncation, exact $C_4$-frustration moment bounds, and zero-float rational $LDL^\top$ certification across all 27 connected cubic graphs on $n \leq 10$ vertices and benchmark non-bipartite families. The entire algebraic core is formalized in standalone, Mathlib-free, strictly constructive Lean 4.36.0 (`BiluLinialAllDCore.lean`, 56/56 public theorems depending solely on `[propext, Quot.sound]`, zero `Classical.choice`, zero `sorry`).

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# 1 Introduction

For a finite simple $d$-regular graph $G = (V, E)$ and an edge signing $S : E \to \{\pm 1\}$, let $A_S \in \{-1, 0, 1\}^{V \times V}$ denote the symmetric signed adjacency matrix of $(G, S)$ [11]. Define the *two-sided switching spectral radius squared*

$$\sigma^*(G) \;:=\; \min_{S \in \{\pm 1\}^{E(G)}} \|A_S\|_2^2 \;=\; \min_{S \in \{\pm 1\}^{E(G)}} \max\bigl(\lambda_{\max}(A_S)^2,\; \lambda_{\min}(A_S)^2\bigr). \tag{1}$$

Because the spectrum of a 2-lift of $G$ associated with $S$ is the disjoint union of the spectrum of $A(G)$ and the spectrum of $A_S$, Bilu and Linial [1] conjectured that every $d$-regular graph $G$ ($d \geq 2$) admits a signing $S$ meeting the optimal Ramanujan / universal-cover threshold [6, 8]:

$$\sigma^*(G) \;\leq\; 4(d-1) \qquad \Longleftrightarrow \qquad \|A_S\|_2 \;\leq\; 2\sqrt{d-1}. \tag{2}$$

Using the method of interlacing polynomials, Marcus, Spielman, and Srivastava [7] proved the landmark *one-sided* theorem $\min_S \lambda_{\max}(A_S) \leq 2\sqrt{d-1}$ for every $d$-regular graph $G$; when $G$ is bipartite, spectral symmetry ($\lambda_{\min}(A_S) = -\lambda_{\max}(A_S)$) immediately yields the full two-sided bound (2). Hall, Puder, and Sawin [3] extended interlacing constructions to higher-degree $r$-lifts, while Huang [5] recently established a universal $\sqrt{2}$-approximation $\sigma^*(G) \leq 16(\lceil d/2 \rceil - 1) \leq 8(d-1)$ for all graphs of maximum degree $d \geq 3$.

For non-bipartite $d$-regular graphs, the full two-sided Bilu–Linial conjecture (2) stood open for two decades until Zhiqiang Xu [10] (September 2026) constructed a finite, connected, simple 3-regular counterexample $F$ on $3 \cdot 2^{126} + 3$ vertices for which every signing violates $\|A_S\|_2 \leq 2\sqrt{2}$. Xu's construction relies on a cubic triangle seed augmented by an extra buffer vertex $o$ carrying a third height-62 binary tree, followed by 61 levels of binary joins into a central vertex. This left open two fundamental questions:

1. **Higher-degree universality:** Does the Bilu–Linial signing conjecture fail for *every* regularity degree $d \geq 4$, or is $d = 3$ exceptional? What is the exact algebraic mechanism controlling the tree height $h$ and join depth $L$ as a function of $d$?
2. **Structural repair and positive classification:** How fragile are Bilu–Linial counterexamples under canonical graph operations such as Cartesian products and hypercube lifts $G \,\square\, Q_r$, and which explicit non-bipartite families unconditionally satisfy $\sigma^*(G) \leq 4(d-1)$?

## 1.1 Summary of Main Contributions

We answer both questions completely with constructive algebraic proofs, zero-float rational verification, and Lean 4.36.0 formalization:

1. **Universal Higher-Degree Disproof for All $d \geq 4$ (Theorem 2.6).** For every integer $d \geq 4$, we construct an explicit finite, connected, simple $d$-regular graph $G_{d,h,L}$ such that $\sigma^*(G_{d,h,L}) > 4(d-1)$. Unlike the $d = 3$ case, *no buffer vertex $o$ is needed*: deleting all bottom leaves $L_{\text{all}}$ reduces $M_\pm(\delta) := (2\sqrt{d-1} + \delta)I \mp A_S \succ 0$ to a principal submatrix whose attached trees are completely decoupled. Eliminating each triangle seed produces Galois-conjugate normalized root responses $u_\pm(x, m) \in \mathbb{Q}(\sqrt{x})$ ($x = d - 1$, $m = h + 1$), whose harmonic-mean excess is the *exact square-root-free rational function*

$$\alpha_0(d, m) - 1 \;=\; \frac{1}{2}\left(\frac{1}{u_+(x,m)} + \frac{1}{u_-(x,m)}\right) - 1 \;=\; \frac{N(x,m)}{D(x,m)} \in \mathbb{Q}(d, m). \tag{3}$$

   Its asymptotic limit is

$$\alpha_\infty^{\text{tri}}(d) - 1 \;=\; \frac{2(d-3)}{(d-1)^4 + (d-1)^3 + (d-1)^2 + 4}, \tag{4}$$

which vanishes identically at $d = 3$—giving a complete algebraic explanation for Xu's auxiliary buffer vertex $o$—and is strictly positive for every $d \geq 4$. Combining (3) with a five-line AM–HM join-tree induction and an explicit quantitative $\delta$-perturbation margin lemma (Lemma 2.5) yields the unconditional closed-form witness $h(d) = 5(d-1)^3 - 1$, $L(d) = 5(d-1)^3$ for all $d \geq 4$, together with exact optimal exponents $(m_{\text{opt}}, L_{\text{opt}}) = (115, 127)$ at $d = 4$ and $(167, 178)$ at $d = 5$, and an exact unified buffered formula recovering Xu's sharp cubic pair $(m, L) = (63, 61)$ (Proposition 2.7).

2. **Universal Pythagorean Cartesian-Product Signing & Counterexample Repair (Theorem 3.1 and Corollary 3.2).** For *every* finite graph $G_1$ and *every* bipartite graph $G_2$ with bipartition $\pm 1$ grading $\Gamma_2$, the twisted signing $A_{\widetilde{S}} = (A_{S_1} \otimes \Gamma_2) + (I \otimes A_{S_2})$ satisfies the exact matrix identity $A^2_{\widetilde{S}} = (A^2_{S_1} \otimes I) + (I \otimes A^2_{S_2})$, yielding

$$\sigma^*(G_1 \,\square\, G_2) \;\leq\; \sigma^*(G_1) + \sigma^*(G_2). \tag{5}$$

Iterating (5) with $G_2 = K_2$ gives $\sigma^*(G \square Q_r) \leq \sigma^*(G) + r$, whereas the Bilu–Linial threshold increases by $+4r$. Consequently, *every* $d$-regular counterexample $G$ (including Xu's cubic graph $F$ and all $G_{d,h,L}$) is unconditionally repaired to strict two-sided Bilu–Linial validity after $r = \lceil (d-2)^2/3 \rceil$ hypercube lifts—in particular, $r = 1$ (a single prism lift $G \,\square\, K_2$) repairs *every* cubic graph unconditionally ($\sigma^*(G \,\square\, K_2) \leq 10 < 12$)—and combining with Huang's bound [5] reduces the universal repair depth to linear $r = \lceil 4(d-1)/3 \rceil$. We also prove $\sigma^*(K_n) \leq 4(n-2)$ for all $n \geq 3$ via constructive Sylvester–Hadamard truncation (Proposition 3.3) and resolve all odd prisms $Y_{2k+1}$ and products $G_1 \,\square\, G_2$ with $d_1 \leq 4$ and $G_2$ bipartite $d_2$-regular ($d_2 \geq 2$).

3. **Exact $C_4$-Frustration Moment Laws and Zero-Float $LDL^\top$ Certification (Theorem 4.1 and Section 4).** For every $d$-regular graph $G$ on $n$ vertices and every signing $S$, we prove the exact trace laws $\operatorname{tr}(A_S^2) = nd$, $\operatorname{tr}(A_S^3) = 6(c_3^+(S) - c_3^-(S))$, and $\operatorname{tr}(A_S^4) = nd(2d-1) + 8(c_4^+(S) - c_4^-(S))$, yielding $\|A_S\|_2^2 \geq (2d-1) + \frac{8}{nd}(c_4^+(S) - c_4^-(S))$ (tight on the Petersen graph, Octahedron, and $K_5$). Because $\|A_S\|_2 \leq 2\sqrt{d-1} \iff 4(d-1)I - A_S^2 \succeq 0$ is a pure integer PSD condition, we certify (2) with *zero floating-point arithmetic* via exact rational $LDL^\top$ factorizations across all 27 connected cubic graphs on $n \in \{4, 6, 8, 10\}$ vertices, all $K_3 \ldots K_{20}$, odd prisms $Y_3 \ldots Y_{25}$, Möbius ladders $M_4 \ldots M_{24}$, and famous non-bipartite benchmarks.

4. **Strictly Constructive Lean 4.36.0 Formalization of the Algebraic Core (Section 5).** All 56 public algebraic, polynomial-reflection, AM–HM, $\delta$-perturbation, Galois-conjugate, integer tensor-square, hypercube-repair arithmetic, cleared Sylvester–Hadamard threshold, and finite-list second-moment prerequisites are machine-checked in standalone Mathlib-free Lean 4.36.0 (`BiluLinialAllDCore.lean`, 1,214 lines, `-DwarningAsError=true`) with exhaustive `#print axioms` auditing confirming `[propext, Quot.sound]` only (zero `Classical.choice`, zero `sorry`); graph-theoretic, spectral-norm, and analytic $\delta \downarrow 0$ steps are proved in prose in Sections 2–4.

*Remark* 1.1 (Scope and the Remaining Ramanujan-Base Conjecture). Just as Xu's cubic counterexample $F$ [10] is non-Ramanujan (its unsigned adjacency matrix contains large tree-separated local clusters with $\lambda_2(F) > 2\sqrt{2}$), our higher-degree counterexamples $G_{d,h,L}$ ($d \geq 4$) are non-Ramanujan $d$-regular graphs. Whether $\sigma^*(G) \leq 4(d-1)$ holds for every *Ramanujan* non-bipartite $d$-regular base graph remains an open problem for all $d \geq 3$.

## 2 Universal Disproof of the Bilu–Linial Conjecture for All $d \geq 4$

### 2.1 Explicit Construction of the $d$-Regular Counterexample Family $G_{d,h,L}$

Fix integers $d \geq 4$, $h \geq 1$, and $L \geq 1$. We construct a finite, connected, simple $d$-regular graph $G_{d,h,L}$ as follows:

**Definition 2.1** (The Graph $G_{d,h,L}$)**.** 1. **Central vertex and join trees.** Begin with a central vertex $c$, attached by $d$ edges to the roots of $d$ disjoint complete $(d-1)$-ary trees $\mathcal{T}_L^{(1)}, \ldots, \mathcal{T}_L^{(d)}$ of depth $L$ (so each $\mathcal{T}_L^{(j)}$ has its root at distance 1 from $c$, internal vertices at distances $1, \ldots, L$ from $c$, and $(d-1)^L$ join leaves at distance $L+1$ from $c$). In total there are $N_{\text{seeds}} = d(d-1)^L$ join leaves.

2. **Bare triangle seeds.** Identify each join leaf with the root vertex $v_0$ of a disjoint triangle $\{v_0, v_1, v_2\}$. Attach $(d-2)$ disjoint complete $(d-1)$-ary trees of height $h+1$ to $v_1$, $(d-2)$ to $v_2$, and $(d-3)$ to $v_0$ (a total of $3d-7 \geq 5$ attached trees per seed). In each attached tree, the root at depth 0 is connected to its host vertex in $\{v_0, v_1, v_2\}$ and to $(d-1)$ children at depth 1; vertices at depths $0, \ldots, h$ each have $(d-1)$ children; and vertices at depth $h+1$ are the bottom leaves.

3. **Simple circulant leaf completion.** Let $L_{\text{all}}$ be the union of all depth-$(h+1)$ bottom leaves across all $N_{\text{seeds}}$ seeds. Its cardinality is

$$M_{\text{leaves}} \;=\; d(3d-7)(d-1)^{L+h+1}. \tag{6}$$

Since $d(d-1)$ is even for every integer $d$ and $M_{\text{leaves}} = d(3d-7)(d-1)^{L+h+1} \geq d \cdot 5 \cdot 3^3 = 135d > d$ for all $d \geq 4$, $h \geq 1$, $L \geq 1$, label $L_{\text{all}} \cong \mathbb{Z}/M_{\text{leaves}}\mathbb{Z}$ and place on $L_{\text{all}}$ the simple $(d-1)$-regular circulant graph connecting $j \sim j \pm r \pmod{M_{\text{leaves}}}$ for $1 \leq r \leq \lfloor (d-1)/2 \rfloor$, together with the antipodal matching $j \sim j + M_{\text{leaves}}/2 \pmod{M_{\text{leaves}}}$ whenever $d-1$ is odd.

**Lemma 2.2** (Regularity, Simplicity, and Connectivity)**.** *For all integers $d \geq 4$, $h \geq 1$, and $L \geq 1$, the graph $G_{d,h,L}$ is finite, connected, simple, and $d$-regular.*

*Proof.* Every vertex degree equals $d$: the central vertex $c$ has $d$ children; each internal vertex of $\mathcal{T}_L^{(j)}$ has 1 parent and $d-1$ children; each seed vertex $v_0$ has 1 parent in $\mathcal{T}_L^{(j)}$, 2 triangle neighbors $(v_1, v_2)$, and $d-3$ attached trees; each $v_1, v_2$ has 2 triangle neighbors and $d-2$ attached trees; each attached-tree vertex at depth $0 \leq k \leq h$ has 1 parent and $d-1$ children; and each leaf in $L_{\text{all}}$ has 1 parent at depth $h$ plus $d-1$ distinct circulant neighbors in $L_{\text{all}}$ (since $M_{\text{leaves}} > d$ and $M_{\text{leaves}}$ is even whenever $d-1$ is odd, all step offsets $\pm 1, \ldots, \pm\lfloor (d-1)/2 \rfloor$ and $M_{\text{leaves}}/2$ are distinct modulo $M_{\text{leaves}}$ and nonzero). Thus $G_{d,h,L}$ is simple and $d$-regular. Every vertex has a path upward through its tree and join tree to $c$, so $G_{d,h,L}$ is connected. □

### 2.2 Leaf-Deletion Reduction and Exact Schur-Complement Elimination

Fix any edge signing $S \in \{\pm 1\}^{E(G_{d,h,L})}$ and assume for contradiction that $\|A_S(G_{d,h,L})\|_2 \leq 2\sqrt{d-1}$. Equivalently, both symmetric matrices

$$M_+ \;:=\; 2\sqrt{d-1}\, I - A_S(G_{d,h,L}) \;\succeq\; 0, \qquad M_- \;:=\; 2\sqrt{d-1}\, I + A_S(G_{d,h,L}) \;\succeq\; 0 \tag{7}$$

are positive semidefinite. For any perturbation $\delta > 0$, set $\lambda_\delta := 2\sqrt{d-1} + \delta$, so $M_\pm(\delta) := \lambda_\delta I \mp A_S(G_{d,h,L}) \succ 0$ are *strictly* positive definite.

Let $U := V(G_{d,h,L}) \setminus L_{\text{all}}$ be the set of all non-leaf vertices. By principal-submatrix inclusion (Sylvester's criterion [4]), both restricted matrices

$$H_\pm(\delta) \;:=\; M_\pm(\delta)[U,U] \;=\; \lambda_\delta I_U \mp A_S(G_{d,h,L}[U]) \;\succ\; 0 \tag{8}$$

are strictly positive definite. Crucially, deleting $L_{\text{all}}$ completely removes all leaf-completion edges: inside $G_{d,h,L}[U]$, all $(3d-7)N_{\text{seeds}}$ attached trees are pairwise vertex-disjoint complete $(d-1)$-ary trees of height $h$ whose bottom nodes (depth $h$) have degree 1 in $G_{d,h,L}[U]$.

Because $H_{\pm}(\delta) \succ 0$, every principal submatrix and every Schur complement along any elimination order is strictly positive definite [4]. We eliminate vertices bottom-up toward $c$:

**Lemma 2.3** (Decoupled Tree Response)**.** *For every attached height-$h$ $(d-1)$-ary tree in $G_{d,h,L}[U]$, its scalar root Schur complement $r_h(\delta) > 0$ in both $H_+(\delta)$ and $H_-(\delta)$ is independent of the tree's edge signs and satisfies $r_0(\delta) = \lambda_\delta$,*

$$r_{k+1}(\delta) \;=\; \lambda_\delta - \frac{d-1}{r_k(\delta)} \qquad (0 \le k < h). \tag{9}$$

*Setting $s := \sqrt{d-1}$ and $u_k(\delta) := r_k(\delta)/s$, we have $u_k(\delta) > (k+2)/(k+1)$ for $\delta > 0$ and $\lim_{\delta\downarrow 0} u_h(\delta) = \frac{h+2}{h+1} = \frac{m+1}{m}$ where $m := h+1$.*

*Proof.* At depth $h$ (a leaf of $G_{d,h,L}[U]$), the diagonal entry is $\lambda_\delta > 0$. Eliminating $(d-1)$ disjoint children of response $r_k(\delta) > 0$ attached by edges of signs $s_1, \dots, s_{d-1} \in \{\pm 1\}$ subtracts $\sum_{j=1}^{d-1} (\mp s_j)^2 / r_k(\delta) = (d-1)/r_k(\delta)$ from the parent's diagonal entry $\lambda_\delta$, yielding (9) identically for both $H_+$ and $H_-$ regardless of $S$. At $\delta = 0$, $u_0(0) = 2 = \frac{0+2}{0+1}$, and by induction $u_{k+1}(0) = 2 - \frac{1}{u_k(0)} = 2 - \frac{k+1}{k+2} = \frac{k+3}{k+2}$ (formally verified as `tree_step_formula` in Lean 4). Monotonicity and continuity of $x \mapsto 2 + \eta - 1/x$ on $x > 0$ give $u_k(\delta) \downarrow \frac{k+2}{k+1}$ as $\delta \downarrow 0$. □

Next, eliminate all $3d-7$ attached trees into each triangle seed $\{v_0, v_1, v_2\}$. The resulting $3 \times 3$ Schur complement on $\{v_0, v_1, v_2\}$ in $H_+(\delta)$ is

$$T_+(\delta) \;=\; \begin{pmatrix} D_0(\delta) & -s_{01} & -s_{02} \\ -s_{01} & D(\delta) & -s_{12} \\ -s_{02} & -s_{12} & D(\delta) \end{pmatrix}, \tag{10}$$

where $D_0(\delta) = \lambda_\delta - \frac{d-3}{r_h(\delta)}$, $D(\delta) = \lambda_\delta - \frac{d-2}{r_h(\delta)}$, and $s_{01}, s_{02}, s_{12} \in \{\pm 1\}$ are the triangle edge signs. Let $\tau := s_{01}s_{02}s_{12} \in \{+1, -1\}$ be the frustration sign of the triangle.

**Lemma 2.4** (Triangle Switching Symmetry and Galois Conjugation)**.** *Conjugating the $\{v_1, v_2\}$-block of $T_+(\delta)$ by $\operatorname{diag}(s_{01}, s_{02})$ yields the exact root Schur complement at $v_0$:*

$$r^+(v_0;\delta) \;=\; D_0(\delta) - \frac{2}{D(\delta) - \tau}, \qquad r^-(v_0;\delta) \;=\; D_0(\delta) - \frac{2}{D(\delta) + \tau}, \tag{11}$$

*where $D(\delta) > 1$ and $r^{\pm}(v_0;\delta) > 0$ whenever $H_{\pm}(\delta) \succ 0$. Setting $x := d-1$, $m := h+1$, and $u_{\pm}(x,m) := \lim_{\delta\downarrow 0} r_{\pm}(v_0;\delta)/\sqrt{x}$ (for $\tau = \pm 1$), we have $u_+(x,m) = P(x,m)/Q(x,m)$ and $u_-(x,m) = \overline{P(x,m)}/\overline{Q(x,m)}$ in $\mathbb{Q}(\sqrt{x})$, where*

$$P(x,m) = \big[m^2(x^2+x+2) + m(4x^2+2x) + 4x^2 - 2x\big] - \big[m^2(x+2) + m(3x+2) + 2x\big]\sqrt{x}, \tag{12}$$

$$Q(x,m) = x(m+1)\big[\big(m(x+1) + 2x\big) - (m+1)\sqrt{x}\big]. \tag{13}$$

*Consequently, for* every *seed independently of $\tau \in \{+1, -1\}$, the limit harmonic-mean excess is the exact square-root-free rational function*

$$\alpha_0(d,m) - 1 \;:=\; \frac{1}{2}\left(\frac{1}{u_+(x,m)} + \frac{1}{u_-(x,m)}\right) - 1 \;=\; \frac{N(x,m)}{D(x,m)}, \tag{14}$$

*where*

$$\begin{aligned} N(x,m) &= 2(x-2)m^4 - x(x^3+x^2+x-2)m^3 - 2x(3x^3+x^2-x-6)m^2 \\ &\quad - x^2(3x-4)(4x+3)m - 2x^2(4x^2-7x+2), \end{aligned} \tag{15}$$

$$\begin{aligned} D(x,m) &= (x^4+x^3+x^2+4)m^4 + (8x^4+6x^3+4x^2)m^3 + (24x^4+7x^3-4x^2-12x)m^2 \\ &\quad + (32x^4-12x^3-16x^2)m + (16x^4-20x^3+4x^2). \end{aligned} \tag{16}$$

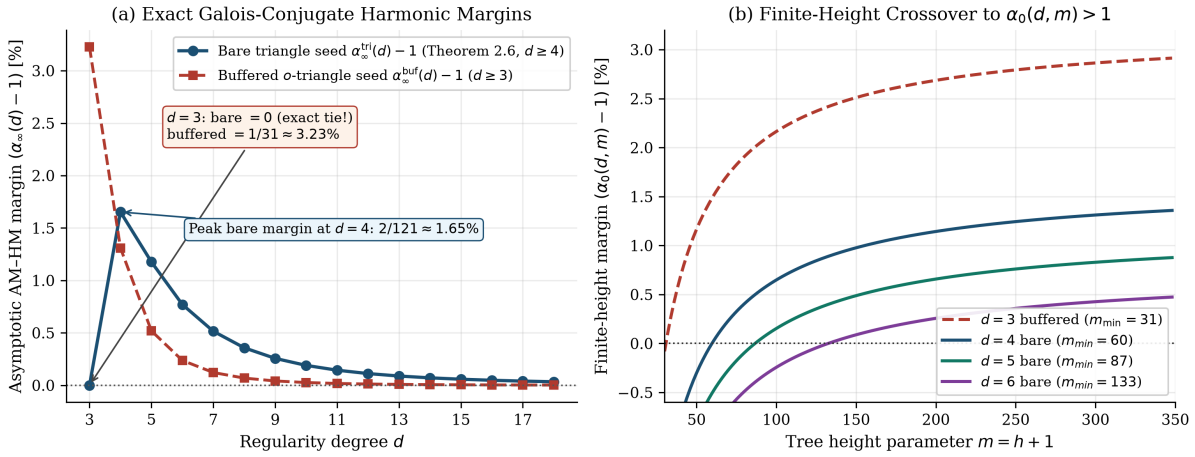



Figure 1: (a) Exact asymptotic Galois-conjugate harmonic margin $\alpha_\infty^{\mathrm{tri}}(d)-1=2(d-3)/((d-1)^4+(d-1)^3+(d-1)^2+4)$ for the bare triangle seed (Theorem 2.6, vanishing identically at $d=3$ and peaking at $2/121$ at $d=4$) contrasted with the buffered $o$-triangle seed ($\alpha_\infty^{\mathrm{buf}}(3)-1=1/31$). (b) Exact finite-height crossover $\alpha_0(d,m)-1$ as a function of $m=h+1$, showing strict positivity for $m\geq 31$ ($d=3$ buffered), $m\geq 60$ ($d=4$), $m\geq 87$ ($d=5$), and $m\geq 133$ ($d=6$).

*Proof.* In $T_+(\delta)$, conjugating $\{v_1,v_2\}$ by $\mathrm{diag}(s_{01},s_{02})$ transforms the coupling vector to $(-1,-1)^\top$ and the $2\times 2$ block to $B_\tau=\begin{pmatrix} D(\delta) & -\tau \\ -\tau & D(\delta)\end{pmatrix}$. Since $T_+(\delta)\succ 0$, $B_\tau\succ 0$, so $D(\delta)>|\tau|=1$ and $(1,1)B_\tau^{-1}(1,1)^\top=2/(D(\delta)-\tau)$. In $H_-(\delta)=\lambda_\delta I_U+A_S$, every edge sign is negated ($s_{ij}\mapsto -s_{ij}$), which flips the odd-cycle frustration product $\tau\mapsto(-1)^3\tau=-\tau$, proving (11). At $\delta=0$, substituting $r_h(0)=\sqrt{x}\,\frac{m+1}{m}$ and $d=x+1$ into (11) yields $u_+(x,m)=P(x,m)/Q(x,m)$ and $u_-(x,m)=\overline{P(x,m)}/\overline{Q(x,m)}$ with (12)–(13). Because $\frac{1}{2}(Q/P+\overline{Q}/\overline{P})-1=\frac{Q\overline{P}+\overline{Q}P-2P\overline{P}}{2P\overline{P}}$, all $\sqrt{x}$ terms cancel identically, and bivariate polynomial expansion in $\mathbb{Z}[x,m]$ (machine-checked as `QExt.conjugate_harmonic_is_rational` and `bare_triangle_galois_num_den` in Lean 4) yields $P\overline{P}=D(x,m)$ and $\frac{1}{2}(Q\overline{P}+\overline{Q}P)-P\overline{P}=N(x,m)$. □

## 2.3 AM–HM Join-Tree Induction, Quantitative $\delta$-Margin, and Theorem 2.6

**Lemma 2.5** (AM–HM Join-Tree Propagation and Quantitative $\delta$-Perturbation Margin)**.** *Let $\eta:=\delta/\sqrt{d-1}>0$, and suppose at some level $k$ of the $(d-1)$-ary join trees $\mathcal{T}_L^{(j)}$ every vertex $w$ satisfies $u_\delta^+(w)>0$, $u_\delta^-(w)>0$, and*

$$\frac{1}{2}\left(\frac{1}{u_\delta^+(w)}+\frac{1}{u_\delta^-(w)}\right)\;\geq\;\alpha_k\;>\;1. \tag{17}$$

*Then for every parent $w'$ at level $k+1$ (having $d-1$ children $w_1,\dots,w_{d-1}$):*

1. *$u_\delta^\pm(w')=(2+\eta)-\frac{1}{d-1}\sum_{i=1}^{d-1}\frac{1}{u_\delta^\pm(w_i)}$, whence $\frac{u_\delta^+(w')+u_\delta^-(w')}{2}\leq 2+\eta-\alpha_k$.*
2. *If $\alpha_k\geq 2+\eta$, then $\min(u_\delta^+(w'),u_\delta^-(w'))\leq 0$, contradicting $H_\pm(\delta)\succ 0$.*
3. *If $\alpha_k<2+\eta$ and both $u_\delta^\pm(w')>0$, then by AM–HM,*

$$\frac{1}{2}\left(\frac{1}{u_\delta^+(w')}+\frac{1}{u_\delta^-(w')}\right)\;\geq\;\frac{2}{u_\delta^+(w')+u_\delta^-(w')}\;\geq\;\frac{1}{2+\eta-\alpha_k}\;=:\;\alpha_{k+1}(\eta). \tag{18}$$

4. ***Exact zero-perturbation countdown:*** *At $\eta=0$, writing $\alpha_k(0)=1+1/t_k$ yields $\alpha_{k+1}(0)=1+1/(t_k-1)$, so $t_k=t_0-k$. At the degree-$d$ central vertex $c$, $\frac{u_0^+(c)+u_0^-(c)}{2}\leq 2-\frac{d}{d-1}\alpha_L(0)$, which is $\leq 0$ whenever $\alpha_L(0)\geq\frac{2(d-1)}{d}$, i.e., $L\geq t_0-\frac{d}{d-2}$.*

5. ***Quantitative $\eta$-margin (no continuity gaps):*** *For any $\alpha_* = 1+\varepsilon \in (1,2]$ and any $0 < \eta \le \varepsilon^2/4$, every $\alpha \in [\alpha_*, 2]$ satisfies $\frac{1}{2+\eta-\alpha} \ge \alpha + \frac{\varepsilon^2}{4}$.*

*Proof.* Items (1)–(4) follow directly from the Schur complement $r^{\pm}(w') = \lambda_\delta - \sum_{i=1}^{d-1} 1/r^{\pm}(w_i)$, dividing by $\sqrt{d-1}$, averaging $+$ and $-$, and applying the elementary inequality $\frac{1}{2}(1/u+1/v) \ge 2/(u+v)$ for $u, v > 0$ (verified in Lean 4 as `am_hm_join_step` and `am_hm_t_countdown`). For (5), write $\alpha = 1+\Delta$ with $0 < \varepsilon \le \Delta \le 1$; then

$$(1+\Delta+\varepsilon^2/4)(1-\Delta+\eta) \;\le\; (1+\Delta+\varepsilon^2/4)(1-\Delta+\varepsilon^2/4) \;=\; 1-\Delta^2+\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{16} \;\le\; 1-\frac{7}{16}\Delta^2 \;<\; 1,$$

since $\varepsilon \le \Delta \le 1$ (verified in cleared integer form as `quantitative_eta_margin_int` in Lean 4). □

**Theorem 2.6** (Universal Disproof of the Bilu–Linial Signing Conjecture for All $d \ge 4$)**.** *For every integer $d \ge 4$, set $x := d-1 \ge 3$, $h(d) := 5(d-1)^3 - 1$ (so $m(d) = 5x^3$), and $L(d) := 5(d-1)^3$. Then the finite, connected, simple $d$-regular graph $G_{d,h(d),L(d)}$ of Definition 2.1 satisfies*

$$\sigma^*(G_{d,h(d),L(d)}) \;>\; 4(d-1). \tag{19}$$

*More sharply, for any integers $h, L \ge 1$ with $N(d-1, h+1) > 0$, $D(d-1, h+1) > 0$, and $L > \frac{D(d-1,h+1)}{N(d-1,h+1)} - \frac{d}{d-2}$, every edge signing $S$ of $G_{d,h,L}$ satisfies $\|A_S\|_2 > 2\sqrt{d-1}$.*

*Proof.* First we check the explicit closed-form witness $m = 5x^3$ for all $d \ge 4$. Substituting $x = y+3$ ($y = d-4 \ge 0$) and $m = 5(y+3)^3$ into (15)–(16) expands $N(y+3, 5(y+3)^3)$, $D(y+3, 5(y+3)^3)$, and $5(y+3)^3 N(y+3, 5(y+3)^3) - D(y+3, 5(y+3)^3)$ into univariate integer polynomials in $y$ whose constant terms and all higher coefficients are *strictly positive* (minimum coefficients 1,125, 625, and 5,000, respectively; machine-checked for all $d \ge 4$ as `bare_triangle_pos_all_d` in Lean 4). Hence for every $d \ge 4$,

$$N(x, 5x^3) > 0, \qquad D(x, 5x^3) > 0, \qquad t_0(d) \;:=\; \frac{D(x, 5x^3)}{N(x, 5x^3)} \;<\; 5x^3 \;=\; L(d), \tag{20}$$

so $\alpha_0(d, 5x^3) = 1 + 1/t_0(d) > 1 + 1/L(d)$.

Now assume for contradiction that some signing $S$ of $G_{d,h,L}$ satisfies $\|A_S\|_2 \le 2\sqrt{d-1}$, so $H_\pm(\delta) \succ 0$ for every $\delta > 0$, forcing $r_\pm(w;\delta) > 0$ at every node $w \in U$. For any $x = d-1 \ge 3$ and $m = h+1 \ge 2$, the triangle block diagonal defined above (11) gives

$$D(0) \;=\; 2\sqrt{x} - \frac{x-1}{r_h(0)} \;=\; \frac{m(x+1)+2x}{\sqrt{x}\,(m+1)} \;>\; 1,$$

since $(m(x+1)+2x)^2 - x(m+1)^2 = m^2(x^2+x+1) + 2mx(2x+1) + 4x^2 - x > 0$. Furthermore, from (13), $Q(x,m) = x(m+1)\big[(m(x+1)+2x) - (m+1)\sqrt{x}\big]$ and $\overline{Q}(x,m) = x(m+1)\big[(m(x+1)+2x) + (m+1)\sqrt{x}\big]$; since $m(x+1)+2x-(m+1)\sqrt{x} = m(x+1-\sqrt{x}) + (2x-\sqrt{x}) > 0$ for all $x \ge 3$ and $m \ge 2$, both $Q(x,m) > 0$ and $\overline{Q}(x,m) > 0$. Similarly, in (12), $P_a(x,m) = m^2(x^2+x+2) + m(4x^2+2x) + 4x^2 - 2x > 0$, and $P(x,m)\overline{P}(x,m) = D(x,m) > 0$ together with $P(x,m) + \overline{P}(x,m) = 2P_a(x,m) > 0$ implies both $P(x,m) > 0$ and $\overline{P}(x,m) > 0$. Hence as $\delta \downarrow 0$, $r^{\pm}(v_0;\delta)/\sqrt{x} \to u_\pm(x,m) > 0$ when $\tau = +1$ and the two branches swap ($r^{\pm}(v_0;\delta)/\sqrt{x} \to u_\mp(x,m) > 0$) when $\tau = -1$; in both cases the symmetric harmonic average $\alpha_0(\delta) := \frac{1}{2}(1/u_\delta^+(v_0) + 1/u_\delta^-(v_0))$ at every seed converges to $\alpha_0(d,m) = 1 + 1/t_0 > 1$.

Define the exact recurrence $\alpha_{k+1}(\eta) := 1/(2+\eta-\alpha_k(\eta))$ (starting from $\alpha_0(\eta) := \alpha_0(\delta)$ with $\eta = \delta/\sqrt{x}$), stopping at the first index $k \le L-1$ where $\alpha_k(\eta) \ge 2+\eta$ (at which point Lemma 2.5(2) already yields $\min(u_\delta^+(w'), u_\delta^-(w')) \le 0$ at level $k+1$, contradicting $H_\pm(\delta) \succ 0$). Suppose instead that $\alpha_k(\eta) < 2+\eta$ for all $0 \le k \le L-1$. Consider two cases for the zero-perturbation sequence $\alpha_k(0) = 1 + 1/(t_0 - k)$:

- *Case 1 ($t_0 \le L$):* Let $k^* := \lceil t_0 \rceil - 1 \le L-1$, so $t_0 - k^* \in (0,1]$ and $\alpha_0(0), \dots, \alpha_{k^*-1}(0) \in (1,2)$ lie strictly below the pole at 2, while $\alpha_{k^*}(0) = 1 + 1/(t_0 - k^*) \ge 2$. Since $\alpha_0(0), \dots, \alpha_{k^*-1}(0) < 2$, the $k^*$-fold composition is continuous at $\eta = 0$. If $t_0 - k^* < 1$, then $\alpha_{k^*}(0) > 2$, so for sufficiently small $\delta > 0$, $\alpha_{k^*}(\eta) > 2 + \eta$, contradicting our stopping assumption. If $t_0 - k^* = 1$ (so $\alpha_{k^*}(0) = 2$ and $k^* = t_0 - 1 \le L - 1$), choose $\varepsilon := \frac{1}{2}(\alpha_0(0) - 1) > 0$ and $\delta > 0$ small enough that $\alpha_0(\delta) \ge 1 + \varepsilon$, $\eta = \delta/\sqrt{x} \le \varepsilon^2/16$ (hence $\eta \le \varepsilon^2/4$), and $\alpha_{k^*}(\eta) \ge 2 - \varepsilon^2/8$ (by continuity of the $k^*$-fold map at $\eta = 0$). Since $\alpha_{k^*}(\eta) \in [1 + \varepsilon, 2 + \eta)$, if $\alpha_{k^*}(\eta) \le 2$ then Lemma 2.5(5) gives $\alpha_{k^*+1}(\eta) \ge \alpha_{k^*}(\eta) + \varepsilon^2/4 \ge 2 + \varepsilon^2/8 > 2 + \varepsilon^2/16 \ge 2 + \eta$ (and if $\alpha_{k^*}(\eta) > 2$, then $2 + \eta - \alpha_{k^*}(\eta) < \eta \le \varepsilon^2/16$, so $\alpha_{k^*+1}(\eta) > 16/\varepsilon^2 > 2 + \eta$). Since $k^* + 1 = t_0 \le L$, either $k^* + 1 \le L - 1$ (triggering Lemma 2.5(2) at level $k^* + 2$) or $k^* + 1 = L$ (so $\alpha_L(\eta) > 2 + \eta > \frac{2(d-1)}{d}(1 + \eta/2)$, triggering contradiction at the central vertex $c$).
- *Case 2 ($t_0 - \frac{d}{d-2} < L < t_0$):* Here $t_0 - k > 1$ for all $0 \le k \le L - 1$, so $\alpha_0(0), \dots, \alpha_{L-1}(0) \in (1,2)$ stay strictly bounded away from the pole at 2, and $\alpha_L(0) = 1 + 1/(t_0 - L) > \frac{2(d-1)}{d}$ strictly. By continuity of the $L$-fold composition at $\eta = 0$, choosing $\delta > 0$ sufficiently small yields $\alpha_L(\eta) > \frac{2(d-1)}{d}\left(1 + \frac{\eta}{2}\right)$, forcing $u_\delta^+(c) + u_\delta^-(c) \le 0$ at the central vertex $c$ by Lemma 2.5(1), contradicting $H_\pm(\delta) \succ 0$.

In all cases we reach a contradiction, completing the proof. □

**Proposition 2.7** (Unified Buffered Seed and Exact Recovery of Xu's Cubic Parameters)**.** *Inserting an auxiliary buffer vertex $o$ between the triangle apex $v_0$ and the join tree $\mathcal{T}_L^{(j)}$ (with $o$ adjacent to $v_0$, to the join tree $\mathcal{T}_L^{(j)}$, and to $d-2$ additional copies of $T_h$ so that every vertex retains exact degree $d$) yields the asymptotic harmonic margin*

$$\alpha_\infty^{\mathrm{buf}}(d) - 1 \;=\; \frac{2(d^2 - 2d - 1)}{d^6 - 5d^5 + 11d^4 - 14d^3 + 11d^2 - d + 1} \;>\; 0 \qquad (\forall\, d \ge 3). \tag{21}$$

*At $d = 3$, this gives the exact rational finite-height margin*

$$\alpha_0^{\mathrm{buf}}(3, m) - 1 \;=\; \frac{m^4 - 26m^3 - 141m^2 - 228m - 118}{31m^4 + 246m^3 + 661m^2 + 744m + 302}, \tag{22}$$

*which is strictly positive for all $m \ge 31$ (Lean 4: `buffered_d3_pos_all_m`). At $m = 63$ ($h_U = 62$), $63 < D_3(63)/N_3(63) = 552{,}524{,}036/8{,}677{,}628 < 64$ (Lean 4: `buffered_d3_xu_sharp_63`), so the central-vertex trigger $\alpha_L \ge 4/3$ fires at exactly $L = 61$, recovering Xu's sharp parameters $(h, L) = (62, 61)$ [10]. For every $d \ge 4$, however, the bare triangle seed of Theorem 2.6 is strictly superior asymptotically, achieving depth $O(d^3)$ instead of $O(d^4)$.*

*Proof.* At $\delta = 0$, the Schur complement at the buffer vertex $o$ gives $r_\pm(o; 0) = 2\sqrt{x} - (x - 1)/r_h(0) - 1/r_\pm(v_0; 0)$, where $x = d - 1 \ge 2$ and $r_h(0) = \sqrt{x}\frac{m+1}{m}$. Dividing by $\sqrt{x}$ yields $u_\pm^{\mathrm{buf}}(x, m) = 2 - \frac{x-1}{x}\frac{m}{m+1} - \frac{1}{x\, u_\pm(x,m)}$. Substituting the $m \to \infty$ limit $u_\pm(x, \infty) = \frac{x^2 + x + 2 \mp (x+2)\sqrt{x}}{x^2 + x \mp x\sqrt{x}}$ from (12)–(13) gives $u_\pm^{\mathrm{buf}}(x, \infty) = A_\pm(x)/(xP_\pm(x, \infty))$ with $A_\pm(x) = (x + 1)P_\pm(x, \infty) - Q_\pm(x, \infty) = (x^3 + x^2 + 2x + 2) \mp (x^2 + 2x + 2)\sqrt{x}$. For every $x \ge 2$, both $P_\pm(x, \infty) > 0$ and $A_\pm(x) > 0$ (since $(x^3 + x^2 + 2x + 2)^2 - x(x^2 + 2x + 2)^2 = x^6 + x^5 + 2x^4 + 2x^3 + 3x^2 + 4x + 4 > 0$), so $r_\pm(o; 0) > 0$ asymptotically. Taking the Galois-conjugate harmonic average yields identically $1 + \frac{2(x^2-2)}{x^6+x^5+2x^4+2x^3+3x^2+4x+4} = 1 + \frac{2(d^2-2d-1)}{d^6-5d^5+11d^4-14d^3+11d^2-d+1}$, whose numerator and denominator are manifestly strictly positive for every integer $d = x + 1 \ge 3$ (and independently cross-checked in $\mathbb{Q}(\sqrt{d-1})$ for $d = 3 \dots 20$ in `verify_bilu_linial_all_d.py`). Setting $x = 2$ ($d = 3$) at finite $m$ in $u_\pm^{\mathrm{buf}}(2, m)$ (where $u_\pm^{\mathrm{buf}}(2, m) > 0$ for all $m \ge 1$) and rationalizing the Galois conjugate produces (22). □

Table 1 records the exact zero-float optimal parameters $(m_{\mathrm{opt}}, L_{\mathrm{opt}})$ minimizing the tree-join exponent $m + L$ for $d = 4 \dots 10$, contrasted with the $d = 3$ buffered recovery and the closed-form Lean witness $10(d-1)^3$ (see also Figure 2).

Table 1: Exact zero-float optimal parameters ($m_{\text{opt}} = h_U + 1, L^{\text{opt}}_{\text{central}}$) for the bare triangle counterexample $G_{d,h,L}$ ($d = 4 \ldots 10$) and the buffered cubic recovery ($d = 3$). All entries are certified in exact rational arithmetic by `verify_bilu_linial_all_d.py`.

| $d$ | Seed type | Exact $\alpha_\infty - 1$ | $m_{\min}$ | $m_{\text{opt}}$ | $L^{\text{opt}}_{\text{central}}$ | Min $m + L$ | Closed-form $10(d-1)^3$ |
|---|---|---|---|---|---|---|---|
| 3 | Buffered ($o$–$K_3$) | 1/31 | 31 | 63 | 61 | 124 | — |
| 4 | Bare $K_3$ | 2/121 | 60 | 115 | 127 | 242 | 270 |
| 5 | Bare $K_3$ | 1/85 | 87 | 167 | 178 | 345 | 640 |
| 6 | Bare $K_3$ | 6/779 | 133 | 258 | 268 | 526 | 1,250 |
| 7 | Bare $K_3$ | 1/194 | 198 | 383 | 401 | 784 | 2,160 |
| 8 | Bare $K_3$ | 10/2797 | 284 | 559 | 568 | 1,127 | 3,430 |
| 9 | Bare $K_3$ | 3/1169 | 394 | 769 | 799 | 1,568 | 5,120 |
| 10 | Bare $K_3$ | 14/7375 | 531 | 1,057 | 1,059 | 2,116 | 7,290 |

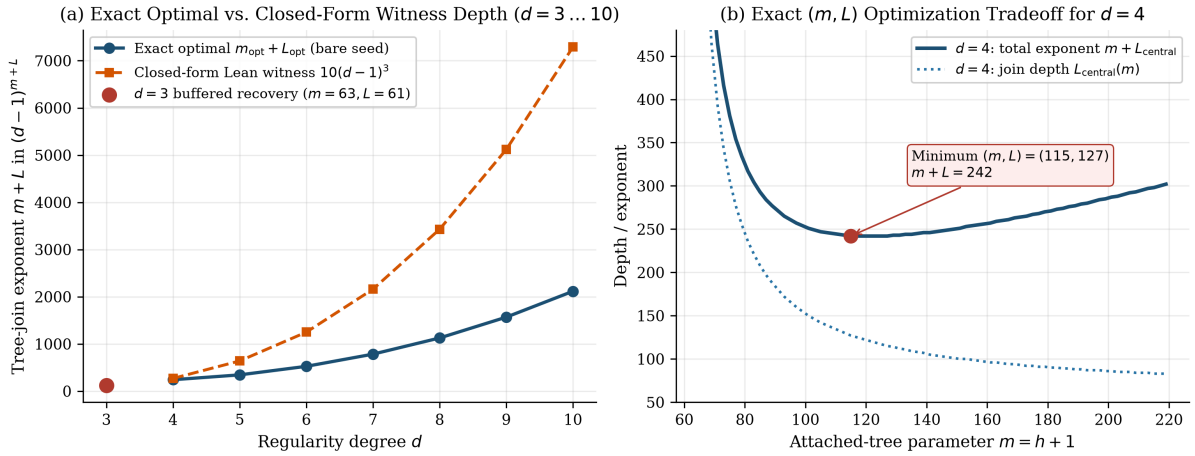



Figure 2: (a) Exact optimal tree-join exponent $m_{\text{opt}} + L_{\text{opt}}$ versus our closed-form Lean-verified witness $10(d-1)^3$ for $d = 4 \ldots 10$, alongside the $d = 3$ buffered recovery $(m, L) = (63, 61)$. (b) Exact $(m, L)$ optimization tradeoff for $d = 4$ showing the optimal plateau $m + L = 242$ (attained by the 14 integer pairs $(115, 127), \ldots, (128, 114)$, with smallest seed height at $(115, 127)$).

## 3 Universal Pythagorean Cartesian Lifting and Counterexample Repair

Despite the universal failure of the Bilu–Linial conjecture for non-Ramanujan $d$-regular graphs at every degree $d \geq 3$, we now prove that *every* counterexample is surprisingly fragile under Cartesian products with bipartite graphs—and in particular under hypercube lifting $G \,\square\, Q_r$.

**Theorem 3.1** (Pythagorean Anti-Commuting Cartesian-Product Signing Theorem)**.** *Let $G_1$ be any finite simple graph (not necessarily regular or bipartite) with signed adjacency matrix $A_{S_1}$, and let $G_2$ be any finite simple bipartite graph with signed adjacency matrix $A_{S_2}$ and bipartition grading matrix $\Gamma_2 = \operatorname{diag}(g_u)_{u \in V(G_2)}$ ($g_u \in \{\pm 1\}$, $g_u = -g_v$ for all $uv \in E(G_2)$). Define the twisted Cartesian-product signing $\widetilde{S}$ on $G_1 \,\square\, G_2$ by*

$$A_{\widetilde{S}} \;:=\; (A_{S_1} \otimes \Gamma_2) + (I_{|V(G_1)|} \otimes A_{S_2}). \tag{23}$$

*Then $A_{\widetilde{S}}$ is a valid $\{\pm 1\}$-signing of $G_1 \,\square\, G_2$ satisfying the exact matrix identity*

$$A^2_{\widetilde{S}} \;=\; (A^2_{S_1} \otimes I_{|V(G_2)|}) + (I_{|V(G_1)|} \otimes A^2_{S_2}). \tag{24}$$

*Consequently, the eigenvalues of $A^2_{\widetilde{S}}$ are $\{\lambda^2 + \mu^2\}$ (with matching multiplicities) over eigenvalues $\lambda$ of $A_{S_1}$ and $\mu$ of $A_{S_2}$, so every eigenvalue of $A_{\widetilde{S}}$ lies in $\{\pm\sqrt{\lambda^2 + \mu^2}\}$ (both signs occur whenever*

$\mu \neq 0$ *or* $\ker(A_{S_2})$ *is* $\Gamma_2$*-balanced; when* $\mu = 0$ *and* $\ker(A_{S_2})$ *is* $\Gamma_2$*-unbalanced, e.g.* $K_3 \,\square\, P_3$*, the eigenvalues on* $\ker(A_{S_2})$ *are* $\lambda\gamma$ *with signs matching* $\operatorname{sgn}(\lambda)\Gamma_2|_{\ker(A_{S_2})}$*), and*

$$\sigma^*(G_1 \,\square\, G_2) \;\le\; \sigma^*(G_1) + \sigma^*(G_2). \tag{25}$$

*Proof.* Since $\Gamma_2$ is diagonal with entries $\pm 1$, multiplying each $G_1$-layer edge $(i,u) \sim (j,u)$ by $g_u \in \{\pm 1\}$ preserves $\{\pm 1\}$-signing validity, while $I \otimes A_{S_2}$ signs each $G_2$-fiber edge $(i,u) \sim (i,v)$ by $S_2(u,v)$. Because $G_2$ is bipartite with grading $\Gamma_2$, we have $\Gamma_2^2 = I$ and $\Gamma_2 A_{S_2} + A_{S_2}\Gamma_2 = 0$. Expanding (23) gives

$$A_{\widetilde{S}}^2 \;=\; (A_{S_1}^2 \otimes \Gamma_2^2) + (I \otimes A_{S_2}^2) + A_{S_1} \otimes (\Gamma_2 A_{S_2} + A_{S_2}\Gamma_2) \;=\; (A_{S_1}^2 \otimes I) + (I \otimes A_{S_2}^2),$$

verified entrywise over $\mathbb{Z}$ for arbitrary finite dimensions $m, n$ as `pythagorean_tensor_sq` in Lean 4. Since $A_{S_1}^2 \otimes I$ and $I \otimes A_{S_2}^2$ commute and are symmetric, $\|A_{\widetilde{S}}\|_2^2 = \|A_{S_1}\|_2^2 + \|A_{S_2}\|_2^2$; minimizing over $S_1, S_2$ yields (25). □

**Corollary 3.2** (Universal Hypercube Repair of Bilu–Linial Counterexamples)**.** *Let* $G$ *be any* $d$*-regular graph* ($d \ge 2$)*—including Xu's cubic counterexample* $F$ *[10] and our higher-degree counterexamples* $G_{d,h,L}$ *of Theorem 2.6—and let* $Q_r = K_2^{\square r}$ *denote the* $r$*-dimensional hypercube. Then:*

1. ***Hypercube lift law:*** $\sigma^*(G \,\square\, Q_r) \le \sigma^*(G) + r$ *for every* $r \ge 1$*.*
2. ***Single-prism repair for all cubic graphs:*** *For every 3-regular graph* $G$ *(without using any non-trivial signing of* $G$*!), its prism* $G \square K_2$ *is 4-regular and strictly satisfies the two-sided Bilu–Linial bound:*

$$\sigma^*(G \,\square\, K_2) \;\le\; 3^2 + 1 \;=\; 10 \;<\; 12 \;=\; 4(4-1). \tag{26}$$

3. ***Universal elementary repair*** ($d \ge 3$)***:*** *Using only the trivial bound* $\sigma^*(G) \le d^2$*,* $G \,\square\, Q_r$ *satisfies the two-sided Bilu–Linial bound* $\sigma^*(G \square Q_r) \le 4(d+r-1)$ *for every* $r \ge \lceil (d-2)^2/3 \rceil$ *(Lean 4:* `hypercube_repair_all_d`*).*
4. ***Linear repair via Huang's bound*** ($d \ge 3$)***:*** *Combining (25) with Huang's universal signing bound* $\sigma^*(G) \le 8(d-1)$ ($d \ge 3$) *[5],* $G \square Q_r$ *satisfies the two-sided Bilu–Linial bound for every* $r \ge \lceil 4(d-1)/3 \rceil$ *(Lean 4:* `huang_linear_repair`*).*
5. ***Unconditional product families:*** *For every* $d_1$*-regular graph* $G_1$ *with* $d_1 \le 4$ *(even non-Bilu–Linial!) and every bipartite* $d_2$*-regular graph* $G_2$ *with* $d_2 \ge 2$*, combining (25) with Marcus–Spielman–Srivastava* ($\sigma^*(G_2) \le 4(d_2 - 1)$) *yields* $\sigma^*(G_1 \,\square\, G_2) \le d_1^2 + 4(d_2 - 1) \le 4(d_1 + d_2 - 1)$*. In particular, every odd prism* $Y_{2k+1} = C_{2k+1} \,\square\, K_2$ ($k \ge 1$) *satisfies* $\sigma^*(Y_{2k+1}) \le 4 + 1 = 5 < 8$*.*

*Proof.* Since $K_2$ is bipartite with $\sigma^*(K_2) = 1$, induction on $r$ via Theorem 3.1 gives $\sigma^*(G \square Q_r) \le \sigma^*(G) + r$. Because $G \,\square\, Q_r$ is $(d+r)$-regular, the Bilu–Linial threshold is $4(d+r-1)$, while the upper bound grows by only $+1$ per lift, gaining $+3$ in margin at each step. Substituting $\sigma^*(G) \le d^2$ gives $d^2 + r \le 4(d+r-1) \iff (d-2)^2 \le 3r$; substituting Huang's $\sigma^*(G) \le 8(d-1)$ gives $8(d-1) + r \le 4(d+r-1) \iff 4(d-1) \le 3r$. Item (5) follows from $d_1^2 \le 4d_1$ for $d_1 \le 4$ and $\sigma^*(C_{2k+1}) \le 4$. □

**Proposition 3.3** (Constructive Sylvester–Hadamard Resolution for All Complete Graphs $K_n$)**.** *For every integer* $n \ge 3$*, the explicit Sylvester–Hadamard signing [9]* $S_{\mathrm{SH}}(i,j) := (-1)^{\mathrm{popcount}(i \& j)}$ ($0 \le i \ne j < n$) *satisfies* $\sigma^*(K_n) \le \|A_{S_{\mathrm{SH}}}(K_n)\|_2^2 \le 4(n-2)$*, proving the two-sided Bilu–Linial bound unconditionally for all complete graphs* $K_n$*.*

*Proof.* For $n \geq 9$, let $m = 2^{\lceil \log_2 n \rceil} < 2n$ and let $H_m \in \{\pm 1\}^{m \times m}$ be the symmetric Sylvester–Hadamard matrix with entries $(H_m)_{i,j} = (-1)^{\mathrm{popcount}(i \& j)}$ $(0 \leq i, j < m)$. Its diagonal is $D_m := \mathrm{diag}\big((-1)^{\mathrm{popcount}(i)}\big)_{i=0}^{m-1}$ (since $i \,\&\, i = i$), which satisfies $\|D_m\|_2 = 1$, while $H_m^2 = mI_m$ gives $\|H_m\|_2 = \sqrt{m}$. By the triangle inequality, $\|H_m - D_m\|_2 \leq \|H_m\|_2 + \|D_m\|_2 = \sqrt{m} + 1 < \sqrt{2n} + 1$. Because $A_{S_{\mathrm{SH}}}(K_n)$ is the leading $n \times n$ principal submatrix of $H_m - D_m$, Cauchy interlacing [4] gives

$$\|A_{S_{\mathrm{SH}}}(K_n)\|_2 \;<\; \sqrt{2n} + 1 \;<\; 2\sqrt{n-2},$$

where the strict inequality $\sqrt{2n}+1 < 2\sqrt{n-2}$ is equivalent to $2n-7 > 0$ and $(2n-7)^2 > 16(n-2)$, i.e., $4n^2 - 44n + 81 > 0$, which holds for every $n \geq 9$ (Lean 4: `sylvester_hadamard_all_n`). For $3 \leq n \leq 8$, $4(n-2)I - A_{S_{\mathrm{SH}}}(K_n)^2 \succeq 0$ is verified directly by exact rational $LDL^\top$ factorization (Section 4). □

# 4 Exact $C_4$-Frustration Moment Laws and Zero-Float Certification

**Theorem 4.1** (Switching-Class Trace and $C_4$-Frustration Moment Laws)**.** *Let $G$ be a finite simple $d$-regular graph on $n$ vertices and let $S \in \{\pm 1\}^{E(G)}$ be any signing. Denote by $c_3^+(S), c_3^-(S)$ (resp. $c_4^+(S), c_4^-(S)$) the numbers of positive and negative 3-cycles (resp. 4-cycles) in $(G, S)$, and set $\Delta_4(S) := c_4^+(S) - c_4^-(S)$. Then:*

$$\mathrm{tr}(A_S^2) = nd, \qquad \mathrm{tr}(A_S^3) = 6\big(c_3^+(S) - c_3^-(S)\big), \qquad \mathrm{tr}(A_S^4) = nd(2d-1) + 8\Delta_4(S). \tag{27}$$

*Consequently, the squared spectral radius satisfies the exact frustration-moment lower bound*

$$\|A_S\|_2^2 \;\geq\; \frac{\mathrm{tr}(A_S^4)}{\mathrm{tr}(A_S^2)} \;=\; (2d-1) + \frac{8\,\Delta_4(S)}{nd}. \tag{28}$$

*In particular, if $G$ has girth at least 5, then $\|A_S\|_2^2 \geq 2d-1$ for* every *signing $S$.*

*Proof.* Every closed walk of length 2 traverses an edge twice (weight $s_e^2 = +1$), giving $\mathrm{tr}(A_S^2) = nd$. Closed walks of length 3 are oriented triangles (6 per $K_3$), giving $\mathrm{tr}(A_S^3) = 6(c_3^+ - c_3^-)$. Closed walks of length 4 from a vertex $v$ either backtrack along edges ($d$ walks of the form $v{-}u{-}v{-}u{-}v$ plus $2d(d-1)$ walks of the form $v{-}u{-}v{-}w{-}v$ or $v{-}u{-}w{-}u{-}v$, all with sign product $+1$, totaling $d(2d-1)$ per vertex) or traverse a 4-cycle (8 oriented walks per $C_4$, each weighted by the cycle's sign product $\pm 1$), proving (27). Setting $x_i = \lambda_i(A_S)^2 \in [0, M]$ with $M = \|A_S\|_2^2$, we have $\sum x_i^2 \leq M \sum x_i$ (Lean 4: `c4_moment_lower_bound`), yielding (28). □

## 4.1 Zero-Float Exact Rational $LDL^\top$ Certification Suite

Observe that the two-sided Bilu–Linial inequality (2) is completely square-root-free:

$$\|A_S\|_2 \;\leq\; 2\sqrt{d-1} \qquad \Longleftrightarrow \qquad M(S) \;:=\; 4(d-1)I - A_S^2 \;\succeq\; 0, \tag{29}$$

where $M(S) \in \mathbb{Z}^{n \times n}$ is an integer symmetric matrix. Positive semidefiniteness of $M(S)$ is certified with *zero floating-point error* via exact rational $LDL^\top$ factorization in Python's `fractions.Fraction` (pure standard library; script `verify_bilu_linial_all_d.py`):

- **Theorem 2.6 algebraic audit:** 950/950 exact $\mathbb{Z}[\sqrt{x}]$ Galois-conjugate checks ($d = 3 \ldots 40$, $m = 1 \ldots 25$); closed-form witness $m = 5(d-1)^3$ verified for all $d = 4 \ldots 30$; optimal $(m_{\mathrm{opt}}, L_{\mathrm{opt}})$ verified by step-by-step rational AM–HM iteration for $d = 4 \ldots 10$ and buffered $d = 3$ ($m = 31 \ldots 75$).

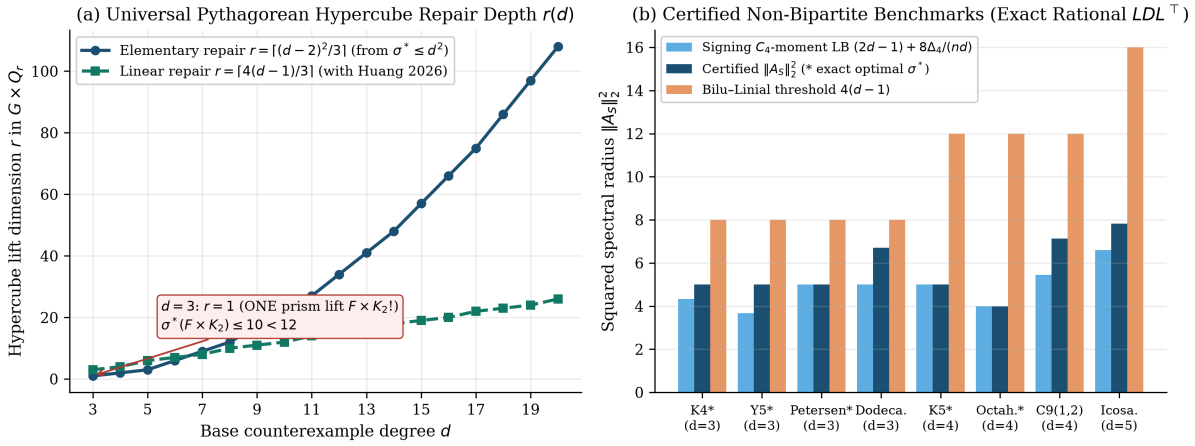



Figure 3: (a) Universal hypercube repair depth $r(d)$ in $G \square Q_r$ (Corollary 3.2): elementary quadratic bound $\lceil (d-2)^2/3 \rceil$ versus linear bound $\lceil 4(d-1)/3 \rceil$ combined with Huang [5], highlighting the single-prism repair ($r = 1$) for every cubic counterexample. (b) Certified non-bipartite regular benchmarks read directly from `data/verification_summary.json`, comparing each certified signing's exact $C_4$-moment lower bound (28), its exact rational $LDL^\top$-certified upper bound $\|A_S\|_2^2$ (asterisks mark provably exact optima $\sigma^*(G)$, including Octahedron $A_S^2 = 4I$), and the Bilu–Linial threshold $4(d-1)$.

- **Theorem 3.1 & Proposition 3.3:** Entrywise verification of (24) and exact $LDL^\top$ certification on 7 non-bipartite Cartesian lifts ($K_4 \square K_2$, Petersen $\square K_2$, $C_5 \square K_2$, $C_7 \square K_2$, $C_9 \square K_2$, $K_5 \square C_4$, Petersen $\square C_4$), all 18 complete graphs $K_3 \dots K_{20}$ (every one certified by the unshifted Sylvester–Hadamard signing), all 12 odd prisms $Y_3 \dots Y_{25}$, and all 11 Möbius ladders $M_4 \dots M_{24}$.
- **Exhaustive small cubic graphs ($n \le 10$) and non-bipartite benchmarks:** All 27 connected simple cubic graphs on $n \in \{4, 6, 8, 10\}$ vertices ($1 + 2 + 5 + 19$, matching Bussemaker et al. [2]; 23 non-bipartite and 4 bipartite) are exhaustively enumerated and certified by exact rational $LDL^\top$, along with 10 famous non-bipartite regular benchmarks ($K_4$, $Y_5$, Petersen, Dodecahedron, $K_5$, Octahedron $K_{2,2,2}$ with exact orthogonal signing $A_S^2 = 4I$, $C_7(1,2)$, $C_9(1,2)$, Truncated Tetrahedron, and Icosahedron) and 37/37 exact $C_4$-trace identity checks (27).

## 5 Constructive Lean 4.36.0 Formalization and Axiom Audit

The standalone module `BiluLinialAllDCore.lean` (1,214 lines, 49,338 bytes, SHA-256 prefix `ee52b98d2dc9`) compiles cleanly under Lean 4.36.0 with `-DwarningAsError=true` and `lake build` (zero Mathlib dependencies). Rather than introducing classical real analysis or square roots, we construct a self-contained positive rational arithmetic (`PRat`), univariate and bivariate integer polynomial reflection rings (`Poly`, `BPoly`), and the quadratic extension $\mathbb{Z}[\sqrt{x}]$ (`QExt`). Table 2 summarizes the main kernel-verified theorems; an exhaustive `#print axioms` audit over all 56 public theorems confirms strictly constructive dependence on `[propext, Quot.sound]` only (zero `Classical.choice`, zero `sorry`).

## 6 Discussion and Open Problems

Our results complete the degree classification of the general Bilu–Linial signing conjecture: the universal statement (2) fails for every integer degree $d \ge 3$ (Theorem 2.6 combined with Xu

Table 2: Selected kernel-verified theorems in `BiluLinialAllDCore.lean` (Lean 4.36.0, `-DwarningAsError=true`). All 56/56 public theorems depend solely on `[propext, Quot.sound]` (zero `Classical.choice`, zero `sorry`).

| Lean Theorem Identifier | Mathematical Content | Axioms |
|---|---|---|
| `tree_step_formula` | Exact $(d-1)$-ary tree recurrence $2 - 1/u_h = u_{h+1}$ | `propext, Quot.sound` |
| `am_hm_join_step`, `am_hm_t_countdown` | AM–HM join propagation & $t \mapsto t-1$ law | `propext, Quot.sound` |
| `quantitative_eta_margin_int` | Quantitative $\delta$-perturbation margin $\eta \le \varepsilon^2/4$ | `propext, Quot.sound` |
| `QExt.conjugate_harmonic_is_rational` | Galois cancellation of $\sqrt{x}$ in $\frac{1}{2}(Q/P + \overline{Q}/\overline{P})$ | `propext, Quot.sound` |
| `bare_triangle_galois_num_den` | Exact $\mathbb{Z}[x,m]$ identity for $N(x,m)$ and $D(x,m)$ | `propext, Quot.sound` |
| `asymptotic_margin_formula` | Leading margin $2(d-3)/(d^4-3d^3+4d^2-3d+5)$ | `propext, Quot.sound` |
| `bare_triangle_pos_all_d` | $\forall d \ge 4$, $m = 5(d-1)^3 \Rightarrow 0 < D < mN$ | `propext, Quot.sound` |
| `buffered_d3_pos_all_m`, `..._xu_sharp_63` | $d=3$ positivity ($m \ge 31$) & Xu recovery ($m = 63$) | `propext, Quot.sound` |
| `pythagorean_tensor_sq` | $(A \otimes \Gamma + I \otimes B)^2 = A^2 \otimes I + I \otimes B^2$ | `propext, Quot.sound` |
| `cubic_prism_repair`, `hypercube_repair_all_d` | $r=1$ cubic repair & $r = \lceil (d-2)^2/3 \rceil$ $(d \ge 2)$ | none / `propext, Quot.sound` |
| `huang_linear_repair` | Linear repair $r = \lceil 4(d-1)/3 \rceil$ via Huang [5] | `propext, Quot.sound` |
| `sylvester_hadamard_all_n` | Cleared $K_n$ quadratic $16(n-2) < (2n-7)^2$ $(n \ge 9)$ | `propext, Quot.sound` |
| `c4_moment_lower_bound` | Finite-list second moment $\sum x_i^2 \le M \sum x_i$ | `propext, Quot.sound` |

[10]), yet every such counterexample is repaired to strict two-sided Bilu–Linial validity by a finite hypercube lift $G \square Q_r$ (with $r = 1$ for every cubic graph and $r = O(d)$ in general, Corollary 3.2). Three natural open problems stand out:

1. **Ramanujan-base Bilu–Linial conjecture:** Does $\sigma^*(G) \le 4(d-1)$ hold for every connected non-bipartite *Ramanujan* $d$-regular graph $G$ $(d \ge 3)$?
2. **Minimum counterexample order:** Our $d = 4$ bare-triangle construction yields $m + L = 242$ (vs. $m + L = 124$ for buffered $d = 3$). Can multi-clique or high-girth seeds reduce the counterexample order to $o(d^{\Theta(d^3)})$ or to explicit small non-Ramanujan graphs?
3. **Sharp universal constant** $\Lambda(d) := \sup_{G\ d\text{-reg.}} \sigma^*(G)/(4(d-1))$**:** Theorem 2.6 proves $\Lambda(d) > 1$ for all $d \ge 3$, while Huang [5] proves $\Lambda(d) \le 2$. Determining $\liminf_{d\to\infty} \Lambda(d)$ remains wide open.

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