The Log-Concave and κ-Concave Steinhaus–Trybuła Barriers: Tangent Stochastic Dominance, Quantile Phase Transitions, and Convolution Transitivity

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Abstract

In the classical Steinhaus–Trybuła paradox, n independent real random variables can form a non-transitive cycle $X_1 \succ \cdots \succ X_n \succ X_1$ with $\min_i \mathbb{P}(X_i > X_{i+1})$ approaching Usiskin's sharp spectrum $1-[4\cos^2(\pi/(n+2))]^{-1}$, rising from the golden ratio $\frac{\sqrt{5}-1}{2} \approx 0.6180$ at $n=3$ to $\frac{3}{4}$ as $n \to \infty$. We uncover a sharp structural dichotomy between unimodality and convexity across Borell's continuum of κ-concave probability measures ($\kappa \in [-\infty, 1]$):

1. **Tangent Stochastic Dominance and Exact κ-Concave Barriers:** Every non-degenerate κ-concave random variable X is pointwise first-order stochastically dominated ($X \preceq_{\mathrm{st}} X^*$) by the extremal shifted power, exponential, or Pareto law tangent to its survival function S_X^κ (or $\log S_X$) at any quantile. For any two independent κ-concave variables with $\mathrm{med}(X) \le \mathrm{med}(Y)$, and consequently for every non-acyclic oriented graph or tournament T of arbitrary size $n \ge 3$, we prove the exact sharp median-crossing ceilings:

$$\tau_\kappa(T) \le M(\kappa) = \begin{cases} \frac{1}{2}, & \kappa = 1 \text{ (uniforms: transitive)}, \\ \frac{13-8\sqrt{2}}{3} \approx 0.56210, & \kappa = \frac{1}{2} \text{ (2D convex bodies)}, \\ \frac{1+2\ln 2}{4} \approx 0.59657, & \kappa = 0 \text{ (all log-concave)}, \\ \frac{1}{2} + \frac{\ln 3}{8} \approx 0.63733, & \kappa = -1 \text{ (Pareto, } S^{-1} \text{ convex)}, \end{cases}$$

   bridging $\frac{1}{2}$ to $\frac{3}{4}$ as $\kappa \to -\infty$ and crossing the golden ratio at $\kappa_* \approx -0.4508$. In particular, log-concave random variables can *never* reach even the 3-variable golden ratio $\frac{\sqrt{5}-1}{2}$ (nor Grünbaum's $1-e^{-1} \approx 0.6321$), regardless of n.

2. **Exact Quantile-Crossing Phase Transition:** For log-concave (X, Y) with $F_X(0) \ge q$ and $S_Y(0) \ge q$ ($\lambda := -\ln(1-q)$), $\sup \mathbb{P}(X > Y)$ undergoes a sharp transition at $\lambda_* \approx 1.25643$ ($(1+2\lambda_*)e^{-\lambda_*} = 1$, $q_* \approx 0.71533$) from the equal-scale branch $(1-q)^2(1-2\ln(1-q))$ to scale collapse $1-q$. As an immediate corollary, whenever one variable is deterministic, $\min_i \mathbb{P}(X_i > X_{i+1}) \le 1 - e^{-\lambda_3} \approx 0.54009$ (attained uniquely by $(0, \mathrm{Exp}(1) - \lambda_3, \lambda_3 - \mathrm{Exp}(1))$).

3. **1D Localization Reduction and Exponential Chain Spectrum:** Via coordinate-wise 1D log-concave localization, $\tau_{\mathrm{LC}}(C_n)$ reduces exactly to truncated exponentials. Optimal scale-ratio chains yield strictly increasing lower bounds 0.54009 ($n=3$), 0.54382 ($n=4$), 0.54691 ($n=5$), converging to $p^*_{\mathrm{Exp}} \approx 0.54912$ (root of $(1+(e+2)c)e^{-2c} = 1$), while one-sided chains converge to $p^+_{\mathrm{Exp}} \approx 0.51955$ ($ep^2(-\ln p) = 1-p$).

4. **Unimodal Zero-Penalty vs. Odd-Cumulant Convolution Transitivity:** Unimodal (even C^∞) random variables suffer *zero* Steinhaus–Trybuła penalty, achieving the full Usiskin spectrum $1 - \frac{1}{4\cos^2(\pi/(n+2))} \to \frac{3}{4}$ for every $n \ge 3$. Conversely, under

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k-fold i.i.d. convolution $S_i^{(k)} = X_i^{(1)} + \cdots + X_i^{(k)}$, every finite log-concave family collapses asymptotically to the transitive lexicographic weak order of its alternating odd cumulants $(\kappa_1, -\kappa_3, +\kappa_5, \dots)$, with zero even-cumulant cross-terms at leading order $k^{-(m-1/2)}$.

The core algebraic factorizations, rational hierarchy inequalities, inductive convex-body margins, and Edgeworth parity lemma (10 theorems) are formally certified in Lean 4.36.0 with zero `sorry`.

1 Introduction

Let $X_1, \dots, X_n$ be independent real-valued random variables. Under the *stochastic precedence* relation $X_i \succ X_j \iff \mathbb{P}(X_i > X_j) > \frac{1}{2}$, Steinhaus and Trybuła [7, 8] discovered that pairwise comparisons can form strict cycles $X_1 \succ X_2 \succ \cdots \succ X_n \succ X_1$. For any oriented graph or tournament $T = (V, A)$ on n vertices and any class $\mathcal{F}$ of probability distributions on $\mathbb{R}$, define its *Steinhaus–Trybuła capacity*

$$\tau_{\mathcal{F}}(T) \;:=\; \sup_{\substack{X_1,\dots,X_n \in \mathcal{F} \\ \text{independent}}} \min_{(i \to j) \in A} \mathbb{P}(X_i > X_j). \tag{1}$$

For the directed n-cycle C_n over unrestricted distributions $\mathcal{F}_{\text{all}}$, Trybuła [8] and Usiskin [9] determined the exact spectrum

$$\tau_{\text{all}}(C_n) \;=\; 1 - \frac{1}{4\cos^2\!\left(\frac{\pi}{n+2}\right)}, \tag{2}$$

which equals $\phi^{-1} = \frac{\sqrt{5}-1}{2} \approx 0.61803$ at $n = 3$, $\frac{2}{3}$ at $n = 4$, and increases strictly to $\frac{3}{4}$ as $n \to \infty$.

Inspecting the extremal distributions achieving (2) reveals a striking structural feature: every variable in Trybuła's and Usiskin's constructions is a *bimodal two-point distribution* placing separated atoms on opposite sides of its competitors. Meanwhile, symmetric distributions (including Gaussians and uniforms) are trivially transitive ($\tau_{\text{sym}}(T) = \frac{1}{2}$ for every non-acyclic T). Between unimodal distributions and symmetric distributions lies Borell's classical continuum of *κ-concave probability measures* [2, 6] ($\kappa \in [-\infty, 1]$), anchored at $\kappa = 0$ by *log-concave* distributions (exponential, Gamma($\alpha \geq 1$), Weibull($\alpha \geq 1$), Beta($\alpha, \beta \geq 1$), Gaussian, and all 1D marginals of uniform measures on convex bodies).

A remarkably simple observation already separates log-concave distributions from $\frac{3}{4}$: in any directed cycle C_n, $\sum_{i=1}^n \mathbb{E}[X_i - X_{i+1}] = 0$ forces at least one arc with $\mathbb{E}[X_i - X_{i+1}] \leq 0$. By the Prékopa–Leindler theorem [6], $Z = X_i - X_{i+1}$ is log-concave, so Grünbaum's inequality [4] yields $\mathbb{P}(X_i > X_{i+1}) \leq 1 - e^{-1} \approx 0.63212$. Yet Grünbaum's bound still exceeds the 3-variable golden ratio $\phi^{-1} \approx 0.61803$ and leaves open both the sharp barrier for log-concave measures and the behavior across the entire κ-concave and unimodal hierarchy.

In this paper we resolve these questions across four interlocking regimes, summarized in Table 1. Rather than bounding means via Grünbaum's theorem, we work directly with *medians* and *quantiles* via a five-line *Tangent Stochastic Dominance Lemma* for κ-concave survival functions, establishing exact sharp ceilings across $\kappa \in [-\infty, 1]$, an exact quantile phase transition for log-concave laws, a 1D localization reduction with explicit exponential-chain limits, a zero-penalty unimodal Usiskin theorem, and a universal odd-cumulant convolution transitivity law.

Distribution Class (κ)	**Sharp Median-Crossing Ceiling $M(\kappa)$**	**Extremal Pair** (med $= 0$)	**Reference**
Uniforms / 1D intervals ($\kappa = 1$)	$\frac{1}{2} = 0.50000$ (strict transitivity)	Unif$[-1, 1]$	Thm. 2.2
2D convex bodies / concave f ($\kappa = \frac{1}{2}$)	$\frac{13 - 8\sqrt{2}}{3} \approx 0.56210$	$\pm\big(\frac{1}{\sqrt{2}} - \text{Beta}(2,1)\big)$	Thm. 2.2
All log-concave laws ($\kappa = 0$)	$\frac{1 + 2\ln 2}{4} \approx 0.59657 < \phi^{-1}$	$\pm\big(\text{Exp}(1) - \ln 2\big)$	Thm. 2.2
Golden-ratio threshold ($\kappa_* \approx -0.4508$, $u = 1$)	$M(\kappa_*; 1) = \frac{\sqrt{5} - 1}{2} \approx 0.61803$	$\pm\big(\text{Pareto}(\gamma_*) - 2^{1/\gamma_*}\big)$	Rem. 2.3
Cauchy-tail Pareto ($\kappa = -1$)	$\frac{1}{2} + \frac{\ln 3}{8} \approx 0.63733$	$\pm\big(\text{Pareto}(1) - 2\big)$	Thm. 2.2
Unimodal / general ($\kappa \to -\infty$)	$\frac{3}{4} = 0.75000$ (zero penalty on C_n!)	Spike + flat tail	Thm. 5.1

Table 1: Exact median-crossing barriers $M(\kappa)$ and universal tournament ceilings $\tau_\kappa(T) \le M(\kappa)$ across Borell's κ-concave continuum. Equal-scale optimality $M(\kappa) = M(\kappa; 1)$ is proved analytically at $\kappa \in \{1, \frac{1}{2}, 0, -1\}$ (satisfying the Lean-certified strict chain $\frac{1}{2} < M(\frac{1}{2}) < M(0) < \phi^{-1} < 1 - e^{-1} < M(-1) < \frac{3}{4}$); for intermediate κ (such as $\kappa_* \approx -0.4508$), the equal-scale value $M(\kappa; 1)$ is reported (see Open Problem 2).

2 Tangent Stochastic Dominance and Exact κ-Concave Barriers

Recall Borell's definition [2]: a Borel probability measure μ on $\mathbb{R}$ is *κ-concave* ($\kappa \in (-\infty, 1]$) if for all Borel sets $A, B \subset \mathbb{R}$ with $\mu(A)\mu(B) > 0$ and every $t \in (0, 1)$,

$$\mu\big(tA + (1-t)B\big) \;\ge\; \begin{cases} \big(t\,\mu(A)^\kappa + (1-t)\,\mu(B)^\kappa\big)^{1/\kappa}, & \kappa \neq 0, \\ \mu(A)^t\,\mu(B)^{1-t}, & \kappa = 0. \end{cases}$$

By Borell's characterization [2], every non-degenerate κ-concave probability measure on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure with an interval support (and every Dirac mass δ_c is degenerate κ-concave for all κ). For $\kappa = 0$ these are precisely the *log-concave* distributions; for $\kappa = 1/k$ ($k \in \mathbb{N}_{\ge 2}$), by Brunn–Minkowski they include all 1D linear marginals of uniform measures on k-dimensional convex bodies $K \subset \mathbb{R}^k$ (in particular, $\kappa = \frac{1}{2}$ corresponds to concave densities on intervals); and for $\kappa = -1/\gamma < 0$ they capture heavy-tailed Pareto-type densities with $f^{\kappa/(1-\kappa)} = f^{-1/(\gamma+1)}$ convex on their support.

For any random variable X with law μ, write its CDF $F_X(x) := \mathbb{P}(X \le x)$ and survival function $S_X(x) := \mathbb{P}(X > x)$. Taking half-lines $A = (x, \infty)$ and $B = (y, \infty)$ in Borell's definition immediately yields $tA + (1-t)B = (tx + (1-t)y, \infty)$, whence directly:

- for $\kappa > 0$, S_X^κ is *concave* on $\{S_X > 0\}$ (and F_X^κ is concave on $\{F_X > 0\}$);
- for $\kappa = 0$, $\log S_X$ is *concave* on $\{S_X > 0\}$ (and $\log F_X$ is concave on $\{F_X > 0\}$);
- for $\kappa = -1/\gamma < 0$ ($\gamma > 0$), $S_X^\kappa = S_X^{-1/\gamma}$ is *convex* on $\{S_X > 0\}$ (and $F_X^\kappa = F_X^{-1/\gamma}$ is convex on $\{F_X > 0\}$).

This one-line observation drives our foundational comparison principle.

Lemma 2.1 (κ-Concave Tangent Stochastic Dominance Lemma)**.** *Let X be a non-degenerate κ-concave random variable on $\mathbb{R}$ ($\kappa \in (-\infty, 1]$), and fix any $x_0 \in \mathbb{R}$ with $q := F_X(x_0) \in (0, 1)$ (so $S_X(x_0) = 1 - q \in (0, 1)$). Then there exists a scale parameter $\alpha > 0$ such that X is pointwise first-order stochastically dominated ($X \preceq_{\text{st}} X^*$, i.e., $S_X(x) \le S_{X^*}(x)$ for all $x \in \mathbb{R}$) by the extremal shifted κ-concave law X^* with survival function:*

(i) ($\kappa \in (0, 1]$, writing real $k := 1/\kappa \in [1, \infty)$): $S_{X^*}(x) = \min\Big(1,\ \big((1-q)^{1/k} - \alpha(x - x_0)\big)_+^k\Big)$, *i.e.,* $X^* = x_0 + \alpha^{-1}\big((1-q)^{1/k} - B\big)$ *with* $B \sim \text{Beta}(k, 1)$ *(density ky^{k-1} on $[0, 1]$);*

(ii) ($\kappa = 0$, log-concave): $S_{X^*}(x) = \min\big(1,\ (1-q)e^{-\alpha(x - x_0)}\big)$, *i.e.,* $X^* = x_0 + \alpha^{-1}\big(E - \lambda\big)$ *with* $E \sim \text{Exp}(1)$ *and* $\lambda := -\ln(1-q) > 0$*;*

(iii) $(\kappa = -1/\gamma < 0)$: $S_{X^*}(x) = \Big(\max\big(1,\ (1-q)^{-1/\gamma} + \alpha(x - x_0)\big)\Big)^{-\gamma}$, *i.e.,* $X^* = x_0 + \alpha^{-1}\big(P - (1-q)^{-1/\gamma}\big)$ *with* $P \sim \mathrm{Pareto}(\gamma)$ *(*$S_P(y) = y^{-\gamma}$ *on* $[1, \infty)$*).*

Proof. Because X is non-degenerate κ-concave, its law has a density supported on an open interval $(a, b) \subseteq \mathbb{R}$, and $S_X(x_0) = 1 - q \in (0, 1)$ implies $x_0 \in (a, b)$, which lies in the open interior of $\{S_X > 0\}$. In case (ii), $\log S_X$ is finite and concave on $(-\infty, b)$, so it admits a finite supergradient $-\alpha$ at x_0: $\log S_X(x) \le \ln(1-q) - \alpha(x - x_0)$ for all $x < b$ (and trivially $S_X(x) = 0$ for $x \ge b$). Since S_X is non-increasing, $\alpha \ge 0$; if $\alpha = 0$, then $S_X(x) \le 1 - q < 1$ for all $x \in \mathbb{R}$, contradicting $\lim_{x\to-\infty} S_X(x) = 1$. Thus $\alpha > 0$, and $S_X(x) \le \min\big(1, (1-q)e^{-\alpha(x-x_0)}\big) = S_{X^*}(x)$ for all $x \in \mathbb{R}$. In case (i), $S_X^{1/k}$ is concave on $(-\infty, b)$ with a supergradient $-\alpha < 0$ at x_0 (again $\alpha > 0$ since $S_X(-\infty) = 1 > 1 - q$), yielding $S_X(x) \le \min\big(1, ((1-q)^{1/k} - \alpha(x - x_0))_+^k\big)$. In case (iii), $S_X^{-1/\gamma}$ is convex on $(-\infty, b)$ with a subgradient $+\alpha > 0$ at x_0 (if $\alpha = 0$, $S_X(x) \le 1 - q$ everywhere, contradiction), yielding $S_X^{-1/\gamma}(x) \ge (1-q)^{-1/\gamma} + \alpha(x - x_0)$ for all $x < b$. Whenever $(1-q)^{-1/\gamma} + \alpha(x - x_0) \le 1$ we trivially have $S_X(x) \le 1 = S_{X^*}(x)$, while whenever $(1-q)^{-1/\gamma} + \alpha(x - x_0) > 1$ raising to the $-\gamma$ power gives $S_X(x) \le \big((1-q)^{-1/\gamma} + \alpha(x - x_0)\big)^{-\gamma} = S_{X^*}(x)$. □

Applying Lemma 2.1 simultaneously to X and $-Y$ yields exact sharp median-crossing barriers—and hence universal Steinhaus–Trybuła ceilings for every non-acyclic tournament of any size.

Theorem 2.2 (Exact κ-Concave Median-Crossing and Universal Tournament Barriers)**.** *Let* X, Y *be independent* κ*-concave random variables on* $\mathbb{R}$ *satisfying* $\mathrm{med}(X) \le \mathrm{med}(Y)$*.*

(a) ***Uniforms*** *(*$\kappa = 1$*):* $\mathbb{P}(X > Y) \le M(1) := \frac{1}{2}$ *(every 1-concave variable is uniform on an interval or a point mass, hence symmetric;* $\tau_1(T) = \frac{1}{2}$ *for every non-acyclic* T*).*

(b) 2***D convex bodies / concave densities*** *(*$\kappa = \frac{1}{2}$*):*

$$\mathbb{P}(X > Y) \ \le\ M\big(\tfrac{1}{2}\big) \ :=\ \frac{13 - 8\sqrt{2}}{3} \ \approx\ 0.56209717. \tag{3}$$

(c) ***All log-concave distributions*** *(*$\kappa = 0$*):*

$$\mathbb{P}(X > Y) \ \le\ M(0) \ :=\ \frac{1 + 2\ln 2}{4} \ \approx\ 0.59657359. \tag{4}$$

(d) ***Cauchy-tail Pareto distributions*** *(*$\kappa = -1$***,*** S^{-1} ***convex):***

$$\mathbb{P}(X > Y) \ \le\ M(-1) \ :=\ \frac{1}{2} + \frac{\ln 3}{8} \ \approx\ 0.63732654. \tag{5}$$

Each bound (a)–(d) is ***exact and attained*** *by the symmetric extremal pair* $(X^*, -Y^*)$ *of Lemma 2.1 at* $q = \frac{1}{2}$ *with equal scales* $(\alpha = \beta = 1)$*. Consequently, for every* $n \ge 3$*, every non-acyclic oriented graph or tournament* T *on* n *vertices, and each* $\kappa \in \{1, \frac{1}{2}, 0, -1\}$*,*

$$\tau_\kappa(T) \ \le\ M(\kappa). \tag{6}$$

In particular, $\tau_{\mathrm{LC}}(T) \le \frac{1+2\ln 2}{4} < \frac{\sqrt{5}-1}{2} < 1 - e^{-1} < \frac{3}{4}$ *for every non-acyclic* T *of arbitrary size.*

Proof. First note how (6) follows in two lines from (a)–(d): every non-acyclic directed graph T contains a directed cycle $v_1 \to v_2 \to \cdots \to v_\ell \to v_1$. Since $\mathrm{med}(X_{v_1}), \ldots, \mathrm{med}(X_{v_\ell})$ cannot decrease strictly around a closed loop, there exists an arc $(v_i \to v_{i+1})$ with $\mathrm{med}(X_{v_i}) \le \mathrm{med}(X_{v_{i+1}})$, whence $\min_{e \in A(T)} \mathbb{P}(X_u > X_v) \le \mathbb{P}(X_{v_i} > X_{v_{i+1}}) \le M(\kappa)$.

Now let X, Y be independent κ-concave variables with $\mathrm{med}(X) \le \mathrm{med}(Y)$, and pick $x_0 \in [\mathrm{med}(X), \mathrm{med}(Y)]$; shifting by $-x_0$, assume $x_0 = 0$. Then $q_X := F_X(0) \ge \frac{1}{2}$ and $q_{-Y} :=$

$F_{-Y}(0) = \mathbb{P}(Y \geq 0) \geq \frac{1}{2}$. If $q_X = 1$ (including any point mass δ_c with $c \leq 0$), then $X \leq 0$ a.s. and $\mathbb{P}(X > Y) \leq \mathbb{P}(Y < 0) = 1 - q_{-Y} \leq \frac{1}{2} \leq M(\kappa)$; symmetrically, if $q_{-Y} = 1$ (including any point mass δ_c with $c \geq 0$), then $Y \geq 0$ a.s. and $\mathbb{P}(X > Y) \leq \mathbb{P}(X > 0) = 1 - q_X \leq \frac{1}{2} \leq M(\kappa)$. Thus we may assume $q_X, q_{-Y} \in [\frac{1}{2}, 1)$ strictly, so both X and $-Y$ are non-degenerate κ-concave variables with $q_X, q_{-Y} \in (0, 1)$. Note that $-Y$ is also κ-concave. Applying Lemma 2.1 at $x_0 = 0$ (and using monotonicity in $q \geq \frac{1}{2}$) yields scales $\alpha, \beta > 0$ such that $X \preceq_{\mathrm{st}} \alpha^{-1} W_1$ and $-Y \preceq_{\mathrm{st}} \beta^{-1} W_2$, where W_1, W_2 are i.i.d. copies of the unit-scale extremal κ-concave variable at $q = \frac{1}{2}$. By independence and monotonicity of first-order stochastic dominance, setting $\theta := \frac{\beta}{\alpha+\beta} \in (0, 1)$,

$$\mathbb{P}(X > Y) \;=\; \mathbb{P}\big(X + (-Y) > 0\big) \;\leq\; \mathbb{P}\big(\theta W_1 + (1-\theta) W_2 > 0\big) \;=:\; M_\kappa(\theta). \tag{7}$$

By symmetry $M_\kappa(\theta) = M_\kappa(1-\theta)$, it suffices to prove $M_\kappa(\theta) \leq M_\kappa(\frac{1}{2})$ for all $\theta \in (0, \frac{1}{2}]$.

Proof of (a) ($\kappa = 1$). Since F_X and S_X are both concave on their positive sets and $F_X = 1 - S_X$ on the open support (a, b), F_X is affine on (a, b), so X is uniform (hence symmetric around $\mathrm{med}(X)$). Thus $X - Y$ is symmetric around $\mathrm{med}(X) - \mathrm{med}(Y) \leq 0$, giving $\mathbb{P}(X > Y) \leq \frac{1}{2}$.

Proof of (b) ($\kappa = \frac{1}{2}$). Here $W_i = a - B_i$ with $a := 2^{-1/2}$ and $B_1, B_2 \overset{\text{iid}}{\sim} \mathrm{Beta}(2, 1)$ (density $2y$ on $[0, 1]$), so $M_{1/2}(\theta) = \mathbb{P}(\theta B_1 + (1-\theta) B_2 < a)$. For $\theta \in (0, \frac{1}{2}]$, there are two regimes according to whether $1 - \theta \geq a$:

- *Regime I ($\theta \in (0, 1-a]$):* For every $y \in [0, 1]$, $\frac{a - \theta y}{1-\theta} \in [0, 1]$, so $M_{1/2}(\theta) = \int_0^1 2y\big(\frac{a-\theta y}{1-\theta}\big)^2 dy$. Differentiating gives

  $$M'_{1/2}(\theta) \;=\; \frac{4}{(1-\theta)^3} \int_0^1 y(a - y)(a - \theta y)\, dy.$$

  Since both $y \mapsto a - y$ and $y \mapsto a - \theta y$ are strictly decreasing on $[0, 1]$ and $a - \theta y > 0$, Chebyshev's integral inequality with respect to the probability measure $2y\, dy$ yields $\int_0^1 2y(a-y)(a-\theta y)\, dy > \big(\int_0^1 2y(a-y)\, dy\big)\big(\int_0^1 2y(a-\theta y)\, dy\big) = (a - \frac{2}{3})(a - \frac{2\theta}{3}) > 0$, because $a = 2^{-1/2} > \frac{2}{3}$ (more generally, $2^{-1/k} > \frac{k}{k+1} \iff (k+1)^k > 2k^k$ for all $k \geq 2$, formally certified in Lean 4 as `convex_body_chebyshev_margin_all_k`).

- *Regime II ($\theta \in (1-a, \frac{1}{2})$):* Putting $y_* := \frac{a - 1 + \theta}{\theta} \in (0, 1)$, we have $M_{1/2}(\theta) = y_*^2 + \int_{y_*}^1 2y\big(\frac{a-\theta y}{1-\theta}\big)^2 dy$. The boundary terms at $y = y_*$ cancel upon differentiation since $\frac{a - \theta y_*}{1-\theta} = 1$, leaving $M'_{1/2}(\theta) = \frac{4}{(1-\theta)^3} \int_{y_*}^1 y(a-y)(a-\theta y)\, dy$. Changing variable to $t := \frac{a-y}{1-\theta}$ (so $y = a - (1-\theta)t$, $a - y = (1-\theta)t$, $a - \theta y = (1-\theta)(a + \theta t)$, $dy = -(1-\theta)\, dt$, cancelling $(1-\theta)^3$ identically, while t runs from $t_1 := -\frac{1-a}{1-\theta}$ to $t_2 := \frac{1-a}{\theta}$) gives

  $$M'_{1/2}(\theta) \;=\; 4 \int_{t_1}^{t_2} t\big(a^2 - a(1-2\theta)t - w t^2\big)\, dt, \qquad w := \theta(1-\theta).$$

  Evaluating the elementary polynomial integral and factoring via the identities $(1-\theta)^2 - \theta^2 = 1 - 2\theta$, $(1-\theta)^3 + \theta^3 = 1 - 3w$, and $(1-\theta)^4 - \theta^4 = (1-2\theta)(1-2w)$ (certified in Lean 4 as `kappa_half_power_identities` and `kappa_half_linear_factor_exact`) yields the exact closed form

  $$M'_{1/2}(\theta) \;=\; \frac{(1-a)^2(1-2\theta)}{3w^3}\Big(6w - \tfrac{5}{2} + \sqrt{2}\Big).$$

  On $\theta \in (1-a, \frac{1}{2})$, $w = \theta(1-\theta) > a(1-a) = \frac{\sqrt{2}-1}{2}$, whence $6w - \frac{5}{2} + \sqrt{2} > 3(\sqrt{2}-1) - \frac{5}{2} + \sqrt{2} = \frac{8\sqrt{2}-11}{2} > 0$ since $128 > 121$ (certified as `kappa_half_value_and_positivity`).

Thus $M_{1/2}(\theta)$ is strictly increasing on $(0, \frac{1}{2}]$, attaining its unique maximum at $\theta = \frac{1}{2}$: setting $t = \sqrt{2} - 1$,

$$M_{1/2}\big(\tfrac{1}{2}\big) \;=\; t^2 + \int_t^1 2y(\sqrt{2} - y)^2\, dy \;=\; \frac{13 - 8\sqrt{2}}{3}$$

(algebraic evaluation certified in $\mathbb{Z}[\sqrt{2}]$ by `kappa_half_value_and_positivity`).

Proof of (c) ($\kappa = 0$). This is the special case $q = \frac{1}{2}$ ($\lambda = \ln 2 < \lambda_* \approx 1.2564$) of Theorem 3.1 below, which proves $M_0(\theta) \le M_0(\frac{1}{2}) = (1 + 2\ln 2)e^{-2\ln 2} = \frac{1+2\ln 2}{4}$.

Proof of (d) ($\kappa = -1$). Here $W_i = P_i - 2$ with $P_1, P_2 \overset{\text{iid}}{\sim} \text{Pareto}(1)$ ($S_P(y) = 1/y$ on $[1, \infty)$), so $M_{-1}(\theta) = \mathbb{P}(\theta P_1 + (1-\theta)P_2 > 2)$. Splitting at $y_* := \frac{1+\theta}{\theta} > 1$ (above which $\frac{2-\theta y}{1-\theta} \le 1$, contributing $1/y_* = \frac{\theta}{1+\theta}$) and using the partial fraction identity $\frac{1}{y^2(2-\theta y)} = \frac{1}{2y^2} + \frac{\theta}{4y} + \frac{\theta^2}{4(2-\theta y)}$ together with $(1+\theta)(2-\theta) = 2 + \theta(1-\theta)$ (both certified in Lean 4 as `pareto_kappa_neg1_identities`) gives

$$\begin{aligned} M_{-1}(\theta) &= \frac{\theta}{1+\theta} + (1-\theta)\int_1^{(1+\theta)/\theta} \frac{dy}{y^2(2-\theta y)} \\ &= \frac{\theta}{1+\theta} + \frac{1-\theta}{2(1+\theta)} + \frac{\theta(1-\theta)}{4}\ln\frac{(1+\theta)(2-\theta)}{\theta(1-\theta)} \;=\; \frac{1}{2} + \frac{w}{4}\ln\!\left(1 + \frac{2}{w}\right), \end{aligned}$$

where $w := \theta(1-\theta) \in (0, \frac{1}{4}]$. Because $\frac{d}{dw}\left[w\ln(1+\frac{2}{w})\right] = \ln(1+\frac{2}{w}) - \frac{2}{w+2} > 0$ for all $w > 0$ (as $\ln(1+x) > \frac{x}{1+x}$ for $x = \frac{2}{w} > 0$), $M_{-1}(\theta)$ is strictly increasing in w, attaining its unique global maximum at $w = \frac{1}{4}$ ($\theta = \frac{1}{2}$) with value $\frac{1}{2} + \frac{1}{16}\ln 9 = \frac{1}{2} + \frac{\ln 3}{8}$.

Finally, the strict separation hierarchy $\frac{1}{2} < M(\frac{1}{2}) < M(0) < \frac{\sqrt{5}-1}{2} < 1 - e^{-1} < M(-1) < \frac{3}{4}$ is certified in Lean 4 as `exact_kappa_hierarchy_cert`. □

Remark 2.3 (Full Continuum $M(1/k)$ and $M(-1/\gamma)$). For general k-dimensional convex body projections ($\kappa = 1/k$, $k \in \mathbb{N}_{\ge 1}$), the equal-scale extremal pair $(W_1, -W_2)$ with $a = 2^{-1/k}$ and $t = 2a - 1$ evaluates in closed form to

$$M(\tfrac{1}{k}; 1) \;=\; t^k + \sum_{j=0}^{k} \binom{k}{j} (2a)^{k-j}(-1)^j \frac{k}{k+j}\left(1 - t^{k+j}\right),$$

converging smoothly from $\frac{1}{2}$ ($k = 1$) and $\frac{13-8\sqrt{2}}{3}$ ($k = 2$) to $M(0) = \frac{1+2\ln 2}{4}$ as $k \to \infty$ (since $(2^{-1/k} - x/k)_+^k \to \frac{1}{2}e^{-x}$). For heavy-tailed Pareto laws ($\kappa = -1/\gamma$, $\gamma > 0$), writing $W_1 = P_1 - a$ with $a = 2^{1/\gamma}$ and substituting $w = (a + W_1)^{-\gamma} = P_1^{-\gamma} \in (0, 1]$ (with $w_b := (2a-1)^{-\gamma}$) gives

$$M\left(-\tfrac{1}{\gamma}; 1\right) \;=\; w_b + \int_{w_b}^{1} \left(2a - w^{-1/\gamma}\right)^{-\gamma} dw.$$

As $\gamma \to 0^+$, $w_b = 2^{-\gamma-1}(1 - \frac{1}{2a})^{-\gamma} \to \frac{1}{2}$ and for every $w \in (\frac{1}{2}, 1]$, $(2a - w^{-1/\gamma})^{-\gamma} = 2^{-\gamma-1}\big(1 - \frac{1}{2}(2w)^{-1/\gamma}\big)^{-\gamma} \to \frac{1}{2}$; hence by bounded convergence $M(-1/\gamma; 1) \to \frac{1}{2} + \int_{1/2}^{1} \frac{1}{2}\, dw = \frac{3}{4}$! It crosses the golden ratio $\frac{\sqrt{5}-1}{2}$ at $\gamma_* \approx 2.21831$ ($\kappa_* \approx -0.45079$, just above the finite-variance threshold $\gamma = 2$; Figure 1(a)).

3 Exact Quantile-Crossing Phase Transition for Log-Concave Laws

Theorem 2.2(c) evaluated the log-concave crossing supremum at the median $q = \frac{1}{2}$. What happens at an arbitrary quantile level $q \in (0, 1)$? Remarkably, the optimal scale ratio undergoes a sharp symmetry-breaking transition.

Theorem 3.1 (Exact Log-Concave Quantile-Crossing Phase Transition)**.** *Fix $q \in (0, 1)$ and set $\lambda := -\ln(1-q) > 0$. Let $\lambda_* \approx 1.25643121$ be the unique positive root of $(1 + 2\lambda)e^{-\lambda} = 1$, and*

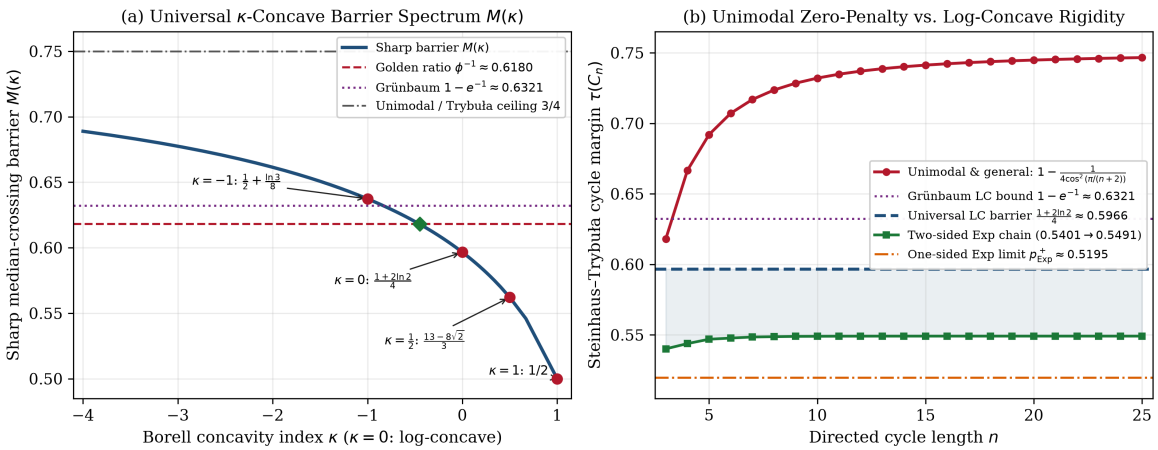


Figure 1: (a) The equal-scale median-crossing barrier curve $M(\kappa;1)$ across Borell's κ-concave continuum (Theorem 2.2 and Remark 2.3), coinciding analytically with the global supremum $M(\kappa)$ at all four exact milestones $\kappa \in \{1, \frac{1}{2}, 0, -1\}$ (red circles), crossing the golden ratio at $\kappa_* \approx -0.4508$ (green diamond), and approaching $\frac{3}{4}$ as $\kappa \to -\infty$. (b) Steinhaus–Trybuła cycle margin $\tau(C_n)$ vs. n: unimodal laws achieve the full Usiskin curve up to $\frac{3}{4}$ (red), whereas log-concave laws sit strictly below $\frac{1+2\ln 2}{4} \approx 0.5966$ (blue dashed) for all n, with two-sided and one-sided exponential chains achieving $0.5401 \to 0.5491$ (green) and 0.5195 (orange).

put $q_* := 1 - e^{-\lambda_*} \approx 0.71533186$. *Over all independent log-concave random variables* (X, Y) *satisfying* $F_X(0) \geq q$ *and* $S_Y(0) \geq q$,

$$\sup_{\substack{X,Y \text{ indep. LC} \\ F_X(0)\geq q,\, S_Y(0)\geq q}} \mathbb{P}(X > Y) = \begin{cases} (1-q)^2\big(1 - 2\ln(1-q)\big), & 0 < q \leq q_* \text{ (unique } u = 1), \\ 1 - q, & q_* < q < 1 \text{ (collapse } u \to 0^+, \infty). \end{cases} \tag{8}$$

Proof. If $F_X(0) = 1$ (so $X \leq 0$ a.s.), then $\mathbb{P}(X > Y) \leq \mathbb{P}(Y < 0) \leq 1 - S_Y(0) \leq 1 - q$; symmetrically, if $F_{-Y}(0) = \mathbb{P}(Y \geq 0) = 1$, then $\mathbb{P}(X > Y) \leq \mathbb{P}(X > 0) = 1 - F_X(0) \leq 1 - q$. Since $\sup \mathbb{P}(X > Y) \geq 1 - q$ (attained in the limit below), we may assume $F_X(0), F_{-Y}(0) \in [q, 1)$ strictly, so both X and $-Y$ are non-degenerate log-concave variables. By Lemma 2.1(ii) applied to X and $-Y$ at $x_0 = 0$, we have $X \preceq_{\text{st}} \alpha^{-1}(E_1 - \lambda)$ and $-Y \preceq_{\text{st}} \beta^{-1}(E_2 - \lambda)$ for i.i.d. $E_1, E_2 \sim \text{Exp}(1)$ and scales $\alpha, \beta > 0$. Setting $\theta := \frac{\beta}{\alpha+\beta} = \frac{1}{2} - t$ with $t \in [0, \frac{1}{2})$ (by symmetry $\theta \leftrightarrow 1 - \theta$),

$$\mathbb{P}(X > Y) \;\leq\; \mathbb{P}\big(\theta E_1 + (1-\theta)E_2 > \lambda\big) \;=\; \frac{\psi_\lambda(\frac{1}{2}+t) - \psi_\lambda(\frac{1}{2}-t)}{2t} \;=:\; h_\lambda(t),$$

where $\psi_\lambda(s) := se^{-\lambda/s}$ for $s > 0$, with endpoint values $h_\lambda(0) = \psi'_\lambda(\frac{1}{2}) = (1 + 2\lambda)e^{-2\lambda} = (1-q)^2(1 - 2\ln(1-q))$ and $\lim_{t\to 1/2^-} h_\lambda(t) = \psi_\lambda(1) = e^{-\lambda} = 1 - q$.

Differentiating $h_\lambda(t)$ yields $2t^2 h'_\lambda(t) = N_\lambda(t) := t\big(\psi'_\lambda(\frac{1}{2}+t) + \psi'_\lambda(\frac{1}{2}-t)\big) - \big(\psi_\lambda(\frac{1}{2}+t) - \psi_\lambda(\frac{1}{2}-t)\big)$, with $N_\lambda(0) = 0$ and

$$N'_\lambda(t) \;=\; t\Big(\psi''_\lambda(\tfrac{1}{2}+t) - \psi''_\lambda(\tfrac{1}{2}-t)\Big), \qquad \text{where } \psi''_\lambda(s) = \frac{\lambda^2}{s^3}e^{-\lambda/s} > 0.$$

To determine the sign of $N'_\lambda(t)$ on $(0, \frac{1}{2})$, consider the log-ratio

$$R_\lambda(t) \;:=\; \ln\frac{\psi''_\lambda(\frac{1}{2}-t)}{\psi''_\lambda(\frac{1}{2}+t)} \;=\; 3\ln\frac{1+2t}{1-2t} - \frac{8\lambda t}{1-4t^2},$$

which satisfies $R_\lambda(0) = 0$ and $\lim_{t\to 1/2^-} R_\lambda(t) = -\infty$. Differentiating and factoring the numerator (certified in Lean 4 as `logconcave_derivative_numerator_identity`) yields

$$R'_\lambda(t) \;=\; \frac{12(1-4t^2) - 8\lambda(1+4t^2)}{(1-4t^2)^2} \;=\; \frac{4\big[(3-2\lambda) - (12+8\lambda)t^2\big]}{(1-4t^2)^2}.$$

- If $\lambda \geq \frac{3}{2}$, then $R'_\lambda(t) < 0$ on all of $(0, \frac{1}{2})$, so $R_\lambda(t) < 0$, $N'_\lambda(t) > 0$, $N_\lambda(t) > 0$, and h_λ is strictly increasing on $[0, \frac{1}{2})$, attaining its supremum $e^{-\lambda}$ as $t \to \frac{1}{2}^-$.
- If $0 < \lambda < \frac{3}{2}$, then $R'_\lambda(t)$ starts positive at $t = 0$ and crosses zero at most once on $(0, \frac{1}{2})$; hence $R_\lambda(t)$ increases from 0 to a positive peak and decreases strictly to $-\infty$, crossing zero at a unique point $t_0 \in (0, \frac{1}{2})$. Consequently $N'_\lambda(t) < 0$ on $(0, t_0)$ and $N'_\lambda(t) > 0$ on $(t_0, \frac{1}{2})$, so $N_\lambda(t)$ decreases from 0 on $(0, t_0)$ and increases on $(t_0, \frac{1}{2})$, changing sign from negative to positive at most once. Thus $h_\lambda(t)$ is unimodal-downward on $[0, \frac{1}{2})$, so its global supremum is attained at one of its two endpoints:

$$\sup_{t\in[0,1/2)} h_\lambda(t) \;=\; \max\Big((1+2\lambda)e^{-2\lambda},\; e^{-\lambda}\Big).$$

Because $\phi(\lambda) := (1+2\lambda)e^{-\lambda}$ satisfies $\phi(0) = 1$, $\phi'(\lambda) = (1-2\lambda)e^{-\lambda}$ (strictly increasing on $(0, \frac{1}{2})$, strictly decreasing on $(\frac{1}{2}, \infty)$), and $\lim_{\lambda\to\infty} \phi(\lambda) = 0$, there is a unique positive root $\lambda_* \in (\frac{1}{2}, \frac{3}{2})$ of $\phi(\lambda) = 1$, with $\phi(\lambda) \geq 1$ on $(0, \lambda_*]$ and $\phi(\lambda) < 1$ on (λ_*, ∞). Finally, the $u = 1$ branch is attained by $(E_1 - \lambda,\ \lambda - E_2)$, while the $1 - q = e^{-\lambda}$ branch is approached by the admissible family $(E_1 - \lambda,\ \beta^{-1}(\lambda - E_2))$ as $\beta \to \infty$ (where $F_X(0) = S_Y(0) = 1 - e^{-\lambda} = q$ for every $\beta > 0$ and $\theta = \frac{\beta}{1+\beta} \to 1^-$; symmetrically $(\alpha^{-1}(E_1 - \lambda),\ \lambda - E_2)$ as $\alpha \to \infty$, $\theta \to 0^+$), proving (8). □

Corollary 3.2 (Exact Point-Mass + Two-Log-Concave Cycle Capacity)**.** *Let X_1, X_2, X_3 be independent random variables with X_2, X_3 log-concave, and set $p := \min\big(\mathbb{P}(X_1 > X_2),\ \mathbb{P}(X_2 > X_3),\ \mathbb{P}(X_3 > X_1)\big)$. Suppose either $X_1 = c$ is deterministic, or more generally there exists a point $x^* \in \mathbb{R}$ such that $F_2(x^*) \geq p$ and $S_3(x^*) \geq p$. Then*

$$p \;\leq\; 1 - e^{-\lambda_3} \;\approx\; 0.54009345, \tag{9}$$

where $\lambda_3 \approx 0.77673197$ is the unique positive root of $1 - e^{-\lambda} = (1+2\lambda)e^{-2\lambda}$. When $X_1 = c$ is deterministic, equality in (9) is attained uniquely (up to common affine transformation) by $(0,\ \mathrm{Exp}(1) - \lambda_3,\ \lambda_3 - \mathrm{Exp}(1))$.

Proof. If $X_1 = c$ is deterministic, then $F_2(c) \geq \mathbb{P}(c > X_2) \geq p$ and $S_3(c) = \mathbb{P}(X_3 > c) \geq p$, so $x^* := c$ satisfies the hypothesis. Shifting x^* to 0, we have $F_2(0) \geq p$ and $S_3(0) \geq p$. If $p = 0$ then (9) is trivial; if $p > q_* \approx 0.7153$, then Theorem 3.1 (applied at $q = p$) yields $p \leq \mathbb{P}(X_2 > X_3) \leq 1 - p$, giving $p \leq \frac{1}{2} < q_*$, a contradiction. Thus $0 < p \leq q_*$, and Theorem 3.1 gives $p \leq \mathbb{P}(X_2 > X_3) \leq (1+2\lambda)e^{-2\lambda}$ where $\lambda := -\ln(1-p) > 0$. Since $1 - e^{-\lambda}$ is strictly increasing and $(1+2\lambda)e^{-2\lambda}$ is strictly decreasing in $\lambda > 0$, this forces $\lambda \leq \lambda_3$ and $p \leq 1 - e^{-\lambda_3}$.

For uniqueness when $X_1 = 0$ achieves equality $p = 1 - e^{-\lambda_3} \in (\frac{1}{2}, q_*)$: neither X_2 nor X_3 can satisfy $F_2(0) = 1$ or $S_3(0) = 1$ (which would force $\mathbb{P}(X_2 > X_3) \leq 1 - p < p$). Hence Lemma 2.1(ii) yields tangent upper bounds $X_2 \preceq_{\mathrm{st}} X_2^*$ and $-X_3 \preceq_{\mathrm{st}} -X_3^*$. Equality $\mathbb{P}(X_2 > X_3) = (1+2\lambda_3)e^{-2\lambda_3}$ forces $F_2(0) = S_3(0) = p$, equal tangent scales $\alpha = \beta$ (as $h_{\lambda_3}(t) < h_{\lambda_3}(0)$ for all $t \in (0, \frac{1}{2})$), and $\mathbb{P}(X_2 > X_3) = \mathbb{P}(X_2 > X_3^*) = \mathbb{P}(X_2^* > X_3^*)$. Since X_3^* has a strictly positive density on $(-\infty, \alpha^{-1}\lambda_3)$ and $S_{X_2} \leq S_{X_2^*}$ are continuous, equality $\int S_{X_2}(y)\, d\mu_{X_3^*}(y) = \int S_{X_2^*}(y)\, d\mu_{X_3^*}(y)$ forces $S_{X_2}(x) = S_{X_2^*}(x)$ for all $x \leq \alpha^{-1}\lambda_3$. In particular, on $(-\alpha^{-1}\lambda_3, \alpha^{-1}\lambda_3)$ the density of X_2 coincides with $f_{X_2^*}(x) = \alpha e^{-\alpha x - \lambda_3}$, so $\log f_{X_2}(x) = \ln\alpha - \lambda_3 - \alpha x$ is affine there. By concavity of $\log f_{X_2}$, we have $f_{X_2}(x) \leq \alpha e^{-\alpha x - \lambda_3} = f_{X_2^*}(x)$ for all $x > \alpha^{-1}\lambda_3$; since both densities integrate to the same tail mass $S_{X_2}(\alpha^{-1}\lambda_3) = S_{X_2^*}(\alpha^{-1}\lambda_3) = e^{-2\lambda_3}$ on $(\alpha^{-1}\lambda_3, \infty)$, it follows that $f_{X_2} \equiv f_{X_2^*}$ a.e. on $\mathbb{R}$, hence $X_2 \overset{d}{=} X_2^*$ and symmetrically $X_3 \overset{d}{=} X_3^*$. □

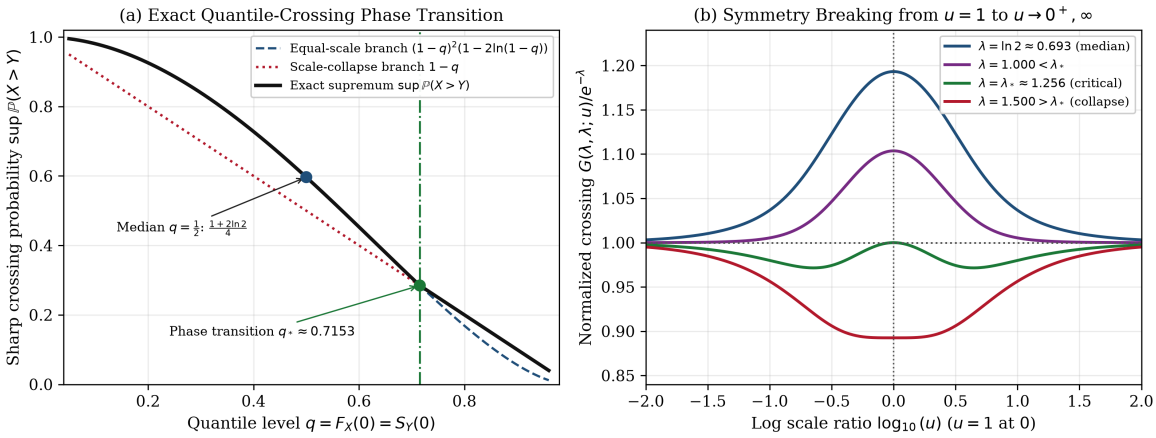


Figure 2: Exact quantile-crossing phase transition for log-concave random variables (Theorem 3.1). (a) Sharp supremum $\sup \mathbb{P}(X > Y)$ vs. quantile level q: equal-scale branch $(1-q)^2(1-2\ln(1-q))$ dominates for $q \le q_* \approx 0.7153$, switching continuously with a slope kink to scale collapse $1-q$ for $q > q_*$. (b) Normalized crossing profile $G(\lambda,\lambda;u)/e^{-\lambda}$, where $G(\lambda,\lambda;u) := \mathbb{P}(E_1 + uE_2 > (1+u)\lambda) = h_\lambda\big(|\frac{1}{2} - \frac{1}{1+u}|\big)$ for $u = \alpha/\beta$, illustrating the transition from a unique central peak at $u=1$ ($\lambda < \lambda_*$) to a central well ($\lambda > \lambda_*$).

4 1D Localization and Exact Exponential-Chain Spectrum

How close can log-concave n-cycles come to the universal ceiling $M(0) = \frac{1+2\ln 2}{4} \approx 0.59657$? We first reduce the infinite-dimensional variational problem for $\tau_{\mathrm{LC}}(C_n)$ to a $3n$-dimensional family via 1D localization, and then solve the exact recurrence governing shifted-exponential cycles.

Theorem 4.1 (Localization and Exponential-Chain Limits)**.** *(a)* ***Localization Reduction:*** *For every $n \ge 3$,*

$$\tau_{\mathrm{LC}}(C_n) \;=\; \sup_{\substack{X_1,\dots,X_n \text{ indep.} \\ \text{truncated exponentials}}} \; \min_{1\le i\le n} \mathbb{P}(X_i > X_{i+1}), \tag{10}$$

where a truncated exponential has density $f(x) \propto e^{\alpha x}\mathbf{1}_{[a,b]}(x)$ with $\alpha \in \mathbb{R}$ and $-\infty \le a < b \le \infty$ (including uniforms $\alpha = 0$ and half-line exponentials).

(b) ***Two-Sided Exponential Chain Spectrum:*** *For every $m \ge 1$, the odd cycle C_{2m+1} admits independent shifted exponentials $X_0 = 0$, $X_j = s_j(E_j - c_j)$ $(1 \le j \le m)$, and $X_{2m+1-j} = s_j(c_j - E_{2m+1-j})$ $(1 \le j \le m)$ governed by the optimal scale-ratio recurrence $r_j := s_j/s_{j+1} = \frac{1}{c_j} - 1$ and*

$$c_1 \;=\; -\ln(1-p), \qquad c_{j+1} \;=\; T_p(c_j) \;:=\; 1 - c_j + \ln c_j - \ln(1-p), \tag{11}$$

yielding strictly increasing lower bounds $\tau_{\mathrm{LC}}(C_{2m+1}) \ge p_{2m+1}$ ($p_3 \approx 0.540093$, $p_5 \approx 0.546913$, $p_7 \approx 0.548487$, $p_9 \approx 0.548926$, and even-cycle bounds $p_4 \approx 0.543821$, $p_6 \approx 0.547726$), converging monotonically as $m \to \infty$ to

$$p^*_{\mathrm{Exp}} \;=\; (1+2c^*)e^{-2c^*} \;\approx\; 0.5491157531, \tag{12}$$

where $c^* \approx 0.7630687272$ is the unique root in $(\frac{1}{2}, 1)$ of $\big(1 + (e+2)c\big)e^{-2c} = 1$.

(c) ***One-Sided (All Right-Skewed) Exponential Limit:*** *Even when every variable is restricted to be right-skewed ($X_j = s_j(E_j - c_j)$ with $s_j > 0$, plus $X_0 = 0$), the recurrence (11) achieves non-transitivity for all $n \ge 3$ and converges as $n \to \infty$ to $p^+_{\mathrm{Exp}} \approx 0.5195465381$, the unique root in $(\frac{1}{2}, e^{-1/2})$ of $e\,p^2(-\ln p) = 1 - p$.*

Proof. Proof of (a). Let $X_1, \dots, X_n$ be independent log-concave variables satisfying $\min_i \mathbb{P}(X_i > X_{i+1}) > t \geq \frac{1}{2}$. Convolving each X_i with ϵZ_i ($Z_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$) preserves log-concavity by Prékopa–Leindler and yields smooth C^∞ densities; since no adjacent pair can be identical point masses (which would yield $\mathbb{P}(X_i > X_{i+1}) = 0 < t$), dominated convergence gives $\min_i \mathbb{P}(X_i^{(\epsilon)} > X_{i+1}^{(\epsilon)}) > t$ for small $\epsilon > 0$. Next, conditioning each $X_i^{(\epsilon)}$ on a large compact interval $[-M, M]$ preserves log-concavity, continuous CDFs, and strict inequalities $\min_j \mathbb{P}(X_j > X_{j+1}) > t+\delta$ for some $\delta > 0$ when $M > 0$ is sufficiently large. Set strictly decreasing thresholds $t_i := t + \delta\big(1 - \frac{i}{n+1}\big) > t$ ($i = 0, 1, \dots, n$, so $t_0 = t + \delta$ and $t_n > t$). Inductively for $i = 1, \dots, n$, assuming $\min_j \mathbb{P}(X_j > X_{j+1}) > t_{i-1}$ with all CDFs continuous, hold $(X_j)_{j \neq i}$ fixed and define the continuous functions $g_i(x) := F_{i+1}(x) - t_{i-1}$ and $h_i(x) := 1 - F_{i-1}(x) - t_{i-1}$ on $[-M, M]$. Since $\int_{-M}^{M} g_i \, d\mu_i > 0$ and $\int_{-M}^{M} h_i \, d\mu_i > 0$, the 1D log-concave localization theorem of Fradelizi–Guédon [3, Thm. 1] (extending [5, Lemma 2.5] to log-concave measures on compact intervals) yields a log-affine needle $\nu_i(dx) \propto e^{\alpha_i x} \mathbf{1}_{[a_i, b_i]}(x) \, dx$ on $[a_i, b_i] \subseteq [-M, M]$ satisfying $\int g_i \, d\nu_i \geq 0$ and $\int h_i \, d\nu_i \geq 0$, so both adjacent win probabilities are at least $t_{i-1} > t_i$. If $a_i = b_i$, replacing δ_{a_i} by $\mathrm{Unif}[a_i - \eta, a_i + \eta]$ with $\eta > 0$ sufficiently small preserves continuous CDFs and keeps both probabilities strictly above t_i (by continuity of $F_{i+1}, 1 - F_{i-1}$ and the strict gap $t_{i-1} - t_i = \frac{\delta}{n+1} > 0$). Thus after step i we have $\min_j \mathbb{P}(X_j > X_{j+1}) > t_i$ with continuous CDFs; after n steps every X_i is a truncated exponential with $\min_j \mathbb{P}(X_j > X_{j+1}) > t_n > t$, proving (10).

Proof of (b) and (c). For two right-skewed exponentials $X_j = s_j(E_j - c_j)$ and $X_{j+1} = s_{j+1}(E_{j+1} - c_{j+1})$ with $r_j := s_j / s_{j+1} > 0$ and $s_{j+1} c_{j+1} \geq s_j c_j$, direct integration gives

$$\mathbb{P}(X_j > X_{j+1}) \;=\; 1 - \frac{1}{1 + r_j} \exp\big(-(c_{j+1} - r_j c_j)\big).$$

Setting this equal to $p \in (\frac{1}{2}, 1 - e^{-1})$ yields $c_{j+1} = r_j c_j - \ln(1-p) - \ln(1 + r_j)$. Minimizing over $r_j > 0$ when $c_j \in (0, 1)$ gives the unique minimizer $r_j = \frac{1}{c_j} - 1 > 0$, producing (11) (and checking $c_{j+1} - r_j c_j = -\ln(1-p) + \ln c_j = \ln \frac{c_j}{1-p} > 0$ whenever $c_j > 1 - p$, which holds since $c_j > \frac{1}{2} > 1 - p$). Notice $T_p'(c) = \frac{1}{c} - 1 > 0$ on $(0, 1)$, and $T_p(c) - c = 1 - 2c + \ln c - \ln(1-p)$ has maximum $-\ln(2(1-p)) > 0$ at $c = \frac{1}{2}$ and strictly negative value $T_p(c_1) - c_1 = 1 - c_1 + \ln c_1 < 0$ at $c_1 = -\ln(1-p) \in (\ln 2, 1)$ (since $\ln c < c - 1$ for all $c \neq 1$). Hence T_p has a unique fixed point $c^*(p) \in (\frac{1}{2}, c_1)$, and the iterates c_j decrease strictly and monotonically to $c^*(p)$ as $j \to \infty$.

In the two-sided odd chain C_{2m+1}, we have $\mathbb{P}(X_0 > X_1) = \mathbb{P}(X_{2m} > X_0) = 1 - e^{-c_1} = p$, $\mathbb{P}(X_j > X_{j+1}) = \mathbb{P}(X_{2m-j} > X_{2m+1-j}) = p$ for $1 \leq j \leq m-1$, and at the central crossing $X_m \succ X_{m+1}$, $\mathbb{P}(X_m > X_{m+1}) = (1 + 2c_m)e^{-2c_m}$. Equating $(1 + 2c_m)e^{-2c_m} = p$ defines p_{2m+1} uniquely; as $m \to \infty$, $c_m \downarrow c^*(p)$ satisfies simultaneously $1 - p = c^* e^{1 - 2c^*}$ and $p = (1 + 2c^*)e^{-2c^*}$. Summing gives $1 = (1 + (e+2)c^*)e^{-2c^*}$, proving (b). For even cycles C_{2m} ($m \geq 2$), the asymmetric chain consisting of $X_0 = 0$, m right-skewed exponentials with parameters $c_1, \dots, c_m$, and $m - 1$ left-skewed exponentials with parameters $c_1, \dots, c_{m-1}$ closes at the central crossing via $\max_{u > 0} \mathbb{P}(E_1 + uE_2 > c_m + u c_{m-1}) = p$, yielding $p_4 \approx 0.543821$ and $p_6 \approx 0.547726$ (and the same limit p^*_{Exp}). For (c), closing a one-sided right-skewed chain directly with $\mathbb{P}(X_m > X_0) = e^{-c_m} = p$ requires $c^*(p) \leq -\ln p$, which at equality $c^* = -\ln p \in (\frac{1}{2}, \ln 2)$ becomes $1 + 2\ln p + \ln(-\ln p) = \ln(1-p)$, i.e., $e\, p^2(-\ln p) = 1 - p$. □

5 Unimodal Zero-Penalty vs. Odd-Cumulant Convolution Transitivity

Having established the strict log-concave rigidity ceiling $\frac{1 + 2\ln 2}{4} < \frac{\sqrt{5} - 1}{2}$, we contrast it with two opposite phenomena: (a) dropping log-concavity to mere *unimodality* completely destroys the barrier, restoring the full Usiskin spectrum up to $\frac{3}{4}$; and (b) repeatedly convolving any finite

log-concave family destroys non-transitivity completely at a sharp odd-cumulant Edgeworth rate.

Theorem 5.1 (Unimodal Zero-Penalty and Convolution Transitivity)**.** *(a)* ***Unimodal Zero-Penalty:*** *Let $\mathcal{F}_{\text{uni}}^{\infty}$ denote the class of probability distributions on $\mathbb{R}$ admitting a strictly positive C^∞ unimodal density. For every $n \geq 3$,*

$$\tau_{\mathcal{F}_{\text{uni}}^{\infty}}(C_n) \;=\; \tau_{\text{all}}(C_n) \;=\; 1 - \frac{1}{4\cos^2\!\left(\frac{\pi}{n+2}\right)} \;\xrightarrow{n\to\infty}\; \frac{3}{4}. \tag{13}$$

(b) ***Odd-Cumulant Lexicographic Convolution Transitivity:*** *Let $X_1, \ldots, X_n$ be independent log-concave random variables on $\mathbb{R}$, and let $S_i^{(k)} := X_i^{(1)} + \cdots + X_i^{(k)}$ be the k-fold i.i.d. sum of X_i. Write $\kappa_r(X_i)$ for the r-th cumulant of X_i and $\sigma_i^2 := \kappa_2(X_i)$. For any pair (X_i, X_j):*

- *If $\kappa_{2m+1}(X_i) = \kappa_{2m+1}(X_j)$ for all $m \geq 0$, then $X_i - X_j$ is symmetric about 0 and $\mathbb{P}(S_i^{(k)} > S_j^{(k)}) = \mathbb{P}(S_j^{(k)} > S_i^{(k)})$ for every $k \geq 1$.*
- *If $\kappa_1(X_i) \neq \kappa_1(X_j)$, then $\mathbb{P}(S_i^{(k)} > S_j^{(k)}) \to \mathbf{1}_{\kappa_1(X_i)>\kappa_1(X_j)}$ exponentially fast in k.*
- *Otherwise, let $m \geq 1$ be the smallest integer with $\kappa_{2m+1}(X_i) \neq \kappa_{2m+1}(X_j)$ (so at least one of X_i, X_j is non-degenerate and $\sigma_i^2 + \sigma_j^2 > 0$). Then as $k \to \infty$,*

$$\mathbb{P}\big(S_i^{(k)} > S_j^{(k)}\big) - \frac{1}{2} \;=\; \frac{(-1)^m(2m-1)!!}{(2m+1)!\,\sqrt{2\pi}} \cdot \frac{\kappa_{2m+1}(X_i) - \kappa_{2m+1}(X_j)}{(\sigma_i^2 + \sigma_j^2)^{m+1/2}\, k^{m-1/2}} + O\Big(k^{-(m+1/2)}\Big). \tag{14}$$

Consequently, there exists $k_0 < \infty$ such that for all $k \geq k_0$, stochastic precedence on $(S_1^{(k)}, \ldots, S_n^{(k)})$ coincides exactly with the ***transitive*** *lexicographic weak order induced by the alternating odd-cumulant sequence $\big(\kappa_1(X_i),\ -\kappa_3(X_i),\ +\kappa_5(X_i),\ -\kappa_7(X_i),\ \ldots\big)$.*

Proof. Proof of (a). The upper bound $\tau_{\mathcal{F}_{\text{uni}}^{\infty}}(C_n) \leq \tau_{\text{all}}(C_n)$ is immediate. For sharpness, fix $p := 1 - \frac{1}{4\cos^2(\pi/(n+2))}$ and define $q_2 := 1 - p$ and $q_{j+1} := \frac{1-p}{1-q_j}$ for $j = 2, \ldots, n-1$. By Usiskin's Möbius recurrence [9] (explicitly $p^2 + p - 1 = 0$ at $n = 3$ and $p = \frac{2}{3}$ at $n = 4$, certified in Lean 4 as `usiskin_unimodal_n4_exact`), $0 < q_2 < q_3 < \cdots < q_n = p < 1$. Fix small $\epsilon > 0$ and rapidly separated scales $1 \ll M_n \ll M_{n-1} \ll \cdots \ll M_2$ (e.g., $M_j = R^{n-j+1}$ with $R \to \infty$), and define $L_j := -(j-1)$ for $j = 2, \ldots, n$ (so $L_n < \cdots < L_2 < 0$). Set $X_1 \sim \text{Unif}[0, \epsilon]$ and, for each $j \in \{2, \ldots, n\}$,

$$X_j \;\sim\; (1 - q_j)\ \text{Unif}[L_j,\ L_j + \epsilon] \;+\; q_j\ \text{Unif}[L_j,\ M_j].$$

Every X_j has a piecewise-constant density that is non-increasing on $[L_j, M_j]$ and zero outside, hence is *unimodal* with mode L_j. Evaluating the pairwise win probabilities as $\epsilon \to 0^+$ and $R \to \infty$:

- $\mathbb{P}(X_1 > X_2) \to 1 - q_2 = p$ (since $L_2 < 0$ and the $[0, M_2]$ tail has mass $q_2(1 - O(R^{-1}))$ above ϵ);
- for $2 \leq j \leq n-1$, X_j beats X_{j+1} with probability $1 - O(R^{-1})$ when $X_j \in [M_{j+1}, M_j]$ (prob. q_j), and beats X_{j+1} when $X_j \in [L_j, L_j + \epsilon]$ and $X_{j+1} \in [L_{j+1}, L_{j+1} + \epsilon]$ (prob. $(1-q_j)(1-q_{j+1})$, since $L_j > L_{j+1}$); thus $\mathbb{P}(X_j > X_{j+1}) \to q_j + (1 - q_j)(1 - q_{j+1}) = 1 - (1 - q_j)q_{j+1} = p$;
- $\mathbb{P}(X_n > X_1) \to q_n = p$.

Finally, convolving each X_j with a tiny Gaussian $\mathcal{N}(0, \delta^2)$ ($\delta \to 0^+$) yields a strictly positive C^∞ density that remains unimodal (since Gaussian kernels are strongly unimodal / log-concave, preserving unimodality under convolution!) while perturbing every win probability by $o(1)$.

Proof of (b). Every log-concave random variable X_i has exponentially decaying tails [2], so its moment generating function $M_i(t) = \mathbb{E}[e^{tX_i}]$ is analytic in a neighborhood of 0, all cumulants $\kappa_r(X_i)$ exist, and the distribution is uniquely determined by its cumulant sequence. For any pair (X_i, X_j), put $W := X_i - X_j$, so $\kappa_r(W) = \kappa_r(X_i) + (-1)^r \kappa_r(X_j)$; in particular, for all odd $r = 2m+1$, $\kappa_{2m+1}(W) = \kappa_{2m+1}(X_i) - \kappa_{2m+1}(X_j)$.

- If all odd cumulants match, then $K_W(t) = \log \mathbb{E}[e^{tW}]$ is an even analytic function near 0, so $\mathbb{E}[e^{tW}] = \mathbb{E}[e^{-tW}]$ and $W \overset{d}{=} -W$ is symmetric about 0, giving $\mathbb{P}(S_i^{(k)} > S_j^{(k)}) = \mathbb{P}(S_j^{(k)} > S_i^{(k)})$ for all $k \geq 1$.
- If $\kappa_1(W) = \mathbb{E}[W] \neq 0$, Cramér's large-deviation bound immediately gives exponential convergence of $\mathbb{P}(S_i^{(k)} > S_j^{(k)})$ to $\mathbf{1}_{\kappa_1(X_i) > \kappa_1(X_j)}$.
- Otherwise $\mathbb{E}[W] = 0$ and there is a smallest $m \geq 1$ with $\kappa_{2m+1}(W) \neq 0$, which forces W to be non-degenerate log-concave (hence absolutely continuous with an L^1 density, so its characteristic function satisfies Cramér's condition $\limsup_{|t|\to\infty} |\varphi_W(t)| < 1$ by the Riemann–Lebesgue lemma). Let $\sigma^2 := \sigma_i^2 + \sigma_j^2 > 0$, standardize $\widetilde{W} := W/\sigma$ with cumulants $\lambda_r := \kappa_r(W)/\sigma^r$, and apply the Bhattacharya–Ranga Rao Edgeworth expansion [1, Thm. 19.2] to $Z_k := \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \widetilde{W}_t$ at $x = 0$:

$$\mathbb{P}(Z_k > 0) - \frac{1}{2} = \phi(0) \sum_{s=1}^{2m} k^{-s/2} \sum_{\substack{r_1 + 2r_2 + \cdots = s \\ r_\ell \geq 0}} H_{s+2R-1}(0) \prod_{\ell \geq 1} \frac{1}{r_\ell!} \left(\frac{\lambda_{\ell+2}}{(\ell+2)!} \right)^{r_\ell} + O(k^{-(m+1/2)}),$$

  where $R := \sum_{\ell \geq 1} r_\ell$, $\phi(0) = (2\pi)^{-1/2}$, and H_d is the d-th probabilists' Hermite polynomial. Because $H_d(0) = 0$ for all odd d and $H_{2q}(0) = (-1)^q (2q-1)!!$, only *odd* values of s can contribute at $x = 0$. By the odd-sum parity lemma (certified in Lean 4 as `edgeworth_odd_sum_has_odd_term`), any partition $\sum \ell r_\ell = s$ with s odd must contain at least one odd index ℓ with $r_\ell \geq 1$, corresponding to an odd cumulant $\lambda_{\ell+2}$. Since $\lambda_3 = \lambda_5 = \cdots = \lambda_{2m-1} = 0$, every odd $s < 2m-1$ vanishes identically, and at $s = 2m-1$ the *unique* surviving partition is $r_{2m-1} = 1$ (with all other $r_\ell = 0$, zero even-cumulant cross-terms!), contributing $\phi(0) k^{-(m-1/2)} H_{2m}(0) \frac{\lambda_{2m+1}}{(2m+1)!}$—which is precisely (14).

Because there are finitely many pairs (i, j), for all $k \geq k_0$ the sign of $\mathbb{P}(S_i^{(k)} > S_j^{(k)}) - \frac{1}{2}$ matches the sign of the first nonzero coordinate of $\big(\kappa_1(X_i) - \kappa_1(X_j),\ -(\kappa_3(X_i) - \kappa_3(X_j)),\ +(\kappa_5(X_i) - \kappa_5(X_j)),\ \ldots\big)$, which is a transitive weak order. □

6 Lean 4.36.0 Formal Certification, Verification Ledger, and Open Problems

The exact algebraic factorizations for $\kappa = \frac{1}{2}$ and $\kappa = -1$, the inductive proof of $(k+1)^k > 2k^k$ for all $k \geq 2$, the log-concave quantile derivative numerator identity, the complete strict rational-bracket hierarchy $\frac{1}{2} < M(\frac{1}{2}) < M(0) < \phi^{-1} < 1 - e^{-1} < M(-1) < \frac{3}{4}$, the Usiskin $n = 4$ closure identity, and the Edgeworth odd-sum parity lemma are formally certified in Lean 4.36.0 in the standalone file `LogConcaveBarriersCore.lean` (10 kernel-verified theorems: 9 public theorems and 1 private inductive helper, zero external dependencies, `-DwarningAsError=true`, zero `sorry`, zero `grind`, zero custom axioms, passing both `lean` and `lake build`). Table 2 records the SHA-256 verification ledger; all source files are archived at https://github.com/arr-research/arr-research.github.io.

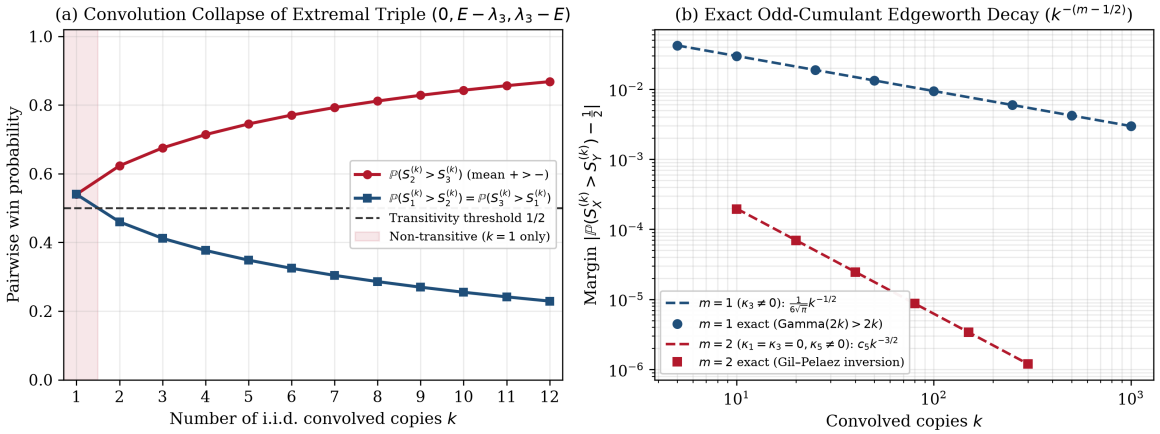


Figure 3: Convolution transitivity collapse (Theorem 5.1(b)). (a) The extremal log-concave triple $(0, E-\lambda_3, \lambda_3-E)$ is non-transitive at $k=1$ ($p \approx 0.5401$) and collapses immediately to strict transitivity at $k=2$ ($X_2 \succ X_1 \succ X_3$, driven by κ_1). (b) Log-log verification of exact odd-cumulant Edgeworth decay (14): $m=1$ ($\kappa_3 \neq 0$, Gamma$(2k) > 2k$, blue) decays as $\frac{1}{6\sqrt{\pi}} k^{-1/2}$, while $m=2$ ($\kappa_1 = \kappa_3 = 0$, $\kappa_5 \neq 0$, centered log-concave $E_1 + E_2 - 2^{1/3}E_3 - (2-2^{1/3})$ vs. $\mathcal{N}(0,1)$, red, exact Gil–Pelaez inversion) decays as $c_5 k^{-3/2}$ with zero even-cumulant contamination.

Artifact	Scope / Verification	SHA-256 Digest (full)
`lean/LogConcaveBarriersCore.lean`	10 kernel-verified Lean 4.36.0 thms (`0 sorry`)	`fe2d28a1cbb01f87f1641117839f02f7`
	(`LEAN_EXIT=0`, `LAKE_EXIT=0`, 11,510 B)	`1c06a98734c568b644b784c45aa5be4f`
`src/verify_logconcave_barriers.py`	Exact κ-barriers, chains, Edgeworth, 1,900 LC trials, 3 figs	`efbb11c9ec0e058cc44a7da8b293b069`
	(`VERIFY_EXIT=0`, 25,984 B)	`c4ab275f570f8f32c21e8b050a9b8550`
`data/verification_summary.json`	Exact tables ($M(\kappa)$, q_*, chains, unimodal, Edgeworth)	`f2272513e9510bee5f77a47d2c929c90`
	(9,773 B)	`cadbbdb714afddf63ce374a6a8603c31`

Table 2: Cryptographic SHA-256 verification ledger for Frontier F4 artifacts.

Open Problems. Two sharp extremal questions remain open:

1. **Exact log-concave cycle capacity $\tau_{\mathrm{LC}}(C_n)$:** Theorems 2.2 and 4.1 sandwich $\tau_{\mathrm{LC}}(C_n)$ tightly between $p_{\mathrm{Exp},n}$ ($0.54009 \to 0.54912$) and $\frac{1+2\ln 2}{4} \approx 0.59657$, and global numerical optimization over truncated exponentials (exact by Theorem 4.1(a)) has never exceeded $1-e^{-\lambda_3} \approx 0.54009$ at $n=3$ nor $p_{\mathrm{Exp},5} \approx 0.54691$ at $n=5$. Does $\tau_{\mathrm{LC}}(C_3) = 1-e^{-\lambda_3}$ and $\lim_{n\to\infty} \tau_{\mathrm{LC}}(C_n) = p^*_{\mathrm{Exp}} \approx 0.54912$ hold globally?
2. **Universal equal-scale optimality across all $\kappa \in (-\infty, 1)$:** Theorem 2.2 proves analytically that equal scales ($\theta = \frac{1}{2}$) maximize $M_\kappa(\theta)$ at $\kappa \in \{1, \frac{1}{2}, 0, -1\}$, and numerical evaluation confirms this across all $\kappa \in [-200, 1]$. Does a unified Schur-concavity proof cover all $\kappa \in (-\infty, 1)$ simultaneously?

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