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ON THE LARGE DAVENPORT CONSTANT OF THE FINITE HEISENBERG GROUP

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ABSTRACT. Let p be an odd prime and let $H_{p^3} = \mathrm{UT}_3(\mathbb{F}_p)$ denote the Heisenberg group of order p^3 and exponent p. While the small Davenport constant $\mathsf{d}(H_{p^3}) = 3p-3$, the Gao constant $E(H_{p^3}) = p^3+3p-3$, and the Loewy length $L(H_{p^3}) = \mathsf{D}_{\mathrm{ord}}(H_{p^3}) = 4p-3$ have recently been determined, the large Davenport constant $\mathsf{D}(H_{p^3})$ has remained open for every odd prime $p \geq 5$, with the best general upper bound being Grynkiewicz's $\mathsf{D}(G) \leq \mathsf{d}(G) + 2|G'| - 1 = 5p-4$.

We prove that for every odd prime $p \geq 3$,

$$3p-1 \;\leq\; \mathsf{D}(H_{p^3}) \;\leq\; 4p-4 \;<\; 4p-3 \;=\; L(H_{p^3}) \;=\; \mathsf{D}_{\mathrm{ord}}(H_{p^3}),$$

yielding a purely conceptual, computer-free proof that $\mathsf{D}(H_{27}) = 8$ and establishing the strict separation $\mathsf{D}(H_{p^3}) < L(H_{p^3})$ for all odd p (in sharp contrast to the non-abelian group $M_{27} = C_9 \rtimes C_3$, where $\mathsf{D}(M_{27}) = 12 > 11 = L(M_{27})$). The lower bound is witnessed by two non-isomorphic extremal minimal product-one families of length $3p-1$, carrying $p-1$ and 0 central elements, respectively. For $p \geq 5$, we sharpen the analysis of the interval $[3p, 4p-4]$ by eliminating all sequences whose abelianization has at least two noncollinear atoms at length $\geq 3p+1$ and all purely collinear-atom sequences supported on $t = 2$ or $t \geq 4$ lines at length $\geq 3p$, reducing the remaining gap strictly to two explicit structural obstructions (O1)–(O2). Finally, via a fiber-doubled lifting construction, we prove the lower bounds $\eta(H_{p^3}) \geq 7p-6$ and $\mathsf{s}(H_{p^3}) \geq 8p-7$ for every odd prime $p \geq 3$.

1. Introduction and Principal Results

For a finite multiplicative group G, let $\mathcal{F}(G)$ denote the free abelian monoid of finite (unordered) sequences over G, written multiplicatively as $S = g_1 \cdot \ldots \cdot g_\ell$ with length $|S| = \ell$ (for a subsequence $U \mid S$, multiset subsequence removal is denoted interchangeably by $S \cdot U^{-1}$ or $S \setminus U$). A sequence S is a *product-one sequence* if $1 \in \pi(S) := \{g_{\sigma(1)} \cdots g_{\sigma(\ell)} : \sigma \in \mathfrak{S}_\ell\}$. The product-one sequences form a Krull–type submonoid $\mathcal{B}(G) \subseteq \mathcal{F}(G)$ whose atoms $\mathcal{A}(\mathcal{B}(G))$ are the *minimal product-one sequences*—those nonempty $S \in \mathcal{B}(G)$ admitting no factorization $S = S_1 \cdot S_2$ with nonempty $S_1, S_2 \in \mathcal{B}(G)$. Two fundamental invariants govern factorization arithmetic over G [4, 7]: the *small Davenport constant* $\mathsf{d}(G)$, the maximum length of a sequence having no nonempty product-one subsequence, and the *large Davenport constant* $\mathsf{D}(G) = \max\{|S| : S \in \mathcal{A}(\mathcal{B}(G))\}$. Always $\mathsf{d}(G)+1 \leq \mathsf{D}(G)$, with equality whenever G is abelian; for non-abelian groups, however, $\mathsf{D}(G)$ can exceed $\mathsf{d}(G)+1$ substantially because an atom $S \in \mathcal{A}(\mathcal{B}(G))$ may contain a proper product-one subsequence whose complement is not product-one.

Let p be an odd prime and let $H_{p^3} = \mathrm{UT}_3(\mathbb{F}_p)$ be the Heisenberg group of order p^3 and exponent p. Over the past few years, several zero-sum invariants of H_{p^3} have been settled. Following the computational determination $\mathsf{d}(H_{27}) = 6$, $\mathsf{D}(H_{27}) = 8$ for $p = 3$ by Cziszter, Domokos, and Szöllősi [1] and subsequent $p = 7$ verification [12], Volkmann [13] gave a uniform proof that $\mathsf{d}(H_{p^3}) = 3p-3$ for all odd primes p, and Qu and Wang [9] established the Gao–Zhuang conjecture $E(H_{p^3}) = p^3+3p-3$. On the group-algebra side, Godara, Joshi, and

ORCID: 0009-0006-7494-9040; correspondence: editor@airr.science. Code and Lean 4 formalization are available at https://github.com/lluiseriksson/heisenberg-large-davenport (tag v1.0.2); code is released under the MIT License and this manuscript under CC-BY-4.0.

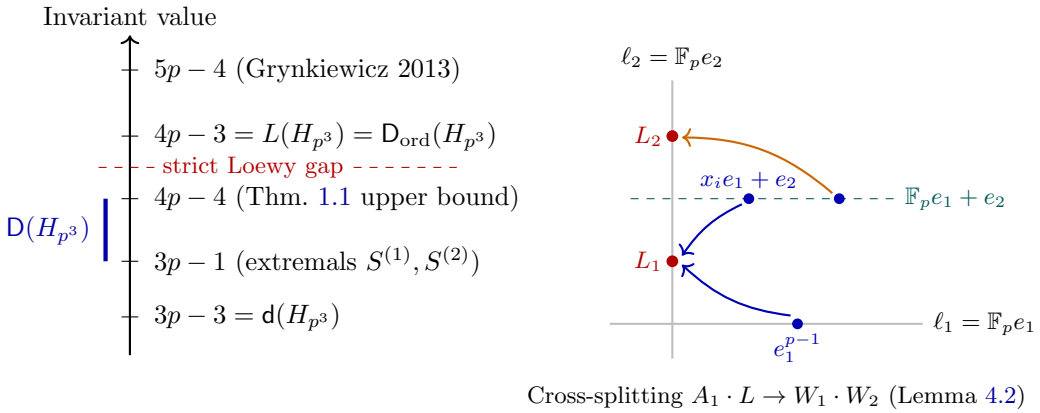

FIGURE 1. Left: separation of $\mathsf{D}(H_{p^3})$ from the Loewy length $L(H_{p^3})$ and the previous bound $5p-4$. Right: cross-splitting between a Property B atom A_1 and a collinear tail $L \subseteq \ell_2$ in the abelianization $\mathbb{F}_p^2$.

Mazumdar [5, 6] proved that the ordered Davenport constant $\mathsf{D}_{\mathrm{ord}}(H_{p^3})$ equals the Loewy length $L(H_{p^3}) := L(\mathbb{F}_p[H_{p^3}]) = 4p-3$.

Yet the large Davenport constant $\mathsf{D}(H_{p^3})$ has remained open for every odd prime $p \geq 5$. Plugging $\mathsf{d}(H_{p^3}) = 3p-3$ and $|H'_{p^3}| = p$ into Grynkiewicz's general bound $\mathsf{D}(G) \leq \mathsf{d}(G)+2|G'|-1$ [7, Theorem 1.1] yields only $\mathsf{D}(H_{p^3}) \leq 5p-4$, leaving open even whether $\mathsf{D}(H_{p^3}) < L(H_{p^3}) = 4p-3$. Meanwhile, the short-subsequence invariants $\eta(H_{p^3})$ and $\mathsf{s}(H_{p^3})$ (minimum sequence lengths forcing a product-one subsequence of length in $[1,p]$ or exactly p, respectively) have also remained open for all odd p.

Our first main theorem lowers the upper bound from $5p-4$ to $4p-4$, gives a purely conceptual proof of $\mathsf{D}(H_{27}) = 8$, constructs two non-isomorphic extremal families of length $3p-1$, and separates $\mathsf{D}(H_{p^3})$ strictly from the Loewy length.

Theorem 1.1. *For every odd prime $p \geq 3$,*

$$3p-1 \;\leq\; \mathsf{D}(H_{p^3}) \;\leq\; 4p-4 \;<\; 4p-3 \;=\; L(H_{p^3}) \;=\; \mathsf{D}_{\mathrm{ord}}(H_{p^3}).$$

In particular, $\mathsf{D}(H_{27}) = 8$ without machine computation. Moreover, for every odd prime $p \geq 3$, $\mathcal{A}(\mathcal{B}(H_{p^3}))$ contains at least two $\mathrm{Aut}(H_{p^3})$-inequivalent minimal product-one sequences of length $3p-1$:

$$S^{(1)} = x^{p-1}(xz) \cdot y^{p-1}(yz) \cdot z^{p-1} \qquad (\textit{carrying } c = p-1 \textit{ central elements}),$$
$$S^{(2)} = x^{p-1}(xz)^{p-1}(x^2z) \cdot y^{p-1}(yz) \qquad (\textit{carrying } c = 0 \textit{ central elements}),$$

where x, y generate H_{p^3} modulo $Z(H_{p^3})$ and $z = [x,y]$.

The strict inequality $\mathsf{D}(H_{p^3}) < L(H_{p^3})$ stands in marked contrast to the other non-abelian group of order 27, namely $M_{27} = C_9 \rtimes C_3$, where Jennings' formula gives $L(M_{27}) = 11$ whereas Cziszter–Domokos–Szöllősi [1] found $\mathsf{D}(M_{27}) = 12 > 11$.

Our methods rely on exponential coordinates $(v,a) \in \mathbb{F}_p^2 \times \mathbb{F}_p$ for H_{p^3}, in which the product of a word becomes an affine functional Φ of its ordered area in $\mathbb{F}_p^2$. Combining Cauchy–Davenport contraction with an arithmetic pairing identity and three-set pigeonhole on the central fiber $Z(H_{p^3}) \cong \mathbb{F}_p$, we show that almost all structural types of length $\geq 3p$ decompose. In fact, for $p \geq 5$, we establish an exactness reduction inside $[3p, 4p-4]$.

Theorem 1.2. *Let $p \geq 5$ be an odd prime, and suppose $S = S_0 \cdot S_Z \in \mathcal{A}(\mathcal{B}(H_{p^3}))$ has length $3p \leq |S| \leq 4p-4$, where S_Z consists of the $c := |S_Z| \leq p-1$ central elements of S and S_0 of the non-central elements. Factor the abelianization $\overline{S_0} = A_1 \cdot \ldots \cdot A_r$ into minimal zero-sum atoms in $\mathcal{B}(\mathbb{F}_p^2)$. Then:*

(a) *If $|S| \geq 3p+1$, then $\overline{S_0}$ cannot contain two noncollinear atoms, nor one noncollinear atom together with a noncollinear complement.*

(b) *For all $|S| \geq 3p$, $\overline{S_0}$ cannot consist solely of collinear atoms supported on $t = 2$ lines or on $t \geq 4$ lines.*

Consequently, every minimal product-one sequence of length in $[3p, 4p-4]$ must belong to one of two residual obstructions:

(O1) Single noncollinear atom with collinear tail: $\overline{S_0} = A_1 \cdot L$ *where $A_1 \in \mathcal{A}(\mathcal{B}(\mathbb{F}_p^2))$ is noncollinear and $L \subseteq \ell$ lies on a single line $\ell \subset \mathbb{F}_p^2$, with $|S \cap M_\ell| \leq 3p-3$ (and if $|S \cap M_\ell| \geq 2p-1$, every minimal product-one subsequence $U \mid (S \cap M_\ell)$ leaves $S_0 \setminus U_0$ collinear or of length $\leq p$). Moreover, if $|S| = 4p-4$ and $|A_1| = 2p-1$, then $|A_1 \cap \ell| = p-1$.*

(O2) Three-line collinear triples and the $3p$ boundary: *Either $\overline{S_0} = L_1 \cdot L_2 \cdot L_3$ is supported on three distinct lines ℓ_1, ℓ_2, ℓ_3 with $|L_j| + c \leq 2p-2$ for each j (excluding (p,p,p) for $c \geq p-4$ and $(p,p,p-1)$ for $c \geq p-3$), or $|S| = 3p$ with either* (i) *$c = 0$ and every atom-partition (W_1, W_2) into noncollinear zero-sums having $\min(|W_1|, |W_2|) \leq p$ (such as the (p,p,p) triple), or* (ii) *$1 \leq c \leq p-3$ and $\overline{S_0} = A_1 \cdot v_1^p \cdot v_2^p$, where A_1 is a noncollinear atom of length $p-c$ and $v_1, v_2 \in \mathbb{F}_p^2$ are linearly independent.*

Finally, lifting the planar extremal configuration for $\eta(\mathbb{F}_p^2) = 3p-2$ by doubling each non-central direction with a central translate yields new lower bounds for the exponent-p invariants $\eta(H_{p^3})$ and $\mathsf{s}(H_{p^3})$.

Theorem 1.3. *For every odd prime $p \geq 3$,*

$$\eta(H_{p^3}) \;\geq\; 7p-6 \qquad \text{and} \qquad \mathsf{s}(H_{p^3}) \;\geq\; 8p-7,$$

witnessed respectively by $T_\eta = x^{p-1}(xz)^{p-1}y^{p-1}(yz)^{p-1}(xy)^{p-1}(xyz)^{p-1}z^{p-1}$ of length $7p-7$ and $T_{\mathsf{s}} = 1^{p-1} \cdot T_\eta$ of length $8p-8$.

2. Symplectic Exponential Coordinates and Order-Value Growth

Throughout, p is an odd prime, $G := H_{p^3} = \mathrm{UT}_3(\mathbb{F}_p)$, and $V := \mathbb{F}_p^2$. Denote by $M(v_1, v_2, u) = \left(\begin{smallmatrix} 1 & v_1 & u \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ the upper-triangular matrices in G. Since 2 is invertible in $\mathbb{F}_p$, the bijection $\psi \colon V \times \mathbb{F}_p \to G$ defined by

$$\psi(v, a) := M\big(v_1, v_2, a + \tfrac{1}{2}v_1v_2\big), \qquad v = (v_1, v_2) \in V,\ a \in \mathbb{F}_p,$$

transfers the group operation to the *symplectic exponential law*

$$(u, a) \cdot (v, b) \;=\; \big(u+v,\ a+b+\tfrac{1}{2}\det(u,v)\big), \tag{1}$$

where $\det(u,v) = u_1v_2 - u_2v_1$. Under (1), $(v,a)^{-1} = (-v, -a)$, $(v,a)^k = (kv, ka)$ (so $\exp(G) = p$), and defining the commutator by $[g,h] := g^{-1}h^{-1}gh$, we have $[(u,a),(v,b)] = (0, \det(u,v)) \in Z(G) = G' = \{(0,a) : a \in \mathbb{F}_p\}$. We fix standard generators $x = ((1,0),0)$, $y = ((0,1),0)$, $z = ((0,0),1)$, so $[x,y] = z$.

For $g = (v,a) \in G$, let $\overline{g} := v \in V$ be its abelianization and $a(g) := a \in \mathbb{F}_p$ its central coordinate; extend $\bar{\cdot}$ elementwise to $\mathcal{F}(G) \to \mathcal{F}(V)$. Let $\mathcal{L}$ denote the $p+1$ one-dimensional subspaces (lines) of V; for each $\ell \in \mathcal{L}$, its preimage $M_\ell := \{g \in G : \overline{g} \in \ell\}$ is a maximal abelian subgroup isomorphic to $C_p \times C_p$ and containing $Z(G)$.

Iterating (1) on any word $(g_1, \dots, g_n)$ with $g_i = (v_i, a_i)$ gives

$$g_1 \cdots g_n \;=\; \Big(\sum_{i=1}^{n} v_i,\;\; \sum_{i=1}^{n} a_i + \Phi(v_1, \dots, v_n)\Big), \tag{2}$$

where the *signed-area functional* on ordered n-tuples over V is

$$\Phi(v_1, \dots, v_n) \;:=\; \frac{1}{2} \sum_{1 \le i < j \le n} \det(v_i, v_j) \;\in\; \mathbb{F}_p.$$

For a sequence $B = v_1 \cdot \ldots \cdot v_n \in \mathcal{F}(V)$, write $\Phi(\mathfrak{S}_n \cdot B) := \{\Phi(v_{\sigma(1)}, \dots, v_{\sigma(n)}) : \sigma \in \mathfrak{S}_n\} \subseteq \mathbb{F}_p$. Whenever $S_0 = g_1 \cdot \ldots \cdot g_n \in \mathcal{F}(G \setminus Z(G))$ has zero-sum abelianization $\overline{S_0} = B$, (2) implies that $\pi(S_0) \subseteq Z(G)$ is the translate $a_0 + \Phi(\mathfrak{S}_n \cdot B)$, where $a_0 = \sum_{i=1}^{n} a(g_i)$. The exponential coordinate identities (1)–(2) and the ensuing order-value growth estimate (Lemma 2.1) are due to Volkmann [13]; we formulate them via the alternating-area functional Φ and include a short self-contained contraction proof for completeness.

Lemma 2.1 (Order-Value Growth; [13])**.** *Let $B = v_1 \cdot \ldots \cdot v_n \in \mathcal{F}(V \setminus \{0\})$ be a zero-sum sequence ($\sum_{i=1}^{n} v_i = 0$) whose support spans V (i.e., B is noncollinear). Then $n \ge 3$ and*

$$|\Phi(\mathfrak{S}_n \cdot B)| \;\ge\; \min(p, n-1).$$

Proof. Since $0 \notin B$ and B is noncollinear, $n \le 1$ is impossible, and $n = 2$ would give $v_2 = -v_1$, contradicting noncollinearity. Thus $n \ge 3$. We proceed by induction on n.

For $n = 3$, $v_1 + v_2 + v_3 = 0$ and noncollinearity imply $\det(v_1, v_2) \neq 0$, while $\det(v_1, v_3) = -\det(v_1, v_2)$ and $\det(v_2, v_3) = \det(v_1, v_2)$. Hence $\Phi(v_1, v_2, v_3) = \frac{1}{2}\det(v_1, v_2)$ and $\Phi(v_2, v_1, v_3) = -\frac{1}{2}\det(v_1, v_2)$. Because p is odd and $\det(v_1, v_2) \neq 0$, these two values are distinct, so $|\Phi(\mathfrak{S}_3 \cdot B)| = 2 = \min(p, 3-1)$.

Now suppose $n \ge 4$. We claim that B contains two terms u, w such that $\det(u, w) \neq 0$ (hence $u + w \neq 0$) and the contracted sequence $B' := (B \cdot u^{-1} \cdot w^{-1}) \cdot (u + w) \in \mathcal{F}(V \setminus \{0\})$ remains noncollinear:

- *Case A:* Some line $\ell_0 \in \mathcal{L}$ contains at least two occurrences v_1, v_2 of B. Since B is noncollinear, there exists an occurrence $u \in B$ with $u \notin \ell_0$. Setting $w := v_1 \in \ell_0 \setminus \{0\}$, we have $\det(u, w) \neq 0$ and $u + w \notin \ell_0$ (in particular $u + w \neq 0$). Since B' contains both $v_2 \in \ell_0 \setminus \{0\}$ and $u + w \notin \ell_0$, B' is noncollinear.
- *Case B:* Every line in $\mathcal{L}$ contains at most one occurrence of B. Since $n \ge 4$, pick any two occurrences u, w of B and two distinct remaining occurrences x, y of $B \cdot u^{-1} \cdot w^{-1}$. Because all four lie on pairwise distinct lines, $\det(u, w) \neq 0$ (so $u + w \neq 0$) and $\det(x, y) \neq 0$; since $x, y \in B'$, B' is noncollinear.

For any ordering $\sigma' = (x_1, \dots, x_{t-1}, u + w, x_{t+1}, \dots, x_{n-1})$ of B', consider the two orderings of B obtained by replacing $u + w$ with the adjacent pairs (u, w) and (w, u):

$$\sigma_+ = (x_1, \dots, x_{t-1}, u, w, x_{t+1}, \dots, x_{n-1}), \qquad \sigma_- = (x_1, \dots, x_{t-1}, w, u, x_{t+1}, \dots, x_{n-1}).$$

By bilinearity of det, for every $i < t$ we have $\det(x_i, u) + \det(x_i, w) = \det(x_i, u + w)$, and for every $j > t$ we have $\det(u, x_j) + \det(w, x_j) = \det(u + w, x_j)$. Only the internal term between u and w changes, giving

$$\Phi(\sigma_+) \;=\; \Phi(\sigma') + \delta, \qquad \Phi(\sigma_-) \;=\; \Phi(\sigma') - \delta, \qquad \delta := \tfrac{1}{2}\det(u, w) \neq 0.$$

Hence $\Phi(\mathfrak{S}_n \cdot B) \supseteq \Phi(\mathfrak{S}_{n-1} \cdot B') + \{+\delta, -\delta\}$. Since $+\delta \neq -\delta$ in $\mathbb{F}_p$, the Cauchy–Davenport theorem and the induction hypothesis give

$$|\Phi(\mathfrak{S}_n \cdot B)| \;\ge\; \min\big(p,\; |\Phi(\mathfrak{S}_{n-1} \cdot B')| + 2 - 1\big) \;\ge\; \min(p, n-1). \qquad \square$$

We now pass from products of S_0 to factorizations of $S = S_0 \cdot S_Z$, where S_Z consists of all central elements of S and S_0 of all non-central elements. Note first that for any $S \in \mathcal{A}(\mathcal{B}(G))$ with $|S| > p$, S_0 is nonempty and S_Z must be product-one free in $Z(G) \cong C_p$: if any nonempty $C \mid S_Z$ had product 1, commuting C to the front of a product-one ordering of S would factor $S = C \cdot (S \cdot C^{-1})$ into two nonempty product-one sequences (impossible since $|S| > |C|$). Hence $c := |S_Z| \le \mathsf{d}(C_p) = p - 1$. Moreover, if $\overline{S_0}$ were collinear on ℓ, then $S \in \mathcal{F}(M_\ell)$ would satisfy $|S| \le \mathsf{D}(M_\ell) = \mathsf{D}(C_p^2) = 2p - 1$. Thus every $S \in \mathcal{A}(\mathcal{B}(G))$ with $|S| \ge 2p$ has noncollinear $\overline{S_0}$ and $c \le p - 1$.

For $C = z_1 \cdot \ldots \cdot z_c \in \mathcal{F}(Z(G))$, write $\Sigma_{\ge 0}(C) := \{a(\prod T) : T \mid C\} \subseteq \mathbb{F}_p$ (including $0 = a(1)$ for $T = \emptyset$) and $\Sigma(C) := \{a(\prod T) : \emptyset \ne T \mid C\}$.

Lemma 2.2 (Arithmetic Pairing and Splitting Criteria)**.** *Let $S = S_0 \cdot S_Z \in \mathcal{A}(\mathcal{B}(G))$ with $c := |S_Z| \le p - 1$ (S_Z product-one free). Then:*

(a) (Central growth) $|\Sigma_{\ge 0}(S_Z)| \ge c + 1$ *and* $|\Sigma(S_Z)| \ge c$*; in particular, if $c = p - 1$ then $\Sigma_{\ge 0}(S_Z) = \mathbb{F}_p$, and if $c = p - 2$ then $|\mathbb{F}_p \setminus \Sigma_{\ge 0}(S_Z)| \le 1$.*

(b) (Arithmetic pairing) *If a sequence $L = (y_1, \ldots, y_m)$ in $\mathbb{F}_p$ sums to $0 \pmod p$ and $L = L_1 \cdot L_2$ with $s_1 := \sum L_1 \not\equiv 0 \pmod p$, then setting $k_1 := (-s_1) \bmod p \in \{1, \ldots, p-1\}$ and $k_2 := (-\sum L_2) \bmod p \in \{1, \ldots, p-1\}$, we have $k_1 + k_2 = p$ and $(k_1 + |L_1|) + (k_2 + |L_2|) = m + p$.*

(c) (Asymmetric splitting) *S_0 cannot be partitioned as $S_0 = W_1 \cdot W_2$ with W_1, W_2 nonempty such that $W_1 \cdot C_1 \in \mathcal{B}(G)$ for some $C_1 \mid S_Z$, $\overline{W_2}$ is zero-sum and noncollinear, and $|W_2| \ge p + 1$.*

(d) (Symmetric three-set pigeonhole splitting) *S_0 cannot be partitioned as $S_0 = W_1 \cdot W_2$ with $\overline{W_1}, \overline{W_2}$ both noncollinear zero-sum sequences satisfying any of:* (i) $|W_1|, |W_2| \ge p + 1$*;* (ii) $|W_1|, |W_2| \le p$ *and* $|S| \ge 3p$*; or* (iii) $|W_1| \ge p + 1$ *and* $|W_2| + c \ge p + 1$.

Proof. (a) Write $S_Z = z_1 \cdot \ldots \cdot z_c$ and define $C_0 := \{0\}$ and $C_i := C_{i-1} \cup (C_{i-1} + a(z_i))$ for $1 \le i \le c$. If $C_{i-1} = \mathbb{F}_p$ for some $i \le c$, then $-a(z_i) \in C_{i-1}$, so $0 \in a(z_i) + C_{i-1} \subseteq \Sigma(z_1 \cdot \ldots \cdot z_i)$, contradicting that S_Z is product-one free. Hence $C_{i-1} \subsetneq \mathbb{F}_p$ and $a(z_i) \ne 0$, so $C_{i-1} + a(z_i) \ne C_{i-1}$ (as $\mathbb{F}_p$ has no nontrivial proper subgroup), giving $|C_i| \ge |C_{i-1}| + 1$. Inductively $|\Sigma_{\ge 0}(S_Z)| \ge |C_c| \ge c + 1$, and $0 \notin \Sigma(S_Z)$ gives $|\Sigma(S_Z)| = |\Sigma_{\ge 0}(S_Z)| - 1 \ge c$.

(b) Since $k_1, k_2 \in \{1, \ldots, p-1\}$ and $k_1 + k_2 \equiv -(s_1 + \sum L_2) \equiv 0 \pmod p$, we have $k_1 + k_2 = p$, whence $(k_1 + |L_1|) + (k_2 + |L_2|) = m + p$.

(c) Suppose such W_1, W_2, C_1 exist, and set $C_2 := S_Z \cdot C_1^{-1}$. By Lemma 2.1, $|\pi(W_2)| \ge \min(p, |W_2| - 1) = p$, so $\pi(W_2) = Z(G)$. In particular, there exists an ordering of W_2 with product $(\prod C_2)^{-1}$, so $W_2 \cdot C_2 \in \mathcal{B}(G)$. Since W_1, W_2 are nonempty, $S = (W_1 \cdot C_1) \cdot (W_2 \cdot C_2)$ factors S into two nonempty product-one sequences, contradicting $S \in \mathcal{A}(\mathcal{B}(G))$.

(d) Set $z_Z := a(\prod S_Z)$, $A := -a(\pi(W_1))$, $B := z_Z + a(\pi(W_2))$, and $C := \Sigma_{\ge 0}(S_Z)$ in $\mathbb{F}_p$. Any $u \in A \cap B \cap C$ corresponds to a subsequence $C_1 \mid S_Z$ with $a(\prod C_1) = u$, whence $-u \in a(\pi(W_1))$ and $-(z_Z - u) \in a(\pi(W_2))$, giving $W_1 \cdot C_1 \in \mathcal{B}(G)$ and $W_2 \cdot (S_Z \cdot C_1^{-1}) \in \mathcal{B}(G)$, a contradiction. In case (i), $A = B = \mathbb{F}_p$, so any C_1 works. In case (iii), $A = \mathbb{F}_p$, while by Lemma 2.1 and part (a), $|B| \ge \min(p, |W_2| - 1)$ and $|C| \ge c + 1$, so $|B| + |C| \ge \min(p + c + 1, |W_2| + c) \ge p + 1$, forcing $B \cap C \ne \emptyset$. In case (ii), $|A| + |B| + |C| \ge (|W_1| - 1) + (|W_2| - 1) + (c + 1) = |S| - 1 \ge 3p - 1 > 2p$; if $A \cap B \cap C$ were empty, each element of $\mathbb{F}_p$ would belong to at most two of A, B, C, yielding $|A| + |B| + |C| \le 2p$, a contradiction. □

3. Extremal Constructions and Lower Bounds

We now establish the lower bounds in Theorem 1.1 and prove Theorem 1.3.

Proposition 3.1. *For every odd prime $p \geq 3$, the sequences*

$$S^{(1)} = x^{p-1}(xz) \cdot y^{p-1}(yz) \cdot z^{p-1} \qquad \text{and} \qquad S^{(2)} = x^{p-1}(xz)^{p-1}(x^2z) \cdot y^{p-1}(yz)$$

are minimal product-one sequences of length $3p-1$ that lie in distinct $\mathrm{Aut}(H_{p^3})$-orbits. In particular, $\mathsf{D}(H_{p^3}) \geq 3p-1$.

Proof. Both sequences clearly have length $3p-1$. Since $Z(G)$ is characteristic in G and $S^{(1)}$ contains $p-1 \geq 2$ central elements whereas $S^{(2)}$ contains 0, no automorphism of G maps $S^{(1)}$ to $S^{(2)}$.

To see that $S^{(1)} \in \mathcal{B}(G)$, evaluate the explicit word

$$w_1 := x^{p-1} \cdot y \cdot (xz) \cdot y^{p-2} \cdot (yz) \cdot z^{p-1}.$$

Using matrix coordinates $x = M(1,0,0)$, $y = M(0,1,0)$, $z = M(0,0,1)$, we compute $x^{p-1}y = M(p-1,1,p-1)$, then $(x^{p-1}y)(xz) = M(p-1,1,p-1)M(1,0,1) = M(0,1,0) = y$, and finally $y \cdot y^{p-2}(yz)z^{p-1} = y^p z^p = 1$. Suppose $S^{(1)} = A \cdot B$ with nonempty $A, B \in \mathcal{B}(G)$. Projecting to V, we have $\overline{S^{(1)}} = (1,0)^p(0,1)^p$, whose only nonempty zero-sum subsequences are $(1,0)^p$, $(0,1)^p$, and $(1,0)^p(0,1)^p$. Since $S^{(1)}_Z = z^{p-1}$ is product-one free, neither $\overline{A}$ nor $\overline{B}$ can be empty. Thus, swapping A, B if needed, $A = x^{p-1}(xz)z^k$ and $B = y^{p-1}(yz)z^{p-1-k}$ for some $0 \leq k \leq p-1$. Because $A \subseteq M_{\mathbb{F}_p(1,0)}$ and $B \subseteq M_{\mathbb{F}_p(0,1)}$ are abelian, A has constant product $x^p z^{1+k} = z^{1+k}$ and B has constant product $y^p z^{p-k} = z^{p-k}$. Then $A \in \mathcal{B}(G)$ forces $k = p-1$, giving $B = y^{p-1}(yz)$ with product $z \neq 1$, contradiction.

For $S^{(2)}$, consider the explicit word

$$w_2 := y \cdot (xz)^{p-1}(x^2z) \cdot y^{p-2}(yz) \cdot x^{p-1}.$$

Since $(xz)^{p-1}(x^2z) = x^{p+1}z^p = x$, $y^{p-2}(yz) = y^{p-1}z = y^{-1}z$, and $x^{p-1} = x^{-1}$, its product in G is

$$y \cdot x \cdot y^{-1}z \cdot x^{-1} = (yxy^{-1}x^{-1})z = [y^{-1}, x^{-1}]z = z^{-1}z = 1.$$

Suppose $S^{(2)} = A \cdot B$ with nonempty $A, B \in \mathcal{B}(G)$. Because $S^{(2)}$ has no central elements, $\overline{A}$ and $\overline{B}$ are nonempty zero-sums in $\overline{S^{(2)}} = (1,0)^{2p-2}(2,0) \cdot (0,1)^p$. Every element of $\overline{S^{(2)}}$ has second coordinate 0 or 1, and $(0,1)$ appears p times, so one factor, say A, must contain zero copies of $(0,1)$ and hence satisfy $A \mid x^{p-1}(xz)^{p-1}(x^2z) \subseteq M_{\mathbb{F}_p(1,0)}$. Write $A = x^\alpha(xz)^\beta(x^2z)^\gamma$ with $0 \leq \alpha, \beta \leq p-1$ and $0 \leq \gamma \leq 1$, not all zero. Since $M_{\mathbb{F}_p(1,0)}$ is abelian, $A \in \mathcal{B}(G)$ gives z-exponent $\beta + \gamma \equiv 0 \pmod p$ and x-exponent $\alpha + \beta + 2\gamma \equiv 0 \pmod p$. The first congruence forces either $\beta = \gamma = 0$ (giving $\alpha \equiv 0$, so $A = \emptyset$) or $\beta = p-1, \gamma = 1$ (giving $\alpha + 1 \equiv 0$, so $\alpha = p-1$). Thus $A = x^{p-1}(xz)^{p-1}(x^2z)$, leaving $B = y^{p-1}(yz)$ whose unique product is $z \neq 1$, contradiction. □

Proof of Theorem 1.3. We first show that T_η (of length $7(p-1) = 7p-7$) contains no nonempty product-one subsequence $U \mid T_\eta$ of length $|U| \leq p$. Let $a, b, c, d \geq 0$ be the numbers of elements of U projecting to $(1,0), (0,1), (1,1), (0,0)$ in V, respectively, so $1 \leq a+b+c+d \leq p$, $0 \leq a, b, c \leq 2p-2$, and $0 \leq d \leq p-1$. Because $\overline{U}$ sums to $(0,0) \in V$, we have $a + c \equiv 0 \pmod p$ and $b + c \equiv 0 \pmod p$.

If $c = 0$, then $p \mid a$ and $p \mid b$, so $a, b \in \{0, p\}$ as $a + b \leq p$.

- If $a = b = 0$, then $1 \leq d \leq p-1$, so $U = z^d$ has product $z^d \neq 1$.
- If $a = p$ (so $b = d = 0$), then $U \mid x^{p-1}(xz)^{p-1}$ consists of $p-k$ copies of x and k copies of xz with $1 \leq k \leq p-1$; because all elements lie in the abelian subgroup $M_{\mathbb{F}_p(1,0)}$, U has unique product $x^p z^k = z^k \neq 1$.
- Symmetrically, if $b = p$ (so $a = d = 0$), then $U \mid y^{p-1}(yz)^{p-1}$ has unique product $z^k \neq 1$.

Suppose now $c \geq 1$. If $1 \leq c \leq p-1$, then $p \mid (a+c)$ and $p \mid (b+c)$ force $a \geq p-c$ and $b \geq p-c$, whence $a+b+c \geq 2p-c \geq p+1$, contradicting $a+b+c+d \leq p$. Therefore $c = p$ and $a = b = d = 0$, so $U \mid (xy)^{p-1}(xyz)^{p-1}$ consists of $p-k$ copies of xy and k copies of xyz with $1 \leq k \leq p-1$. Since $xyz = (xy)z$ commutes with xy and $(xy)^p = 1$, U has unique product $(xy)^p z^k = z^k \neq 1$. Thus $\eta(H_{p^3}) \geq |T_\eta| + 1 = 7p-6$.

Finally, $T_{\mathsf{s}} = 1^{p-1} \cdot T_\eta$ has length $8(p-1) = 8p-8$. If $W \mid T_{\mathsf{s}}$ were a product-one subsequence of length p, deleting all copies of 1 from W (at most $p-1$ elements) would leave a nonempty product-one subsequence $U \mid T_\eta$ of length $|U| \leq p$, contradicting what was just proved. Hence $\mathsf{s}(H_{p^3}) \geq |T_{\mathsf{s}}| + 1 = 8p-7$. □

4. Heavy-Line Elimination and Structure-Theorem Cross-Splitting

To push $\mathsf{D}(H_{p^3})$ down to $4p-4$, we must eliminate maximal abelian subgroups containing too many terms of S, and we must cross-split configurations consisting of a maximal-length planar atom paired with a collinear tail.

Lemma 4.1 (Heavy-Line Elimination)**.** *Let $S = S_0 \cdot S_Z \in \mathcal{A}(\mathcal{B}(G))$ have length $|S| \geq 4p-3$. Then for every line $\ell \in \mathcal{L}$,*

$$|S \cap M_\ell| \;\leq\; 2p-2.$$

Proof. Suppose to the contrary that $|S \cap M_\ell| \geq 2p-1$ for some $\ell \in \mathcal{L}$. Recall that $M_\ell \cong C_p \times C_p$ has Davenport constant $\mathsf{D}(M_\ell) = 2p-1$ and, by Reiher's theorem [10] (Kemnitz's conjecture), $\eta(M_\ell) = 3p-2$. Set $S' := S \cdot U^{-1} = S_0' \cdot S_Z'$ for a chosen nonempty product-one subsequence $U = U_0 \cdot U_Z \mid (S \cap M_\ell)$. Since S_Z is product-one free, $U_0 \neq \emptyset$. Notice that $S_0 \setminus M_\ell \neq \emptyset$ because $\overline{S_0}$ is noncollinear, and $\overline{S_0'} = \overline{S_0} \cdot \overline{U_0}^{-1}$ is zero-sum.

Case 1: $|S \cap M_\ell| \geq 3p-2$. By $\eta(M_\ell) = 3p-2$, choose $U \mid (S \cap M_\ell)$ nonempty product-one of length $|U| \leq p$. Then $|(S_0 \cap M_\ell) \setminus U_0| \geq |S \cap M_\ell| - |U| - |S_Z'| \geq (3p-2) - p - (p-1) = p-1 \geq 2$, so S_0' contains elements both on ℓ and off ℓ ($S_0 \setminus M_\ell \subseteq S_0'$), making $\overline{S_0'}$ noncollinear. Moreover, $|S_0'| = |S'| - |S_Z'| \geq (4p-3-p) - (p-1) = 2p-2 \geq p+1$. By Lemma 2.2(c) (with $W_1 = U_0, C_1 = U_Z, W_2 = S_0'$), S splits, contradiction.

Case 2: $2p-1 \leq |S \cap M_\ell| \leq 3p-3$. Pick a minimal product-one subsequence $U = U_0 \cdot U_Z \mid (S \cap M_\ell)$, so $|U| \leq 2p-1$. Notice U_0 and S_Z are disjoint subsets of $S \cap M_\ell$, so $|U_0| + c \leq |S \cap M_\ell| \leq 3p-3$, which yields

$$|S_0'| = |S| - c - |U_0| \geq (4p-3) - (3p-3) = p.$$

Suppose first that $\overline{S_0'}$ is *noncollinear* and $|S_0'| \geq p+1$; then Lemma 2.2(c) immediately splits S, contradiction. It remains to check the two boundary ways this can fail:

Subcase 2a: $|S_0'| = p$. This forces $|S| = 4p-3$, $|S \cap M_\ell| = 3p-3$, and $U_0 = S_0 \cap M_\ell$. Since $|U_0| + |U_Z| = |U| \leq 2p-1$ and $|U_0| + c = 3p-3$ with $c \leq p-1$, we must have $c \in \{p-2, p-1\}$.

- If $c = p-2$, then $|U_0| = 2p-1$, forcing $U_Z = \emptyset$, so $U = U_0$ is a minimal zero-sum sequence of length $2p-1$ in $M_\ell \cong C_p^2$. Fixing any $g \in U_0$, the sequence $U_0 \cdot g^{-1}$ is zero-sum free of length $2p-2 = \mathsf{d}(C_p^2)$, whence every nonzero element of M_ℓ is a nonempty subsequence product of $U_0 \cdot g^{-1}$ (for if $h \notin \Sigma(U_0 \cdot g^{-1})$, appending $-h$ would give a zero-sum-free sequence of length $2p-1$, impossible). Picking $z_1 \in S_Z$, there exists nonempty $V_0 \mid (U_0 \cdot g^{-1}) \subsetneq U_0$ with product z_1^{-1}. Replacing U by $V := V_0 \cdot z_1$ leaves $g \in (S_0 \cap M_\ell) \setminus V_0$ on ℓ alongside $S_0 \setminus M_\ell$ ($|S_0 \setminus M_\ell| = p$) off ℓ, so the new complement $S_0 \cdot V_0^{-1}$ is noncollinear of length $\geq p+1$, contradicting Lemma 2.2(c).
- If $c = p-1$, then $|U_0| = 2p-2$ and by Lemma 2.2(a) $\Sigma(S_Z) = Z(G) \setminus \{1\}$. Projecting U_0 to $\ell \cong C_p$ gives a zero-sum sequence of length $2p-2 > p = \mathsf{D}(C_p)$, which therefore factors as $\overline{V_0} \cdot \overline{V_0'}$ with both $\overline{V_0}, \overline{V_0'}$ nonempty zero-sum on ℓ. Then $\prod V_0 = z^a \in Z(G)$: if

$a = 0$ set $V := V_0$, while if $a \neq 0$ pick $C_1 \mid S_Z$ with $\prod C_1 = z^{-a}$ and set $V := V_0 \cdot C_1$. In either case $V \mid (S \cap M_\ell)$ is product-one and $S_0 \cdot V_0^{-1}$ contains V_0' on ℓ as well as $S_0 \setminus M_\ell$ (size p) off ℓ, hence is noncollinear of length $\geq p+2$, contradicting Lemma 2.2(c).

Subcase 2b: $|S_0'| \geq p+1$ and $\overline{S_0'}$ is collinear. Because $S_0 \setminus M_\ell \subseteq S_0'$ is nonempty and disjoint from ℓ, $\overline{S_0'}$ must lie on a second line $\ell' \neq \ell$, which forces $U_0 = S_0 \cap M_\ell$ and $S_0 \subseteq M_\ell \cup M_{\ell'}$. Set $n_\ell := |S_0 \cap M_\ell|$ and $n_{\ell'} := |S_0 \cap M_{\ell'}| = |S_0'| \geq p+1$. Note that $n_\ell = |U_0| \leq 2p-1$.

- If $c \geq 1$ and $n_\ell = 2p-1$, then $|U_0| + |U_Z| = |U| \leq 2p-1$ automatically forces $U_Z = \emptyset$ and $U = U_0$ minimal in M_ℓ; the exact argument of Subcase 2a ($c = p-2$) finds a product-one $V = V_0 \cdot z_1 \mid (S \cap M_\ell)$ with $V_0 \subsetneq S_0 \cap M_\ell$, leaving $(S_0 \cap M_\ell) \setminus V_0$ on ℓ and $S_0 \cap M_{\ell'}$ of size $n_{\ell'} \geq p+1$ on ℓ', so $S_0 \cdot V_0^{-1}$ is noncollinear of size $\geq p+2$, splitting S.
- Otherwise either $c = 0$ or $n_\ell \leq 2p-2$. Suppose first that $n_\ell \geq p+1$ (note $n_{\ell'} \geq p+1$ already). Since $\mathsf{D}(\ell) = \mathsf{D}(\ell') = p$, both $\overline{S_0 \cap M_\ell}$ and $\overline{S_0 \cap M_{\ell'}}$ factor into at least two zero-sum atoms on ℓ and ℓ'. Grouping atoms on ℓ into $\overline{S_0 \cap M_\ell} = A_1 \cdot A_2$ minimizing $|A_1| - |A_2| \geq 0$ (with $|A_2| \geq 2$) and likewise $\overline{S_0 \cap M_{\ell'}} = B_1 \cdot B_2$ minimizing $|B_1| - |B_2| \geq 0$ (with $|B_2| \geq 2$), observe that if A_1 is a single atom (so $|A_1| \leq p$) then $|A_2| \geq n_\ell - p$, whereas if every atom on ℓ has size $\leq n_\ell/2$ then minimality of $|A_1| - |A_2|$ implies $|A_1| - |A_2| \leq |A_1|/2$ (moving the smallest atom of A_1 to A_2 cannot decrease the gap), whence $|A_2| \geq n_\ell/3$; in all cases $|A_1| \geq n_\ell/2$ and $|A_2| \geq \min(\max(2, n_\ell - p), n_\ell/3)$, and analogously for (B_1, B_2). Set $W_1 := A_1 \cdot B_2$ and $W_2 := A_2 \cdot B_1$ (lifted to S_0); both are noncollinear zero-sums.
  - If $c = 0$, then $n_\ell + n_{\ell'} = |S| \geq 4p-3$ and $n_\ell \leq 2p-1$ force $n_{\ell'} \geq 2p-2$. Thus $|A_2| + |B_1| \geq 2 + (p-1) = p+1$, while $|A_1| + |B_2| \geq \frac{n_\ell}{2} + \min(n_{\ell'} - p, \frac{n_{\ell'}}{3})$ exceeds p because $\frac{n_\ell}{2} + n_{\ell'} - p \geq (4p-3) - \frac{2p-1}{2} - p = 2p - \frac{5}{2} > p$ and $\frac{n_\ell}{2} + \frac{n_{\ell'}}{3} \geq \frac{4p-3}{3} + \frac{p+1}{6} > p$.
  - If $c \geq 1$ and $n_\ell \leq 2p-2$, then $n_\ell + c \geq 2p-1$ gives $|A_1| + |B_2| + c \geq \lceil n_\ell/2 \rceil + 2 + c \geq \lceil (2p-1-c)/2 \rceil + 2 + c \geq p+1$, while $n_{\ell'} = |S| - (n_\ell + c) \geq (4p-3) - (2p-2+c) = 2p-1-c$ yields $|A_2| + |B_1| + c \geq 2 + \lceil n_{\ell'}/2 \rceil + c \geq p+1$.

  Thus both $|W_1| + c \geq p+1$ and $|W_2| + c \geq p+1$ (and if both $|W_1|, |W_2| \leq p$, then $|S| \geq 4p-3 \geq 3p$), so Lemma 2.2(d) splits S.

  Finally, if $n_\ell \leq p$, then $n_\ell + c \geq 2p-1$ forces $c = p-1$ and $n_\ell = p$, whence $n_{\ell'} = |S| - (2p-1) \geq 2p-2 \geq p+1$. Since $c = p-1$, Lemma 2.2(a) gives $\Sigma_{\geq 0}(S_Z) = \mathbb{F}_p$. Picking a minimal zero-sum atom $A \mid \overline{S_0 \cap M_{\ell'}}$ of minimum length (so $|A| \leq \lfloor n_{\ell'}/2 \rfloor \leq p-1$ if $n_{\ell'} \leq 2p-1$, and $|A| \leq p$ if $n_{\ell'} \geq 2p$) leaves $|(S_0 \cap M_{\ell'}) \setminus A| \geq \lceil n_{\ell'}/2 \rceil \geq p-1$. Lifting A to $S_0 \cap M_{\ell'}$ yields $A \cdot C_1 \in \mathcal{B}(G)$ for some $C_1 \mid S_Z$, while the remaining non-central sequence $(S_0 \cap M_\ell) \cdot ((S_0 \cap M_{\ell'}) \setminus A)$ is noncollinear of length $\geq p + (p-1) = 2p-1 \geq p+1$, contradicting Lemma 2.2(c).

□

Next we eliminate the critical boundary where S_0 consists of a single length-$(2p-1)$ planar atom paired with a collinear tail on a disjoint line. By the structure theorem for Property B over $C_p \oplus C_p$ (conjectured by Gao and Geroldinger [2], proved for primes by Reiher [11] and combined with [3]), every minimal zero-sum sequence $A \in \mathcal{A}(\mathcal{B}(V))$ of length $2p-1$ admits a basis (e_1, e_2) of V in which

$$A \;=\; e_1^{p-1} \cdot \prod_{i=1}^{p} (x_i e_1 + e_2), \qquad x_1, \ldots, x_p \in \mathbb{F}_p, \quad \sum_{i=1}^{p} x_i \equiv 1 \pmod{p}. \tag{3}$$

Lemma 4.2 (Structure-Theorem Cross-Splitting)**.** *Let $S = S_0 \cdot S_Z \in \mathcal{A}(\mathcal{B}(G))$ with $|S| \geq 4p-3$ (or $|S| = 4p-4$ when $p \geq 5$). Suppose $\overline{S_0} = A_1 \cdot L$, where $A_1 \in \mathcal{A}(\mathcal{B}(V))$ has $|A_1| = 2p-1$ and*

$L \subseteq \ell$ is a nonempty zero-sum sequence on a single line $\ell \in \mathcal{L}$ with $|A_1 \cap \ell| \le p-2$. Then S is not minimal product-one (a contradiction).

Proof. Write A_1 in the form (3) for a basis (e_1, e_2) of V. If $\ell = \mathbb{F}_p e_1$, then $|A_1 \cap \ell| \ge p-1$, contradicting $|A_1 \cap \ell| \le p-2$. Hence $\ell = \mathbb{F}_p(se_1+e_2)$ for some $s \in \mathbb{F}_p$; replacing e_2 by $e_2' := se_1+e_2$ and x_i by $x_i - s$ (note $\sum(x_i - s) \equiv 1 - ps \equiv 1 \pmod p$), we may assume $\ell = \mathbb{F}_p e_2$. By hypothesis, $z_0 := |\{i : x_i = 0\}| = |A_1 \cap \ell| \le p-2$. Thus at least two indices i have $x_i \neq 0$. Notice also that $x_1, \dots, x_p$ cannot all be equal, since $\sum_{i=1}^p x_i \equiv 1 \not\equiv 0 \pmod p$.

Let $m := |L|$, so $m + c = |S| - (2p-1) \ge 2p-2$ when $|S| \ge 4p-3$ (and $m + c = 2p-3$ when $|S| = 4p-4$). Write $L = (y_1 e_2) \cdots (y_m e_2)$ with $y_j \in \mathbb{F}_p^\times$ and $\sum_{j=1}^m y_j \equiv 0 \pmod p$. For any nonempty proper subsequence $L_1 \subsetneq L$ with complement $L_2 := L \cdot L_1^{-1}$ and $\sum L_1 = Y_1 e_2 \neq 0$, let $k := (-Y_1) \bmod p \in \{1, \dots, p-1\}$. Choose a k-element subset $I \subsetneq \{1, \dots, p\}$ such that both I and $I^c := \{1, \dots, p\} \setminus I$ contain at least one index i with $x_i \neq 0$ (possible because $p - z_0 \ge 2$, and for $k = 1$ or $k = p-1$ we choose the singleton index from $\{i : x_i \neq 0\}$). Let $\alpha := (-\sum_{i \in I} x_i) \bmod p \in \{0, \dots, p-1\}$, and set

$$\overline{W_1} := e_1^{\alpha} \cdot \prod_{i \in I} (x_i e_1 + e_2) \cdot L_1, \qquad \overline{W_2} := e_1^{p-1-\alpha} \cdot \prod_{i \notin I} (x_i e_1 + e_2) \cdot L_2.$$

Clearly $\overline{W_1} \cdot \overline{W_2} = \overline{S_0}$. Both $\overline{W_1}$ and $\overline{W_2}$ are zero-sum in V: their e_2-coordinates are $(k + Y_1) \equiv 0 \pmod p$ and $((p-k) - Y_1) \equiv 0 \pmod p$, while their e_1-coordinates are $\alpha + \sum_{i \in I} x_i \equiv 0 \pmod p$ and $(p-1-\alpha) + \sum_{i \notin I} x_i \equiv (p-1) + 1 \equiv 0 \pmod p$. Moreover, both $\overline{W_1}$ and $\overline{W_2}$ are noncollinear because each contains $L_r \subseteq \mathbb{F}_p e_2 \setminus \{0\}$ together with an element $x_i e_1 + e_2 \notin \mathbb{F}_p e_2$. Note two endpoint properties:

- If $k = 1$ (so $I = \{i\}$ with $x_i \neq 0$), then $\alpha = (-x_i) \bmod p \in \{1, \dots, p-1\}$, hence $\alpha \ge 1$.
- If $k = p-1$ (so $I^c = \{j\}$ with $x_j \neq 0$), then $\sum_{i \in I} x_i \equiv 1 - x_j \pmod p$, so $\alpha \equiv x_j - 1 \pmod p$ and $(p - 1 - \alpha) \equiv -x_j \pmod p$, hence $p - 1 - \alpha \ge 1$.

Look at the shifted lengths $|W_1| + c = \alpha + k + |L_1| + c$ and $|W_2| + c = (p-1-\alpha) + (p-k) + (m - |L_1|) + c$. For any $1 \le t \le m-1$ with $p \nmid t$, there exists a subsequence $L_1 \subsetneq L$ of size t with $\sum L_1 \neq 0$: otherwise every t-subset of L would have sum 0, swapping one element inside and outside shows all y_j are equal, and then $t y_1 \equiv 0 \pmod p$ contradicts $y_1 \neq 0$.

Case 1: $|S| \ge 4p-3$ (so $m + c \ge 2p-2$). If $c = p-1$, take $t = 1$ (since $m \ge p-1 \ge 2$ and $y_1 \neq 0$): then $|W_1| + c = \alpha + k + 1 + (p-1) \ge p+1$ and $|W_2| + c \ge 0 + 1 + (m-1) + (p-1) \ge 2p-2 \ge p+1$. If $c \le p-2$, set $t := p - 1 - c \in \{1, \dots, p-1\}$. Note $m - t = m + c - p + 1 \ge p - 1 \ge 2$, so $1 \le t \le m-2$. Choose L_1 of size t with $\sum L_1 \neq 0$. Then:

- $|W_1| + c = \alpha + k + p - 1$: if $k = 1$, $\alpha \ge 1$ gives $\ge p+1$; if $k \ge 2$, $\alpha \ge 0$ gives $\ge p+1$.
- $|W_2| + c = (p-1-\alpha) + (p-k) + (m+c-p+1) + c \ge (p-1-\alpha) + (p-k) + (p-1) + c$: if $c \ge 1$, this is $\ge 0 + 1 + (p-1) + 1 = p+1$; if $c = 0$, then $k \le p-2$ gives $\ge 0 + 2 + (p-1) = p+1$, while $k = p-1$ gives $p - 1 - \alpha \ge 1$, hence $\ge 1 + 1 + (p-1) = p+1$.

Since $|S| \ge 4p-3 \ge 3p$, Lemma 2.2(d) yields a splitting of S in all subcases (whether both $|W_1|, |W_2| \le p$, both $\ge p+1$, or one $\ge p+1$ and the other $+c \ge p+1$).

Case 2: $|S| = 4p-4$, $p \ge 5$ (so $m + c = 2p-3$). If $c = p-1$ (so $m = p-2$), take $t = 1$ ($m - t = p - 3 \ge 2$): then $|W_1| + c = \alpha + k + p \ge p+1$ and $|W_2| + c \ge 0 + 1 + (p-3) + (p-1) = 2p-3 \ge p+2$. If $2 \le c \le p-2$, set $t := p - 1 - c \in \{1, \dots, p-3\}$ (so $m - t = p - 2 \ge 3$): then $|W_1| + c = \alpha + k + p - 1 \ge p+1$ (using $\alpha \ge 1$ if $k = 1$) and $|W_2| + c \ge 0 + 1 + (p-2) + c \ge p+1$. If $c = 1$ ($m = 2p-4 \ge p+1$, $t = p-2$), then $|W_1| + 1 \ge p+1$ as above, while $|W_2| + 1 \ge (p-1-\alpha) + (p-k) + (p-2) + 1 = (p-1-\alpha) + (p-k) + p - 1 \ge p+1$ (using $p - 1 - \alpha \ge 1$ if $k = p-1$). Finally, if $c = 0$ ($m = 2p-3$), choose $t = p-1$ ($m - t = p-2$). Any $(p-1)$-element subsequence $L_1 \subset L$ with $Y_1 \not\equiv 0, 1, 2 \pmod p$ has $k = (-Y_1) \bmod p \in \{1, \dots, p-3\}$, giving

$|W_1| \geq p+1$ (as $k=1 \implies \alpha \geq 1$) and $|W_2| \geq 3+(p-2) = p+1$. Thus we only fail if every $(p-1)$-element subsequence of L has sum in $\{0,1,2\} \pmod p$. Let $m_{\max}$ denote the maximum multiplicity of any element in $(y_1, \ldots, y_m)$:

- If $m_{\max} \leq 2p-6$, list the $m = 2p-3$ terms $y_1, \ldots, y_m$ with equal values contiguous. Since $m-2 = 2p-5 > 3$, the three pairs (y_1, y_{m-2}), (y_2, y_{m-1}), and (y_3, y_m) are disjoint, and each pair consists of distinct elements (else $m_{\max} \geq m-2 = 2p-5$). Fixing $p-4$ of the remaining $2p-9 \geq p-4$ terms and picking one term from each pair produces $(p-1)$-subset sums forming a translate $s_0 + \{0, d_1\} + \{0, d_2\} + \{0, d_3\}$ with $d_j \in \mathbb{F}_p^\times$; applying Cauchy–Davenport twice gives at least $\min(p, 4) = 4$ distinct sums, so at least one avoids $\{0,1,2\} \pmod p$.
- $m_{\max} = 2p-3$ ($L = a^{2p-3}$) is impossible because $(2p-3)a \equiv -3a \not\equiv 0 \pmod p$ for $p \geq 5$.
- If $m_{\max} = 2p-4$ ($L = a^{2p-4}b$, $b \neq a$), then $b \equiv 4a \pmod p$ and the $(p-1)$-subset sums are $(p-1)a \equiv -a$ and $(p-2)a + b \equiv 2a$; since $-a, 2a \not\equiv 0$, landing in $\{0,1,2\}$ forces $\{-a, 2a\} = \{1,2\}$, so $-a = 2, 2a = 1$ in $\mathbb{F}_p$, forcing $p = 5, a = 3, b = 2$ ($L = 3^6 \cdot 2$).
- If $m_{\max} = 2p-5$ ($L = a^{2p-5}b_1b_2$, $b_1, b_2 \neq a$), then $b_1 + b_2 \equiv 5a \pmod p$, and L contains $(p-1)$-subsequences a^{p-1} (sum $-a$), $a^{p-2}b_1$ (sum $b_1 - 2a$), $a^{p-2}b_2$ (sum $b_2 - 2a$), and $a^{p-3}b_1b_2$ (sum $-3a + b_1 + b_2 \equiv 2a$). Requiring $-a, 2a \in \{0,1,2\}$ again forces $p = 5, a = 3$; then $b_1 - 1, b_2 - 1 \in \{0,1,2\} \pmod 5$ with $b_1, b_2 \neq 3$ forces $b_1, b_2 \in \{1,2\}$, contradicting $b_1 + b_2 \equiv 5a \equiv 0 \pmod 5$.

In the sole remaining configuration ($p = 5$, $L = 3^6 \cdot 2$), take $L_1 = 3^2 \cdot 2$ ($|L_1| = 3$, $Y_1 = 3$, $k = 2$). If $z_0 = 3$ (exactly two nonzero x_i's), let $I = \{i, j\}$ contain one nonzero and one zero index; if $z_0 \leq 2$ (at least three nonzero x_i's), pick two nonzero indices with $x_i + x_j \not\equiv 0 \pmod 5$ (impossible for all pairs among three nonzero elements to sum to $0 \pmod 5$). In either case both I and I^c contain a nonzero index and $\alpha \geq 1$, whence $|W_1| = \alpha + 2 + 3 \geq 6 = p+1$ and $|W_2| = (4-\alpha) + 3 + 4 \geq 7 \geq p+1$, splitting S. □

5. Proof of Theorems 1.1 and 1.2

We first assemble the proof of Theorem 1.1 and then refine the argument inside $[3p, 4p-4]$ for Theorem 1.2.

Proof of Theorem 1.1. The lower bound $\mathsf{D}(H_{p^3}) \geq 3p-1$ and the two non-isomorphic extremal families $S^{(1)}, S^{(2)}$ are established in Proposition 3.1. For the Loewy length, the Jennings–Brauer–Zassenhaus series of $G = H_{p^3}$ is $G_1 = G$, $G_2 = [G,G]G_1^p = Z(G)$ (since $\exp(G) = p$), and $G_3 = [G_2, G]G_1^p = \{1\}$. Thus the rank invariants are $d_1 = \dim_{\mathbb{F}_p}(G_1/G_2) = 2$ and $d_2 = \dim_{\mathbb{F}_p}(G_2/G_3) = 1$, and Jennings' theorem [8] gives

$$L(H_{p^3}) \;=\; 1 + (p-1)(1 \cdot 2 + 2 \cdot 1) \;=\; 4p-3,$$

matching $\mathsf{D}_{\mathrm{ord}}(H_{p^3}) = 4p-3$ proved in [5, 6]. In contrast, for $M_{27} = \langle a, b \mid a^9 = b^3 = 1, bab^{-1} = a^4 \rangle$, the Jennings series has $G_1/G_2 \cong C_3^2$ ($d_1 = 2$), $G_2 = G_3 = \langle a^3 \rangle$ ($d_2 = 0$, $d_3 = 1$), giving $L(M_{27}) = 1 + 2(1 \cdot 2 + 3 \cdot 1) = 11$, while Cziszter–Domokos–Szöllősi [1] showed $\mathsf{D}(M_{27}) = 12 > 11$.

It remains to prove $\mathsf{D}(H_{p^3}) \leq 4p-4$. Suppose for contradiction that $S = S_0 \cdot S_Z \in \mathcal{A}(\mathcal{B}(G))$ has length $|S| \geq 4p-3$. By Lemma 4.1, $|S \cap M_\ell| \leq 2p-2$ for every $\ell \in \mathcal{L}$, and as noted after Lemma 2.1, $c = |S_Z| \leq p-1$ and $\overline{S_0}$ is noncollinear of length $n = |S| - c \geq 3p-2 > 2p-1 = \mathsf{D}(V)$. Factor $\overline{S_0} = A_1 \cdot \ldots \cdot A_r$ into minimal zero-sum atoms in $\mathcal{B}(V)$, where $r \geq 2$ and $2 \leq |A_i| \leq 2p-1$.

Case 1: At least two atoms are noncollinear, say A_1 and A_2.

- If some noncollinear atom, say A_1, has $|A_1| \geq p+1$, set $W_1 := A_1$ and $W_2 := A_2 \cdot \ldots \cdot A_r$ (lifted to S_0). Both are noncollinear zero-sums, and since $|A_1| \leq 2p-1$, we have $|W_2| + c = |S| - |A_1| \geq (4p-3) - (2p-1) = 2p-2 \geq p+1$ (for all $p \geq 3$). Lemma 2.2(d) splits S.

- Otherwise $|A_i| \le p$ for all $1 \le i \le r$. Initialize $W_1 := A_1$, $W_2 := A_2 \cdot \ldots \cdot A_r$. If $|W_1| + c \ge p+1$ already, then $|W_2| + c = |S| - |W_1| \ge (4p-3) - p = 3p - 3 \ge p+1$, so Lemma 2.2(d) splits S (note if $|W_2| \le p$ as well, $|S| \ge 4p - 3 \ge 3p$). Otherwise $|W_1| + c \le p$; transfer atoms $A_3, A_4, \ldots$ one by one into W_1 until the first step $k \ge 3$ where $|W_1^{\text{new}}| + c \ge p+1$ (this must occur before exhausting $A_3, \ldots, A_r$, as otherwise $|S| = |W_1| + |A_2| + c \le 2p < 4p-3$). Because $A_1 \subseteq W_1^{\text{new}}$ and $A_2 \subseteq W_2^{\text{new}}$, both parts remain noncollinear. Moreover, $|W_1^{\text{old}}| + c \le p$ and $|A_k| \le p$ yield $|W_1^{\text{new}}| \le 2p - c$, whence

$$|W_2^{\text{new}}| + c \;=\; |S| - |W_1^{\text{new}}| \;\ge\; |S| - 2p + c \;\ge\; 2p - 3 + c.$$

  If $p \ge 5$, then $2p - 3 + c \ge 7 > p+1$ and Lemma 2.2(d) applies. If $p = 3$, then $|S| \ge 9$, and $2p - 3 + c = 3 + c \ge 4 = p+1$ whenever $c \ge 1$. Thus the only remaining case is $p = 3$, $c = 0$, $|S| = 9$, and $|W_2^{\text{new}}| = 3$. Since $|W_1^{\text{new}}| + |W_2^{\text{new}}| = 9$ and every atom in $\mathcal{B}(\mathbb{F}_3^2)$ has size 2 or 3, $|W_2^{\text{new}}| = 3$ forces $W_2^{\text{new}} = A_2$ (a single atom of size 3) and $|W_1^{\text{new}}| = |W_1^{\text{old}}| + |A_k| = 6$; as $|W_1^{\text{old}}| \le 3$ and $|A_k| \le 3$, necessarily $|W_1^{\text{old}}| = |A_k| = 3$, whence $W_1^{\text{old}} = A_1$ (since A_1 is noncollinear of size 3). Setting $B_1 := A_1$, $B_2 := A_2$, and $B_3 := A_k$, we have $\overline{S_0} = B_1 \cdot B_2 \cdot B_3$ with $|B_1| = |B_2| = |B_3| = 3$, where B_1, B_2 are noncollinear size-3 atoms and B_3 is a size-3 atom (noncollinear or collinear). Every noncollinear size-3 atom in $\mathbb{F}_3^2$ selects one nonzero vector from each of three of the four lines in $\mathcal{L}$.
  - If some line $\ell \in \mathcal{L}$ contains an inverse pair $v, -v$ belonging to two of B_1, B_2, B_3 (say $v \in B_1$ and $-v \in B_2$, or $v \in B_1$ and $-v \in B_3$), set $W_1 := (B_i \cdot B_j) \cdot (v \cdot (-v))^{-1}$ (where $v \in B_i, -v \in B_j$) and $W_2 := v \cdot (-v) \cdot B_m$ (where B_m is the remaining atom). Then $|W_1| = 4$ and $|W_2| = 5$. Moreover, since B_i is noncollinear (at least one of B_i, B_j is B_1 or B_2, say B_i is noncollinear), $B_i \cdot v^{-1}$ consists of two linearly independent vectors, so W_1 is noncollinear; and B_m cannot lie entirely in ℓ (else $|S \cap M_\ell| \ge 5$, contradicting Lemma 4.1), so W_2 is also noncollinear. Lemma 2.2(d)(i) splits S.
  - If no mutually inverse pair $\{v, -v\}$ occurs in $\overline{S_0}$, then all vectors of $\overline{S_0}$ on each line $\ell \in \mathcal{L}$ are identical. Since B_1 and B_2 each meet 3 of the 4 lines of $\mathcal{L}$, they share at least 2 lines and hence share 2 vectors v_1, v_2, forcing $B_1 = B_2 = \{v_1, v_2, v_3\}$ with $v_3 := -v_1 - v_2$. Similarly, if B_3 is noncollinear, $B_3 = \{v_1, v_2, v_3\}$ as well, so $\overline{S_0}$ regroups into three collinear atoms $v_1^3 \cdot v_2^3 \cdot v_3^3$, which is eliminated by the 3-line $(3,3,3)$ argument in Case 3 below. If instead B_3 is collinear ($B_3 = u^3$), Lemma 4.1 forbids $u \in \{v_1, v_2, v_3\}$, so u lies on the fourth line $\mathbb{F}_3(v_1 - v_2)$ and equals $\pm(v_1 - v_2)$. Replacing v_1 with v_2 if needed so that $u = v_1 - v_2$, the regrouping $\overline{W_1} := v_1^2 \cdot v_2 \cdot u$ (size 4, noncollinear) and $\overline{W_2} := v_2 \cdot v_3^2 \cdot u^2$ (size 5, noncollinear) yields two noncollinear zero-sums of sizes $4, 5 \ge p+1$, splitting S by Lemma 2.2(d)(i).

Case 2: Exactly one atom is noncollinear, say A_1, while $L := A_2 \cdot \ldots \cdot A_r$ is a product of collinear atoms.

- If L is *noncollinear* (uses ≥ 2 lines): if $|A_1| \ge p+1$, then $|L| + c = |S| - |A_1| \ge (4p-3) - (2p-1) = 2p - 2 \ge p+1$, so Lemma 2.2(d) splits S with $W_1 = A_1, W_2 = L$. If $|A_1| \le p$, then $|A_i| \le p$ for all i: since L uses ≥ 2 lines, pick atoms A_2, A_3 on distinct lines (so $|A_2| + |A_3| \le 2p$). Setting $W_2 := A_2 \cdot A_3$ (noncollinear) and $W_1 := A_1 \cdot A_4 \cdot \ldots \cdot A_r$ (noncollinear since $A_1 \subseteq W_1$), we have $|W_1| + c = |S| - |A_2| - |A_3| \ge (4p-3) - 2p = 2p - 3$, which is $\ge p+1$ for $p \ge 5$, and also $\ge p+1 = 4$ for $p = 3$ unless $c = 0$, $|S| = 9$, $r = 3$, and $|A_1| = |A_2| = |A_3| = 3$. In that lone $p = 3$ configuration, $\overline{S_0} = A_1 \cdot v^3 \cdot w^3$ with v, w lying on distinct lines $\ell_1 = \mathbb{F}_3 v$, $\ell_2 = \mathbb{F}_3 w$:

– If $\overline{S_0}$ contains an inverse pair $\{u,-u\}$, then one element (say $-u$) belongs to A_1 and the other (u) to A_j for some $j\in\{2,3\}$; letting A_m be the remaining collinear atom ($m\in\{2,3\}\setminus\{j\}$), the partition $\overline{W_1'}:=(A_1\cdot A_j)\cdot(u\cdot(-u))^{-1}$ (size 4, containing the two independent vectors of $A_1\cdot(-u)^{-1}$) and $\overline{W_2'}:=u\cdot(-u)\cdot A_m$ (size 5, meeting ℓ_j and ℓ_m) splits S via Lemma 2.2(d)(i).
– Otherwise $\overline{S_0}$ has no inverse pair, so A_1 equals v if it meets ℓ_1 and w if it meets ℓ_2. Since A_1 meets 3 of the 4 lines of $\mathbb{F}_3^2$: either A_1 meets both ℓ_1 and ℓ_2, forcing $A_1=v\cdot w\cdot(-v-w)$ and $\overline{S_0}=v^4\cdot w^4\cdot(-v-w)$—where choosing coordinates $v=(1,0)$, $w=(0,1)$ and writing the lifts of $v^4,w^4,(-v-w)$ as $g_i=((1,0),a_i)$ $(1\le i\le 4)$, $h_j=((0,1),b_j)$ $(1\le j\le 4)$, and $u_0=((2,2),c_0)$, every triangle $g_i\cdot h_j\cdot u_0$ leaves a noncollinear complement of size $6\ge 4$, so Lemma 2.2(c) forces $\pi(g_i\cdot h_j\cdot u_0)=\{a_i+b_j+c_0\pm 2 \pmod 3\}$ to miss $0 \pmod 3$, whence $a_1=a_2=a_3=a_4$ and $g_1g_2g_3=1$ splits S via Lemma 2.2(c)—or A_1 misses one of ℓ_1,ℓ_2 (say ℓ_2), forcing $A_1=v\cdot(v+w)\cdot(v-w)$ and $\overline{S_0}=v^4\cdot w^3\cdot(v+w)\cdot(v-w)$, which splits via $\overline{W_1'}:=v^2\cdot w\cdot(v-w)$ (size 4, noncollinear) and $\overline{W_2'}:=v^2\cdot w^2\cdot(v+w)$ (size 5, noncollinear) by Lemma 2.2(d)(i).

Now if $|W_2|+c\le p$ while $|W_1|+c\ge p+1$ (note for $p=3, c=0$ one already has $|W_2|=|A_2|+|A_3|\ge 4=p+1$), transferring atoms $A_4,\dots,A_r$ into W_2 one by one (keeping $A_1\subseteq W_1$) crosses $|W_2|+c\ge p+1$ with $|W_2^{\text{new}}|\le 2p-c$ and $|W_1^{\text{new}}|+c=|S|-|W_2^{\text{new}}|\ge 2p-3+c\ge p+1$ (since $p\ge 5$ or $c\ge 1$ at $p=3$).

- If L is *collinear* on a single line $\ell\in\mathcal{L}$: if $|A_1|\le 2p-2$, then $|S\cap M_\ell|\ge |L|+c=|S|-|A_1|\ge 2p-1$, contradicting Lemma 4.1. Hence $|A_1|=2p-1$. By Lemma 4.1, $|A_1\cap\ell|+|L|+c\le 2p-2$, while $|L|+c=|S|-(2p-1)\ge 2p-2$; thus $A_1\cap\ell=\emptyset$ and Lemma 4.2 yields a contradiction.

Case 3: Every atom $A_1,\dots,A_r$ is collinear. Group the atoms by their supporting lines $\ell_1,\dots,\ell_t\in\mathcal{L}$ ($t\ge 2$ since $\overline{S_0}$ is noncollinear): $\overline{S_0}=L_1\cdot\ldots\cdot L_t$ with $m_j:=|L_j|\ge 2$, ordered $m_1\ge m_2\ge\cdots\ge m_t\ge 2$. By Lemma 4.1, $m_j+c\le 2p-2$ for every j.

- *Subcase $t=2$:* $(m_1+c)+(m_2+c)=|S|+c\ge 4p-3+c>4p-4$, contradicting $m_j+c\le 2p-2$.
- *Subcase $t\ge 4$:* Start with $W_1:=L_1\cdot L_t$ and $W_2:=L_2\cdot L_3$, both noncollinear, and sequentially assign $L_4,\dots,L_{t-1}$ to whichever of W_1,W_2 currently has smaller length (choosing W_2 on ties). Three outcomes are exhaustive:
  – If W_2 never exceeds W_1 (so $W_1=L_1\cdot L_t$, $W_2=L_2\cdot\ldots\cdot L_{t-1}$, and $|W_2|\le|W_1|$ at the end, including $t=4$ when $|W_2|\le|W_1|$), then $|W_2|+c\le p$ would imply $m_1+m_t=|S|-(|W_2|+c)\ge 3p-3$; since $m_1\le 2p-2-c$, we get $m_t\ge p-1+c$, and thus $|W_2|+c\ge m_2+m_3+c\ge 2m_t+c\ge 2p-2+3c\ge p+1$, contradiction.
  – If W_1 remains strictly smaller at every step (so $W_1=L_1\cdot L_4\cdot\ldots\cdot L_t$, $W_2=L_2\cdot L_3$, and $|W_1|<|W_2|$ at the end, including $t=4$ when $|W_1|<|W_2|$), then $|W_1|+c\le p$ with $m_t\ge 2$ implies $m_1\le|W_1|-m_t\le p-c-2$, giving $|S|=(|W_1|+c)+m_2+m_3\le p+2m_1\le 3p-4-2c<4p-3$, contradiction.
  – Otherwise at some step $k\in\{4,\dots,t-1\}$ adding L_k to the smaller (or tied) bucket matches or crosses the other, after which $\big||W_1|-|W_2|\big|\le m_k\le m_4$ holds inductively through all subsequent steps. At the end $\big||W_1|-|W_2|\big|\le m_4\le(|S|-c)/4$, whence

$$2\min(|W_1|,|W_2|)+2c\;=\;|S|+c-\big||W_1|-|W_2|\big|\;\ge\;\frac{3|S|+5c}{4}\;\ge\;\frac{12p-9}{4}\;>\;2p$$

  for all $p\ge 3$, so $\min(|W_1|,|W_2|)+c\ge p+1$ and Lemma 2.2(d) splits S.
- *Subcase $t=3$:* Here $\overline{S_0}=L_1\cdot L_2\cdot L_3$ on three lines ℓ_1,ℓ_2,ℓ_3.

– Suppose first that L_1 is *not* a single atom (which holds automatically if $m_1 \geq p+1$; note $m_1 \geq 4$ since every zero-sum atom has length ≥ 2), so $L_1 = B_1 \cdot B_2$ as a product of atoms on ℓ_1 minimizing $|B_1| - |B_2| \geq 0$. As noted in Lemma 4.1, $|B_1| \geq m_1/2$ and $|B_2| \geq \min(\max(2, m_1 - p), m_1/3)$. Put $W_1 := B_1 \cdot L_3$ and $W_2 := B_2 \cdot L_2$, both noncollinear zero-sums. Since $m_2 \leq m_1 \leq 2p-2-c$, we have $m_3 \geq |S| - c - 2m_1 \geq 4p-3-c-2m_1$, so

$$|W_1| + c \;\geq\; \frac{m_1}{2} + (4p-3-c-2m_1) + c \;=\; 4p-3-\frac{3m_1}{2} \;\geq\; p + \frac{3c}{2}.$$

For $c \geq 1$, $|W_1| + c \geq p + \frac{3}{2}$, so $\geq p+2$; for $c = 0$, equality $4p-3-\frac{3}{2}m_1 = p$ would force $m_1 = m_2 = 2p-2$ and $m_3 = 1$, impossible since $m_3 \geq 2$. Hence $|W_1| + c \geq p+1$. For W_2: if $|B_2| \geq m_1 - p$ and $m_3 \leq p$, then $|W_2| + c \geq (m_1 - p) + m_2 + c = |S| - m_3 - p \geq 2p-3$ ($\geq p+1$ for $p \geq 5$, while for $p = 3$ the bounds $4 \leq m_1 \leq 4 - c$ force $m_1 = 4$, $c = 0$, $|B_2| = 2$, and $2m_2 \geq 9 - 4 = 5$, so $m_2 \geq 3$ and $|W_2| + c \geq 2 + 3 = 5 \geq 4$); if $m_3 \geq p+1$, then $m_2 + c \geq p+1$ gives $|W_2| + c \geq 2 + p + 1 \geq p+1$. Finally, if $|B_2| \geq m_1/3$, notice $m_2 + c \leq p-2$ would imply $|S| \leq m_1 + 2(p-2-c) + c \leq (2p-2-c) + 2p - 4 - c = 4p - 6 - 2c < 4p-3$, impossible; hence $m_2 + c \geq p-1$ and $|W_2| + c \geq 2 + (p-1) = p+1$. Lemma 2.2(d) therefore splits S.

– Otherwise L_1 is a single atom, so $m_3 \leq m_2 \leq m_1 \leq p$. Since $m_1 + m_2 + m_3 + c = |S| \geq 4p-3$ and $c \leq p-1$, this forces $m_1 = m_2 = p$, $m_3 \geq p-1$ (so $m_1 + m_2 + m_3 \geq 3p-1$), and $c \geq p-3$. If *any* L_j ($j \in \{1,2,3\}$) is reducible as $L_j = B_1 \cdot B_2$ with nonempty zero-sum factors B_1, B_2 ($|B_1|, |B_2| \geq 2$), write $\{j,k,r\} = \{1,2,3\}$ and set $W_1 := B_1 \cdot L_k$ and $W_2 := B_2 \cdot L_r$ (both noncollinear zero-sums). Then $|W_1| + |W_2| = m_1 + m_2 + m_3 \geq 3p-1$ gives $\max(|W_1|, |W_2|) \geq \lceil (3p-1)/2 \rceil \geq p+1$ for all $p \geq 3$, while $|W_i| + c \geq 2 + (p-1) + (p-3) = 2p-2 \geq p+1$ for both $i \in \{1,2\}$ and all $p \geq 3$, splitting S via Lemma 2.2(d).

– Consequently L_1, L_2, L_3 are all single atoms of size $\leq p$, with either $(m_1, m_2, m_3) = (p,p,p)$ or $(p,p,p-1)$, and $c \in \{p-3, p-2, p-1\}$ (where $(p,p,p-1)$ requires $c \geq p-2$). For every $i \in \{1,2,3\}$, the complementary pair $L_j \cdot L_k$ is noncollinear of length $m_j + m_k = |S| - c - m_i \geq (4p-3) - (p-1) - p = 2p-2 \geq p+1$. By Lemma 2.2(c), no $C_i \mid S_Z$ can satisfy $L_i \cdot C_i \in \mathcal{B}(G)$, so $-a(\prod L_i) \notin \Sigma_{\geq 0}(S_Z)$ for $i = 1,2,3$. Since $c = p-1$ gives $\Sigma_{\geq 0}(S_Z) = \mathbb{F}_p$ by Lemma 2.2(a), $c = p-1$ is impossible. Thus $c \in \{p-3, p-2\}$ (with $(p,p,p-1)$ only at $c = p-2$).
Because $m_1 = m_2 = p$, any minimal zero-sum sequence of length p in C_p is constant: indeed, if $v_1 \neq v_2$ appear in a zero-sum-free sequence T of length $p-1$ in C_p, Cauchy–Davenport yields $|\Sigma(T)| \geq 3 + (p-3) = p$, forcing $0 \in \Sigma(T)$, contradiction. Thus $\overline{L_1} = v_1^p$ and $\overline{L_2} = v_2^p$. In the $\mathbb{F}_p$-basis (v_1, v_2) of V, we have $\overline{L_1} = (1,0)^p$, $\overline{L_2} = (0,1)^p$, and $\ell_3 = \mathbb{F}_p(a,b)$ for some $a, b \in \mathbb{F}_p^\times$.

  * If $(m_1, m_2, m_3) = (p,p,p)$, then $\overline{L_3} = (a,b)^p$. For each $r \in \{1, \ldots, p-1\}$, let $t_1(r) := (-ra) \bmod p \in \{1, \ldots, p-1\}$ and $t_2(r) := (-rb) \bmod p \in \{1, \ldots, p-1\}$, and partition $\overline{S_0}$ into the two noncollinear zero-sums

$$\overline{W_1^{(r)}} = (1,0)^{t_1(r)}(0,1)^{t_2(r)}(a,b)^r, \qquad \overline{W_2^{(r)}} = (1,0)^{p-t_1(r)}(0,1)^{p-t_2(r)}(a,b)^{p-r}.$$

For $p \geq 5$, choose $r = 2 \leq p-1$; since $c \geq p-3$, we have $|W_1^{(2)}| + c \geq 1+1+2+(p-3) = p+1$ and $|W_2^{(2)}| + c \geq 1+1+(p-2)+(p-3) = 2p-3 \geq p+2$, splitting S. For $p = 3$ ($c \in \{0,1\}$): if $c = 1$ then $r = 1$ gives $|W_1^{(1)}| + 1 \geq 4$ and $|W_2^{(1)}| + 1 \geq 5$. If $c = 0$ ($|S| = 9$), every $(a,b) \neq (2,2)$ in $(\mathbb{F}_3^\times)^2$ splits directly

into sizes $(4,5)$: take $r=2$ when $(a,b)=(1,1)$ (so $t_1(2)=t_2(2)=1$, $|W_1^{(2)}|=4$) and $r=1$ when $\{a,b\}=\{1,2\}$ (so $\{t_1(1),t_2(1)\}=\{1,2\}$, $|W_1^{(1)}|=4$). In the remaining case $(a,b)=(2,2)$, every triangle formed from one element each of L_1,L_2,L_3 leaves a noncollinear complement of size $6\geq 4$. By Lemma 2.2(c), every one of the 27 such triangles must miss $1\in Z(H_{27})$. Writing the elements of L_1,L_2,L_3 as $g_i=((1,0),a_i)$, $h_j=((0,1),b_j)$, and $w_k=((2,2),c_k)$ $(1\leq i,j,k\leq 3)$, we have $\pi(g_i\cdot h_j\cdot w_k)=\{a_i+b_j+c_k\pm 2 \pmod 3\}$; missing $0 \pmod 3$ forces $a_i+b_j+c_k\equiv 0 \pmod 3$ for all i,j,k. Thus $a_1=a_2=a_3$, giving $\prod L_1=z^{3a_1}=1$, contradicting $-a(\prod L_1)\notin\Sigma_{\geq 0}(\emptyset)=\{0\}$.

∗ If $(m_1,m_2,m_3)=(p,p,p-1)$ at $c=p-2$, write $\ell_3=\mathbb{F}_p(1,\lambda)$ $(\lambda\in\mathbb{F}_p^\times)$ and $\overline{L_3}=(x_1,\lambda x_1)\cdots(x_{p-1},\lambda x_{p-1})$ with $x_j\in\mathbb{F}_p^\times$ and $\sum x_j\equiv 0 \pmod p$. Set $t_1:=(-x_1) \bmod p\in\{1,\dots,p-1\}$ and $t_2:=(-\lambda x_1) \bmod p\in\{1,\dots,p-1\}$, and put $\overline{W_1}:=(1,0)^{t_1}(0,1)^{t_2}(x_1,\lambda x_1)$ and $\overline{W_2}:=\overline{S_0}\cdot\overline{W_1}^{-1}$ (both noncollinear zero-sums). Then $|W_1|+c=(t_1+t_2+1)+(p-2)\geq p+1$ and $|W_2|+c=(3p-2-t_1-t_2)+(p-2)=4p-4-t_1-t_2\geq 2p-2\geq p+1$, splitting S via Lemma 2.2(d).

In all cases we reach a contradiction, proving $\mathsf{D}(H_{p^3})\leq 4p-4$. For $p=3$, $3p-1=4p-4=8$, giving $\mathsf{D}(H_{27})=8$. □

Proof of Theorem 1.2. Let $p\geq 5$ and $S=S_0\cdot S_Z\in\mathcal{A}(\mathcal{B}(H_{p^3}))$ with $3p\leq|S|\leq 4p-4$. Factor $\overline{S_0}=A_1\cdot\ldots\cdot A_r$ in $\mathcal{B}(V)$.

For (a), suppose $|S|\geq 3p+1$:

- If $\overline{S_0}$ has ≥ 2 noncollinear atoms A_1,A_2: if $|A_1|\geq p+1$, then $|A_2\cdots A_r|+c=|S|-|A_1|\geq(3p+1)-(2p-1)=p+2>p$, splitting S via Lemma 2.2(d). If all $|A_i|\leq p$, the greedy transfer in Case 1 of Theorem 1.1 yields $|W_1^{\text{new}}|+c\geq p+1$ and $|W_2^{\text{new}}|+c=|S|-|W_1^{\text{new}}|\geq(3p+1)-2p+c=p+1+c\geq p+1$, splitting S.
- If $\overline{S_0}=A_1\cdot L$ has one noncollinear atom A_1 and a noncollinear complement L: if $|A_1|\geq p+1$, $|L|+c\geq(3p+1)-(2p-1)=p+2$, splitting S; if $|A_1|\leq p$, start from $W_2=A_2\cdot A_3$ (two atoms of L on distinct lines) and $W_1=A_1\cdot A_4\cdot\ldots\cdot A_r$, which satisfies $|W_1|+c\geq(3p+1)-2p=p+1$. If $|W_2|+c\geq p+1$ already, Lemma 2.2(d) splits S; otherwise greedy transfer of $A_4,\dots,A_r$ into W_2 stops with $|W_2^{\text{new}}|+c\geq p+1$ and $|W_2^{\text{new}}|\leq 2p-c$, whence the post-transfer complement satisfies $|W_1^{\text{new}}|+c=|S|-|W_2^{\text{new}}|\geq(3p+1)-2p+c=p+1+c\geq p+1$, again splitting S.

Even at $|S|=3p$, the only configurations outside (O1) that escape this partition argument are: (i) $c=0$ when the greedy transfer (or every atom-partition into noncollinear zero-sums) ends with $\min(|W_1|,|W_2|)\leq p$ (in sizes $(p,2p)$), or (ii) Case 2 with $1\leq c\leq p-3$, $r=3$, $|A_1|=p-c$, and $|A_2|=|A_3|=p$ (so $A_2=v_1^p, A_3=v_2^p$ for linearly independent v_1,v_2). Indeed, if $r\geq 4$ in Case 2 with $|A_2|=|A_3|=p$, then either A_4 lies on a third line (giving $|A_4|\leq p-|A_1|-c\leq p-3$ and $|A_2|+|A_4|\leq 2p-3$, whence greedy transfer from $W_2=A_2\cdot A_4$ succeeds for $c\geq 1$ and reduces at $c=0$ to case (i)), or A_4 shares the line of A_2 or A_3, say A_2, in which case $W_1:=A_1\cdot A_2$ and $W_2:=A_3\cdot A_4\cdot\ldots\cdot A_r$ are noncollinear zero-sums with $|W_1|\geq p+3$ and $|W_2|+c=2p-|A_1|\geq p+2$ (as $|A_1|+|A_4|+c\leq p$ forces $|A_1|\leq p-2$). Both boundaries lie in (O2).

For (b), suppose $\overline{S_0}=L_1\cdot\ldots\cdot L_t$ consists entirely of collinear atoms on $t\geq 2$ lines, ordered $m_1\geq\cdots\geq m_t\geq 2$, with $|S|\geq 3p$.

- *Elimination of $t=2$:* If $m_2\leq p$, set $k:=p+1-m_2\in\{1,\dots,p-1\}$. Since $m_1+c=|S|-m_2\geq 2p$, we have $m_1\geq 2p-c\geq p+1>k$. Deleting k elements from L_1 leaves $(m_1-k)+c=|S|-p-1\geq 2p-1=\mathsf{D}(M_{\ell_1})$ elements in M_{ℓ_1}, yielding a nonempty product-one subsequence $U=U_0\cdot U_Z$ ($U_0\neq\emptyset$ as S_Z is product-one free) disjoint from

the k deleted elements. Thus $S_0 \cdot U_0^{-1}$ is noncollinear of length $\geq k + m_2 = p + 1$, splitting S by Lemma 2.2(c). If instead $m_1 \geq m_2 \geq p + 1$: whenever $m_1 + c \geq 2p$, deleting 1 element from L_1 leaves $(m_1 - 1) + c \geq 2p - 1$ elements in M_{ℓ_1} and yields U whose complement has length $\geq 1 + m_2 \geq p + 2$, splitting S. If $m_1 + c \leq 2p - 1$ (so $m_1 \leq 2p-1$), factor $L_1 = B_1 \cdot B_2$ and $L_2 = C_1 \cdot C_2$ as minimal-difference atom splits with $|B_1| \geq |B_2|$ and $|C_1| \geq |C_2|$, so $|B_1| \geq m_1/2$, $|C_1| \geq m_2/2$, $|B_2| \geq \min(m_1 - p, m_1/3)$, and $|C_2| \geq \min(m_2 - p, m_2/3)$. Putting $W_1 := B_1 \cdot C_2$ and $W_2 := B_2 \cdot C_1$, if $|C_2| \geq m_2 - p$ then $|W_1| + c \geq (m_1 + m_2 + c) - \frac{m_1}{2} - p \geq 3p - \frac{2p-1}{2} - p = p + \frac{1}{2} > p$, while if $|C_2| \geq m_2/3$ then $|W_1| + c \geq \frac{m_1+m_2+c}{3} + \frac{m_1}{6} + \frac{2c}{3} \geq p + \frac{p+1}{6} > p$; symmetrically $|W_2| + c \geq p + 1$. Lemma 2.2(d) splits S.

- *Elimination of $t \geq 4$:* Apply the greedy balancing of Case 3 of Theorem 1.1 starting from $W_1 = L_1 \cdot L_t$ and $W_2 = L_2 \cdot L_3$. If a crossing/matching occurs at some step $k \in \{4, \ldots, t-1\}$, then $2\min(|W_1|, |W_2|) + 2c \geq \frac{3|S|+5c}{4} \geq \frac{9p}{4} > 2p$, so $\min(|W_1|, |W_2|) + c \geq p + 1$, splitting S. If W_1 stays strictly smaller throughout (so $W_1 = L_1 \cdot L_4 \cdot \ldots \cdot L_t$, $W_2 = L_2 \cdot L_3$, and $|W_1| < |W_2|$), then $|W_1| + c \leq p$ gives $m_1 \leq p - c - 2$ and $|S| \leq p + 2m_1 \leq 3p - 4 - 2c < 3p$, impossible; hence $|W_2| + c > |W_1| + c \geq p + 1$ and S splits. Finally, if W_2 never exceeds W_1 (so $W_2 = L_2 \cdot \ldots \cdot L_{t-1}$ and $|W_2| \leq |W_1|$) and $|W_2| + c \leq p$, then $2m_t \leq |W_2| \leq p - c$ gives $m_t \leq \frac{p-c}{2}$, while $m_1 + m_t = |S| - (|W_2| + c) \geq 2p$ yields $m_1 \geq \frac{3p+c}{2} > p$. Note $m_2 + \cdots + m_t = |W_2| + m_t \leq \frac{3(p-c)}{2}$. If $m_2 + \cdots + m_t \leq p$, deleting $k := p + 1 - (m_2 + \cdots + m_t) \geq 1$ elements from L_1 leaves $(m_1 - k) + c = |S| - p - 1 \geq 2p - 1 = \mathsf{D}(M_{\ell_1})$ elements in M_{ℓ_1}, splitting S as above. Otherwise $m_2 + \cdots + m_t \geq p+1$, so $|W_2| \geq p+1-m_t$; factoring $L_1 = B_1 \cdot B_2$ into minimal-difference atoms ($|B_1| \geq m_1/2$, $|B_2| \geq \min(m_1 - p, m_1/3)$) and setting $W_1' := B_1 \cdot L_t$, $W_2' := B_2 \cdot W_2$ gives $|W_1'| + c \geq \frac{m_1}{2} + m_t + c \geq \frac{2p-m_t}{2} + m_t + c = p + \frac{m_t}{2} + c \geq p + 1$ (since $m_t \geq 2$), and $|W_2'| + c \geq p + 1$ (if $|B_2| \geq m_1 - p$, $|W_2'| + c \geq |S| - p - m_t \geq 2p - \frac{p}{2} > p$; if $|B_2| \geq m_1/3$, $|W_2'| + c \geq \frac{3p+c}{6} + (p + 1 - \frac{p-c}{2}) + c = p + 1 + \frac{5c}{3} \geq p + 1$). Again S splits.

Consequently, any $S \in \mathcal{A}(\mathcal{B}(H_{p^3}))$ of length $3p \leq |S| \leq 4p - 4$ must fall under (O1) or (O2). We verify the remaining fine conditions stated inside (O1) and (O2):

- In (O1), $|S \cap M_\ell| \leq 3p - 3$ because Case 1 of Lemma 4.1 goes through verbatim whenever $|S| \geq 3p$: any nonempty product-one $U = U_0 \cdot U_Z \mid (S \cap M_\ell)$ with $|U| \leq p$ leaves at least $(3p - 2) - p - (p - 1) = p - 1 \geq 4$ elements of $S_0 \cap M_\ell$ alongside $S_0 \setminus M_\ell \neq \emptyset$, giving $|S_0 \cdot U_0^{-1}| \geq 3p - p - (p - 1) = p + 1$ and splitting S via Lemma 2.2(c). Likewise, when $|S \cap M_\ell| \geq 2p - 1$, any minimal product-one $U = U_0 \cdot U_Z \mid (S \cap M_\ell)$ has $U_0 \neq \emptyset$ and $S_0 \setminus U_0 \neq \emptyset$, so Lemma 2.2(c) prevents $S_0 \setminus U_0$ from being simultaneously noncollinear and of length $\geq p + 1$. Moreover, if $|S| = 4p - 4$ and $|A_1| = 2p - 1$, write A_1 in the Property B basis (3): if $\ell = \mathbb{F}_p e_1$ then $|A_1 \cap \ell| = p - 1$, whereas if $\ell = \mathbb{F}_p(se_1 + e_2)$ then $\sum_{i=1}^{p} x_i \equiv 1 \not\equiv ps \pmod p$ forbids $|A_1 \cap \ell| = p$ and Lemma 4.2 (Case 2) forbids $|A_1 \cap \ell| \leq p - 2$, again forcing $|A_1 \cap \ell| = p - 1$.
- In (O2) for $t = 3$, if $m_1 + c \geq 2p - 1 = \mathsf{D}(M_{\ell_1})$, then either $m_2 + m_3 \geq p + 1$ (so any nonempty product-one $U = U_0 \cdot U_Z \mid (S \cap M_{\ell_1})$ leaves $S_0 \cdot U_0^{-1}$ containing $L_2 \cdot L_3$, which is noncollinear of length $\geq p + 1$ and splits S by Lemma 2.2(c)), or $m_2 + m_3 \leq p$, in which case setting $k := p+1-(m_2+m_3) \in \{1, \ldots, p-3\}$ (so $m_1 \geq 3p-(p-1)-p = p+1 > k$) and deleting k elements from L_1 leaves $(m_1-k)+c = |S|-p-1 \geq 2p-1 = \mathsf{D}(M_{\ell_1})$ elements in M_{ℓ_1}, yielding $U = U_0 \cdot U_Z$ ($U_0 \neq \emptyset$) whose complement $S_0 \cdot U_0^{-1}$ contains those k elements and $L_2 \cdot L_3$, hence is noncollinear of length $\geq p + 1$ and splits S. Thus $|L_j| + c \leq 2p - 2$ for each j. Finally, consider the excluded (p, p, p) triple at $c \geq p - 4$ and $(p, p, p - 1)$ at $c \geq p-3$ (both with $p \geq 5$, $m_1+m_2+m_3 \geq 3p-1$, and $c \geq p-4$). If any L_j ($j \in \{1, 2, 3\}$)

is reducible as $L_j = B_1 \cdot B_2$ ($|B_1|, |B_2| \geq 2$), setting $\{j,k,r\} = \{1,2,3\}$ yields noncollinear zero-sums $W_1 := B_1 \cdot L_k$ and $W_2 := B_2 \cdot L_r$ satisfying $\max(|W_1|, |W_2|) \geq \lceil (3p-1)/2 \rceil \geq p+2$ and $|W_i| + c \geq 2 + (p-1) + (p-4) = 2p-3 \geq p+2$ for $i = 1,2$, splitting S via Lemma 2.2(d). Hence L_1, L_2, L_3 are all single atoms, so $L_1 = v_1^p$ and $L_2 = v_2^p$, and Case 3 of Theorem 1.1 in two-exponent coordinates applies: for (p,p,p) at $c \geq p-4$, $r = 2$ yields $|W_1^{(2)}| + c \geq p+1$ and $|W_2^{(2)}| + c \geq 2p-4 \geq p+1$ unless $t_1(2) = t_2(2) = 1$, in which case $r = 4 \leq p-1$ gives $t_1(4) = t_2(4) = 2$, $|W_1^{(4)}| + c \geq 8 + (p-4) = p+4$, and $|W_2^{(4)}| + c \geq (3p-8) + (p-4) = 4p-12 \geq p+3$; for $(p,p,p-1)$ at $c \geq p-3$, pick $(x_1, \lambda x_1) \in \overline{L_3}$ so that $t_1 + t_2 \geq 3$ (automatic if $\lambda \neq 1$, since $t_1 = 1 \implies t_2 = \lambda \bmod p \neq 1$; and choosing $x_1 \neq p-1$ if $\lambda = 1$, possible as $(p-1)^2 \equiv 1 \not\equiv 0 \pmod p$), giving $|W_1| + c \geq (3+1) + (p-3) = p+1$ and $|W_2| + c \geq 4p - 5 - 2(p-1) = 2p-3 \geq p+2$. □

6. Mechanized Formalization and Computational Probes

To stress-test every algebraic and combinatorial step independently, we developed two verification artifacts archived in the project repository (https://github.com/lluiseriksson/heisenberg-large-davenport, release `v1.0.2`):

1. *Zero-import Lean 4.36.0 formalization* (`lean/HeisenbergDavenport.lean`): mechanizes the exponential group law (1) for arbitrary odd moduli $p = 2k+1 \geq 3$ (`hmul_assoc`, `hpow_p_eq_one`, `commutator_eq_det`), the exact commutator-area identity (2) (`hprod_eq_area`), the contraction shift identity of Lemma 2.1 (`areaSum_contract_pair`), the arithmetic pairing and constructive three-set pigeonhole cores of Lemma 2.2 (`arithmetic_pairing_complement`, `three_set_pigeonhole`), the explicit product-one proofs for $S^{(1)}$ and $S^{(2)}$ across all $p = 2k+1$ (`s1Word_prod_one`, `s2Word_prod_one`), and the short-subsequence exclusion core of Theorem 1.3 (`t_eta_central_nonzero`). The file compiles with zero imports, zero `sorry`, zero `Classical.choice`, and zero warnings under `lean -DwarningAsError=true`.
2. *Exact Python verification*: using exact mixed-radix dynamic programming over subsequence multiplicities in `code/verify_heisenberg_davenport.py` (paired with the pytest suite `code/test_heisenberg_davenport.py`), we verified:
   - exact minimality of $S^{(1)}$ and $S^{(2)}$ (length $3p-1$) and the short-subsequence exclusion of T_η (length $7p-7$) and T_{s} (length $8p-8$) for $p \in \{3,5,7,11\}$;
   - an exhaustive orbit check over all 80 $\mathrm{GL}_2(\mathbb{F}_3)$-orbits (17,738 shear-reduced lift sequences across $c \in \{0,1,2\}$) of length 9 in H_{27}, confirming zero minimal product-one sequences of length 9 (corroborating our conceptual proof of $\mathsf{D}(H_{27}) = 8$);
   - targeted hill-climbing searches over lifts of representative (O1) and (O2) configurations in H_{125} ($p = 5$, lengths $15, 16$) and H_{343} ($p = 7$, length 21), every trial terminating with strictly positive split count (≥ 6 splits for (O1), ≥ 16 splits for 3-line (O2)).

These results point naturally to the conjecture that neither (O1) nor (O2) can be realized in H_{p^3} for $p \geq 5$, and that the lower bounds of Theorems 1.1 and 1.3 are exact.

Conjecture 6.1. For every odd prime $p \geq 3$, $\mathsf{D}(H_{p^3}) = 3p-1$.

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