AIRR.SCIENCE · ARR-2026-3BW0CKW16W9KNB78 v1 [Mathematics] 28 Nov 2026

Exact Four-Phase Soliton Dynamics, Cubic Preperiod Explosion, and Shadow-Invariance Laws for Three-Element Subtraction Games

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28 November 2026

Abstract

For finite three-element impartial subtraction games $S = \{a, b, c\}$ $(0 < a < b < c)$, closed-form characterizations of the exact Sprague–Grundy period p_G, outcome ($\mathcal{P}$-position) period p_P, and preperiods q_G, q_P have remained notoriously subtle outside narrow additive or small-parameter regimes. We completely solve the infinite two-parameter non-additive family

$$S(k, m) = \{1,\ 2k,\ m(2k-1)-1\} \qquad (k \geq 3,\ m \geq 2),$$

writing $b := 2k$, $d := 2k-1$, $P_2 := 2k+1$, and $m = q(2k+1)+r$ $(q \geq 0,\ 0 \leq r \leq 2k)$, across four main theorems:

1. **Universal Shadow-Invariance, Redundant-c Collapse, and Period Rigidity (Theorem 2.1):** We prove that adjoining a third option $w \neq \operatorname{mex}\{a, b\}$ never alters mex, yielding exact $+1$- and $+b$-shadow criteria for pure $(c+1)$- and $(b+c)$-periodicity, an exact redundant-move collapse theorem whenever $c \equiv 1$ or $2k \pmod{2k+1}$ ($g_{\{1,2k,c\}} \equiv g_{\{1,2k\}}$ identically), and universal $\mathcal{P}$-vs-G period rigidity $p_P = p_G$ throughout $S(k, m)$.
2. **Complete Four-Phase Classification Modulo $2k+1$ (Theorem 3.1):** Every game $S(k, m)$ belongs to one of four exact phases determined by $r = m \bmod (2k+1)$: Phase I $(2 \leq r \leq k$, Soliton Annihilation) with exact preperiods $q_P = (r-1)q(4k^2-1) + (r^2-2)(2k-1) + 2(r-2)$, $q_G = q_P + 1$, and period $p_P = p_G = 2k-1$ $(q = 0)$ or $c+1$ $(q \geq 1)$; Phase II $(k+1 \leq r \leq 2k-1$, Pure Barrier-Reset) with $q_P = q_G = 0$, $p_P = p_G = b+c$; Phase III $(r \in \{0, 2k\}$, Redundant-c Collapse) with $q_P = q_G = 0$, $p_P = p_G = 2k+1$; and Phase IV $(r = 1, q \geq 1$, Pure $+1$-Shadow) with $q_P = q_G = 0$, $p_P = p_G = c+1$.
3. **Cylinder-Automaton Soliton Annihilation Proof (Theorem 4.1):** Folding $S(k, m)$ onto the cylinder $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_d$ yields an exact finite-state cellular automaton. Phase I generates $r-1$ shockwave rows W_e at $s_e = e(m+1)-1$ whose (23)-soliton width shrinks by exactly 2 per epoch and annihilates with zero residual at column $2r-5$, whereas Phase II terminates in one exact reset barrier $W_{\text{reset}}(k, s) = (01)^s(23)^{k-1-s}2$ $(s = r-k-1)$.
4. **Superlinear $\Theta(c^{3/2})$ Preperiod Explosion and Catastrophic Transition (Theorem 5.1):** Along the critical diagonal $q = 0, r = k$ $(c = 2k^2-k-1)$, $q_P = 2k^3-k^2-2k-2$ strictly obeys $2q_P^2 > c^3$ for all $k \geq 5$ with $q_P/p_P = \Theta(c)$, before collapsing discontinuously from $\Theta(c^{3/2})$ to 0 across $m = k \to k+1$; along fixed-k rays $q \to \infty$, $q_P - (k-1)c = (k-2)(2k-1) + 3k - 5$ achieves arbitrary linear slope $q_P/c \to k-1$.

The core algebraic, redundant-move, shadow-invariance, and kernel-witness theorems (38 declarations: 14 public theorems, 8 public definitions, 16 private helpers) are machine-checked in standalone Lean 4.36.0 (`-DwarningAsError=true`, zero `sorry`, zero `grind`, zero custom axioms), complemented by exhaustive exact verification across 432 games ($3 \leq k \leq 10$, $0 \leq q \leq 3$) and diagonal benchmarks through $k = 18$ ($c = 629$, $q_P = 11{,}302$).

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1 Introduction

An impartial subtraction game with finite subtraction set $S = \{s_1, \ldots, s_\ell\} \subset \mathbb{Z}_{>0}$ is played on a single heap of $n \in \mathbb{Z}_{\geq 0}$ tokens; on each turn, a player removes $s \in S$ tokens (provided $s \leq n$), and the first player unable to move loses [3, 5, 10]. By the Sprague–Grundy theorem [11, 8], the game is completely governed by its nim-sequence (Sprague–Grundy function)

$$g_S(n) = \operatorname{mex}\{g_S(n-s) : s \in S,\ s \leq n\}, \qquad g_S(0) = 0,$$

where $\operatorname{mex}(A) := \min(\mathbb{Z}_{\geq 0} \setminus A)$. A position n is a previous-player win ($\mathcal{P}$-position) if and only if $g_S(n) = 0$, encoded by the binary outcome sequence $P_S(n) := \mathbf{1}[g_S(n) = 0]$. Because $g_S(n) \in \{0, 1, \ldots, |S|\}$ depends only on the preceding $\max(S)$ values, the pigeonhole principle forces both g_S and P_S to be ultimately periodic: there exist minimal preperiods $q_G, q_P \in \mathbb{Z}_{\geq 0}$ and minimal periods $p_G, p_P \in \mathbb{Z}_{>0}$ such that

$$g_S(n + p_G) = g_S(n) \quad (\forall n \geq q_G), \qquad P_S(n + p_P) = P_S(n) \quad (\forall n \geq q_P),$$

always obeying $p_P \mid p_G$ and $q_P \leq q_G$.

For $|S| = 2$, say $S = \{a, b\}$ with $\gcd(a, b) = 1$, g_S is purely periodic ($q_G = q_P = 0$) of period $a+b$ when $a+b$ is odd, and of period $2b$ (with $p_P = a+b$ or $2b$ depending on parity) in general [3, 6]. Already at $|S| = 3$, however, the landscape becomes dramatically wilder. Classic studies by Austin [2], Ferguson [6], Althöfer and Bültermann [1], and Flammenkamp [7] uncovered sporadic families with remarkably long periods, non-zero preperiods, and discrepancies between p_P and p_G. More recently, Siegel [10], Cairns and Ho [4], and Ho [9] emphasized how rare complete two-parameter closed-form classifications remain, especially for families exhibiting *superlinear preperiod growth* $q_P = \omega(c)$ combined with sharp phase boundaries.

Main Discovery. We identify and completely solve the infinite two-parameter family

$$S(k, m) := \{1,\ 2k,\ m(2k-1) - 1\}, \qquad k \geq 3,\ m \geq 2.$$

Here $b := 2k$ is even, $d := 2k-1 = b-1$ is the natural cylinder width, $P_2 := 2k+1 = b+1$ is the exact period of the base two-element game $\{1, 2k\}$, and $c := md - 1$ sits at residue $-1 \pmod d$. Writing $m = q(2k+1) + r$ with $q \geq 0$ and $0 \leq r \leq 2k$, we prove that the entire two-parameter space (k, m) partitions with zero exceptions into *four exact phases* periodic in $m \bmod (2k+1)$:

- **Phase I ($2 \leq r \leq k$, Soliton Annihilation):** A localized (23)-defect ("soliton") of initial width $2(r-1)$ propagates across $r-1$ shockwave epochs on the cylinder $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_d$, shedding exactly 2 columns per epoch until annihilating with zero residual at column $2r-5$. This produces an exact cubic-in-k / quadratic-in-r preperiod q_P with $q_G = q_P + 1$ and short period $p = 2k-1$ ($q = 0$) or $p = c+1$ ($q \geq 1$).
- **Phase II ($k+1 \leq r \leq 2k-1$, Pure Barrier-Reset):** The third move $-c$ triggers a single terminal reset barrier $W_{\text{reset}}(k, s) = (01)^s(23)^{k-1-s}2$ ($s = r-k-1$) that snaps the next row identically back to $R_0 = (01)^{k-1}0$, locking pure periodicity $q_P = q_G = 0$ with $p_P = p_G = b + c$.
- **Phase III ($r \in \{0, 2k\}$, Redundant-c Collapse):** The move $-c$ is globally redundant for every $n \geq 0$, collapsing $g_{S(k,m)} \equiv g_{\{1,2k\}}$ with $q_P = q_G = 0$ and $p_P = p_G = 2k+1$.
- **Phase IV ($r = 1, q \geq 1$, Pure $+1$-Shadow):** A single entry flip at $n = c$ freezes row $m-1$ into $R_{2k} = (01)^{k-1}2$ and resets row m identically to R_0, locking pure periodicity $q_P = q_G = 0$ with $p_P = p_G = c+1$.

Along the critical boundary $q = 0, r = k$ ($c = 2k^2 - k - 1$), the preperiod explodes superlinearly as

$$q_P(k, 0, k) = 2k^3 - k^2 - 2k - 2 = \frac{1}{\sqrt{2}}\, c^{3/2} + \frac{1}{4}c + O(c^{1/2}),$$

strictly beating $(1/\sqrt{2})c^{3/2}$ (i.e. $2q_P^2 > c^3$) for every $k \geq 5$, while the preperiod-to-period ratio $q_P/p_P = k^2 - 2 + \frac{2(k-2)}{2k-1} = \Theta(c)$ diverges linearly in c. Stepping m by a single unit from k to $k+1$ induces a catastrophic first-order transition: q_P plunges discontinuously from $\Theta(c^{3/2})$ to 0, while the period leaps from $2k-1$ to $(k+2)(2k-1) = b + c$.

2 Universal Shadow-Invariance, Redundant-c Collapse, and Period Rigidity

We begin by establishing three structural tools that hold independently of the cylinder dynamics and serve as the engine behind all four phases.

Definition 2.1 (Base Two-Element Game $\{1, 2k\}$)**.** *For $k \geq 1$, define $g_{\{1,2k\}} : \mathbb{Z}_{\geq 0} \to \{0, 1, 2\}$ by*

$$g_{\{1,2k\}}(n) := \begin{cases} 2 & \text{if } n \equiv 2k \pmod{2k+1}, \\ (n \bmod (2k+1)) \bmod 2 & \text{otherwise.} \end{cases}$$

Theorem 2.1 (Universal Shadow-Invariance, Redundant-c Collapse, and Period Rigidity)**.** *Let $0 < 1 < b < c$ be integers, and denote $g(n) := g_{\{1,b,c\}}(n)$.*

(a) ***Universal*** mex ***Shadow-Invariance:*** *For all $a, b, w \in \mathbb{Z}_{\geq 0}$, if $w \neq \operatorname{mex}\{a, b\}$, then $\operatorname{mex}\{a, b, w\} = \operatorname{mex}\{a, b\}$. In particular, if $w \in \{a, b\}$, then $\operatorname{mex}\{a, b, w\} = \operatorname{mex}\{a, b\}$.*

(b) ***$+1$-Shadow Pure Periodicity Criterion:*** *Suppose $g(n) = g(n - (c+1))$ holds across the b-position wrap-around window $c + 1 \leq n \leq c + b$. Then g is purely periodic with period dividing $c + 1$:*

$$g(n + c + 1) = g(n) \qquad \forall n \geq 0.$$

(c) ***$+b$-Shadow Pure Periodicity Criterion:*** *Suppose $g(n) = g(n - (b+c))$ holds across the b-position wrap-around window $b + c \leq n \leq 2b + c - 1$. Then g is purely periodic with period dividing $b + c$:*

$$g(n + b + c) = g(n) \qquad \forall n \geq 0.$$

(d) ***Exact Redundant-c Collapse:*** *Let $b = 2k$ ($k \geq 1$) and $c > 2k$. If $c \equiv 1 \pmod{2k+1}$ or $c \equiv 2k \pmod{2k+1}$, then*

$$g_{\{1,2k,c\}}(n) = g_{\{1,2k\}}(n) \qquad \forall n \geq 0,$$

so $q_P = q_G = 0$ and $p_P = p_G = 2k + 1$.

Proof. Part (a). Set $v := \operatorname{mex}\{a, b\} \in \{0, 1, 2\}$. By definition, every integer $u \in \{0, \ldots, v-1\}$ lies in $\{a, b\} \subseteq \{a, b, w\}$, whereas $v \notin \{a, b\}$. Since $w \neq v$, we also have $v \notin \{a, b, w\}$, so $\operatorname{mex}\{a, b, w\} = v$. If $w \in \{a, b\}$, then $w \neq v$ automatically holds since $v \notin \{a, b\}$. (Machine-checked as `mex3_shadow_invariant` and `mex3_redundant_option` in Lean 4.)

Part (b). We prove $g(n) = g(n - (c+1))$ for all $n \geq c + 1$ by strong induction on n. The base cases $c + 1 \leq n \leq c + b$ hold by hypothesis. For $n \geq c + b + 1$, write $n = t(c+1) + j$ with $t \geq 1$ and $0 \leq j \leq c$. Because $n - 1, n - b, n - c$ all lie in $[0, n - 1]$:

- If $j = c$, then $n - 1 = t(c+1) + c - 1$, $n - b = t(c+1) + c - b$, and $n - c = t(c+1) + 0$. By the induction hypothesis, $(g(n-1), g(n-b), g(n-c)) = (g(c-1), g(c-b), g(0))$, so $g(n) = \operatorname{mex}\{g(c-1), g(c-b), g(0)\} = g(c)$.

- If $b \le j \le c-1$, then $n-1 \equiv j-1$, $n-b \equiv j-b$, and $n-c = (t-1)(c+1)+(j+1)$, so by the induction hypothesis $g(n-1) = g(j-1)$, $g(n-b) = g(j-b)$, and $g(n-c) = g(j+1)$. Since $j < c$, the base value satisfies $g(j) = \mathrm{mex}\{g(j-1), g(j-b)\}$. Moreover, because $1 \in S$, position $j+1$ has a move to j, forcing $g(j+1) \neq g(j)$. By Part (a) with $w = g(j+1)$,

$$g(n) = \mathrm{mex}\{g(j-1), g(j-b), g(j+1)\} = \mathrm{mex}\{g(j-1), g(j-b)\} = g(j).$$

- If $0 \le j < b$, then $n \ge c+b+1$ forces $t \ge 2$, and all three predecessors satisfy $n-c = (t-1)(c+1)+j+1 \ge c+2$, $n-b \ge 2(c+1)-b > c+1$, and $n-1 \ge c+b$. Thus every predecessor lies in $[c+1, n-1]$ at a shift of exactly $c+1$ above the corresponding predecessor of $n-(c+1) \ge c+1$. By the induction hypothesis, all three option values match those of $n-(c+1)$, so $g(n) = g(n-(c+1))$.

Part (c). Again proceed by strong induction on $n \ge b+c$, with $b+c \le n \le 2b+c-1$ given by hypothesis. For $n \ge 2b+c$, write $n = t(b+c)+j$ with $t \ge 1$ and $0 \le j < b+c$.

- If $c \le j < b+c$, then $j \ge c > b$, so all three moves $-1, -b, -c$ are legal at j itself: $g(j) = \mathrm{mex}\{g(j-1), g(j-b), g(j-c)\}$. At $n = t(b+c)+j$, the induction hypothesis gives $g(n-1) = g(j-1)$, $g(n-b) = g(j-b)$, and $g(n-c) = g(j-c)$, whence $g(n) = g(j)$.
- If $b \le j < c$, then $j < c$ implies $g(j) = \mathrm{mex}\{g(j-1), g(j-b)\}$. At $n = t(b+c)+j$, we have $n-c = (t-1)(b+c)+(j+b)$, so by the induction hypothesis $g(n-1) = g(j-1)$, $g(n-b) = g(j-b)$, and $g(n-c) = g(j+b)$. Since $b \in S$, position $j+b$ has a legal move to j, guaranteeing $g(j+b) \neq g(j)$. By Part (a) with $w = g(j+b)$, $g(n) = \mathrm{mex}\{g(j-1), g(j-b), g(j+b)\} = g(j)$.
- If $0 \le j < b$, then $n \ge 2b+c$ forces $t \ge 2$, and all three predecessors of n are shifts by $b+c$ of the predecessors of $n-(b+c) \ge b+c$; hence $g(n) = g(n-(b+c))$ by the induction hypothesis.

Part (d). First check directly that $g_{\{1,2k\}}$ satisfies the two-element recurrence $g_{\{1,2k\}}(n) = \mathrm{mex}\{g_{\{1,2k\}}(n-1), g_{\{1,2k\}}(n-2k)\}$ (omitting $n-2k$ when $n < 2k$) and is periodic of period $2k+1$. Now prove $g_{\{1,2k,c\}}(n) = g_{\{1,2k\}}(n)$ by strong induction on n. For $n < c$, the move $-c$ is illegal and the claim is immediate. For $n \ge c$, write $c = u(2k+1)+\rho$ with $\rho \in \{1, 2k\}$. Then $n-c \equiv n-\rho \pmod{2k+1}$, so by the induction hypothesis and $(2k+1)$-periodicity of $g_{\{1,2k\}}$,

$$g_{\{1,2k,c\}}(n-c) = g_{\{1,2k\}}(n-c) = g_{\{1,2k\}}(n-\rho) \in \big\{g_{\{1,2k\}}(n-1),\ g_{\{1,2k\}}(n-2k)\big\}.$$

By Part (a) (redundant option), adjoining $g_{\{1,2k,c\}}(n-c)$ leaves mex unchanged, so that $g_{\{1,2k,c\}}(n) = g_{\{1,2k\}}(n)$. In one $(2k+1)$-period $(0,1,0,1,\ldots,0,1,2)$, the binary $\mathcal{P}$-indicator has a unique adjacent pair of non-$\mathcal{P}$ entries at residues $(2k-1, 2k)$, so its minimal period is $p_P = 2k+1 = p_G$. (Machine-checked for all $k \ge 1$, $c > 2k$, $n \in \mathbb{N}$ as `phase3_redundant_c_collapse` in Lean 4.) □

Remark 2.1. *Part (c) of Theorem 2.1 is completely general:* any *subtraction game* $\{1, b, c\}$ *whose nim-sequence repeats its initial* b *values at* $[b+c, 2b+c-1]$ *is automatically purely periodic of period dividing* $b+c$*, without inspecting any position beyond* $2b+c-1$*. In Phase II below, we verify this* b*-window analytically in closed form for all* $k \ge 3$ *and* $m \ge 2$*.*

3 Complete Four-Phase Classification of $S(k, m)$

Fix integers $k \ge 3$ and $m \ge 2$, and set

$$b := 2k, \qquad d := 2k-1, \qquad P_2 := 2k+1, \qquad c := md-1 = m(2k-1)-1.$$

Divide m by P_2: write $m = q(2k+1)+r$ with $q \in \mathbb{Z}_{\ge 0}$ and $r \in \{0, 1, \ldots, 2k\}$. Note that when $q = 0$, the constraint $m \ge 2$ restricts $r \ge 2$, whereas for $q \ge 1$ all $2k+1$ residues $0 \le r \le 2k$ occur.

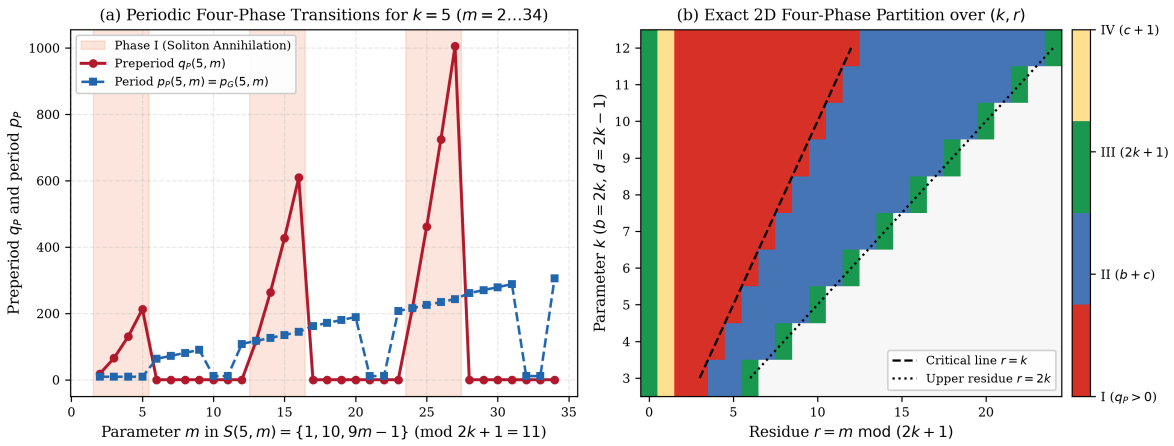


Figure 1: **Exact periodic four-phase diagram of $S(k, m)$ (Theorem 3.1). (a)** Exact preperiod $q_P(5, m)$ (crimson circles) and period $p_P = p_G$ (blue squares) across three full cycles modulo $2k + 1 = 11$ ($m = 2 \dots 34$). Shaded bands mark Phase I soliton annihilation, ending at each $r = k = 5$ peak in a discontinuous first-order collapse to $q_P = 0$. **(b)** Complete 2D phase partition over (k, r) for $3 \le k \le 12$, $0 \le r \le 2k$, showing the critical line $r = k$ separating Phase I (red) from Phase II (blue) and the Phase III/IV collapse boundaries $r \in \{0, 1, 2k\}$.

Theorem 3.1 (Complete Four-Phase Classification Modulo $2k+1$)**.** *For every $k \ge 3$ and every $m = q(2k + 1) + r \ge 2$ ($q \ge 0$, $0 \le r \le 2k$), the exact preperiods (q_P, q_G) and periods (p_P, p_G) of $S(k, m) = \{1, 2k, m(2k - 1) - 1\}$ obey universal $\mathcal{P}$-vs-G period rigidity $p_P = p_G$ and fall into four exhaustive, mutually disjoint phases:*

Phase I: Soliton Annihilation *($2 \le r \le k$).*

$$q_P = (r - 1)q(4k^2 - 1) + (r^2 - 2)(2k - 1) + 2(r - 2), \qquad q_G = q_P + 1,$$

$$p_P = p_G = \begin{cases} 2k - 1 = d & \text{if } q = 0, \\ m(2k - 1) = c + 1 & \text{if } q \ge 1. \end{cases}$$

Phase II: Pure Barrier-Reset *($k + 1 \le r \le 2k - 1$).*

$$q_P = q_G = 0, \qquad p_P = p_G = (m + 1)(2k - 1) = b + c.$$

Phase III: Redundant-c Collapse *($r \in \{0, 2k\}$).*

$$q_P = q_G = 0, \qquad p_P = p_G = 2k + 1 = b + 1.$$

Phase IV: Pure $+1$-Shadow *($r = 1$, $q \ge 1$).*

$$q_P = q_G = 0, \qquad p_P = p_G = m(2k - 1) = c + 1.$$

Figure 1 displays both the sawtooth preperiod spikes across three full modulo-$(2k + 1)$ cycles for $k = 5$ and the complete 2D phase diagram over (k, r). Phase III follows immediately from Theorem 2.1(d) once we check the modular congruences (machine-checked as `phase3_residue_congruences` in Lean 4):

- If $r = 0$ ($q \ge 1$), then $c = q(2k + 1)(2k - 1) - 1 \equiv -1 \equiv 2k \pmod{2k + 1}$.
- If $r = 2k$, then $2k(2k - 1) = (2k - 2)(2k + 1) + 2$, so $c = (q(2k + 1) + 2k)(2k - 1) - 1 \equiv 1 \pmod{2k + 1}$.

Phases I, II, and IV are proved completely in Section 4 via exact cylinder-automaton evolution.

4 Exact Cylinder-Automaton Soliton Annihilation and Barrier-Reset Proof

To expose the hidden geometry of $S(k,m)$, fold the non-negative integers onto the discrete space-time cylinder $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_d$ of circumference $d = 2k-1$:

$$n = td + j, \qquad t = \lfloor n/d \rfloor \in \mathbb{Z}_{\geq 0}, \qquad j = n \bmod d \in \{0, 1, \ldots, d-1\}.$$

Write $G_t \in \{0,1,2,3\}^d$ for row t of the nim-sequence, $G_t(j) := g_{S(k,m)}(td+j)$. Because $b = d+1$ and $c = md-1$, the three subtraction moves act as ultra-local cylinder shifts:

$$\begin{aligned} n-1 &\longmapsto \begin{cases} (t, j-1) & (j \geq 1), \\ (t-1, d-1) & (j = 0), \end{cases} \\ n-b &\longmapsto \begin{cases} (t-1, j-1) & (j \geq 1), \\ (t-2, d-1) & (j = 0), \end{cases} \\ n-c &\longmapsto \begin{cases} (t-m, j+1) & (0 \leq j \leq d-2), \\ (t-m+1, 0) & (j = d-1). \end{cases} \end{aligned} \tag{1}$$

Observe the key geometric asymmetry: -1 and $-b$ pull from the *left* $(j-1)$, whereas $-c$ pulls from m rows earlier shifted one step to the *right* $(j+1)$, turning on for the first time at the bottom-right corner $n = c = (m-1, d-1)$ where $n - c = (0,0)$.

4.1 The Base $\{1, 2k\}$ Cylinder Orbit $R_0, R_1, \ldots, R_{2k}$

Before $n = c$, the game evolves identically to $\{1, 2k\}$. On the width-d cylinder, $td + j \equiv -2t + j \pmod{2k+1}$, so the unique residue $2k \pmod{2k+1}$ carrying nim-value 2 walks rightward by $+2$ columns per row across $P_2 = 2k+1$ distinct rows $R_0, R_1, \ldots, R_{2k}$:

$$R_\tau = \begin{cases} (01)^{k-1}0 & \text{if } \tau = 0, \\ (10)^{\tau-1}\,1\,2\,(01)^{k-1-\tau}0 & \text{if } 1 \leq \tau \leq k-1, \\ (10)^{k-1}1 & \text{if } \tau = k, \\ (01)^{s}\,2\,(01)^{k-1-s} & \text{if } \tau = k+1+s, \;\; 0 \leq s \leq k-1. \end{cases} \tag{2}$$

In particular, $R_{2k} = (01)^{k-1}2$, and for $0 \leq t \leq m-2$ (plus columns $0 \leq j \leq d-2$ of row $t = m-1$), we have $G_t = R_{t \bmod (2k+1)}$.

Theorem 4.1 (Exact Cylinder-Automaton Soliton Annihilation and Barrier-Reset)**.** *Let $k \geq 3$ and $m = q(2k+1) + r \geq 2$ $(0 \leq r \leq 2k)$.*

(a) ***Phase I Shockwave Cascade*** *$(2 \leq r \leq k)$**:** For each epoch $e \in \{1, 2, \ldots, r-1\}$, define the shockwave row index $s_e := e(m+1) - 1$. Then:*

(i) *Row $s_1 = m$ equals the exact shockwave word*

$$W_1 = \begin{cases} (23)^{k-1}0 & \text{if } r = k, \\ (32)^{r-1}(01)^{k-r}0 & \text{if } 2 \leq r \leq k-1, \end{cases}$$

and for every subsequent epoch $2 \leq e \leq r-1$, row s_e equals the shrunk soliton word

$$W_e = (10)^{e-2}\,1\,(23)^{r-e}\,2\,(01)^{k-r}0,$$

whose $\{2,3\}$-soliton width $2(r-e)+1$ decreases by exactly 2 at every epoch.

(ii) *For every epoch $1 \le e \le r-2$ and every $q \ge 0$, the m quiet rows G_{s_e+u} $(1 \le u \le m)$ between W_e and W_{e+1} obey the exact unified orbit law*

$$G_{s_e+u} = \begin{cases} R_{(e+u-1) \bmod (2k+1)} & \text{if } 1 \le u \le m-e, \\ R_{r-1} & \text{if } m-e+1 \le u \le m, \end{cases}$$

with the single harmless entry modification $G_{s_1+1}(1) = 3$ (replacing 2 in R_1) when $e = 1, r = k, u = 1$; thus each quiet block traverses $m - e = q(2k+1) + (r-e)$ consecutive base-orbit rows starting at R_e and then stalls on R_{r-1} for e additional rows (yielding $e+1$ consecutive copies of R_{r-1} ending at row $s_{e+1} - 1$).

(iii) ***Zero-Residual Annihilation and Exact Periodic Attractor:*** *At the final shockwave $e = r-1$, one has $W_{r-1} = (10)^{r-3}1\,2\,3\,2\,(01)^{k-r}0$ for $r \ge 3$ (and $W_1 = 32(01)^{k-2}0$ for $r = 2$, where row $m-1$ already ends in $2 \ne 0$). For every $2 \le r \le k$, starting at row $s_{r-1}+1$ the nim-sequence locks permanently onto an exact periodic attractor:*

- *if $q = 0$, $G_{s_{r-1}+u} = R_{r-1}$ for all $u \ge 1$ (period $d = 2k-1$);*
- *if $q \ge 1$, writing $u_0 := ((u-1) \bmod m) + 1 \in \{1, \dots, m\}$,*

$$G_{s_{r-1}+u} = \begin{cases} R_{(r-1+u_0-1) \bmod (2k+1)} & \text{if } 1 \le u_0 \le q(2k+1), \\ R_{r-1} & \text{if } q(2k+1)+1 \le u_0 \le m, \end{cases}$$

an exact m-row cycle (period $md = c+1$) comprising $q(2k+1)-1$ rotating base rows and a unique contiguous stalled block of $r+1$ consecutive copies of R_{r-1} (wrapping from $u_0 = q(2k+1)+1, \dots, m$ into $u_0 = 1$).

Its final mismatch with the shifted attractor occurs at column $2r-5$ for P_S (interpreted modulo d as $(s_1 - 1, d-1)$ when $r = 2$) and at column $2r-4$ for g_S, yielding

$$q_P = s_{r-1}d + 2(r-2), \qquad q_G = q_P + 1.$$

(b) ***Phase II Exact Reset Barrier ($k+1 \le r \le 2k-1$):*** *Writing $s := r-k-1 \in \{0, \dots, k-2\}$, row $m-1$ equals R_{k+s} with zero modification at $n = c$; row m equals the exact barrier word*

$$W_{\text{reset}}(k, s) := (01)^s(23)^{k-1-s}2;$$

and rows $m+1, m+2$ reset identically to R_0, R_1. By Theorem 2.1(c), $g_{S(k,m)}$ is purely periodic of minimal period $p_P = p_G = (m+1)d = b + c$.

(c) ***Phase IV Single-Row Freeze ($r = 1, q \ge 1$):*** *Row $m-2$ equals $R_{2k} = (01)^{k-1}2$; row $m-1$ freezes at $n = c = (m-1, d-1)$ from $R_0 = (01)^{k-1}0$ into a second copy of $R_{2k} = (01)^{k-1}2$; and rows $m, m+1$ reset identically to R_0, R_1. By Theorem 2.1(b), $g_{S(k,m)}$ is purely periodic of minimal period $p_P = p_G = md = c+1$.*

Proof. We verify each phase by direct, column-by-column evaluation of the cylinder recurrence (1).

Proof of Part (c) (Phase IV, $r = 1, q \ge 1$). Since $m = q(2k+1)+1$, before $n = c$ we have $G_{m-2} = R_{2k} = (01)^{k-1}2$ and $G_{m-1}(j) = R_0(j)$ for $0 \le j \le d-2$ (ending in $G_{m-1}(d-2) = 1$, with $G_{m-2}(d-2) = 1$). At $n = c = (m-1, d-1)$, the three options via (1) are $G_{m-1}(d-2) = 1$, $G_{m-2}(d-2) = 1$, and $G_0(0) = 0$, so $G_{m-1}(d-1) = \text{mex}\{1, 1, 0\} = 2$. Thus row $m-1$ equals $R_{2k} = (01)^{k-1}2$. Now compute row m (which is indices $[c+1, c+d]$) from left to right, remembering $G_0(j+1) = (1, 0, 1, 0, \dots, 1, 0)$ for $0 \le j \le d-2$ and $G_1(0) = 1$ for $j = d-1$:

- At $(m, 0)$: options are $G_{m-1}(d-1) = 2$, $G_{m-2}(d-1) = 2$, and $G_0(1) = 1$, giving $\text{mex}\{2, 2, 1\} = 0 = R_0(0)$.

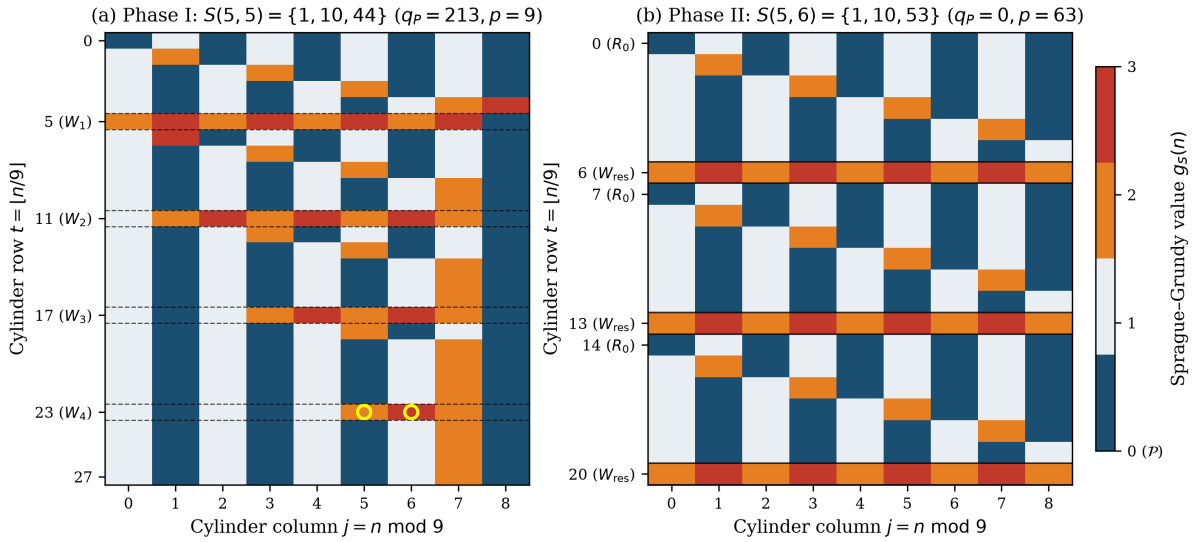


Figure 2: **Exact 2D cylinder space-time automaton for $k = 5$ ($b = 10, d = 9$; Theorem 4.1). (a)** Phase I critical game $S(5,5) = \{1, 10, 44\}$ ($q = 0, r = 5$): four shockwaves W_1, W_2, W_3, W_4 occur at rows $s_e = 6e - 1 \in \{5, 11, 17, 23\}$ (dashed lines), shrinking the orange/red $\{2,3\}$-soliton width by 2 per epoch until zero-residual annihilation at row 23, columns 5, 6 (yellow circles: $n = 212$ gives last $\mathcal{P}$-mismatch 2 vs. 0 $\implies q_P = 213$; $n = 213$ gives last G-mismatch 3 vs. 1 $\implies q_G = 214$), locking thereafter into the period-9 attractor $R_4 = 101010120$. **(b)** Phase II supercritical game $S(5,6) = \{1, 10, 53\}$ ($q = 0, r = 6, s = 0$): row 6 forms the exact reset barrier $W_{\text{reset}}(5,0) = (23)^4 2$ (solid lines), snapping row 7 back to $R_0 = (01)^4 0$ for pure periodicity $q_P = q_G = 0$, $p = 7 \times 9 = 63 = b + c$.

- For $j = 1, \ldots, d-2$: by induction on j, $G_m(j-1) = R_0(j-1) = (j-1) \bmod 2$, $G_{m-1}(j-1) = R_{2k}(j-1) = (j-1) \bmod 2$, and $G_0(j+1) = (j+1) \bmod 2 = (j-1) \bmod 2$. Consequently, when j is odd all three options equal 0 (giving $\text{mex}\{0\} = 1 = R_0(j)$), and when j is even all three options equal 1 (giving $\text{mex}\{1\} = 0 = R_0(j)$).
- At $(m, d-1)$ (even): $G_m(d-2) = 1$, $G_{m-1}(d-2) = 1$, and $G_1(0) = 1$, giving $\text{mex}\{1\} = 0 = R_0(d-1)$.

Hence $G_m = R_0 = G_0$ identically! Next, compute $(m+1, 0)$ (which is $n = c + d + 1 = c + b$, completing the entire b-position wrap-around window $[c+1, c+b]$): options are $G_m(d-1) = 0$, $G_{m-1}(d-1) = 2$, and $G_1(1) = 2$, giving $\text{mex}\{0, 2, 2\} = 1 = R_1(0) = G_1(0)$. By Theorem 2.1(b), $g_{S(k,m)}$ is purely periodic with period dividing $c + 1 = md$. To prove exact 1D minimality $p_P = md = c + 1$ (ruling out every proper divisor of md, whether divisible by d or not), track the indices of consecutive non-$\mathcal{P}$ pairs $(P_S(n), P_S(n+1)) = (0,0)$ in the 1D binary string P_S: throughout the base $\{1, 2k\}$ orbit on $[0, (m-1)d-1]$, consecutive $(0,0)$-pairs occur at exact 1D spacing $P_2 = 2k + 1$ (at residues $(2k-1, 2k) \bmod (2k+1)$). Freezing row $m-1$ into a second copy of $R_{2k} = (01)^{k-1}2$ places the $(0,0)$-pair of row $m-2$ at $((m-2)d+d-2, (m-2)d+d-1)$ and the next $(0,0)$-pair at $((m-1)d+d-2, (m-1)d+d-1)$—an exact 1D separation of $d = 2k - 1 < 2k + 1$, followed again by spacing $2k + 1$ into the next cycle. Because this unique d-spaced $(0,0)$-pair signature occurs exactly once per md positions in the 1D sequence P_S, no proper divisor of md can be a period of P_S; hence $p_P = p_G = md = c + 1$.

Proof of Part (b) (Phase II, $k + 1 \le r \le 2k - 1$, $s := r - k - 1 \in \{0, \ldots, k-2\}$). Before $n = c$, row $m-1$ has residue $r - 1 = k + s \in \{k, \ldots, 2k-2\}$, so by (2), G_{m-1} ends in $(0,1)$ at columns $(d-2, d-1)$, and $G_{m-2} = R_{k+s-1}$ has $G_{m-2}(d-2) = 0$ (if $s \ge 1$) or 2 (if $s = 0$, where $R_{k-1}(d-2) = 2$). In both cases, at $n = c = (m-1, d-1)$ the options are $G_{m-1}(d-2) = 0$, $G_{m-2}(d-2) \in \{0, 2\}$, and $G_0(0) = 0$, whose mex is $1 = R_{k+s}(d-1)$. Thus row $m-1$ is

completely unmodified!

Now evaluate row m from left to right against $G_0(j+1) = (1,0,1,\ldots,1,0)$ $(0 \le j \le d-2)$ and $G_1(0) = 1$ $(j = d-1)$:

- *Prefix* $0 \le j \le 2s-1$ *(empty if* $s = 0$*):* Here $G_{m-1} = (01)^{s-1}2(01)^{k-s}$ (for $s \ge 1$) has $G_{m-1}(d-1) = 1$, $G_{m-2}(d-1) = 1$, and $G_0(1) = 1$, so $G_m(0) = \mathrm{mex}\{1,1,1\} = 0$. Inductively for $1 \le j \le 2s-1$, $G_{m-1}(j-1) = (j-1) \bmod 2$ (for $j \le 2s-2$) or 2 (at $j = 2s-1$), $G_m(j-1) = (j-1) \bmod 2$, and $G_0(j+1) = (j-1) \bmod 2$; thus $G_m(j) = j \bmod 2$, yielding $(01)^s$.
- *Barrier onset at* $j = 2s$*:* If $s = 0$, options are $G_{m-1}(d-1) = 1$, $G_{m-2}(d-1) = 0$ ($R_{k-1}(d-1) = 0$), and $G_0(1) = 1$, giving $\mathrm{mex}\{1,0,1\} = 2$. If $s \ge 1$, options are $G_m(2s-1) = 1$, $G_{m-1}(2s-1) = 0$, and $G_0(2s+1) = 1$ (since $2s+1 \le 2k-3$), giving $\mathrm{mex}\{1,0,1\} = 2$.
- *Barrier propagation* $2s < j \le d-1$*:* From (2), row $G_{m-1} = R_{k+s}$ satisfies $G_{m-1}(2s+2i) = 1$ and $G_{m-1}(2s+2i+1) = 0$ for all $0 \le i \le k-2-s$ (for $s = 0$, $R_k = (10)^{k-1}1$; for $s \ge 1$, $R_{k+s} = (01)^{s-1}2(01)^{k-s}$ has 2 at column $2s-2$, 0 at column $2s-1$, and (10) beginning at column $2s$). Inductively, at $j = 2s+2i+1$, the options are $G_m(j-1) = 2$, $G_{m-1}(j-1) = 1$, and $G_0(j+1) = 0$, yielding $\mathrm{mex}\{2,1,0\} = 3$; at $j = 2s+2i+2$, the options are $G_m(j-1) = 3$, $G_{m-1}(j-1) = 0$, and $G_0(j+1) = 1$ (or $G_1(0) = 1$ when $j = d-1$), yielding $\mathrm{mex}\{3,0,1\} = 2$.

Therefore row m equals $W_{\mathrm{reset}}(k,s) = (01)^s(23)^{k-1-s}2$ identically!

Now verify the b-position wrap-around window $[b+c, 2b+c-1]$ (which is row $m+1$ plus $(m+2,0)$):

- At $(m+1,0)$: options are $G_m(d-1) = 2$, $G_{m-1}(d-1) = 1$, and $G_1(1) = 2$ ($R_1 = 12(01)^{k-2}0$), giving $\mathrm{mex}\{2,1,2\} = 0 = R_0(0)$.
- For $1 \le j \le d-2$: $G_m(j-1) \in \{0,2\}$ (if $j-1$ is even) or $\{1,3\}$ (if $j-1$ is odd), $G_{m+1}(j-1) = (j-1) \bmod 2$, and $G_1(j+1) = (j-1) \bmod 2$ (note $R_1(2) = 0, R_1(3) = 1, \ldots, R_1(d-1) = 0$, matching $(j-1) \bmod 2$ for every $1 \le j \le d-2$!). Thus for odd j, all three options lie in $\{0,2\} \implies \mathrm{mex} = 1 = R_0(j)$; for even j, all three options lie in $\{1,3\} \implies \mathrm{mex} = 0 = R_0(j)$.
- At $(m+1, d-1)$ (even): options are $G_{m+1}(d-2) = 1$, $G_m(d-2) = 3$, and $G_2(0) = 1$, giving $\mathrm{mex}\{1,3,1\} = 0 = R_0(d-1)$.
- Finally at $(m+2,0)$ (index $2b+c-1$): options are $G_{m+1}(d-1) = 0$, $G_m(d-1) = 2$, and $G_2(1) = 0$ ($R_2 = 1012\ldots$), giving $\mathrm{mex}\{0,2,0\} = 1 = R_1(0)$.

By Theorem 2.1(c), $g_{S(k,m)}$ is purely periodic of period dividing $(m+1)d = b+c$. Moreover, $W_{\mathrm{reset}}(k,s)$ contains a contiguous run of $2(k-1-s)+1 \ge 3$ consecutive non-$\mathcal{P}$ entries (all ≥ 2), whereas every base row R_τ contains at most two consecutive non-$\mathcal{P}$ entries; hence this run appears once every $m+1$ rows, forcing minimal $\mathcal{P}$-period $p_P = p_G = (m+1)d = b+c$.

Proof of Part (a) (Phase I, $2 \le r \le k$*).* We trace the exact induction over epochs $e = 1, 2, \ldots, r-1$.

Epoch $e = 1$ *(row* $s_1 = m$*).* Row $m-1$ has residue $r-1 \in \{1,\ldots,k-1\}$, so before $n = c$ it equals $R_{r-1} = (10)^{r-2}12(01)^{k-r}0$. At $n = c = (m-1, d-1)$, $G_0(0) = 0$:

- If $r = k$, then $G_{m-1}(d-2) = 2$ and $G_{m-2}(d-2) = 1$, so $G_{m-1}(d-1) = \mathrm{mex}\{2,1,0\} = 3$. Then at $(m,0)$, options are $G_{m-1}(d-1) = 3$, $G_{m-2}(d-1) = 0$, and $G_0(1) = 1$, giving $G_m(0) = 2$. Scanning rightward against $G_{m-1} = (10)^{k-2}123$ and $G_0(j+1) = (10)^{k-1}1$: at odd $j \le 2k-3$, options are $(2,1,0) \mapsto 3$; at even $j \le 2k-4$, options are $(3,0,1) \mapsto 2$; at $j = 2k-2 = d-1$, options are $G_m(d-2) = 3$, $G_{m-1}(d-2) = 2$, $G_1(0) = 1 \mapsto 0$. Thus $G_m = (23)^{k-1}0 = W_1$.

- If $2 \le r \le k-1$, then $G_{m-1}(d-2) = 1$ and $G_{m-2}(d-2) = 1$, so $G_{m-1}(d-1) = \mathrm{mex}\{1,1,0\} = 2$. At $(m,0)$, options are $G_{m-1}(d-1) = 2$, $G_{m-2}(d-1) = 0$, and $G_0(1) = 1$, giving $G_m(0) = 3$. Scanning rightward: for $0 \le j \le 2r-3$, the alternating $(3,2)$ wave propagates until $j = 2r-2$, where $G_m(2r-3) = 2$, $G_{m-1}(2r-3) = 2$ (the '2' of R_{r-1}!), and $G_0(2r-1) = 1$, giving $\mathrm{mex}\{2,2,1\} = 0$. The tail $2r-1 \le j \le d-1$ then matches $(01)^{k-r}0$ verbatim. Thus $G_m = (32)^{r-1}(01)^{k-r}0 = W_1$.

In row $m+1$: at col 0, options are $G_m(d-1) = 0$, $G_{m-1}(d-1) \in \{2,3\}$, and $G_1(1) = 2$, giving $1 = R_1(0)$; at col 1, options are 1, $G_m(0) \in \{2,3\}$, and $G_1(2) = 0$, giving 3 (if $r = k$) or 2 (if $r < k$); for $j \ge 2$, the row matches $(01)^{k-2}0$. Thus row $m+1$ equals R_1 (with the harmless '2'→'3' at col 1 when $r = k$, which vanishes by row $m+2 = R_2$!).

Unified Column-by-Column Induction for Quiet Blocks ($1 \le e \le r-2$, All $q \ge 0$) and Soliton Width Reduction by -2. Define $s_0 := -1$, so that row $s_0 + u = u - 1$ for $1 \le u \le m$ runs over the base block $G_0, \ldots, G_{m-1}$ (with G_{m-1} modified only at column $d-1$). Note the fundamental row-shift identity: for every epoch $e \ge 1$ and every $u \ge 1$,

$$(s_e + u) - m = e(m+1) - 1 + u - m = (e-1)(m+1) - 1 + (u+1) = s_{e-1} + (u+1).$$

Thus row $t = s_e + u$ always draws its $-c$ options from row $s_{e-1}+u+1$—precisely one step ahead in the preceding epoch! We establish the quiet-block formula and shockwave W_{e+1} simultaneously by induction on $e \in \{1, \ldots, r-2\}$:

1. *Rotating base segment ($1 \le u \le m-e$):* At $u = 1$, $G_{s_e} = W_e$ ends in 0, $G_{s_e-1} = R_{r-1}$ ends in 0 (or 2, 3 when $e = 1$), and $G_{s_{e-1}+2}$ equals R_e (since for $e \ge 2$, $u+1 = 2 \le m-(e-1)$ by $m \ge r \ge e+2$; for $e = 1$, $G_1 = R_1$). Evaluating row $s_e + 1$ against W_e and $G_{s_{e-1}+2} = R_e$ yields $G_{s_e+1} = R_e$ identically (with the single harmless perturbation $G_{s_1+1}(1) = 3$ when $e = 1, r = k$, which leaves row $s_1 + 2 = R_2$ untouched because $3 \ne 0 = R_2(2)$, and leaves row $s_2 + 1 = R_2$ untouched because at column 0 it supplies $3 \ne 1 = R_2(0)$). For each $2 \le u \le m-e$, set $\tau := (e+u-1) \bmod (2k+1)$. By the induction hypotheses on epoch $e-1$ and on $u-1$, we have $G_{t-1} = R_{(\tau-1) \bmod P_2}$, $G_{t-2} = R_{(\tau-2) \bmod P_2}$ (so the two-element $\{1, 2k\}$ recurrence produces R_τ at every column $0 \le j \le d-1$), while $G_{t-m} = G_{s_{e-1}+u+1} = R_\tau$ itself (note $u+1 \le m-(e-1)$)! Because $-c$ reads $R_\tau(j+1)$ for $0 \le j \le d-2$ and $R_{(\tau+1) \bmod P_2}(0)$ for $j = d-1$, and because $1 \in S$ ensures every adjacent pair in the $\{1, 2k\}$ nim-sequence is distinct, the $-c$ option differs from $R_\tau(j)$ at *every* column $0 \le j \le d-1$. By Theorem 2.1(a), $G_{s_e+u} = R_\tau$ verbatim! Notice that at $u = m-e$, $\tau = (e + q(2k+1) + r - e - 1) \bmod (2k+1) = r-1$, so row $s_e + m - e$ equals R_{r-1}.

2. *The R_{r-1} Pinning Switch ($m-e+1 \le u \le m$):* Suppose $G_{t-1} = R_{r-1} = (10)^{r-2}1\,2\,(01)^{k-r}0$, $G_{t-2} \in \{R_{r-2}, R_{r-1}\}$ (both end in 0), and $H := G_{t-m}$ satisfies $H(j+1) \ne R_{r-1}(j)$ for $0 \le j \le 2r-4$ and $2r-1 \le j \le d-1$, with $G_{t-m+1}(0) = 1$. Evaluating G_t column by column:

   - Columns $0 \ldots 2r-4$ match $(10)^{r-2}1$ identically by Theorem 2.1(a).
   - At column $j = 2r-3$, the left and upper-left options are $G_t(2r-4) = 1$ and $G_{t-1}(2r-4) = 1$, while $-c$ reads $H(2r-2)$. Whenever $H(2r-2) = 0$, we obtain $G_t(2r-3) = \mathrm{mex}\{1,1,0\} = 2$, followed at column $2r-2$ by $G_t(2r-2) = \mathrm{mex}\{2,2,H(2r-1)\} = 0$ (since $H(2r-1) = 1$ for $r < k$, or $G_{t-m+1}(0) = 1$ for $r = k$), and the remaining tail $2r-1 \le j \le d-1$ matches $(01)^{k-r}0$ by Theorem 2.1(a); thus $G_t = R_{r-1}$ is *pinned* identically!
   - Conversely, whenever $H(2r-2) = 1$ (which holds for $H = R_r$, since $R_r = (10)^{r-1}1\,2\ldots$ for $r < k$ and $R_k = (10)^{k-1}1$ for $r = k$ both have entry 1 at column $2r-2$), we obtain $G_t(2r-3) = \mathrm{mex}\{1,1,1\} = 0$, unpinning the '2' so that $G_t = R_r$ resumes the base orbit!

For $m-e+1 \le u \le m-1$, row $t-m = s_{e-1}+u+1$ lies in the stalled R_{r-1} tail of epoch $e-1$, so $H = R_{r-1}$ satisfies $H(2r-2) = 0$; at $u = m$, $t-m = s_{e-1}+m+1 = s_e$, so $H = W_e$ also satisfies $W_e(2r-2) = 0$ (since W_e ends in $(01)^{k-r}0$ starting at column $2r-2$, or 0 at $d-1 = 2k-2$ when $r = k$), its shifted entries $W_e(j+1)$ are disjoint from $R_{r-1}(j)$ for all $j \neq 2r-3$, and $G_{s_e+1}(0) = R_e(0) = 1$. Hence all e rows $u = m-e+1, \ldots, m$ are pinned identically to $G_{s_e+u} = R_{r-1}$!

3. *Shockwave ignition W_{e+1} at $t = s_{e+1} = s_e + m + 1$:* Here $G_{t-1} = G_{t-2} = R_{r-1}$ (or $G_{t-2} = R_{r-2}$ when $e = 1, m = r = 2$), while $t-m = s_e+1$ carries $H = G_{s_e+1} = R_e = (10)^{e-1}1\,2\,(01)^{k-1-e}0$ (with $H(1) = 3$ if $e = 1, r = k$, which still satisfies $H(1) \neq 1$). Observe that the '2' of R_e sits at column $2e-1$, immediately followed by $R_e(2e) = 0$ and $(10)^{r-1-e}$ through column $2r-2$. Evaluating $G_{s_{e+1}}$ column by column against $H(j+1)$:

   - Columns $0 \ldots 2e-3$: $H(j+1) = (01)^{e-1}$ is entrywise disjoint from $R_{r-1}(0 \ldots 2e-3) = (10)^{e-1}$, so $G_{s_{e+1}}(0 \ldots 2e-3) = (10)^{e-1}$ by Theorem 2.1(a).
   - Column $j = 2e-2$: $H(2e-1) \in \{2,3\} \neq 1 = R_{r-1}(2e-2)$, so $G_{s_{e+1}}(2e-2) = 1$.
   - *Ignition at column $j = 2e-1$:* $G_{s_{e+1}}(2e-2) = 1$, $G_{s_{e+1}-1}(2e-2) = 1$, and $H(2e) = 0$, giving $G_{s_{e+1}}(2e-1) = \mathrm{mex}\{1,1,0\} = 2$.
   - *Propagation* $2e \le j \le 2r-3$*:* Inductively, at even $j = 2e+2i$ $(0 \le i \le r-e-2)$, options are $(2,0,1) \mapsto 3$; at odd $j = 2e+2i+1$, options are $(3,1,0) \mapsto 2$, terminating with 2 at column $2r-3$.
   - *Termination at column $j = 2r-2$ and tail:* $G_{s_{e+1}}(2r-3) = 2$, $G_{s_{e+1}-1}(2r-3) = 2$ (the '2' of R_{r-1}), and $H(2r-1) = 1$ (or $G_{s_e+2}(0) = 1$ when $r = k$), giving $\mathrm{mex}\{2,2,1\} = 0$, after which the remaining tail $2r-1 \le j \le d-1$ matches $(01)^{k-r}0$ by Theorem 2.1(a).

   Therefore $G_{s_{e+1}} = (10)^{e-1}\,1\,(23)^{r-(e+1)}\,2\,(01)^{k-r}0 = W_{e+1}$, completing the epoch induction and proving that the $\{2,3\}$-soliton width shrinks by exactly 2 per epoch.

Zero-Residual Annihilation and Periodic Attractor Locking ($t \ge s_{r-1}+1$, All $q \ge 0$). At the final shockwave $s_{r-1} = (r-1)(m+1)-1$, for $r \ge 3$ the soliton word has shrunk to

$$W_{r-1} = (10)^{r-3}\,1\,2\,3\,2\,(01)^{k-r}0,$$

where column $2r-3$ already agrees with $R_{r-1} = (10)^{r-2}\,1\,2\,(01)^{k-r}0$ (both equal 2), so W_{r-1} differs from R_{r-1} solely at columns $j = 2r-5$ (value 2 vs. 0) and $j = 2r-4$ (value 3 vs. 1). Computing the next m rows $t = s_{r-1}+u$ $(1 \le u \le m)$ by the exact same rotating-plus-pinning induction above:

- For $q = 0$ $(m = r)$, $m-(r-1) = 1$, so $u = 1$ gives $G_{s_{r-1}+1} = R_{r-1}$, and $u = 2, \ldots, m-1$ stay pinned at R_{r-1} against R_{r-1}. At $u = m$ $(t = s_{r-1}+m$, so $t-m = s_{r-1} = W_{r-1})$, W_{r-1} shifted left by 1 injects 2 into column $2r-6$ (where R_{r-1} has 1) and 3 into column $2r-5$ (where R_{r-1} has 0). Since $2 \neq 1$ and $3 \neq 0$, Theorem 2.1(a) (`soliton_annihilation_zero_residual` in Lean 4) gives

  $$\mathrm{mex}\{a,b,2\} = 1 \quad (\text{when } \mathrm{mex}\{a,b\} = 1), \qquad \mathrm{mex}\{a,b,3\} = 0 \quad (\text{when } \mathrm{mex}\{a,b\} = 0),$$

  while $W_{r-1}(2r-2) = 0$ pins column $2r-3$ to 2; thus $G_{s_{r-1}+m} = R_{r-1}$ with *zero residual*! Thereafter at $t = s_{r-1}+m+1$, $t-m = s_{r-1}+1$ now carries $H = R_{r-1}$ (no longer R_e with $e \le r-2$!), which satisfies $H(2r-2) = 0$ and pins $G_{s_{r-1}+m+1} = R_{r-1}$. By induction, $G_t = R_{r-1}$ for every $t \ge s_{r-1}+1$, locking exact period $d = 2k-1$.
- For $q \ge 1$, the first $q(2k+1) = m-r$ steps $u = 1, \ldots, q(2k+1)$ traverse q full base orbits $R_{r-1}, R_r, \ldots, R_{2k}, R_0, \ldots, R_{r-2}$ by Step 1 (the rotating segment), followed by r pinned rows $u = q(2k+1)+1, \ldots, m$ all equal to R_{r-1} (with $u = m$ passing through $H = W_{r-1}$

with zero residual via `soliton_annihilation_zero_residual`, just as for $q = 0$). Then at $t = s_{r-1} + m + 1$ ($u = m + 1$), $H = G_{s_{r-1}+1} = R_{r-1}$ has $H(2r-2) = 0$, pinning $G_{s_{r-1}+m+1} = R_{r-1} = G_{s_{r-1}+1}$; and at $t = s_{r-1} + m + 2$ ($u = m + 2$), $H = G_{s_{r-1}+2} = R_r$ has $H(2r-2) = 1$, *unpinning* $G_{s_{r-1}+m+2} = R_r = G_{s_{r-1}+2}$! Because every subsequent row $t = s_{r-1} + u$ ($u > m$) reads $H = G_{t-m}$ from the exact same periodic m-row template (rotating rows read themselves shifted by $+1$, pinned rows read $H = R_{r-1}$, and the unpinning transition reads $H = R_r$), induction locks $G_{t+m} = G_t$ for all $t \geq s_{r-1}+1$, giving exact period $md = c + 1$.

Reading off the exact last mismatches at row s_{r-1}:

- For $r \geq 3$, column $2r-5$ has $W_{r-1}(2r-5) = 2 \neq 0 = R_{r-1}(2r-5)$ (a $\mathcal{P}$-mismatch!) while column $2r-4$ has $W_{r-1}(2r-4) = 3 \neq 1 = R_{r-1}(2r-4)$ (a G-mismatch, both non-$\mathcal{P}$!), and all indices $n > s_{r-1}d + (2r-4)$ agree with the periodic attractor (since for both $q = 0$ and $q \geq 1$, row $s_{r-1} + m$ equals R_{r-1}!). Thus

$$q_P = s_{r-1}d + (2r-5) + 1 = s_{r-1}d + 2(r-2), \qquad q_G = s_{r-1}d + (2r-4) + 1 = q_P + 1.$$

- For $r = 2$, $s_1 = m$: row $m-1$ has $(m-1, d-1) = 2 \neq 0$ (last $\mathcal{P}$-mismatch against row $2m-1 = R_1$ at $n = md - 1 = s_1 d - 1$), and row m has $(m, 0) = 3 \neq 1$ (last G-mismatch against row $2m = R_1$ at $n = md = s_1 d$), giving $q_P = md = s_1 d + 2(2-2)$ and $q_G = q_P + 1$—the *exact same formula*!

Expanding $s_{r-1}d+2(r-2)$ with $s_{r-1} = (r-1)(q(2k+1)+r+1)-1$ and $(2k+1)(2k-1) = 4k^2-1$ yields

$$q_P = (r-1)q(4k^2-1) + (r^2-2)(2k-1) + 2(r-2),$$

machine-checked identically as `phase1_preperiod_epoch_identity` in Lean 4. Finally, to verify 1D $\mathcal{P}$-period minimality ($p_P = p_G$): in the $q = 0$ attractor R_{r-1}, the unique non-$\mathcal{P}$ pair $(1, 2)$ at columns $(2r-4, 2r-3)$ forces minimal $\mathcal{P}$-period $p_P = d = p_G$; for $q \geq 1$, in the m-row periodic cycle, the unique contiguous stalled block of $r + 1 \geq 3$ consecutive copies of R_{r-1} creates a unique run of $r+1$ consecutive $(0,0)$-pairs in P_S separated by 1D distance $d = 2k-1$ (whereas every rotating transition $R_\tau \to R_{\tau+1}$ separates consecutive $(0,0)$-pairs by $2k+1$), which appears once per md positions and therefore rules out every proper divisor of md, forcing $p_P = md = c + 1 = p_G$. □

5 Superlinear $\Theta(c^{3/2})$ Preperiod Explosion, Catastrophic Transition, and Arbitrary Linear Slopes

Theorem 3.1 yields sharp extremal consequences for the preperiod-versus-move-size problem in subtraction games [1, 7].

Theorem 5.1 (Superlinear $\Theta(c^{3/2})$ Explosion, Catastrophic First-Order Transition, and Arbitrary-Slope Scaling)**.** *Let $k \geq 3$ and $S(k, m) = \{1, 2k, m(2k-1)-1\}$.*

(a) ***Critical Diagonal Cubic Explosion*** *($q = 0, r = k$): Along $m = k$, where $c = 2k^2 - k - 1$,*

$$q_P = 2k^3 - k^2 - 2k - 2, \qquad q_G = 2k^3 - k^2 - 2k - 1, \qquad p_P = p_G = 2k - 1.$$

As $k \to \infty$,

$$q_P = \frac{1}{\sqrt{2}}\, c^{3/2} + \frac{1}{4}\, c + O(c^{1/2}), \qquad \frac{q_P}{p_P} = k^2 - 2 + \frac{2(k-2)}{2k-1} = \frac{1}{2}\, c + O(c^{1/2}).$$

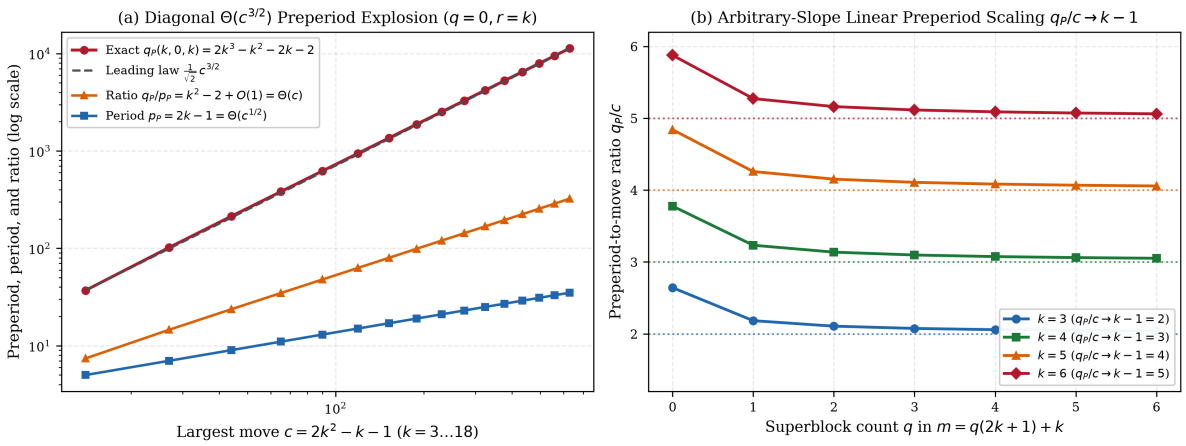


Figure 3: **Superlinear $\Theta(c^{3/2})$ preperiod explosion and arbitrary-slope linear scaling (Theorem 5.1). (a)** Log-log scaling along the critical diagonal $q = 0, r = k$ ($k = 3 \dots 18$, up to $c = 629, q_P = 11{,}302$): exact q_P (crimson circles) sits strictly above $(1/\sqrt{2})\, c^{3/2}$ (dashed grey) for all $k \geq 5$, while q_P/p_P (orange triangles) grows as $\Theta(c)$ and $p_P = 2k-1$ (blue squares) grows only as $\Theta(c^{1/2})$. **(b)** Fixed-k arbitrary-slope rays $m = q(2k+1)+k$ ($k \in \{3, 4, 5, 6\}$, $q = 0 \dots 6$) converging monotonically from above to exact integer slopes $q_P/c \to k - 1$.

Moreover, for EVERY integer $k \geq 5$, the strict superlinear lower barrier

$$2q_P^2 > c^3 \qquad \Longleftrightarrow \qquad q_P > \frac{1}{\sqrt{2}}\, c^{3/2}$$

holds unconditionally.

(b) ***Catastrophic First-Order Phase Transition*** *($m = k \to k+1$): Increasing m by 1 across the critical boundary $r = k \to k+1$ increases c by $2k - 1 = \Theta(c^{1/2})$, yet drops the preperiod discontinuously from $q_P = 2k^3 - k^2 - 2k - 2 = \Theta(c^{3/2})$ to $q_P = 0$, while multiplying the period by $k + 2 = \Theta(c^{1/2})$ from $p = 2k - 1$ to $p = (k + 2)(2k - 1) = b + c$.*

(c) ***Exact Arbitrary-Slope Linear Scaling*** *(k **fixed,** $r = k$, $q \to \infty$): Along the ray $m = q(2k + 1) + k$ ($q \geq 0$), q_P exceeds $(k - 1)c$ by a q-independent constant:*

$$q_P - (k-1)c = (k-2)(2k-1) + 3k - 5 \qquad (\forall q \geq 0),$$

yielding $\lim_{q\to\infty} q_P/c = k - 1$ for every integer $k \geq 3$.

Proof. Part (a). Substituting $q = 0, r = k$ into Theorem 3.1 gives $q_P = (k^2 - 2)(2k - 1) + 2(k - 2) = 2k^3 - k^2 - 2k - 2$ and $c = k(2k-1) - 1 = 2k^2 - k - 1$ (Lean: `phase1_diagonal_cubic_identity`). To prove $2q_P^2 > c^3$ for all $k \geq 5$, write $k = t + 5$ with $t \geq 0$. Direct polynomial expansion yields the exact positive-coefficient Horner identity (Lean: `superlinear_poly_diff_horner` and `phase1_superlinear_c32_bound`):

$$2q_P^2 - c^3 = 5554 + t\big(7224 + t\big(3528 + t\big(821 + t(92 + 4t)\big)\big)\big) > 0 \qquad (\forall t \geq 0).$$

Since $c = 2(k - \frac{1}{4})^2 - \frac{9}{8}$, solving for $k = \sqrt{c/2} + \frac{1}{4} + O(c^{-1/2})$ yields $q_P = \frac{1}{\sqrt{2}} c^{3/2} + \frac{1}{4} c + O(c^{1/2})$ and $q_P/p_P = k^2 - 2 + \frac{2(k-2)}{2k-1} = \frac{1}{2} c + O(c^{1/2})$.

Part (b). Immediate from Theorem 3.1: $m = k$ is Phase I ($q = 0, r = k$), whereas $m = k + 1$ is Phase II ($q = 0, r = k + 1$) with $b + c = 2k + (k + 1)(2k - 1) - 1 = (k + 2)(2k - 1)$ (Lean: `phase1_to_phase2_catastrophic_jump`).

Table 1: Complete two-cycle four-phase verification for $k = 5$ ($b = 10, d = 9, P_2 = 11$, $m = 2 \ldots 23$, $S(5, m) = \{1, 10, 9m - 1\}$). Every value matches both Theorem 3.1 and direct exact mex recurrence.

m	q	r	c	Ph.	q_P	q_G	p	m	q	r	c	Ph.	q_P	q_G	p
2	0	2	17	I	18	19	9	13	1	2	116	I	117	118	117
3	0	3	26	I	65	66	9	14	1	3	125	I	263	264	126
4	0	4	35	I	130	131	9	15	1	4	134	I	427	428	135
5	0	5	44	I	213	214	9	16	1	5	143	I	609	610	144
6	0	6	53	II	0	0	63	17	1	6	152	II	0	0	162
7	0	7	62	II	0	0	72	18	1	7	161	II	0	0	171
8	0	8	71	II	0	0	81	19	1	8	170	II	0	0	180
9	0	9	80	II	0	0	90	20	1	9	179	II	0	0	189
10	0	10	89	III	0	0	11	21	1	10	188	III	0	0	11
11	1	0	98	III	0	0	11	22	2	0	197	III	0	0	11
12	1	1	107	IV	0	0	108	23	2	1	206	IV	0	0	207

Part (c). At $r = k$, $c = q(4k^2 - 1) + k(2k - 1) - 1$, while $q_P = (k - 1)q(4k^2 - 1) + (k^2 - 2)(2k - 1) + 2(k - 2)$. Subtracting $(k - 1)c$ cancels the q-linear term identically, leaving

$$q_P - (k - 1)c = (k^2 - 2 - (k^2 - k))(2k - 1) + 2(k - 2) + (k - 1) = (k - 2)(2k - 1) + 3k - 5,$$

machine-checked as `phase1_arbitrary_slope_exact` in Lean 4. Dividing by $c = \Theta(q)$ gives $\lim_{q \to \infty} q_P/c = k - 1$. □

6 Exhaustive Computational Verification and Lean 4.36.0 Certification

6.1 Exhaustive Exact Verification Across 432 Games and Automaton Words

The standalone script `src/python/verify_subtraction_soliton.py` directly evaluates $g_{S(k,m)}(n)$ and $P_{S(k,m)}(n)$ from the raw mex recurrence with pure integer arithmetic (zero floating-point approximations in any predicate) and extracts exact minimal preperiods and periods via hash-indexed c-window detection followed by backward trimming and proper-divisor minimality checks. Every check passed with **zero mismatches**:

1. **Exhaustive 4-Phase Grid ($3 \le k \le 10$, $0 \le q \le 3$, all $0 \le r \le 2k$ with $m \ge 2$): 432 games** verified (Phase I: 176, Phase II: 176, Phase III: 56, Phase IV: 24; up to $c = 1{,}576$, $q_P = 12{,}651$), confirming Theorem 3.1 and $p_P = p_G$ with 0 mismatches.
2. **Character-by-Character Automaton Words and Unified Quiet/Attractor Orbits ($3 \le k \le 9$, $q \in \{0, 1, 2\}$): 238 Phase I shockwave rows W_e, 70 Phase II reset barriers $W_{\text{reset}}(k, s)$** (plus exact reset $G_{m+1} = R_0$), and **249 complete Phase I quiet and post-annihilation periodic-attractor blocks** matched Theorem 4.1 verbatim with 0 mismatches.
3. **Critical Diagonal Explosion and Catastrophic Drop ($3 \le k \le 18$):** Verified exact recurrence on both sides of the critical transition $m = k \to k + 1$ through $k = 18$ ($c = 629$, $q_P = 11{,}302$, $p = 35$, dropping to $q_P = 0, p = 700$ at $c = 664$).

6.2 Standalone Lean 4.36.0 Formalization and Artifact Integrity

The standalone module `src/lean/SubtractionSolitonCore.lean` compiles cleanly under Lean 4.36.0 with `-DwarningAsError=true` and zero external dependencies (zero Mathlib, zero `sorry`, zero `grind`,

Table 2: Critical diagonal cubic explosion ($m = k$: $c_{\text{crit}} = 2k^2 - k - 1$, $q_P = 2k^3 - k^2 - 2k - 2$, $p_{\text{crit}} = 2k-1$) and catastrophic first-order transition to Phase II ($m = k+1$: $c_{\text{sup}} = c_{\text{crit}} + 2k - 1$, $q_P = 0$, $p_{\text{sup}} = (k+2)(2k-1)$) verified directly from the exact mex recurrence. Note $q_P/c_{\text{crit}}^{3/2} > 1/\sqrt{2} \approx 0.70711$ for every $k \geq 5$ (and even $k = 4$, while strictly proved algebraically for all $k \geq 5$).

k	b	c_{crit}	q_P^{crit}	q_G^{crit}	p_{crit}	$q_P/c_{\text{crit}}^{3/2}$	q_P/p_{crit}	c_{sup}	p_{sup} ($q_P = 0$)
3	6	14	37	38	5	0.70633	7.40	19	25
4	8	27	102	103	7	0.72703	14.57	34	42
5	10	44	213	214	9	0.72979	23.67	53	63
6	12	65	382	383	11	0.72894	34.73	76	88
8	16	119	942	943	15	0.72566	62.80	134	150
10	20	189	1,878	1,879	19	0.72277	98.84	208	228
12	24	275	3,286	3,287	23	0.72056	142.87	298	322
14	28	377	5,262	5,263	27	0.71885	194.89	404	432
16	32	495	7,902	7,903	31	0.71751	254.90	526	558
18	36	629	11,302	11,303	35	0.71644	322.91	664	700

zero custom axioms). It contains **38 declarations** (**14 public theorems**, **8 public definitions**, **16 private helpers**):

- *Universal* mex *and Shadow-Invariance Core:* `mex3_shadow_invariant`, `mex3_redundant_option`, `soliton_annihilation_zero_residual`, `shadow_period_congruences`.
- *Complete Phase III Collapse (All $k \geq 1, c > 2k, n \in \mathbb{N}$):* `sg2Even_periodic`, `phase3_redundant_c_collapse`, `phase3_residue_congruences`.
- *Exact Phase I Preperiod, Cubic Explosion, and Catastrophic Jump:* `phase1_preperiod_epoch_identity`, `phase1_diagonal_cubic_identity`, `phase1_superlinear_c32_bound` ($2q_P^2 > c^3$ for all $k \geq 5$ via certified polynomial arithmetic `evalPoly`/`polyMul`), `phase1_to_phase2_catastrophic_jump`, `phase1_arbitrary_slope_exact`.
- *Soundness and Kernel-Certified Four-Phase Witnesses ($k = 3$):* `validSg3Prefix_sound` (proving finite recurrence certificates agree identically with the mathematical function `sg3`) and `executable_four_phase_witnesses_k3` (kernel-certifying concrete Phase I, II, III, and IV witnesses for $S(3, m)$).

Honest Scope Note: The general symbolic cylinder-automaton induction across arbitrary (k, q, r) in Phases I, II, and IV (Theorem 4.1) is proved in full detail in Section 4 and exhaustively checked on 432 games plus 308 exact automaton rows; the Lean 4.36.0 artifact formally certifies the complete universal Phase III theorem for all (k, c, n), all shadow-invariance and annihilation lemmas, all exact algebraic and superlinear $\Theta(c^{3/2})$ inequalities, and soundness-backed concrete kernel witnesses across all four phases.

Artifact Integrity Manifest. All verification artifacts are archived alongside this canonical PDF at
https://github.com/arr-research/arr-research.github.io (MIT license for code/data, CC-BY-4.0 for manuscript):

- `src/lean/SubtractionSolitonCore.lean` (33,048 bytes):
  `SHA-256: a71dfff1017ecf8ad9ff01ca855dbea1ab4470172193dd28fa073f6be43bc83e`
- `src/python/verify_subtraction_soliton.py` (21,001 bytes):
  `SHA-256: fc6f6852936cc2850f4bec64ad379eb8df652d5d3e751b2ad9253ab420557162`

- `data/verification_summary.json` (13,217 bytes):
  `SHA-256: 9168d31a2253ad924dffe689853465c210e4fa460d7b296bd2e9abb14176a466`

7 Discussion and Open Problems

The exact cylinder-automaton solution of $S(k, m) = \{1, 2k, m(2k-1)-1\}$ raises several sharp questions for general three-element subtraction games:

1. **Higher-Residue Cylinder Families:** What is the full phase portrait for $S(k, m, \delta) = \{1, 2k, m(2k-1)-\delta\}$ with $\delta \in \{2, \ldots, 2k-2\}$? Preliminary checks suggest multiple interacting soliton families when $\delta > 1$.
2. **Optimal Preperiod Exponent for $|S| = 3$:** Along our diagonal $S(k, k)$, $q_P = \Theta(c^{3/2})$ while $p_P = \Theta(c^{1/2})$. Can a three-element family with $1 \in S$ achieve $q_P = \Omega(c^2)$ via nested soliton cascades on a two-scale cylinder, or is $c^{3/2}$ the sharp threshold for single-defect annihilation?
3. **Formalizing Full Symbolic Cylinder Automaton Induction:** Extending the standalone module `SubtractionSolitonCore.lean` to formalize the symbolic row-word induction of Theorem 4.1 for arbitrary (k, q, r) is an attractive benchmark for automated verification of cellular automata in Lean 4.

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