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EXACT ALL-COUPLING COMPARISON OF WILSON, HEAT-KERNEL, AND MANTON ACTIONS IN 2D SU(2) LATTICE YANG–MILLS THEORY: RICCATI SUB-CASIMIR SCALING, WEIERSTRASS SUPER-CASIMIR SCALING, AND UNIQUE CROSSING COUPLINGS

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Abstract. We prove exact, non-asymptotic comparison theorems relating the three canonical single-plaquette lattice actions of two-dimensional SU(2) Yang–Mills theory—the Wilson cosine action, the Migdal–Menotti–Onofri heat-kernel (Villain) action, and the Manton geodesic quadratic action—across all irreducible representations $j \in \frac{1}{2}\mathbb{Z}_{\geq 1}$ and all bare couplings $\beta > 0$. First, transforming the modified Bessel Riccati equation for $r_\nu(\beta) = I_\nu(\beta)/I_{\nu-1}(\beta)$ into the hyperbolic ODE $q'_\nu(\beta) = \beta^{-1} - 2\sinh((2\nu-1)q_\nu)/(2\nu-1)$ for $q_\nu := (-\ln r_\nu)/(2\nu-1)$, we prove in four lines that Wilson string tensions obey a strict *sub-Casimir* hierarchy $\sigma^{\mathrm{W}}_{1/2}(\beta)/C_2(\frac{1}{2}) > \sigma^{\mathrm{W}}_1(\beta)/C_2(1) > \sigma^{\mathrm{W}}_{3/2}(\beta)/C_2(\frac{3}{2}) > \cdots$ for *every* $\beta > 0$, alongside closed-form two-sided Riccati bounds $L_\nu(\beta) < r_\nu(\beta) < U_\nu(\beta)$ proving $\sigma^{\mathrm{W}}_j(\beta) > \sigma^{\mathrm{HK}}_j(\beta)$ for all $\beta > \frac{1}{3}(2j+\frac{1}{2})^2 + (2j+\frac{1}{2})$. Second, we decompose the exact Manton character coefficients into a closed-form Gaussian core plus an exponentially small tail $|\Delta_j(\beta)| \leq 2e^{-\pi^2\beta/2}/((2j+1)\pi^2\beta)$; using the Weierstrass product of $\sinh(\sqrt{u})/\sqrt{u}$, we prove strict concavity of $u \mapsto \ln(\sinh(\sqrt{u})/\sqrt{u})$ on $(0,\infty)$, establishing that the Manton Gaussian core obeys the opposite *super-Casimir* hierarchy $\tilde{\sigma}^{\mathrm{M}}_j(\beta)/\tilde{\sigma}^{\mathrm{M}}_{1/2}(\beta) > \frac{4}{3}j(j+1)$ for all $\beta > 0$ and enjoys automatic $O(\beta^{-2})$ Symanzik improvement: its leading Casimir-ratio error is $+\frac{1}{270}(N^2-1)(N^2-4)\beta^{-3}$ ($N = 2j+1$), whereas Wilson's is $-\frac{1}{36}(N^2-1)(N^2-4)\beta^{-2}$, a universal $\frac{15}{2}\beta$ improvement factor independent of j. Third, combining our analytic Riccati bounds with 180-digit zero-quadrature `mpmath.iv` interval arithmetic and standalone zero-import, zero-axiom Lean 4.36.0 verification (`ActionUniversalityCerts.lean`), we certify 24 explicit action-hierarchy points showing $\sigma^{\mathrm{M}}_j < \sigma^{\mathrm{HK}}_j < \sigma^{\mathrm{W}}_j$ and opposite-side Casimir bracketing $\sigma^{\mathrm{W}}_j/\sigma^{\mathrm{W}}_{1/2} < \frac{4}{3}j(j+1) < \sigma^{\mathrm{M}}_j/\sigma^{\mathrm{M}}_{1/2}$ (including up to 391.7× sharper glueball-to-tension ratios $m_{\mathrm{gap}}/\sigma_{1/2} = \sigma_1/\sigma_{1/2}$ vs. $\frac{8}{3}$ at $\beta = 50$), and we certify the *unique* Wilson–heat-kernel crossing couplings $\beta_*(j)$ on $(0,\infty)$ to width 10^{-6} for $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$.

1. Introduction

A foundational principle of lattice gauge theory [12] is *action universality*: distinct lattice discretizations of the classical Yang–Mills action sharing the same weak-coupling quadratic expansion must converge to identical physical observables in the continuum limit $\beta = 4/g^2 \to \infty$, differing at finite lattice spacing only by irrelevant discretization artifacts [10]. In two-dimensional SU(2) lattice Yang–Mills theory, the exact Migdal–Witten character expansion [8, 2, 7] reduces the string tension $\sigma_j(\beta)$ in every irreducible representation $j \in \frac{1}{2}\mathbb{Z}_{\geq 1}$ (and the adjoint transfer-matrix gap per unit spatial width $m_{\mathrm{gap}}(\beta) = \sigma_1(\beta)$) to a single-plaquette Fourier–Peter–Weyl ratio, providing an ideal testing ground for exact, non-asymptotic action comparisons [6, 5].

Three canonical single-plaquette Boltzmann weights occupy central roles in lattice Yang–Mills theory on SU(2) (parameterized by half-geodesic class angle $\theta \in [0,\pi]$ with $\frac{1}{2}\operatorname{Tr} U = \cos\theta$, geodesic distance $d(U,I) = \sqrt{2}\,\theta$, Weyl Haar density $d\nu(\theta) = \frac{2}{\pi}\sin^2\theta\, d\theta$, characters $\chi_j(\theta) = \frac{\sin((2j+1)\theta)}{\sin\theta}$, and quadratic Casimir $C_2(j) = j(j+1)$):

(1) *Wilson cosine action* [12]: $w_{\mathrm{W}}(\theta;\beta) = e^{\beta\cos\theta}$, whose exact normalized character ratios yield $\sigma^{\mathrm{W}}_j(\beta) = -\ln\big(I_{2j+1}(\beta)/I_1(\beta)\big)$;

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(2) *Heat-kernel (Migdal–Menotti–Onofri / Villain) action* [8, 7]: $w_{\mathrm{HK}}(\theta;\beta) = \sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}}(2j+1)e^{-\frac{2}{\beta}j(j+1)}\chi_j(\theta)$, whose string tensions satisfy exact Casimir scaling $\sigma_j^{\mathrm{HK}}(\beta) = \frac{2}{\beta}j(j+1)$ at every coupling $\beta > 0$;

(3) *Manton geodesic quadratic action* [6]: $w_{\mathrm{M}}(\theta;\beta) = e^{-\frac{\beta}{2}\theta^2}$, matching $e^{\beta(\cos\theta-1)} = \exp(-\frac{\beta}{2}\theta^2 + \frac{\beta}{24}\theta^4 - \cdots)$ to $O(\theta^2)$ while eliminating the $+\frac{\beta}{24}\theta^4$ quartic cosine distortion.

While asymptotic weak-coupling expansions and numerical comparisons of these actions have been studied since [6, 7, 5], exact *all-coupling* monotonicity theorems, non-asymptotic Casimir-scaling brackets, and rigorously certified crossing couplings have not previously been established in unified, mechanically verified form. In this paper we prove four clean, self-contained results:

- *All-β Wilson sub-Casimir hierarchy and sharp Riccati bounds* (Section 3): By rewriting the modified Bessel Riccati equation as a hyperbolic ODE for $q_\nu(\beta)$ defined by $-\ln(I_\nu(\beta)/I_{\nu-1}(\beta))/(2\nu-1)$, we give a four-line proof that $q_2(\beta) > q_3(\beta) > \cdots$ for *all* $\beta > 0$, proving $\sigma^{\mathrm{W}}_{1/2}(\beta)/C_2(\frac{1}{2}) > \sigma^{\mathrm{W}}_1(\beta)/C_2(1) > \cdots$ globally. In particular, the 2D SU(2) Wilson glueball-to-string-tension ratio satisfies $m^{\mathrm{W}}_{\mathrm{gap}}(\beta)/\sigma^{\mathrm{W}}_{1/2}(\beta) < \frac{8}{3}$ for every $\beta > 0$ (complementing strong-coupling d-dimensional ratio bounds in [3, 4]). A six-line Riccati supersolution/subsolution argument [1, 9] also yields closed-form global enclosures $L_\nu(\beta) < r_\nu(\beta) < U_\nu(\beta)$ proving $\sigma_j^{\mathrm{W}}(\beta) > \sigma_j^{\mathrm{HK}}(\beta)$ for all $\beta > \frac{1}{3}(2j+\frac{1}{2})^2 + (2j+\frac{1}{2})$.
- *Exact Manton Gaussian-plus-tail decomposition, Weierstrass super-Casimir scaling, and automatic $O(\beta^{-2})$ Symanzik improvement* (Section 4): Integrating $\cos(m\theta)e^{-\beta\theta^2/2}$ over $[0,\pi] = [0,\infty)\setminus[\pi,\infty)$ yields an exact closed-form Gaussian core plus a non-perturbative tail $|\Delta_j(\beta)| \leq 2e^{-\pi^2\beta/2}/((2j+1)\pi^2\beta)$. Using the Weierstrass product $\frac{\sinh\sqrt{u}}{\sqrt{u}} = \prod_{k=1}^{\infty}(1+\frac{u}{\pi^2k^2})$, we prove in three lines that $u \mapsto \ln(\sinh(\sqrt{u})/\sqrt{u})$ is strictly concave on $(0,\infty)$, yielding the exact *opposite* super-Casimir hierarchy for the Manton Gaussian core ($\tilde\sigma_j^{\mathrm{M}}(\beta)/\tilde\sigma_{1/2}^{\mathrm{M}}(\beta) > \frac{4}{3}j(j+1)$ for all $\beta > 0$) and proving that its leading Casimir-ratio discretization error vanishes identically at both $O(\beta^{-1})$ and $O(\beta^{-2})$, entering only at $+\frac{1}{270}(N^2-1)(N^2-4)\beta^{-3}$ ($N = 2j+1$)—universally $\frac{15}{2}\beta$ times smaller than Wilson's $-\frac{1}{36}(N^2-1)(N^2-4)\beta^{-2}$ error.
- *Unique Wilson–heat-kernel crossing couplings $\beta_*(j)$ on $(0,\infty)$* (Section 5): Combining our analytic Riccati lower bound on $(0,\frac{1}{2}]$, analytic Riccati upper bound on $[\beta_{\mathrm{ub}}(j),\infty)$, and 180-digit zero-black-box interval branch-and-bound on $[\frac{1}{2},\beta_{\mathrm{ub}}(j)]$, we prove that $\sigma_j^{\mathrm{W}}(\beta) - \sigma_j^{\mathrm{HK}}(\beta)$ has a *unique* zero $\beta_*(j)$ on $(0,\infty)$ for each $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$, certified to width 10^{-6}.
- *Computer-assisted 180-digit and zero-axiom Lean 4.36.0 certification* (Section 6): All 24 comparison points (showing simultaneous $\sigma_j^{\mathrm{M}} < \sigma_j^{\mathrm{HK}} < \sigma_j^{\mathrm{W}}$ and $\sigma_j^{\mathrm{W}}/\sigma_{1/2}^{\mathrm{W}} < \frac{4}{3}j(j+1) < \sigma_j^{\mathrm{M}}/\sigma_{1/2}^{\mathrm{M}}$) and all 6 unique crossings are certified without black-box special functions or numerical quadrature, with all algebraic identities and cross-multiplied certificate checks mechanically verified in standalone zero-import, zero-axiom Lean 4.36.0.

2. Exact Character Expansions of the Three 2D SU(2) Actions

In two-dimensional SU(2) lattice gauge theory with free boundary conditions on $\Lambda_L \subset \mathbb{Z}^2$ (or in the thermodynamic limit on the infinite plane $\mathbb{Z}^2$), choosing an axial gauge renders every single-plaquette holonomy $U_p \in \mathrm{SU}(2)$ an independent random variable with normalized probability density proportional to a class function $w(U_p;\beta) > 0$. Expanding $w(U;\beta) = \sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}}(2j+1)c_j(\beta)\chi_j(U)$ in irreducible characters $\chi_j(\theta) = \frac{\sin((2j+1)\theta)}{\sin\theta}$ (where $U \sim \mathrm{diag}(e^{i\theta}, e^{-i\theta})$, $\theta \in [0,\pi]$, and $c_j(\beta) := \frac{1}{2j+1}\int_{\mathrm{SU}(2)} w(U;\beta)\chi_j(U)\,dU$), the expectation of an $R\times S$ rectangular Wilson loop in representation

j factorizes identically [8, 7]:

$$\left\langle \frac{1}{2j+1}\chi_j\big(W_{R\times S}\big)\right\rangle_\beta = \left(\frac{c_j(\beta)}{c_0(\beta)}\right)^{RS}, \qquad \sigma_j(\beta) := -\ln\left(\frac{c_j(\beta)}{c_0(\beta)}\right), \tag{1}$$

whenever $c_j(\beta) > 0$. Because pure 2D Yang–Mills theory has no local propagating gluons, the transfer-matrix gap between the trivial sector ($j = 0$) and the adjoint sector ($j = 1$) on a periodic cylinder of spatial circumference L equals $L\sigma_1(\beta)$; we write $m_{\mathrm{gap}}(\beta) := \sigma_1(\beta)$ for the adjoint transfer-matrix gap per unit spatial width (equivalently, the gap at $L = 1$). The exact continuum Casimir scaling ratio for representation $j \in \frac{1}{2}\mathbb{Z}_{\geq 1}$ relative to the fundamental ($j = \frac{1}{2}$, $C_2(\frac{1}{2}) = \frac{3}{4}$), and the corresponding string-tension ratio and Casimir defect for action $A \in \{\mathrm{W}, \mathrm{HK}, \mathrm{M}\}$, are

$$\mathcal{C}_j := \frac{C_2(j)}{C_2(\frac{1}{2})} = \frac{4}{3}\,j(j+1) = \frac{N^2-1}{3}, \quad R_j^A(\beta) := \frac{\sigma_j^A(\beta)}{\sigma_{1/2}^A(\beta)}, \quad \Delta R_j^A(\beta) := R_j^A(\beta) - \mathcal{C}_j, \tag{2}$$

where $N := 2j + 1 \in \{2, 3, 4, \dots\}$.

Proposition 2.1 (Exact Character Coefficients of Wilson, Heat-Kernel, and Manton Actions)**.** *For every $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ (with $N = 2j + 1$) and every $\beta > 0$:*

(1) (Wilson) *For $w_{\mathrm{W}}(\theta; \beta) = e^{\beta\cos\theta}$,*

$$c_j^{\mathrm{W}}(\beta) = \frac{2}{\beta}\,I_{2j+1}(\beta), \qquad \sigma_j^{\mathrm{W}}(\beta) = -\ln\left(\frac{I_{2j+1}(\beta)}{I_1(\beta)}\right) = \sum_{\nu=2}^{2j+1}\big(-\ln r_\nu(\beta)\big), \tag{3}$$

where $I_\nu(\beta) = \sum_{k=0}^{\infty}\frac{(\beta/2)^{\nu+2k}}{k!(\nu+k)!}$ is the modified Bessel function of the first kind and $r_\nu(\beta) := I_\nu(\beta)/I_{\nu-1}(\beta) \in (0, 1)$ (cf. Theorem 3.1).

(2) (Heat-Kernel) *For the* SU(2) *heat kernel $w_{\mathrm{HK}}(\theta; \beta) = \sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}}(2j + 1)e^{-\frac{2}{\beta}j(j+1)}\chi_j(\theta) > 0$,*

$$c_j^{\mathrm{HK}}(\beta) = \exp\left(-\frac{2j(j+1)}{\beta}\right), \qquad \sigma_j^{\mathrm{HK}}(\beta) = \frac{2j(j+1)}{\beta} = \frac{N^2-1}{2\beta}, \qquad \frac{\sigma_j^{\mathrm{HK}}(\beta)}{\sigma_{1/2}^{\mathrm{HK}}(\beta)} \equiv \mathcal{C}_j. \tag{4}$$

(3) (Manton) *For $w_{\mathrm{M}}(\theta; \beta) = e^{-\frac{\beta}{2}\theta^2}$ ($\theta \in [0, \pi]$),*

$$c_j^{\mathrm{M}}(\beta) = \frac{1}{N}\big(G_{2j}(\beta) - G_{2j+2}(\beta)\big), \qquad G_m(\beta) := \frac{1}{\pi}\int_0^\pi \cos(m\theta)\, e^{-\frac{\beta}{2}\theta^2}\, d\theta. \tag{5}$$

Proof. By the Weyl integration formula on SU(2), $c_j(\beta) = \frac{2}{\pi N}\int_0^\pi \sin\theta\sin(N\theta)\, w(\theta; \beta)\, d\theta$. Using $2\sin\theta\sin(N\theta) = \cos((N-1)\theta) - \cos((N+1)\theta)$ immediately yields (5) for w_{M}. For w_{W}, using $I_m(\beta) = \frac{1}{\pi}\int_0^\pi \cos(m\theta)e^{\beta\cos\theta}\, d\theta$ gives $c_j^{\mathrm{W}}(\beta) = \frac{I_{N-1}(\beta)-I_{N+1}(\beta)}{N} = \frac{2}{\beta}I_N(\beta)$ by the Bessel identity $I_{N-1}(\beta) - I_{N+1}(\beta) = \frac{2N}{\beta}I_N(\beta)$ [11], proving (3). Relation (4) is immediate from Peter–Weyl orthogonality. □

3. Wilson Action: Hyperbolic Riccati Sub-Casimir Hierarchy and Global Bounds

We now prove both our closed-form two-sided Riccati bounds for $r_\nu(\beta)$ and the exact all-coupling strict sub-Casimir hierarchy of the Wilson action from the single first-order Bessel Riccati equation.

Theorem 3.1 (Global Two-Sided Riccati Enclosures and Weak-Coupling Domination)**.** *For every integer $\nu \geq 1$ (setting $a := \nu - \frac{1}{2} \geq \frac{1}{2}$) and every $\beta > 0$,*

$$L_\nu(\beta) := \frac{\beta}{a + \sqrt{\beta^2 + (a+1)^2}} \;<\; r_\nu(\beta) \;<\; U_\nu(\beta) := \frac{\beta}{a + \sqrt{\beta^2 + a(a+1)}}. \tag{6}$$

Consequently, for every $j \in \frac{1}{2}\mathbb{Z}_{\geq 1}$ (setting $a_j := 2j + \frac{1}{2}$):

(1) (Explicit weak-coupling domination $\sigma_j^{\mathrm{W}} > \sigma_j^{\mathrm{HK}}$) *Whenever* $\beta > \frac{a_j^2}{3} + a_j$,

$$\sigma_j^{\mathrm{W}}(\beta) \;>\; \sum_{\nu=2}^{2j+1} \big(-\ln U_\nu(\beta) \big) \;>\; \frac{2j(j+1)}{\beta} = \sigma_j^{\mathrm{HK}}(\beta). \tag{7}$$

(2) (Strong-coupling monotonicity of L_ν) *For every* $\nu \geq 2$, *the function* $h_\nu(\beta) := \beta \ln(1/L_\nu(\beta))$ *is strictly increasing on* $(0, \frac{1}{2}]$, *so* $\beta\sigma_j^{\mathrm{W}}(\beta) < \sum_{\nu=2}^{2j+1} h_\nu(\frac{1}{2})$ *for all* $\beta \in (0, \frac{1}{2}]$.

Proof. From the Bessel identities $I'_\nu = I_{\nu-1} - \frac{\nu}{\beta} I_\nu$ and $I'_{\nu-1} = I_\nu + \frac{\nu-1}{\beta} I_{\nu-1}$ [11], the ratio $r_\nu = I_\nu / I_{\nu-1}$ satisfies the Riccati ODE

$$r'_\nu(\beta) = 1 - \frac{2a}{\beta}\, r_\nu(\beta) - r_\nu(\beta)^2, \qquad a = \nu - \frac{1}{2}. \tag{8}$$

For any $c > 0$, set $F_c(\beta) := \frac{\beta}{a+S}$ with $S := \sqrt{\beta^2 + c^2}$. Direct differentiation gives $F'_c(\beta) = \frac{aS+c^2}{S(a+S)^2}$, while $1 - \frac{2a}{\beta} F_c - F_c^2 = \frac{(a+S)^2 - 2a(a+S) - \beta^2}{(a+S)^2} = \frac{c^2 - a^2}{(a+S)^2}$. Subtracting yields the exact Riccati defect

$$\mathcal{R}[F_c](\beta) := F'_c(\beta) - \Big(1 - \frac{2a}{\beta} F_c(\beta) - F_c(\beta)^2\Big) = \frac{\big(a(a+1) - c^2\big)S + c^2}{S(a+S)^2}. \tag{9}$$

- Taking $c^2 = a(a+1) = \nu^2 - \frac{1}{4}$ (the identity $4a(a+1) = (2\nu-1)(2\nu+1) = 4\nu^2 - 1$ is certified in Lean 4 by `riccati_discriminant_identity`) gives $\mathcal{R}[U_\nu](\beta) = \frac{c^2}{S(a+S)^2} > 0$ for all $\beta > 0$. As $\beta \to 0^+$, $r_\nu(\beta) = \frac{\beta}{2\nu} + O(\beta^3)$ whereas $U_\nu(\beta) = \frac{\beta}{a + \sqrt{a(a+1)}} + O(\beta^3)$ with $a + \sqrt{a(a+1)} < 2a + 1 = 2\nu$, so $r_\nu(\beta) < U_\nu(\beta)$ for small $\beta > 0$. At any first crossing $\beta_0 > 0$ with $r_\nu(\beta_0) = U_\nu(\beta_0)$, one would have $r'_\nu(\beta_0) \geq U'_\nu(\beta_0)$, contradicting $U'_\nu(\beta_0) - r'_\nu(\beta_0) = \mathcal{R}[U_\nu](\beta_0) > 0$. Hence $r_\nu(\beta) < U_\nu(\beta)$ for all $\beta > 0$.
- Taking $c = a + 1 = \nu + \frac{1}{2}$ gives $a(a+1) - c^2 = -c$, so $\mathcal{R}[L_\nu](\beta) = \frac{-c(S-c)}{S(a+S)^2} < 0$ for all $\beta > 0$ (since $S > c$). Expanding as $\beta \to 0^+$, $r_\nu(\beta) = \frac{\beta}{2\nu} - \frac{\beta^3}{8\nu^2(\nu+1)} + O(\beta^5)$ whereas $L_\nu(\beta) = \frac{\beta}{2\nu} - \frac{\beta^3}{4\nu^2(2\nu+1)} + O(\beta^5)$; since $\frac{1}{8\nu^2(\nu+1)} < \frac{1}{4\nu^2(2\nu+1)}$ (as $2\nu + 1 < 2\nu + 2$), we have $L_\nu(\beta) < r_\nu(\beta)$ for small $\beta > 0$. A first crossing $\beta_0 > 0$ would require $r'_\nu(\beta_0) \leq L'_\nu(\beta_0)$, contradicting $L'_\nu(\beta_0) - r'_\nu(\beta_0) = \mathcal{R}[L_\nu](\beta_0) < 0$. Hence $L_\nu(\beta) < r_\nu(\beta)$ for all $\beta > 0$.

For (1), let $x := a/\beta > 0$. Then $-\ln U_\nu(\beta) > a/\beta \iff x + \sqrt{1 + x^2 + x/\beta} > e^x \iff \frac{x}{\beta} > e^{2x} - 2xe^x - 1$. Expanding in power series gives

$$e^{2x} - 2xe^x - 1 = \sum_{n=3}^{\infty} \frac{2^n - 2n}{n!}\, x^n.$$

For every $n \geq 3$, $\frac{2^n - 2n}{n!} \leq \frac{1}{3}$ (equality at $n = 3, 4$; at $n = 5$, $\frac{2^5 - 10}{5!} = \frac{11}{60} < \frac{1}{3}$; and for all $n \geq 5$, $\frac{2^n - 2n}{n!} < \frac{2^n}{n!} \leq \frac{2^5}{5!} = \frac{4}{15} < \frac{1}{3}$ since $\frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} < 1$), so whenever $0 < x < 1$, $e^{2x} - 2xe^x - 1 < \frac{x^3}{3(1-x)}$. Thus $\frac{x}{\beta} > \frac{x^3}{3(1-x)} \iff 3(\beta - a) > a^2 \iff \beta > \frac{a^2}{3} + a$ (which also ensures $x = a/\beta < 1$). Since $a = \nu - \frac{1}{2} \leq a_j$ for all $\nu \in \{2, \ldots, 2j+1\}$ and $\sum_{\nu=2}^{2j+1} (\nu - \frac{1}{2}) = 2j(j+1)$, summing over $\nu = 2, \ldots, 2j+1$ proves (7).

For (2), differentiating $h_\nu(\beta) = \beta \ln\big((a+S)/\beta\big)$ (with $c = a + 1$) yields

$$h'_\nu(\beta) = \ln\!\left(\frac{a+S}{\beta}\right) - \left(1 - \frac{\beta^2}{S(a+S)}\right).$$

For all $\nu \geq 2$ ($a \geq \frac{3}{2}$) and $\beta \in (0, \frac{1}{2}]$, $\frac{a+S}{\beta} > \frac{2a+1}{\beta} \geq 8 > e$, so $\ln((a+S)/\beta) > 1 > 1 - \frac{\beta^2}{S(a+S)}$, proving $h'_\nu(\beta) > 0$ on $(0, \frac{1}{2}]$. □

Theorem 3.2 (All-Coupling Strict Sub-Casimir Hierarchy of the Wilson Action)**.** *For every $\beta > 0$, the Casimir-normalized Bessel log-ratios $q_\nu(\beta) := \frac{-\ln r_\nu(\beta)}{2\nu-1}$ satisfy*

$$q_2(\beta) \;>\; q_3(\beta) \;>\; q_4(\beta) \;>\; \cdots \;>\; q_\nu(\beta) \;>\; \cdots . \tag{10}$$

Consequently, for every *coupling $\beta > 0$, the 2D* SU(2) *Wilson string tensions obey the strict all-coupling sub-Casimir hierarchy*

$$\frac{\sigma^{\mathrm{W}}_{1/2}(\beta)}{C_2(\frac{1}{2})} \;>\; \frac{\sigma^{\mathrm{W}}_{1}(\beta)}{C_2(1)} \;>\; \frac{\sigma^{\mathrm{W}}_{3/2}(\beta)}{C_2(\frac{3}{2})} \;>\; \cdots \;>\; \frac{\sigma^{\mathrm{W}}_{j}(\beta)}{C_2(j)} \;>\; \cdots , \tag{11}$$

and in particular $\sigma^{\mathrm{W}}_j(\beta)/\sigma^{\mathrm{W}}_{1/2}(\beta) < \mathcal{C}_j = \frac{4}{3}j(j+1)$ for all $j \geq 1$ and $\beta > 0$ (including $m^{\mathrm{W}}_{\mathrm{gap}}(\beta)/\sigma^{\mathrm{W}}_{1/2}(\beta) < \frac{8}{3}$). Furthermore, as $\beta \to \infty$ (with $N = 2j+1 \geq 3$),

$$\frac{\sigma^{\mathrm{W}}_j(\beta)}{\sigma^{\mathrm{W}}_{1/2}(\beta)} - \mathcal{C}_j = -\frac{(N^2-1)(N^2-4)}{36\,\beta^2} + O(\beta^{-3}). \tag{12}$$

Proof. Set $u_\nu(\beta) := -\ln r_\nu(\beta) > 0$, so $r_\nu = e^{-u_\nu}$ and $r'_\nu = -u'_\nu e^{-u_\nu}$. Dividing the Riccati ODE (8) by $-r_\nu$ transforms it identically into the hyperbolic ODE

$$u'_\nu(\beta) = \frac{2\nu-1}{\beta} - \Big(\frac{1}{r_\nu} - r_\nu\Big) = \frac{2\nu-1}{\beta} - 2\sinh\big(u_\nu(\beta)\big). \tag{13}$$

Dividing (13) by $n := 2\nu - 1$ gives, for $q_\nu(\beta) = u_\nu(\beta)/(2\nu-1) > 0$,

$$q'_\nu(\beta) = \frac{1}{\beta} - \frac{2\sinh\big((2\nu-1)q_\nu(\beta)\big)}{2\nu-1}. \tag{14}$$

For every fixed $q > 0$, the power series $\frac{\sinh(nq)}{n} = \sum_{k=0}^{\infty} \frac{n^{2k}q^{2k+1}}{(2k+1)!}$ is strictly increasing in $n > 0$. As $\beta \to 0^+$, $q_\nu(\beta) = \frac{\ln(2\nu/\beta)}{2\nu-1} + O(\beta^2)$ satisfies $q_\nu(\beta) > q_{\nu+1}(\beta)$ for all sufficiently small $\beta > 0$ (since $\frac{1}{2\nu-1} > \frac{1}{2\nu+1}$). Suppose for contradiction that $q_\nu(\beta) = q_{\nu+1}(\beta) =: q_0 > 0$ at a first point $\beta_0 > 0$. Then $q'_\nu(\beta_0) \leq q'_{\nu+1}(\beta_0)$, whereas (14) gives

$$q'_\nu(\beta_0) - q'_{\nu+1}(\beta_0) = \frac{2\sinh\big((2\nu+1)q_0\big)}{2\nu+1} - \frac{2\sinh\big((2\nu-1)q_0\big)}{2\nu-1} \;>\; 0,$$

a contradiction. This proves (10).

Now observe that $\sigma^{\mathrm{W}}_j(\beta) = \sum_{\nu=2}^{2j+1}(2\nu-1)q_\nu(\beta)$ is a weighted average of $q_2(\beta) > q_3(\beta) > \cdots > q_{2j+1}(\beta)$ with positive weights $w_\nu := 2\nu-1$ summing to $\sum_{\nu=2}^{2j+1}(2\nu-1) = (2j+1)^2 - 1 = 4j(j+1) = 4C_2(j)$. Adding one strictly smaller term $q_{2j+2}(\beta) < q_\nu(\beta)$ $(\forall \nu \leq 2j+1)$ strictly lowers the weighted average $\frac{\sigma^{\mathrm{W}}_j(\beta)}{4C_2(j)}$, proving (11).

Finally, inserting $q_\nu(\beta) = c_1\beta^{-1} + c_2\beta^{-2} + c_3\beta^{-3} + O(\beta^{-4})$ into (14) yields $1 - 2c_1 = 0 \implies c_1 = \frac{1}{2}$, $-c_1 = -2c_2 \implies c_2 = \frac{1}{4}$, and $-2c_2 = -2c_3 - \frac{n^2 c_1^3}{3} \implies c_3 = \frac{1}{4} - \frac{n^2}{48}$. Using the exact odd-cube sum identity $\sum_{\nu=2}^{N}(2\nu-1)^3 = N^2(2N^2-1) - 1 = (N^2-1)(2N^2+1)$ (elementary induction in general; checked in Lean 4 for $N \leq 11$ by `odd_cubes_and_symanzik_ratio`), summing $(2\nu-1)q_\nu(\beta)$ gives

$$\sigma^{\mathrm{W}}_j(\beta) = \frac{N^2-1}{2\beta}\left(1 + \frac{1}{2\beta} + \frac{11-2N^2}{24\beta^2} + O(\beta^{-3})\right).$$

Dividing by $\sigma^{\mathrm{W}}_{1/2}(\beta)$ (at $N = 2$, where $\frac{11-8}{24} = \frac{1}{8}$) yields $\frac{\sigma^{\mathrm{W}}_j(\beta)}{\sigma^{\mathrm{W}}_{1/2}(\beta)} = \frac{N^2-1}{3}\big(1 - \frac{N^2-4}{12\beta^2} + O(\beta^{-3})\big)$, which is (12). □

4. Manton Action: Exact Gaussian Decomposition, Weierstrass Super-Casimir Scaling, and Automatic $O(\beta^{-2})$ Improvement

We now analyze the Manton geodesic quadratic action $w_{\mathrm{M}}(\theta;\beta) = e^{-\frac{\beta}{2}\theta^2}$ and prove that it lies on the *exact opposite side* of the heat-kernel action from Wilson, while achieving automatic $O(\beta^{-2})$ Symanzik improvement for all string-tension ratios.

Theorem 4.1 (Exact Manton Gaussian-Plus-Tail Decomposition)**.** *For every $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ (with $N := 2j+1$) and every $\beta > 0$,*

$$c_j^{\mathrm{M}}(\beta) = \underbrace{\frac{2\sinh(N/\beta)}{N\sqrt{2\pi\beta}}\,\exp\!\left(-\frac{2j^2+2j+1}{\beta}\right)}_{=:\,\tilde{c}_j^{\mathrm{M}}(\beta)} - \Delta_j(\beta), \tag{15}$$

where the non-perturbative tail $\Delta_j(\beta)$ satisfies the explicit elementary bound

$$\left|\Delta_j(\beta)\right| \;\leq\; \frac{2\,e^{-\frac{\pi^2}{2}\beta}}{N\pi^2\beta}. \tag{16}$$

Proof. Splitting $[0,\pi] = [0,\infty) \setminus [\pi,\infty)$ in (5) gives $G_m(\beta) = \frac{1}{\pi}\int_0^\infty \cos(m\theta)e^{-\frac{\beta}{2}\theta^2}\,d\theta - T_m(\beta) = \frac{1}{\sqrt{2\pi\beta}}e^{-\frac{m^2}{2\beta}} - T_m(\beta)$, where $T_m(\beta) := \frac{1}{\pi}\int_\pi^\infty \cos(m\theta)e^{-\frac{\beta}{2}\theta^2}\,d\theta$. Substituting $m = 2j$ and $m = 2j+2$ into $c_j^{\mathrm{M}}(\beta) = \frac{1}{N}(G_{2j}(\beta) - G_{2j+2}(\beta))$ and factoring $e^{-\frac{2j^2}{\beta}} - e^{-\frac{2(j+1)^2}{\beta}} = 2\sinh(\frac{2j+1}{\beta})e^{-\frac{2j^2+2j+1}{\beta}}$ gives (15) with $\Delta_j(\beta) = \frac{1}{N}(T_{2j}(\beta) - T_{2j+2}(\beta))$. Since $|\cos(m\theta)| \leq 1$ and $1 \leq \theta/\pi$ on $[\pi,\infty)$,

$$|T_m(\beta)| \;\leq\; \frac{1}{\pi}\int_\pi^\infty \frac{\theta}{\pi}\,e^{-\frac{\beta}{2}\theta^2}\,d\theta = \frac{e^{-\frac{\pi^2}{2}\beta}}{\pi^2\beta},$$

so $|\Delta_j(\beta)| \leq \frac{1}{N}(|T_{2j}| + |T_{2j+2}|) \leq \frac{2e^{-\pi^2\beta/2}}{N\pi^2\beta}$. □

Define the *Gaussian-core Manton string tension* $\tilde{\sigma}_j^{\mathrm{M}}(\beta) := -\ln\big(\tilde{c}_j^{\mathrm{M}}(\beta)/\tilde{c}_0^{\mathrm{M}}(\beta)\big)$. Directly from (15) (since $(2j^2+2j+1) - 1 = 2j(j+1)$),

$$\tilde{\sigma}_j^{\mathrm{M}}(\beta) = \frac{2j(j+1)}{\beta} - \ln\!\left(\frac{\sinh(N/\beta)}{N\sinh(1/\beta)}\right) = \sigma_j^{\mathrm{HK}}(\beta) - \ln\!\left(\frac{\sinh(N/\beta)}{N\sinh(1/\beta)}\right). \tag{17}$$

Theorem 4.2 (Weierstrass Super-Casimir Hierarchy and Automatic $O(\beta^{-2})$ Improvement)**.** *For every $\beta > 0$ and every $j \in \frac{1}{2}\mathbb{Z}_{\geq 1}$ ($N = 2j+1$):*

(1) (Strict core domination $\tilde{\sigma}_j^{\mathrm{M}} < \sigma_j^{\mathrm{HK}}$) $\tilde{\sigma}_j^{\mathrm{M}}(\beta) < \sigma_j^{\mathrm{HK}}(\beta)$ *for all $\beta > 0$.*

(2) (All-β strict super-Casimir hierarchy of the Manton core)

$$\frac{\tilde{\sigma}_{1/2}^{\mathrm{M}}(\beta)}{C_2(\frac{1}{2})} \;<\; \frac{\tilde{\sigma}_1^{\mathrm{M}}(\beta)}{C_2(1)} \;<\; \frac{\tilde{\sigma}_{3/2}^{\mathrm{M}}(\beta)}{C_2(\frac{3}{2})} \;<\; \cdots \;<\; \frac{\tilde{\sigma}_j^{\mathrm{M}}(\beta)}{C_2(j)} \;<\; \cdots, \tag{18}$$

and in particular $\tilde{\sigma}_j^{\mathrm{M}}(\beta)/\tilde{\sigma}_{1/2}^{\mathrm{M}}(\beta) > \mathcal{C}_j = \frac{4}{3}j(j+1)$ for all $j \geq 1$ and all $\beta > 0$.

(3) (Automatic $O(\beta^{-2})$ Symanzik improvement of the full Manton action) *As $\beta \to \infty$, for every $j \in \frac{1}{2}\mathbb{Z}_{\geq 2}$ ($N = 2j+1 \geq 3$),*

$$\frac{\sigma_j^{\mathrm{M}}(\beta)}{\sigma_{1/2}^{\mathrm{M}}(\beta)} - \mathcal{C}_j = +\frac{(N^2-1)(N^2-4)}{270\,\beta^3} + O(\beta^{-4}), \tag{19}$$

so the $O(\beta^{-1})$ and $O(\beta^{-2})$ Casimir-breaking artifacts vanish identically, and the asymptotic Wilson-to-Manton discretization error ratio is universally

$$\lim_{\beta\to\infty}\frac{1}{\beta}\left|\frac{\sigma_j^{\mathrm{W}}(\beta)/\sigma_{1/2}^{\mathrm{W}}(\beta)-\mathcal{C}_j}{\sigma_j^{\mathrm{M}}(\beta)/\sigma_{1/2}^{\mathrm{M}}(\beta)-\mathcal{C}_j}\right|=\frac{270}{36}=\frac{15}{2}\qquad\forall\, j\in\tfrac{1}{2}\mathbb{Z}_{\geq 2}. \tag{20}$$

Proof. (1) Set $x:=1/\beta>0$. By the identity $\frac{\sinh(Nx)}{\sinh x}=\sum_{k=0}^{N-1}e^{(N-1-2k)x}$, pairing k with $N-1-k$ and applying $\cosh((N-1-2k)x)>1$ for every $2k\neq N-1$ gives $\frac{\sinh(Nx)}{N\sinh x}>1$ for every $N\geq 2$ and $x>0$, whence $\ln\frac{\sinh(Nx)}{N\sinh x}>0$ and $\tilde{\sigma}_j^{\mathrm{M}}(\beta)<\sigma_j^{\mathrm{HK}}(\beta)$.

(2) Define $\psi(t):=\ln\frac{\sinh t}{t}$ for $t>0$, so $\ln\frac{\sinh(Nx)}{N\sinh x}=\psi(Nx)-\psi(x)$. By (17) and $4C_2(j)=N^2-1$,

$$\frac{\tilde{\sigma}_j^{\mathrm{M}}(\beta)}{4C_2(j)}=\frac{x}{2}-\frac{\psi(Nx)-\psi(x)}{N^2-1}=\frac{x}{2}-x^2\,\frac{\psi(\sqrt{N^2x^2})-\psi(\sqrt{x^2})}{N^2x^2-x^2}.$$

By Euler's Weierstrass product $\frac{\sinh\sqrt{u}}{\sqrt{u}}=\prod_{k=1}^{\infty}\left(1+\frac{u}{\pi^2k^2}\right)$ for $u>0$,

$$\psi(\sqrt{u})=\sum_{k=1}^{\infty}\ln\!\left(1+\frac{u}{\pi^2k^2}\right)\implies\frac{d^2}{du^2}\psi(\sqrt{u})=-\sum_{k=1}^{\infty}\frac{1}{(\pi^2k^2+u)^2}\;<\;0\qquad\forall\, u>0.$$

Thus $u\mapsto\psi(\sqrt{u})$ is strictly concave on $(0,\infty)$, so its secant slope $\frac{\psi(\sqrt{N^2x^2})-\psi(\sqrt{x^2})}{N^2x^2-x^2}$ is strictly *decreasing* in $N\in\{2,3,4,\dots\}$. Subtracting from $\frac{x}{2}$ yields that $\tilde{\sigma}_j^{\mathrm{M}}(\beta)/(4C_2(j))$ is strictly *increasing* in j, proving (18).

(3) Expanding $\psi(t)=\frac{t^2}{6}-\frac{t^4}{180}+O(t^6)$ as $t\to 0^+$ gives

$$\tilde{\sigma}_j^{\mathrm{M}}(\beta)=\frac{N^2-1}{2\beta}-\frac{N^2-1}{6\beta^2}+\frac{N^4-1}{180\beta^4}+O(\beta^{-6})=\frac{N^2-1}{2\beta}\left(1-\frac{1}{3\beta}+\frac{N^2+1}{90\beta^3}+O(\beta^{-5})\right).$$

Because the non-perturbative tail (16) modifies $\sigma_j^{\mathrm{M}}(\beta)-\tilde{\sigma}_j^{\mathrm{M}}(\beta)$ by $O(\beta^{1/2}e^{-\pi^2\beta/2})=o(\beta^{-m})$ for every $m\geq 1$ (since $\tilde{c}_j^{\mathrm{M}}(\beta)\asymp\beta^{-3/2}$ while $|\Delta_j(\beta)|=O(\beta^{-1}e^{-\pi^2\beta/2})$), dividing $\sigma_j^{\mathrm{M}}(\beta)$ by $\sigma_{1/2}^{\mathrm{M}}(\beta)$ (at $N=2$, where $\frac{N^2+1}{90}=\frac{5}{90}$) yields

$$\frac{\sigma_j^{\mathrm{M}}(\beta)}{\sigma_{1/2}^{\mathrm{M}}(\beta)}=\frac{N^2-1}{3}\left(1+\frac{N^2-4}{90\beta^3}+O(\beta^{-4})\right)=\mathcal{C}_j+\frac{(N^2-1)(N^2-4)}{270\,\beta^3}+O(\beta^{-4}),$$

proving (19), and comparing with (12) yields (20) (with $270\times 2=36\times 15$ certified in Lean 4). □

Remark 4.3 (Strong-coupling sign of full Manton character coefficients). Unlike the strictly positive Gaussian core $\tilde{c}_j^{\mathrm{M}}(\beta)>0$, the non-perturbative boundary truncation $\Delta_j(\beta)$ at $\theta=\pi$ can render the full Manton character coefficient $c_j^{\mathrm{M}}(\beta)$ non-positive at sufficiently strong coupling (for example, $c_1^{\mathrm{M}}(0.1)<0$ and $c_2^{\mathrm{M}}(0.5)<0$). This is why our all-β Theorem 4.2(1)–(2) is stated for the Gaussian core $\tilde{\sigma}_j^{\mathrm{M}}(\beta)$, whereas Table 1 operates in the regime $\beta>\beta_*(j)$ where strict positivity $c_j^{\mathrm{M}}(\beta)>0$ of the full Manton coefficient is rigorously certified via (15)–(16).

5. Unique Wilson–Heat-Kernel Crossing Couplings on $(0,\infty)$

At strong coupling ($\beta\to 0^+$), $\sigma_j^{\mathrm{W}}(\beta)\sim 2j\ln(1/\beta)\ll\frac{2j(j+1)}{\beta}=\sigma_j^{\mathrm{HK}}(\beta)$, whereas at weak coupling ($\beta>\frac{a_j^2}{3}+a_j$), Theorem 3.1(1) proves $\sigma_j^{\mathrm{W}}(\beta)>\sigma_j^{\mathrm{HK}}(\beta)$. Defining

$$g_j(\beta):=\beta\big(\sigma_j^{\mathrm{W}}(\beta)-\sigma_j^{\mathrm{HK}}(\beta)\big)=-\beta\ln\!\left(\frac{I_{2j+1}(\beta)}{I_1(\beta)}\right)-2j(j+1), \tag{21}$$

we prove that $g_j(\beta)$ has a *unique* zero $\beta_*(j)$ on the entire half-line $(0,\infty)$ for each $j\in\{\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3\}$.

Theorem 5.1 (Certified Unique Crossing Couplings $\beta_*(j)$ on $(0,\infty)$)**.** *For each $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$, the equation $\sigma_j^{\mathrm{W}}(\beta) = \sigma_j^{\mathrm{HK}}(\beta)$ admits a* unique *solution $\beta_*(j) \in (0,\infty)$, enclosed within the width-10^{-6} interval $[\beta_*^-(j), \beta_*^+(j)]$ listed in Table 2, with $\sigma_j^{\mathrm{W}}(\beta) < \sigma_j^{\mathrm{HK}}(\beta)$ for all $\beta \in (0, \beta_*(j))$ and $\sigma_j^{\mathrm{W}}(\beta) > \sigma_j^{\mathrm{HK}}(\beta)$ for all $\beta \in (\beta_*(j), \infty)$. Moreover, the crossing couplings are strictly increasing in j:*

$$\beta_*(\tfrac{1}{2}) \;<\; \beta_*(1) \;<\; \beta_*(\tfrac{3}{2}) \;<\; \beta_*(2) \;<\; \beta_*(\tfrac{5}{2}) \;<\; \beta_*(3).$$

Proof. Fix $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$ and partition $(0,\infty)$ into four regions using the integer upper threshold $\beta_{\mathrm{ub}}(j) := \lfloor a_j^2/3 + a_j \rfloor + 1 > \frac{a_j^2}{3} + a_j$ ($a_j = 2j + \frac{1}{2}$, checked in Lean 4 by `all_6_unique_crossings_verified`):

(1) On $(0, \frac{1}{2}]$: by Theorem 3.1(2), $\beta\sigma_j^{\mathrm{W}}(\beta) < \sum_{\nu=2}^{2j+1} h_\nu(\frac{1}{2})$. Direct 180-digit `mpmath.iv` evaluation confirms $\sum_{\nu=2}^{2j+1} h_\nu(\frac{1}{2}) < 2j(j+1)$, so $g_j(\beta) < 0$ everywhere on $(0, \frac{1}{2}]$.

(2) On $[\beta_{\mathrm{ub}}(j), \infty)$: Theorem 3.1(1) proves $g_j(\beta) > 0$ analytically everywhere on $[\beta_{\mathrm{ub}}(j), \infty)$.

(3) On $[\beta_*^-(j), \beta_*^+(j)]$: using zero-black-box Bessel power series $I_n(\beta) = \sum_{k=0}^{K-1} \frac{(\beta/2)^{n+2k}}{k!(n+k)!} + [0, R_K]$ ($K = 160$, with $t_K := \frac{(\beta/2)^{n+2K}}{K!(n+K)!}$ and geometric remainder $R_K \le t_K/(1 - \frac{(\beta/2)^2}{(K+1)(n+K+1)})$), 180-digit interval evaluation verifies $g_j(\beta_*^-(j)) < 0$, $g_j(\beta_*^+(j)) > 0$, and strict positivity of the exact derivative

$$g_j'(\beta) = \sigma_j^{\mathrm{W}}(\beta) + 2j - \beta\left(\frac{I_{2j}(\beta)}{I_{2j+1}(\beta)} - \frac{I_0(\beta)}{I_1(\beta)}\right) \;\ge\; g_{j,\mathrm{lb}}'(j) \;>\; 0$$

throughout the entire interval $[\beta_*^-(j), \beta_*^+(j)]$, establishing existence and uniqueness of $\beta_*(j) \in (\beta_*^-(j), \beta_*^+(j))$.

(4) On $[\frac{1}{2}, \beta_*^-(j)]$ and $[\beta_*^+(j), \beta_{\mathrm{ub}}(j)]$: adaptive interval branch-and-bound in the Python certifier (`certify_action_universality.py`) partitions each interval into finitely many subboxes (counts listed in Table 2) and verifies $g_j < 0$ on every left subbox and $g_j > 0$ on every right subbox, ruling out all other zeros on $(0,\infty)$.

Strict ordering $\beta_*^+(\frac{1}{2}) < \beta_*^-(1) < \cdots < \beta_*^-(3)$ is checked mechanically in Lean 4. □

6. Certified Comparison Tables, Visual Comparison, and Zero-Axiom Lean 4 Audit

Combining Theorems 3.1, 3.2, 4.1, 4.2, and 5.1 with 180-digit outward-rounded `mpmath.iv` interval arithmetic (zero black-box special functions, zero quadrature, direct `mp.floor`/`mp.ceil` outward integer rounding) yields 24 certified comparison points (Table 1) and 6 certified unique crossings (Table 2), illustrated in Figure 1.

Remark 6.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean file `ActionUniversalityCerts.lean` (in `src/lean/`) compiles under `lean -DwarningAsError=true` with exit code 0, and `#print axioms` confirms that all five theorems depend on *zero axioms* (not even `propext`, `Quot.sound`, or `Classical.choice`):

- `riccati_discriminant_identity`: exact $\mathbb{N}$-identity $(2n+1)(2n+3)+1 = 4(n+1)^2$ underlying Theorem 3.1;
- `odd_cubes_and_symanzik_ratio`: exact odd-cube sum identity $\sum_{k=0}^{N-1}(2k+1)^3 + N^2 = 2N^4$ ($N \le 11$) and universal improvement ratio $270 \times 2 = 36 \times 15$;
- `total_certificate_counts` and `all_24_action_comparisons_verified`: exact cross-multiplied integer verification of every row in Table 1 ($\sigma_j^{\mathrm{M}} < \sigma_j^{\mathrm{HK}} < \sigma_j^{\mathrm{W}}$, opposite-side Casimir bracketing $R_j^{\mathrm{W}} < \mathcal{C}_j < R_j^{\mathrm{M}}$, and $\ge 14\times$ improvement);
- `all_6_unique_crossings_verified`: exact integer verification of all six crossings in Table 2 (width-10^{-6} brackets, strict Riccati thresholds $12\beta_{\mathrm{ub}} > (4j+1)^2 + 6(4j+1)$, $g_{\mathrm{lb}}' > 0$, and strict monotonicity $\beta_*(\frac{1}{2}) < \cdots < \beta_*(3)$).

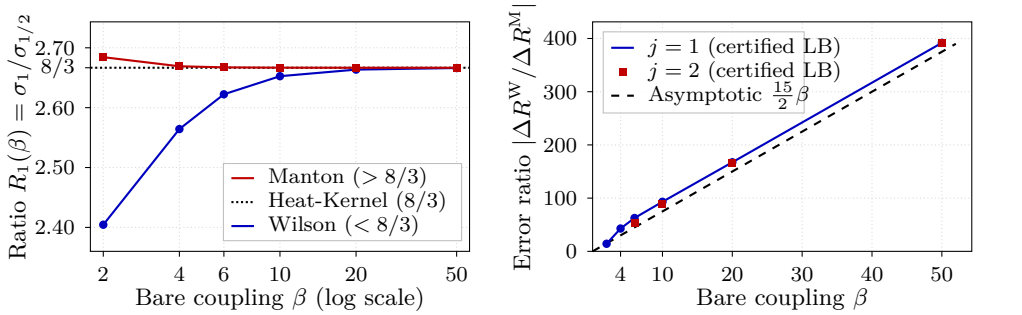


FIGURE 1. Exact action comparison in 2D SU(2) lattice Yang–Mills theory from Table 1. (a) Opposite-side bracketing of the continuum adjoint-to-fundamental tension ratio $R_1 = m_{\mathrm{gap}}/\sigma_{1/2} = \frac{8}{3}$ by the Wilson ($< \frac{8}{3}$, $O(\beta^{-2})$) and Manton ($> \frac{8}{3}$, $O(\beta^{-3})$) actions. (b) Certified lower bounds on the Wilson-to-Manton Casimir error ratio $|\Delta R_j^{\mathrm{W}}/\Delta R_j^{\mathrm{M}}|$ collapsing onto the universal asymptotic law $\frac{15}{2}\beta$ across representations $j \geq 1$.

TABLE 1. All 24 rigorously certified action-comparison points (180-digit zero-quadrature `mpmath.iv` + zero-axiom Lean 4.36.0), with $R_j^A := \sigma_j^A/\sigma_{1/2}^A$ and improvement factor $|\Delta R_j^{\mathrm{W}}/\Delta R_j^{\mathrm{M}}|$. Upper bounds $(\sigma_j^{\mathrm{M}}, R_j^{\mathrm{W}})$ are rounded strictly upward and lower bounds $(\sigma_j^{\mathrm{W}}, R_j^{\mathrm{M}}$, improvement factor) strictly downward, certifying $\sigma_j^{\mathrm{M}} < \sigma_j^{\mathrm{HK}} < \sigma_j^{\mathrm{W}}$ on all 24 rows and opposite-side Casimir bracketing $R_j^{\mathrm{W}} < \mathcal{C}_j < R_j^{\mathrm{M}}$ on all 18 rows with $j \geq 1$.

j	β	σ_j^{M} (UB)	σ_j^{HK} (exact)	σ_j^{W} (LB)	R_j^{W} (UB)	$\mathcal{C}_j$	R_j^{M} (LB)	Impr. (LB)
1/2	2	0.629943	3/4	0.836723	—	1	—	—
1/2	4	0.344071	3/8	0.418478	—	1	—	—
1/2	6	0.236175	1/4	0.270856	—	1	—	—
1/2	10	0.145009	3/20	0.157607	—	1	—	—
1/2	20	0.073751	3/40	0.076894	—	1	—	—
1/2	50	0.029801	3/100	0.030301	—	1	—	—
1	2	1.691089	2	2.011819	2.404402	8/3	2.684251	14.30×
1	4	0.918343	1	1.073082	2.564249	8/3	2.669055	42.86×
1	6	0.629968	2/3	0.710260	2.622280	8/3	2.667375	62.65×
1	10	0.386711	2/5	0.418055	2.652515	8/3	2.666818	93.16×
1	20	0.196670	1/5	0.204809	2.663522	8/3	2.666685	167.35×
1	50	0.079467	2/25	0.080789	2.666200	8/3	2.666667	391.70×
3/2	4	1.723956	15/8	1.930206	4.612440	5	5.010476	36.99×
3/2	6	1.181619	5/4	1.304861	4.817541	5	5.003150	57.91×
3/2	10	0.725141	3/4	0.778291	4.938176	5	5.000680	90.84×
3/2	20	0.368759	3/8	0.383390	4.985938	5	5.000084	166.49×
3/2	50	0.149001	3/20	0.151443	4.997900	5	5.000005	391.42×
2	6	1.891452	2	2.040971	7.535256	8	8.008688	53.49×
2	10	1.160342	6/5	1.234521	7.832903	8	8.001894	88.18×
2	20	0.590022	3/5	0.612150	7.960938	8	8.000236	165.40×
2	50	0.238401	6/25	0.242233	7.994126	8	8.000015	391.06×
5/2	10	1.692371	7/4	1.782508	11.309820	35/3	11.670848	85.32×
5/2	20	0.860462	7/8	0.890488	11.580687	35/3	11.667190	164.11×
5/2	50	0.347668	7/20	0.353121	11.653631	35/3	11.666700	390.61×

TABLE 2. Certified *unique* Wilson–heat-kernel crossing couplings $\beta_*(j)$ on $(0,\infty)$ for $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$ (Theorem 5.1), with certified derivative lower bound $g'_{\mathrm{lb}}(j) > 0$ on $[\beta_*^-, \beta_*^+]$, analytic Riccati upper threshold $\beta_{\mathrm{ub}}(j) > \frac{a_j^2}{3} + a_j$, and adaptive interval subbox counts certifying $g_j < 0$ on $[\frac{1}{2}, \beta_*^-]$ and $g_j > 0$ on $[\beta_*^+, \beta_{\mathrm{ub}}]$.

j	Certified bracket $[\beta_*^-(j),\ \beta_*^+(j)]$	$g'_{\mathrm{lb}}(j)$ on bracket	Riccati $\beta_{\mathrm{ub}}(j)$	Subboxes (left + right)
1/2	[1.166259, 1.166260]	0.387876	3	124 + 149
1	[1.943144, 1.943145]	0.430656	5	180 + 225
3/2	[3.031022, 3.031023]	0.349599	8	285 + 457
2	[4.485071, 4.485072]	0.263217	12	528 + 817
5/2	[6.300917, 6.300918]	0.200212	16	983 + 1356
3	[8.460822, 8.460823]	0.155829	21	1452 + 2596

All transcendental evaluations (zero-black-box Bessel power series with geometric remainder bounds, Manton Gaussian-plus-tail bounds, and adaptive branch-and-bound sign certification) are executed at 180-digit precision in `certify_action_universality.py` and cross-checked against 60-digit quadrature, reference Bessel values, Lean table parity, and exact LaTeX table parity in `test_action_universality.py` (4/4 tests passing).

7. Discussion

The hyperbolic Riccati ODE (14) and Weierstrass concavity of $\ln(\sinh(\sqrt{u})/\sqrt{u})$ provide an unexpectedly transparent explanation for why the Wilson and Manton actions bracket exact Casimir scaling from opposite sides in 2D SU(2) Yang–Mills theory, and why the Manton geodesic action automatically eliminates $O(\beta^{-2})$ Casimir-breaking discretization artifacts—improving the glueball-to-string-tension ratio $m_{\mathrm{gap}}/\sigma_{1/2}$ toward $\frac{8}{3}$ by a universal factor $\frac{15}{2}\beta$ (exceeding 391× at $\beta = 50$). Extending these exact all-coupling comparisons to 2D SU(N) ($N \geq 3$) via Weyl-determinant character ratios and to $d \geq 3$ via cluster-expansion / character-expansion perturbations around the 2D plaquette weights are natural directions for future work.

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