Noncommuting Block Measurements Generate Quantum Diffusion

A three-obstruction convergence hierarchy, a noncommutative connectivity gap, and exact algebraic architectures

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Abstract

Repeated projective measurements are usually discussed either in the ballistic Zeno scaling or after reduction to classical outcome probabilities. We study a different regime in which each nonselective measurement retains a full matrix algebra inside every degenerate outcome block. Let $\mathcal{E}(X) = \sum_a P_a X P_a$, let $\mathcal{E}_\tau$ be its rotation by $e^{-i\tau H}$, and close one cycle by $\Phi_\tau = \mathcal{E}\mathcal{E}_\tau\mathcal{E}$. We prove, uniformly on compact time intervals and in diamond norm,

$$\Phi^n_{\sqrt{t/n}} \longrightarrow e^{-tK}\mathcal{E}, \qquad K = \mathcal{E}\,\mathrm{ad}_H(1-\mathcal{E})\,\mathrm{ad}_H\,\mathcal{E} = C_H^* C_H.$$

The limit is a genuinely quantum Markov semigroup on $\bigoplus_a \mathcal{B}(\mathcal{H}_a)$, with explicit jumps $\sqrt{2}P_b H P_a$. Beyond upper bounds, we identify three exact local obstructions for the unprocessed physical product. The cubic map M_H produces a nonzero $n^{-1/2}$ coefficient; after it vanishes, the quartic product obstruction J_H produces a nonzero n^{-1} coefficient; after both vanish, the quintic obstruction P_H produces a nonzero $n^{-3/2}$ coefficient. If all three vanish, the error is $O(n^{-2})$. All three coefficient transforms are injective. An exact three-record algebraic example has $M_H = J_H = 0$ but $P_H \neq 0$, proving that the third branch is attained. The fixed algebra is $\mathcal{A} \cap \{H_{\mathrm{off}}\}'$, and the least singular value of the commutator frame C_H defines a noncommutative connectivity gap. In the primitive case that gap is the exact exponential mixing exponent in diamond norm and is Lipschitz robust under Hamiltonian perturbations. Rank-one blocks reduce exactly to twice the squared-coupling graph Laplacian. An eight-dimensional rational architecture with four qubit blocks is certified irreducible; its gap is the unique root in $(0.4545, 0.4547)$ of an explicit quartic. Symbolic certificates, semidefinite diamond-norm replays, and convergence tests accompany the paper.

1 Problem and result

Fix a finite-dimensional Hilbert space

$$\mathcal{H} = \bigoplus_{a=1}^{r} \mathcal{H}_a, \qquad d = \dim \mathcal{H}, \qquad \mathcal{A} = \bigoplus_{a=1}^{r} \mathcal{B}(\mathcal{H}_a) \subset \mathcal{B}(\mathcal{H}). \tag{1}$$

The trace-preserving conditional expectation onto $\mathcal{A}$ is the block pinching

$$\mathcal{E}(X) = \sum_{a=1}^{r} P_a X P_a. \tag{2}$$

Unlike a rank-one pinching, $\mathcal{E}$ need not destroy entanglement with a reference system: each record value a retains the quantum state on $\mathcal{H}_a$. This is the minimal extension in which sequential nonselective measurements cease to be an ordinary finite-state Markov chain.

Let $H = H^*$, $U_\tau = e^{-i\tau H}$, and

$$\mathcal{E}_\tau = \mathrm{Ad}_{U_\tau} \circ \mathcal{E} \circ \mathrm{Ad}_{U_\tau^*}, \qquad \Phi_\tau = \mathcal{E} \circ \mathcal{E}_\tau \circ \mathcal{E}. \tag{3}$$

Thus a cycle begins and ends in the same record algebra while visiting a nearby incompatible one. Write $\mathrm{ad}_H(X) = [H, X]$, use the Hilbert–Schmidt inner product, and set

$$C_H = (1 - \mathcal{E})\,\mathrm{ad}_H\big|_{\mathcal{A}}, \qquad K_H = C_H^* C_H = \mathcal{E}\,\mathrm{ad}_H(1 - \mathcal{E})\,\mathrm{ad}_H\,\mathcal{E}. \tag{4}$$

The paper establishes the following chain of statements.

(i) The closed cycle has no linear term and its quadratic defect is exactly K_H.

(ii) The diffusive scaling $\tau = \sqrt{t/n}$ produces the quantum semigroup e^{-tK_H} in diamond norm, with explicit uniform errors.

(iii) $-K_H$ has a block-jump GKLS representation, so the limit is not merely a formal contraction.

(iv) $\ker K_H$ and its spectral gap are computable from an off-block commutator frame. This replaces classical graph connectivity by a noncommutative condition.

(v) Three exact superoperator obstructions classify four direct-product levels: $\Theta(n^{-1/2})$, $\Theta(n^{-1})$, $\Theta(n^{-3/2})$, and $O(n^{-2})$. Exact examples show that the first three levels are nonempty.

All channel statements below are maps on the full $\mathcal{B}(\mathcal{H})$: the initial $\mathcal{E}$ is retained on the right. Consequently the diamond norm includes arbitrary reference systems and no operator-space norm on an abstract direct sum is being silently substituted.

2 The local expansion

Put $D = \mathrm{ad}_H$ and $Q = 1 - \mathcal{E}$. Since $\mathrm{Ad}_{U_\tau} = e^{-i\tau D}$ on operators,

$$\Phi_\tau = \mathcal{E} e^{-i\tau D} \mathcal{E} e^{i\tau D} \mathcal{E}. \tag{5}$$

Define the retained first and second commutator maps and their obstruction by

$$A_H = \mathcal{E} D \mathcal{E}, \qquad B_H = \mathcal{E} D^2 \mathcal{E}, \qquad M_H = \frac{i}{2}[A_H, B_H]. \tag{6}$$

For the next level put $A_H^{(r)} = \mathcal{E} D^r \mathcal{E}$ on $\mathcal{A}$ and define

$$N_H = \frac{1}{12} A_H^{(4)} + \frac{1}{4}(A_H^{(2)})^2 - \frac{1}{6}(A_H^{(1)} A_H^{(3)} + A_H^{(3)} A_H^{(1)}), \tag{7}$$

$$J_H = N_H - \frac{1}{2} K_H^2. \tag{8}$$

The correction $K_H^2/2$ is essential: N_H is the fourth Taylor coefficient of the cycle, while J_H is its fourth-order defect relative to the target exponential. The fifth obstruction is

$$P_H = \frac{i}{24}[A_H^{(4)}, A_H^{(1)}] + \frac{i}{12}[A_H^{(2)}, A_H^{(3)}]. \tag{9}$$

Theorem 2.1 (diamond Taylor law and exact cubic obstruction)**.** *For every real τ,*

$$\Phi_\tau = \mathcal{E} - \tau^2 K_H + R_\tau, \qquad \|R_\tau\|_\diamond \le \frac{32}{3}\|H\|^3 |\tau|^3. \tag{10}$$

More precisely,

$$\Phi_\tau = \mathcal{E} - \tau^2 K_H + \tau^3 M_H + S_\tau, \qquad \|S_\tau\|_\diamond \le \frac{32}{3}\|H\|^4 |\tau|^4. \tag{11}$$

At fourth order one has the exact refinement

$$\Phi_\tau = \mathcal{E} - \tau^2 K_H + \tau^3 M_H + \tau^4 N_H + T_\tau, \qquad \|T_\tau\|_\diamond \le \frac{128}{15}\|H\|^5|\tau|^5. \tag{12}$$

At fifth order,

$$\Phi_\tau = \mathcal{E} - \tau^2 K_H + \tau^3 M_H + \tau^4 N_H + \tau^5 P_H + U_\tau, \qquad \|U_\tau\|_\diamond \le \frac{256}{45}\|H\|^6|\tau|^6. \tag{13}$$

Thus the local error is fourth order if and only if $M_H = 0$. A physically transparent sufficient condition is

$$\mathcal{E}(H) \in Z(\mathcal{A}), \tag{14}$$

under which

$$\|R_\tau\|_\diamond \le \frac{32}{3}\|H\|^4|\tau|^4. \tag{15}$$

Condition (14) allows different scalar energies on different blocks.

Proof. Differentiate (5). The two first-order terms cancel after compression by $\mathcal{E}$. The second-order coefficient is

$$\mathcal{E}D\mathcal{E}D\mathcal{E} - \mathcal{E}D^2\mathcal{E} = -\mathcal{E}DQD\mathcal{E} = -K_H.$$

The diamond norm of D is at most $2\|H\|$. The mth derivative of (5) is a sum of 2^m terms after the binomial count, whence

$$\sup_s \|\Phi_s^{(m)}\|_\diamond \le 2^m(2\|H\|)^m.$$

Taylor's integral remainder with $m = 3$ gives (10). Direct multiplication gives the cubic coefficient

$$\frac{i}{2}(\mathcal{E}D\mathcal{E}D^2\mathcal{E} - \mathcal{E}D^2\mathcal{E}D\mathcal{E}).$$

This is M_H in (6). Applying the fourth-derivative estimate and dividing by 4! gives (11); hence fourth order is equivalent to $M_H = 0$. For $X \in \mathcal{A}$, $\mathcal{E}D\mathcal{E}(X) = [\mathcal{E}(H), X]$; it vanishes under (14), so $M_H = 0$ and (15) follows.

For completeness, multiplication of the two compressed exponential series in (5) gives at order four

$$\frac{1}{12}\mathcal{E}D^4\mathcal{E} + \frac{1}{4}\mathcal{E}D^2\mathcal{E}D^2\mathcal{E} - \frac{1}{6}(\mathcal{E}D\mathcal{E}D^3\mathcal{E} + \mathcal{E}D^3\mathcal{E}D\mathcal{E}),$$

which is (7). The fifth-derivative estimate divided by 5! is $2^5(2\|H\|)^5/5! = 128\|H\|^5/15$, proving (12). At order five the pure $A_H^{(5)}$ terms cancel, while the remaining pairs give

$$\frac{i}{24}(A_H^{(4)}A_H^{(1)} - A_H^{(1)}A_H^{(4)}) + \frac{i}{12}(A_H^{(2)}A_H^{(3)} - A_H^{(3)}A_H^{(2)}),$$

which is (9). The sixth-derivative estimate gives $(4\|H\|)^6/6! = 256\|H\|^6/45$, proving (13). $\square$

The positivity of K_H is structural rather than perturbative:

$$\langle X, K_H X\rangle_2 = \|C_H X\|_2^2 \ge 0. \tag{16}$$

This identity will simultaneously produce the GKLS generator, fixed algebra, spectral gap, and robustness estimate.

3 Uniform diffusive product limit

Theorem 3.1 (ancilla-complete diffusive Zeno limit). *Let $h = \|H\|$, $t \geq 0$, $n \geq 1$, and $\tau_n = \sqrt{t/n}$. Then*

$$\left\|\Phi_{\tau_n}^n - e^{-tK_H}\mathcal{E}\right\|_\diamond \leq \frac{32}{3}h^3\frac{t^{3/2}}{\sqrt{n}} + 8h^4\frac{t^2}{n}e^{4h^2t/n}. \tag{17}$$

If $M_H = 0$—in particular, under (14)—the first term may be replaced by

$$\frac{32}{3}h^4\frac{t^2}{n}. \tag{18}$$

In particular, $\Phi_{\sqrt{t/n}}^n \to e^{-tK_H}\mathcal{E}$ in diamond norm, uniformly for t in compact subsets of $[0,\infty)$. The general rate is $O(n^{-1/2})$ and every cubic-free architecture has rate $O(n^{-1})$.

Proof. Both Φ_τ and $e^{-sK_H}\mathcal{E}$ are channels (the latter follows independently from Theorem 4.1). The channel telescoping inequality therefore has no growth factor. Set $s = t/n$. Theorem 2.1 bounds

$$n\|\Phi_{\sqrt{s}} - (\mathcal{E} - sK_H)\|_\diamond$$

by the first term of (17), or by (18). Moreover

$$\|K_H\|_\diamond \leq \|D\|_\diamond^2 \leq 4h^2 =: \kappa.$$

The exponential remainder satisfies

$$\|e^{-sK_H} - 1 + sK_H\|_\diamond \leq \frac{s^2\kappa^2}{2}e^{s\kappa}.$$

After a second telescoping over n factors this contributes $\kappa^2t^2e^{\kappa t/n}/(2n)$, which is the second term in (17). ☐

Theorem 3.2 (sharp three-obstruction hierarchy). *For every $t > 0$, define*

$$G_t = t^{3/2}\int_0^1 e^{-(1-s)tK_H}M_He^{-stK_H}\mathcal{E}\,\mathrm{d}s. \tag{19}$$

Then

$$\sqrt{n}(\Phi_{\sqrt{t/n}}^n - e^{-tK_H}\mathcal{E}) \longrightarrow G_t \tag{20}$$

in diamond norm, and $M_H \mapsto G_t$ is injective. If $M_H = 0$, define the second coefficient

$$F_t = t^2\int_0^1 e^{-(1-s)tK_H}J_He^{-stK_H}\mathcal{E}\,\mathrm{d}s. \tag{21}$$

Then

$$n(\Phi_{\sqrt{t/n}}^n - e^{-tK_H}\mathcal{E}) \longrightarrow F_t \tag{22}$$

in diamond norm, and $J_H \mapsto F_t$ is injective. Define also

$$R_t = t^{5/2}\int_0^1 e^{-(1-s)tK_H}P_He^{-stK_H}\mathcal{E}\,\mathrm{d}s. \tag{23}$$

If also $J_H = 0$, then

$$n^{3/2}(\Phi_{\sqrt{t/n}}^n - e^{-tK_H}\mathcal{E}) \longrightarrow R_t \tag{24}$$

in diamond norm, and $P_H \mapsto R_t$ is injective. Consequently,

$$
\begin{array}{cl}
M_H \neq 0 & \|\Phi^n_{\sqrt{t/n}} - e^{-tK_H}\mathcal{E}\|_\diamond = \|G_t\|_\diamond n^{-1/2} + o(n^{-1/2}), \\
M_H = 0,\ J_H \neq 0 & \|\Phi^n_{\sqrt{t/n}} - e^{-tK_H}\mathcal{E}\|_\diamond = \|F_t\|_\diamond n^{-1} + o(n^{-1}), \\
M_H = J_H = 0,\ P_H \neq 0 & \|\Phi^n_{\sqrt{t/n}} - e^{-tK_H}\mathcal{E}\|_\diamond = \|R_t\|_\diamond n^{-3/2} + o(n^{-3/2}), \\
M_H = J_H = P_H = 0 & \|\Phi^n_{\sqrt{t/n}} - e^{-tK_H}\mathcal{E}\|_\diamond = O(n^{-2}).
\end{array}
\tag{25}
$$

This classifies the first four orders of the unprocessed single-channel iterate (3); it is not a claim about multi-product extrapolation schemes.

Proof. On $\mathcal{A}$, equations (11) and the exponential series give

$$\Phi_{\sqrt{t/n}} = e^{-tK_H/n} + \frac{t^{3/2}}{n^{3/2}} M_H + O_\diamond(n^{-2}).$$

A telescoping expansion around $e^{-tK_H/n}$ shows that the terms containing two cubic insertions and all fourth-order remainders are $O_\diamond(n^{-1})$. After multiplication by $\sqrt{n}$, the single-insertion sum is the Riemann sum

$$t^{3/2} \frac{1}{n} \sum_{k=0}^{n-1} e^{-(n-1-k)tK_H/n} M_H e^{-ktK_H/n} \mathcal{E},$$

which converges in diamond norm to (19). To prove injectivity, diagonalize the positive Hilbert–Schmidt operator K_H. In its eigenbasis, the (i, j) entry of M_H is multiplied by

$$f_{ij}(t) = \int_0^1 e^{-(1-s)t\lambda_i - st\lambda_j}\, \mathrm{d}s > 0.$$

Every multiplier is strictly positive, so $G_t = 0$ if and only if $M_H = 0$. Norm convergence in (20) yields the first line of (25).

Assume next that $M_H = 0$. Equations (12) and (8) give, on $\mathcal{A}$,

$$\Phi_{\sqrt{t/n}} = e^{-tK_H/n} + \frac{t^2}{n^2} J_H + O_\diamond(n^{-5/2}).$$

The same single-insertion expansion, now multiplied by n, converges to (21); two quartic insertions and the Riemann-sum error vanish after this scaling. Diagonalizing K_H shows that every matrix entry of J_H is multiplied by the same strictly positive $f_{ij}(t)$, so the second transform is injective.

If $J_H = 0$, equations (13) and (9) give

$$\Phi_{\sqrt{t/n}} = e^{-tK_H/n} + \frac{t^{5/2}}{n^{5/2}} P_H + O_\diamond(n^{-3}).$$

A third single-insertion argument, multiplied by $n^{3/2}$, converges to (23); the same positive Schur multipliers prove injectivity. If $P_H = 0$, channel telescoping gives $O(n\, n^{-3}) = O(n^{-2})$. This proves (25). □

Corollary 3.3 (orientation memory is prelimit)**.** *The cycles $\Phi^n_{+\sqrt{t/n}}$ and $\Phi^n_{-\sqrt{t/n}}$ have the same limit. If $M_H \neq 0$, then*

$$\sqrt{n}(\Phi^n_{+\sqrt{t/n}} - \Phi^n_{-\sqrt{t/n}}) \longrightarrow 2G_t,$$

so their diamond distance is sharply $\Theta(n^{-1/2})$. If $M_H = 0$, the common quartic term is orientation-even and

$$n^{3/2}(\Phi^n_{+\sqrt{t/n}} - \Phi^n_{-\sqrt{t/n}}) \longrightarrow 2R_t,$$

with R_t given by (23). Thus the orientation distance is $O(n^{-3/2})$ and is sharply $\Theta(n^{-3/2})$ exactly when $P_H \neq 0$.

This is a useful separation of scales. Orientation can be operationally visible at finite cycle number, while the second-order diffusion forgets it. A surviving orientation effect must therefore be sought either in the first correction or in an architecture whose macroscopic scaling retains odd orders.

4 The emergent quantum Markov semigroup

Write $H_{ab} = P_a H P_b$. For $\rho = \bigoplus_a \rho_a \in \mathcal{A}$, define

$$(\mathcal{L}_H \rho)_a = 2 \sum_{b \neq a} H_{ab} \rho_b H_{ba} - \left\{ \sum_{b \neq a} H_{ab} H_{ba}, \rho_a \right\}. \tag{26}$$

Theorem 4.1 (block-jump representation)**.** *On $\mathcal{A}$, $\mathcal{L}_H = -K_H$. It has a GKLS representation with jump operators*

$$L_{ba} = \sqrt{2}\, P_b H P_a, \qquad b \neq a. \tag{27}$$

Hence $(e^{-tK_H})_{t \geq 0}$ is completely positive, trace preserving, unital, and Hilbert–Schmidt self-adjoint on $\mathcal{A}$.

Proof. Insert (27) into $\sum_{a \neq b} (L_{ba} \rho L_{ba}^* - \frac{1}{2}\{L_{ba}^* L_{ba}, \rho\})$ and take its ath diagonal block; this gives (26). Direct expansion of $-\mathcal{E}[H, (1 - \mathcal{E})[H, \rho]]$ gives the same expression. The remaining properties follow from the GKLS form, while Hilbert–Schmidt self-adjointness follows from $K_H = C_H^* C_H$. □

The jump L_{ba} transfers a full density operator rather than a scalar population. When $\dim \mathcal{H}_a > 1$, coherences internal to a record block are transported, rotated, or destroyed according to the matrices H_{ab}. This is the point at which the model is irreducibly quantum.

5 Noncommutative connectivity and exact mixing exponent

Let $H_{\text{off}} = H - \mathcal{E}(H)$ and let $\mathcal{F}_H$ denote the fixed algebra of e^{-tK_H}.

Theorem 5.1 (fixed algebra and frame gap)**.** *One has*

$$\mathcal{F}_H = \ker K_H = \ker C_H = \mathcal{A} \cap \{H_{\text{off}}\}'. \tag{28}$$

Let Π_H be the trace-preserving Hilbert–Schmidt conditional expectation onto $\mathcal{F}_H$, and define

$$\gamma_H = \min \left\{ \frac{\|C_H X\|_2^2}{\|X\|_2^2} : 0 \neq X \in \mathcal{A},\ X \perp \mathcal{F}_H \right\}. \tag{29}$$

Then γ_H is the least positive eigenvalue of K_H. If $\mathcal{F}_H = \mathbb{C}\mathbf{1}$, the semigroup is primitive and $\Pi_H = \Omega$, where $\Omega(X) = \operatorname{Tr}(X)\mathbf{1}/d$.

Proof. Equation (16) shows $K_H X = 0$ iff $C_H X = 0$. For block-diagonal X, $(1 - \mathcal{E})[H, X] = [H_{\text{off}}, X]$, which proves (28). It is a unital $*$-subalgebra. The remaining statements are the finite-dimensional spectral theorem applied to the positive operator $C_H^* C_H$; when the fixed algebra is scalar, trace symmetry gives the unique faithful invariant state $\mathbf{1}/d$ and relaxation to it. □

The terminology "noncommutative connectivity" is literal: in the scalar-block case (28) reduces to constancy on connected graph components, whereas matrix blocks require consistency of internal operators around coupling cycles.

Theorem 5.2 (diamond mixing exponent)**.** *For all $t \geq 0$,*

$$\frac{1}{2}e^{-\gamma_H t} \leq \frac{1}{2}\|e^{-tK_H}\mathcal{E} - \Pi_H\|_\diamond \leq \frac{d}{2}e^{-\gamma_H t}. \tag{30}$$

Thus γ_H is the exact asymptotic diamond-norm exponent. In the primitive case $\Pi_H = \Omega$.

Proof. On $\mathcal{F}_H^\perp$, the Hilbert–Schmidt operator norm of e^{-tK_H} is $e^{-\gamma_H t}$. For any map T on $\mathcal{B}(\mathcal{H})$, $\|T\|_\diamond \leq d\|T\|_{2\to 2}$: apply Cauchy–Schwarz on $\mathcal{H} \otimes \mathcal{H}$ and use $\|T \otimes \mathrm{id}_d\|_{2\to 2} = \|T\|_{2\to 2}$. This gives the upper bound. For the lower bound choose a Hermitian eigenoperator $X \perp \mathcal{F}_H$ with eigenvalue γ_H and use $X/\|X\|_1$ as an admissible trace-norm input. Its output has trace norm $e^{-\gamma_H t}$. □

Proposition 5.3 (robustness)**.** *Assume H is primitive. For a Hermitian perturbation Δ, define*

$$\underline{\gamma}_{H+\Delta} = \min_{0 \neq X \perp \mathbf{1}} \frac{\|C_{H+\Delta}X\|_2^2}{\|X\|_2^2}.$$

Then

$$\sqrt{\underline{\gamma}_{H+\Delta}} \geq \sqrt{\gamma_H} - 2\|\Delta\|. \tag{31}$$

In particular, $2\|\Delta\| < \sqrt{\gamma_H}$ guarantees that $H + \Delta$ remains primitive and that its gap obeys the squared right-hand side of (31). Only the off-block part of Δ contributes to $C_{H+\Delta} - C_H$.

Proof. On the fixed domain $\mathcal{A} \ominus \mathbb{C}\mathbf{1}$, the reverse triangle inequality gives

$$\|C_{H+\Delta}X\|_2 \geq \|C_H X\|_2 - \|(1-\mathcal{E})[\Delta, X]\|_2 \geq (\sqrt{\gamma_H} - 2\|\Delta\|)\|X\|_2.$$

Positivity of the right-hand side excludes any new fixed vector. □

6 Rank-one reduction and what it misses

Suppose every block is one-dimensional and use the record basis $\{|a\rangle\}$. Let

$$(L_H)_{ab} = -|H_{ab}|^2 \quad (a \neq b), \qquad (L_H)_{aa} = \sum_{b \neq a} |H_{ab}|^2. \tag{32}$$

Corollary 6.1 (classical graph boundary)**.** *On diagonal matrices, $K_H = 2L_H$. Hence the block theorem recovers the squared-coupling graph diffusion and its gap $2\lambda_2(L_H)$.*

For rank-one blocks, (26) is an ordinary continuous-time Markov chain. For larger blocks, replacing H_{ab} by $\|H_{ab}\|$ loses the cycle consistency encoded in (28). A connected support graph is necessary but not sufficient to identify the fixed algebra: matrix-valued edge holonomies can preserve or destroy internal quantum symmetries.

The third branch of Theorem 3.2 is not vacuous. Let w be the unique root in $(2.6104, 2.6105)$ of

$$f(w) = 22849w^4 - 40560w^3 - 70146w^2 + 50544w + 6561, \tag{33}$$

put

$$p = \sqrt{\frac{25}{2} - \frac{19}{4}w}, \qquad q = -\sqrt{20 + \frac{2}{3}w}, \qquad (a, b, c) = \frac{1}{3}(p + q, q - 2p, p - 2q),$$

and consider the three-record Hamiltonian

$$H_\diamond = \begin{pmatrix} a & 1 & 3/2 \\ 1 & b & i\sqrt{w} \\ 3/2 & -i\sqrt{w} & c \end{pmatrix}. \tag{34}$$

Proposition 6.2 (exact attainment of the third branch)**.** *For $H_\diamond$ with three rank-one blocks,*

$$M_{H_\diamond} = J_{H_\diamond} = 0, \qquad P_{H_\diamond} \neq 0.$$

Consequently, for every $t > 0$ its direct diffusive product error is sharply $\|R_t\|_\diamond n^{-3/2} + o(n^{-3/2})$.

Proof. Rank-one compression gives $A_H^{(1)} = 0$, hence $M_H = 0$. Set

$$A = \frac{25}{2} - \frac{19}{4}w, \quad B = 20 + \frac{2}{3}w, \quad C = 8w + \frac{13}{2} - \frac{27}{4w}.$$

A direct three-state calculation reduces the independent entries of $J_H = 0$ to $p^2 = A$, $q^2 = B$, and $(q-p)^2 = C$. The first two hold by definition, while

$$4AB - (A + B - C)^2 = -\frac{f(w)}{144w^2} = 0.$$

On the isolating interval $A, B > 0$ and $(A + B - C)/2 < 0$; since $p > 0$ and $q < 0$, this fixes $pq = (A + B - C)/2$ and proves the third identity. Finally, direct substitution in (9) gives $(P_H)_{23} = 15\sqrt{w}/4 \neq 0$. A Sturm certificate for (33) and these identities is included in the supplement. □

7 An exact irreducible four-block architecture

Take four blocks $\mathcal{H}_a \cong \mathbb{C}^2$ and the rational block Hamiltonian

$$H_* = \begin{pmatrix} 0 & I & A & I \\ I & 0 & I & B \\ A & I & 0 & 0 \\ I & B & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{35}$$

Proposition 7.1 (exact irreducibility certificate)**.** *For H_*, $\mathcal{F}_{H_*} = \mathbb{C}\mathbf{1}$ and $\operatorname{rank} C_{H_*} = 15$. The characteristic polynomial of K_{H_*} factors over $\mathbb{Z}$ as*

$$\begin{aligned} \chi_{K_*}(x) = x(x-8) & \\ &\times (x^4 - 26x^3 + 215x^2 - 626x + 580) \\ &\times (x^4 - 26x^3 + 215x^2 - 594x + 228) \\ &\times (x^6 - 44x^5 + 700x^4 - 5200x^3 + 18880x^2 - 31296x + 18048), \end{aligned} \tag{36}$$

Its gap is the unique root of

$$x^4 - 26x^3 + 215x^2 - 594x + 228 \tag{37}$$

in

$$\frac{909}{2000} < \gamma_* < \frac{4547}{10000}, \qquad \gamma_* = 0.45459953167349515\ldots. \tag{38}$$

Proof. If $X = \bigoplus_a X_a$ commutes with H_*, the three identity edges force $X_1 = X_2 = X_3 = X_4 =: Y$. The two remaining edges require $[A, Y] = [B, Y] = 0$. Since A has simple spectrum, Y is diagonal; commuting also with B makes its diagonal entries equal. Thus Y is scalar. Exact Gaussian elimination gives $\operatorname{rank} C_{H_*} = 15$. Rational determinant computation gives (36); a Sturm sequence proves that (37) has no root in $(0, 909/2000)$ and exactly one in the interval (38). These steps are replayed by the symbolic certificate in the supplement. □

The construction separates two notions that coincide for scalar graphs. Support connectivity synchronizes blocks, but primitive quantum mixing additionally needs an irreducible family of matrix cycle constraints. This supplies a small exact benchmark for algorithms that seek primitive block-measurement architectures.

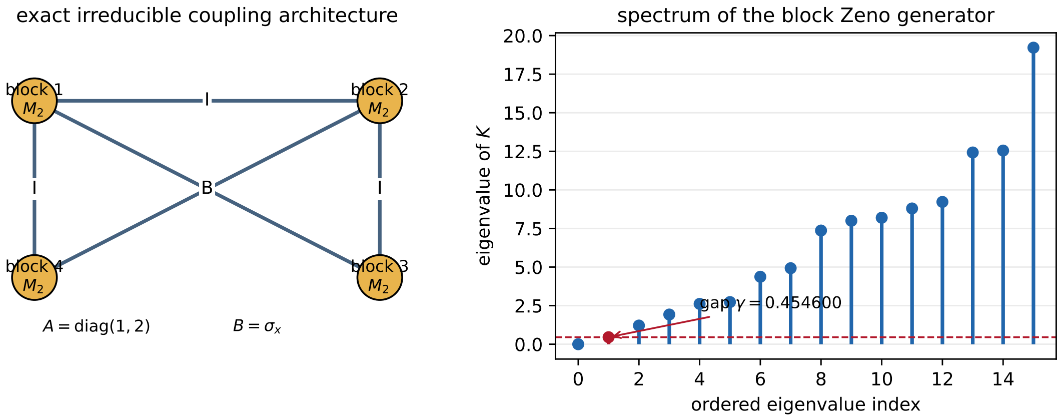

Figure 1: Left: the block architecture of (35). Identity edges synchronize the internal observables; the noncommuting A and B cycle constraints reduce their common commutant to scalars. Right: the exactly certified 16-dimensional spectrum of K_{H_*}.

8 Finite-cycle numerics and resource ledger

The accompanying verifier uses column-major superoperators and three independent kinds of checks:

(a) matrix identities $K = C^*C$ and $\mathcal{L} = -K$ on the block algebra;

(b) convergence slopes for local and product errors;

(c) a primal semidefinite program for the diamond norm, including a reference of dimension d.

For the deterministic two-block tests, the fitted local slopes are 3.05410 (generic) and 3.99925 (horizontal). After subtracting the exact $\tau^3 M_H$ term, both local slopes become fourth order: 3.99876 and 3.99925. The corresponding product slopes are -0.50103 and -1.00063; the forward/reverse difference has slope -0.49910. The cubic obstruction norms are 2.12511 and 0, respectively. For the generic case at $t = 0.4$, $\|G_t\|_{2\to2} = 0.211242$ and the scaled residual has fitted slope -0.50021. In the horizontal case, $\|J_H\|_{2\to2} = 4.55176$ and $\|F_t\|_{2\to2} = 0.0900448$; the residual after scaling by n has slope -1.00061, stronger than the general $o(1)$ conclusion because this example has an additional parity cancellation.

For the exact algebraic example (34), $\|P_H\|_{2\to2} = 12.2896$ and $\|R_{0.1}\|_{2\to2} = 0.0122095$. Over $n = 512, 1024, 2048, 4096$, the unscaled product error has fitted slope -1.51292, while the residual after multiplying by $n^{3/2}$ and subtracting $R_{0.1}$ has slope -0.500756. Thus deterministic tests resolve all three sharp branches of Theorem 3.2. For the horizontal four-dimensional instance, the computed half-diamond errors for $n = 8, 32, 128$ are

$$5.76696 \times 10^{-3}, \quad 1.41550 \times 10^{-3}, \quad 3.52269 \times 10^{-4}, \tag{39}$$

while half of the explicit theorem bounds is respectively 0.35940, 0.081960, and 0.0200545. The bounds are deliberately dimension-free and conservative; the SDP confirms the predicted rate in an ancilla-complete metric.

The limit does not make repeated measurement free. For fixed macroscopic diffusive time t, n cycles use a rotation scale $|\tau| = \sqrt{t/n}$. Therefore the total coherent dwell required by a direct implementation is proportional to

$$n|\tau| = \sqrt{nt}, \tag{40}$$

up to whether the forward and reverse rotations are counted separately. To make the finite-cycle approximation at most ε for fixed H, t, Theorem 3.2 gives $n = \Theta(\varepsilon^{-2})$, $\Theta(\varepsilon^{-1})$, or $\Theta(\varepsilon^{-2/3})$ when respectively the first nonzero obstruction is M_H, J_H, or P_H. If all three vanish, $n = O(\varepsilon^{-1/2})$ is sufficient. The centrality condition (14) is a directly checkable cubic cancellation, after which the

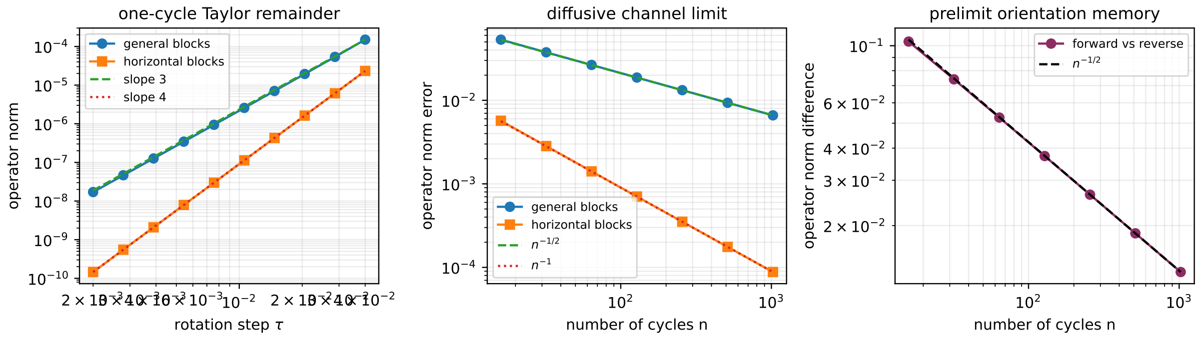

Figure 2: Deterministic stress tests for the first two branches. The cubic obstruction produces the sharp $n^{-1/2}$ correction; the central-block example has $M_H = 0$, $J_H \neq 0$, and produces the sharp n^{-1} correction. The exact third branch is reported in the text.

next obstructions decide further gains. The hierarchy is therefore an architecture-level resource statement, not only a numerical slope.

9 Relation to prior work and novelty boundary

The ballistic quantum Zeno limit and its projected Hamiltonian dynamics are classical results; see the review by Facchi and Pascazio [2] and rigorous product formulas such as Exner and Ichinose [3]. General quantum operations and kicks admit a broad Zeno theory [5], differently scaled noncommuting products have been studied in their own right [6], and optimal $O(n^{-1})$ bounds are available for ordinary Zeno sequences in an abstract infinite-dimensional setting [12]. We do not claim any of those frameworks as new.

The closest scaling precedent is Facchi and Ligabò [4]: at the $N^{1/2}$ threshold, a selective rank-one survival probability acquires a nontrivial diffusive limit. Here that scalar probability is promoted to the full unconditional block channel, with an explicit GKLS generator, uniform diamond control, a fixed-algebra theorem, and a quantitative matrix-valued gap. Rank-one sequential measurements have also been described using doubly stochastic Markov chains [7]; Corollary 6.1 identifies the precise boundary where that classical reduction applies.

Alternating and noncommuting conditional expectations are central to quantum functional inequalities [8]. Our question is different: the two expectations coalesce as $\tau \to 0$, and their second-order defect is retained over n cycles. Multi-product Zeno formulas can cancel successive errors and reach arbitrary algebraic orders through combinations of several sequences and post-processing [13]; Theorem 3.2 instead derives the first three nonzero correction transforms for the unprocessed iterate of the single CPTP channel (3). Standard criteria for irreducibility of quantum Markov semigroups remain the appropriate background [9, 10]. The contribution here is the specialized commutator frame (4), its block-architecture interpretation, its exact gap-to-diamond law, and exact algebraic witnesses for both primitivity and the third convergence branch. Adaptive time-dependent Zeno measurements can instead generate geometric coherent dynamics [11]; our measurements are nonselective and generate trace-symmetric diffusion.

Accordingly, the priority claim is limited to the following conjunction, to the best of our knowledge as of 10 August 2026:

(i) the full block-record diffusive product limit in diamond norm with explicit finite-n constants;
(ii) the exact cubic, quartic, and quintic product obstructions and the sharp $\Theta(n^{-1/2})$, $\Theta(n^{-1})$, $\Theta(n^{-3/2})$, $O(n^{-2})$ hierarchy;
(iii) the block-jump generator and commutator-frame gap as one theorem package;
(iv) the exact rational primitive architecture and perturbative certificate.

No claim is made that the Zeno effect, Chernoff products, GKLS form, quantum-semigroup irreducibility, classical graph mixing, or alternating projections are individually new.

10 Limitations and open problems

The results are finite-dimensional and assume ideal projective block measurements. Instrument-level outcome records, measurement duration, readout errors, and correlated environmental noise are outside the theorem. The constants in Theorem 3.1 are not optimized, and architectures with $M_H = J_H = P_H = 0$ may have a sharper sixth- or higher-order classification than the stated bound. Higher-order multi-product or post-processed formulas lie outside this physical single-channel architecture. The dimension factor in (30) converts a Hilbert–Schmidt gap into a worst-case ancilla-assisted bound and may be sharpened for specific block geometries.

Three next problems appear concrete. First, classify the sixth-order product obstruction on the joint zero set $M_H = J_H = P_H = 0$. Second, replace the pinching by a general trace-preserving conditional expectation and identify which parts of the frame construction survive without a matrix-block presentation. Third, compute logarithmic Sobolev constants for (26), which would control relative entropy rather than only diamond distance.

11 Reproducibility statement

The supplement contains the complete Python source, pinned package versions, deterministic seed 20260810, JSON outputs, and two vector figures. The numerical verifier reconstructs every superoperator from H and the block dimensions, checks the exact formulas for M_H, N_H, J_H, and P_H, verifies convergence to G_t, F_t, and R_t, checks complete positivity and trace preservation of the limiting channels, solves the primal Watrous diamond-norm SDP, and tests the rank-one identity. A separate SymPy program certifies both algebraic examples using exact integers, rationals, polynomial identities, and Sturm intervals. No numerical eigensolver is used for the irreducibility or root-count certificates. SHA-256 hashes are recorded in the manifest.

AI assistance disclosure. AI tools assisted with literature triage, symbolic/numerical code, and manuscript drafting. The named author is responsible for the claims. Every theorem-facing computational assertion is reproducible from the supplied source; the analytical proofs are included above and in Appendix A.

A Proof details and norm bookkeeping

A.1 Channel conventions

For a linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$,

$$\|T\|_\diamond = \sup_{m\geq 1} \|T \otimes \mathrm{id}_m\|_{1\to 1};$$

in dimension d the supremum is attained for $m = d$. A channel has diamond norm one. Left or right multiplication by H has completely bounded trace norm $\|H\|$, hence $\|\operatorname{ad}_H\|_\diamond \leq 2\|H\|$. The conditional expectations $\mathcal{E}$ and Π_H are channels and Hilbert–Schmidt orthogonal projections.

A.2 Taylor derivatives

Writing $F(\tau) = \mathcal{E}e^{-i\tau D}\mathcal{E}e^{i\tau D}\mathcal{E}$, Leibniz' rule gives

$$F^{(m)}(\tau) = \sum_{j=0}^{m} \binom{m}{j} \mathcal{E}(-iD)^j e^{-i\tau D}\mathcal{E}(iD)^{m-j} e^{i\tau D}\mathcal{E}$$

with the powers inserted on the two sides of the middle $\mathcal{E}$. Therefore

$$\|F^{(m)}(\tau)\|_\diamond \le \sum_{j=0}^{m} \binom{m}{j} \|D\|_\diamond^m = 2^m \|D\|_\diamond^m.$$

For $m = 3$ this is $64h^3$ and Taylor division by 3! gives $32h^3/3$. For $m = 4$ it is $256h^4$ and division by 4! again gives $32h^4/3$.

A.3 Why block centrality is sufficient

For $X \in \mathcal{A}$,

$$\mathcal{E}[H, X] = [\mathcal{E}(H), X].$$

The center of $\mathcal{A}$ consists of $\bigoplus_a \lambda_a I_{\mathcal{H}_a}$, so different block scalars are allowed. If $\mathcal{E}(H)$ is central, $\mathcal{E}D\mathcal{E} = 0$. The two uncancelled-looking cubic monomials each contain this factor and vanish. No equality of the λ_a is used.

A.4 Product comparison

Let $A_s = \Phi_{\sqrt{s}}$, $B_s = e^{-sK_H}\mathcal{E}$. Since both are contractions in diamond norm,

$$\|A_s^n - B_s^n\|_\diamond \le n\|A_s - B_s\|_\diamond.$$

Insert $\mathcal{E} - sK_H$ between them. The first difference is controlled by Theorem 2.1. For the second, the exponential power series and $\|K_H\|_\diamond \le 4h^2$ give the stated bound. Setting $s = t/n$ proves uniformity on $[0, T]$ directly.

A.5 Spectrum-to-diamond conversion

Let $T_t = e^{-tK_H}\mathcal{E} - \Pi_H$. Its Hilbert–Schmidt norm is $e^{-\gamma_H t}$. For $Z \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$,

$$\|(T_t \otimes \mathrm{id})(Z)\|_1 \le d\|(T_t \otimes \mathrm{id})(Z)\|_2 \le de^{-\gamma_H t}\|Z\|_2 \le de^{-\gamma_H t}\|Z\|_1.$$

This proves the upper half of (30). Hermiticity preservation of K_H allows the gap eigenvector to be chosen Hermitian, which proves the lower half without a dimension-dependent normalization.

B Exact spectral data

The ordered eigenvalues of K_{H_*}, rounded only for display, are

$$0, 0.454599532, 1.209862210, 1.923717458, 2.614380960, 2.731408082,$$
$$4.371410896, 4.925803281, 7.366074024, 8, 8.194862754, 8.796685623,$$
$$9.220870692, 12.424734434, 12.548188836, 19.217401218.$$

The exact certificate constructs the 16×16 rational matrix of K_{H_*} in block matrix-unit coordinates. It verifies symmetry, rank 15, the identity null vector, the factorization (36), and the Sturm counts around (38).

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