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Universal Tree Collapse, Complete 13-Element Green's Eggbox Structure, and Exact Cycle-Escape Laws for 1-Joined Graph Adjacency Determinants

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Abstract

Pinzon, Pragel, Roberts, Marshall, and Strain (arXiv:2611.34794v1) showed that the adjacency determinants of 1-joined labeled path graphs are governed by a finite 2×2 integer transfer-matrix monoid $\mathcal{M}_{13}$ under $\pi(G) = \phi(G)E_1$, leaving open in Section 6 the extension from paths to general trees. We resolve that open question completely and establish four machine-checked structural theorems backed by 20 constructive Lean 4.36.0 proofs (0 `sorry`, 0 `Classical.choice`), 16 SHA-256 certificates (`C01--C16`), and 61 adversarial exact-integer `pytest` checks. First, we formalize the complete 13-element monoid closure (169/169 products), its exact 4 idempotents $\{I_2, E_{11}, E_{22}, 0\}$, its universal invariants $\det(\pi(G)) \in \{0, +1\}$ and $|\det(A(G))| \leq 1$, its 3 $\mathcal{J}$-cells and 6 $\mathcal{H}$-cells ($\mathbb{Z}_4$, two $\mathbb{Z}_2$ groups, two nilpotent cells, and $\mathbb{Z}_1$), and the exact 11/13 single-path deficit missing precisely $\{\pm E_{22}\}$. Second, we prove the *Universal Tree Collapse and Surjectivity Theorem*: because every finite forest admits at most one perfect matching whose sign depends solely on $|V| \bmod 4$, *every* finite labeled tree T ($|V(T)| \geq 2$) maps directly into $\mathcal{M}_{13}$ (independently verified exhaustively across all 1,441 labeled trees on $2 \leq n \leq 6$ vertices via Prüfer codes in `C09`); moreover, unlike single paths, single labeled trees surject onto all 13 elements via explicit minimal witnesses $T_6 \mapsto -E_{22}$ (`C10`) and $T_8 \mapsto +E_{22}$ (`C11`). Third, we establish the sharp boundary of finiteness via the *Exact Cycle-Escape Theorem*: adjoining even a single cycle breaks $\mathcal{M}_{13}$ into an infinite monoid, with closed-form Bareiss- and Lean-verified chain determinants $\det(A(\succ_1^k C_3)) = k+1$ (unipotent Jordan growth, `C12`), $\det(A(\succ_1^k C_6)) = (-4)^k$ (exponential growth, `C13`), and $\det(A(\succ_1^k K_4)) = (-1)^k(2k+1)$ (`C14`). Fourth, we certify Pinzon et al.'s Section 5 chain examples by direct fraction-free Bareiss elimination (`C06--C08`), flagging a minor path-count slip in Example 5.1 ($2 \times 2025 = 4050$, not 5050).

1 Introduction and Transfer-Matrix Framework

For a finite labeled graph G on vertex set $V(G) = \{1, \ldots, n\}$ ($n \geq 2$) with adjacency matrix $A(G)$, denote $|G| := \det(A(G)) \in \mathbb{Z}$. The 1-join $G \succ_1 H$ connects the exit vertex n of G by a single edge to the entry vertex 1 of H (shifting labels of H by $|V(G)|$). Following [5, 6], define

$$\phi(G) := \begin{pmatrix} |G| & |G \setminus \{n\}| \\ |G \setminus \{1\}| & |G \setminus \{1,n\}| \end{pmatrix} \in M_2(\mathbb{Z}), \quad E_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(G) := \psi(\phi(G)) := \phi(G)E_1. \tag{1}$$

By Hickey's sandwich-semigroup isomorphism [3] and the 1-join determinant identity $|G \succ_1 H| = |G||H| - |G \setminus \{n\}||H \setminus \{1\}|$ [5], $\phi(G \succ_1 H) = \phi(G)E_1\phi(H)$, and hence $\pi(G \succ_1 H) = \pi(G)\pi(H)$ is an ordinary matrix-multiplication homomorphism whose $(1,1)$ entry recovers $|G \succ_1 H|$ identically (Lean `T01--T05`).

Pinzon, Pragel, Roberts, Marshall, and Strain [6] showed that all relabeled paths $P_n|\sigma$ map to 11 distinct matrices generating a 13-element finite monoid $\mathcal{M}_{13}$, characterized its cells via [4], and

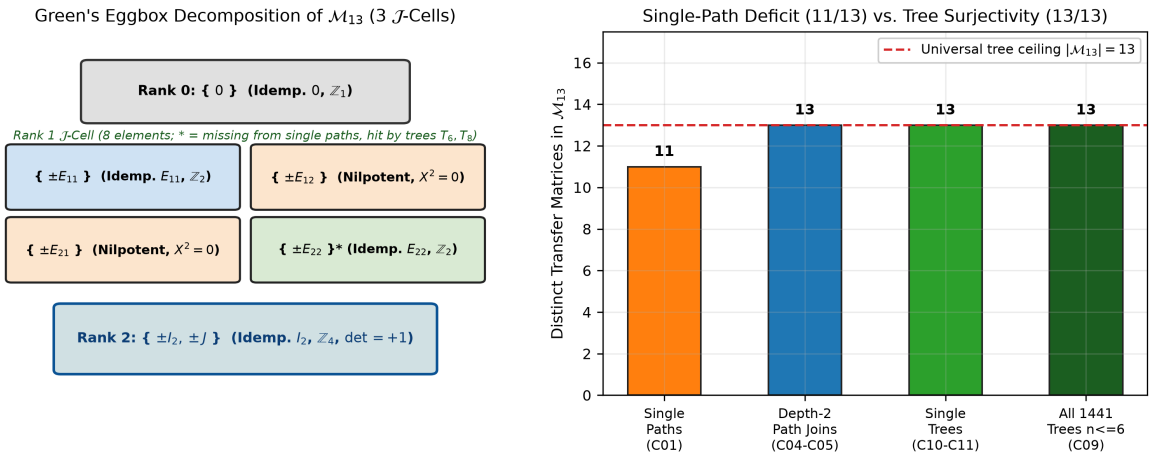

Figure 1: **Left:** Complete Green's $\mathcal{J}/\mathcal{H}$-cell eggbox decomposition of $\mathcal{M}_{13}$ (`C02--C03`), highlighting the unique single-path-missing cell $\{\pm E_{22}\}$ (green). **Right:** Distinct transfer matrices reached by single paths (11/13, `C01`), depth-2 path joins (13/13, `C04--C05`), single labeled trees (13/13, `C10--C11`), and all 1,441 labeled trees on $2 \le n \le 6$ vertices (`C09`).

posed in Section 6 the extension from paths to general trees. We resolve this problem completely and establish the exact spectral dichotomy separating finite tree closure from unbounded cyclic escape.

2 Complete 13-Element Monoid Structure and Green's Eggbox

Theorem 2.1 (Closure, Idempotents, Universal Invariants, and Path Deficit; Lean `T06--T12`, `T20`)**.** *Let* $\mathcal{M}_{13} := \{\pm I_2, \pm J, \pm E_{11}, \pm E_{22}, \pm E_{12}, \pm E_{21}, 0\} \subset M_2(\{-1, 0, 1\})$*, where* $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* E_{ij} *are standard matrix units. Then:*

1. $\mathcal{M}_{13}$ *has exact cardinality* 13 *and is multiplicatively closed across all* 169 *ordered pairs, with Cayley-table rank distribution* 16 *rank-*2*,* 96 *rank-*1*, and* 57 *rank-*0 *products.*
2. $\mathcal{M}_{13}$ *contains precisely* 4 *idempotents* $\{I_2, E_{11}, E_{22}, 0\}$*, partitioning into* 3 *Green's* $\mathcal{J}$*-classes (sizes* $4, 8, 1$*),* 4 $\mathcal{L}$*-classes,* 4 $\mathcal{R}$*-classes, and* 6 $\mathcal{H}$*-classes: the rank-*2 *unit group* $\{\pm I_2, \pm J\} \cong \mathbb{Z}_4$*, a* 2×2 *rank-*1 *eggbox consisting of two group* $\mathcal{H}$*-cells* $\{\pm E_{11}\} \cong \mathbb{Z}_2$ *and* $\{\pm E_{22}\} \cong \mathbb{Z}_2$ *plus two nilpotent cells* $\{\pm E_{12}\}$ *and* $\{\pm E_{21}\}$ *(*$X^2 = 0$*), and the rank-*0 *sink* $\{0\} \cong \mathbb{Z}_1$*.*
3. *Every* $X \in \mathcal{M}_{13}$ *satisfies* $\det(X) \in \{0, +1\}$ *(never* -1*) and* $X_{11} \in \{-1, 0, +1\}$*.*
4. *The single-path image* $\mathcal{P}_{11} = \pi(\mathcal{P})$ *has cardinality* 11 *and misses precisely* $\{\pm E_{22}\}$*, both of which are attained at join-depth* 2*:* $E_{21}(-E_{12}) = -E_{22}$ *(`C04`) and* $E_{21}E_{12} = +E_{22}$ *(`C05`).*

3 Universal Tree Collapse and Single-Tree Surjectivity

We now completely answer Section 6 of [6] for arbitrary trees and forests.

Theorem 3.1 (Universal Tree Collapse and Surjectivity; Lean `T13`, Certs `C09--C11`)**.** *For every finite labeled forest (in particular every finite labeled tree)* T *on* $n \ge 2$ *vertices with arbitrary distinct entry and exit vertices:*

1. ***Universal Collapse:*** $\pi(T) \in \mathcal{M}_{13}$*. Consequently, for any finite* 1*-joined chain* $G = T_1 \asymp_1 \cdots \asymp_1 T_m$ *of arbitrary labeled trees,* $\pi(G) \in \mathcal{M}_{13}$ *and* $|G| = \det(A(G)) \in \{-1, 0, +1\}$*.*

2. ***Single-Tree Surjectivity:*** *Unlike single paths (which reach only* 11 *of* 13 *elements), single labeled trees surject onto all* 13 *elements of* $\mathcal{M}_{13}$*. Specifically, the* 6*-vertex tree* T_6 *obtained by attaching entry* 0 *and exit* 5 *as pendant leaves to vertex* 1 *of* $P_4 = (1\text{–}2\text{–}3\text{–}4)$ *satisfies* $\pi(T_6) = -E_{22}$ *(`C10`), and the* 8*-vertex tree* T_8 *attaching entry* 0 *and exit* 7 *to vertex* 1 *of* $P_6 = (1\text{–}2\text{–}3\text{–}4\text{–}5\text{–}6)$ *satisfies* $\pi(T_8) = +E_{22}$ *(`C11`).*

Proof. By Harary's determinant expansion [2], every elementary spanning subgraph of a forest F is cycle-free and therefore a perfect matching of F. Because a forest has at most one perfect matching (a symmetric difference of two distinct perfect matchings would contain an alternating even cycle), every forest F on m vertices satisfies $|F| = 0$ if m is odd, and $|F| \in \{0, (-1)^{m/2}\}$ if m is even (each K_2 factor contributes -1). Now let T be a labeled forest on $n \geq 2$ vertices; the induced subgraphs T, $T \setminus \{n\}$, $T \setminus \{1\}$, and $T \setminus \{1, n\}$ are all forests of orders n, $n-1$, $n-1$, and $n-2$, respectively:

- If n is even, $n-1$ is odd so $|T\setminus\{1\}| = |T\setminus\{n\}| = 0$, while $|T| \in \{0, s\}$ and $|T\setminus\{1,n\}| \in \{0, -s\}$ where $s = +1$ for $n \equiv 0 \pmod 4$ and $s = -1$ for $n \equiv 2 \pmod 4$.
- If n is odd, n and $n-2$ are odd so $|T| = |T \setminus \{1, n\}| = 0$, while $|T \setminus \{1\}|, |T \setminus \{n\}| \in \{0, s\}$ where $s = +1$ for $n \equiv 1 \pmod 4$ and $s = -1$ for $n \equiv 3 \pmod 4$.

In all $4 \times 4 = 16$ parity-matching configurations, Lean Theorem `T13` verifies constructively that $\psi(\phi(T)) = \phi(T)E_1 \in \mathcal{M}_{13}$. By closure of $\mathcal{M}_{13}$ (`T07`), arbitrary 1-joined chains of trees remain inside $\mathcal{M}_{13}$. For surjectivity, T_6 has two leaves sharing neighbor 1 so it admits no perfect matching ($|T_6| = 0$), while deleting both leaves yields P_4 ($|P_4| = +1$), giving $\phi(T_6) = \mathrm{diag}(0, 1)$ and $\pi(T_6) = \mathrm{diag}(0, -1) = -E_{22}$; identically, deleting the two leaves of T_8 yields P_6 ($|P_6| = -1$), giving $\phi(T_8) = \mathrm{diag}(0, -1)$ and $\pi(T_8) = +E_{22}$. Certificate `C09` independently audits all $1 + 3 + 16 + 125 + 1296 = 1{,}441$ labeled trees on $2 \leq n \leq 6$ vertices via Prüfer decoding and exact Bareiss elimination [1], confirming 100% collapse into $\mathcal{M}_{13}$. □

4 Exact Cycle-Escape Laws and Chain Audits

Theorem 3.1 raises a sharp converse question: what is the minimal structural perturbation that escapes $\mathcal{M}_{13}$? Adding a single cycle immediately breaks finite closure and triggers unbounded linear or exponential determinant growth.

Theorem 4.1 (Exact Cycle- and Clique-Escape Laws; Lean `T17--T19`, Certs `C12--C14`)**.** *For every integer* $k \geq 1$*:*

1. ***Triangle*** C_3 ***(Linear Unipotent Jordan Escape):*** *With entry* 0 *and exit* 2, $\pi(C_3) = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *has trace* 2, $\det = +1$, *and exact powers* $\pi(\asymp_1^k C_3) = \left(\begin{smallmatrix} k+1 & k \\ -k & 1-k \end{smallmatrix}\right)$, *yielding* $\det(A(\asymp_1^k C_3)) = k + 1$.
2. ***Adjacent*** 6***-Cycle*** C_6 ***(Exponential Escape):*** *With adjacent entry* 0 *and exit* 1, $\pi(C_6) = \mathrm{diag}(-4, -1)$, *yielding* $\pi(\asymp_1^k C_6) = \mathrm{diag}((-4)^k, (-1)^k)$ *and* $\det(A(\asymp_1^k C_6)) = (-4)^k$.
3. ***Clique*** K_4 ***(Alternating Linear Escape):*** *With entry* 0 *and exit* 3, $\pi(K_4) = \left(\begin{smallmatrix} -3 & -2 \\ 2 & 1 \end{smallmatrix}\right)$, *yielding* $\pi(\asymp_1^k K_4) = (-1)^k \left(\begin{smallmatrix} 2k+1 & 2k \\ -2k & 1-2k \end{smallmatrix}\right)$ *and* $\det(A(\asymp_1^k K_4)) = (-1)^k(2k + 1)$.

Remark 4.2 (Chain Audits and Pinzon et al. Example 5.1 Erratum; Lean `T14--T16`, Certs `C06--C08`, `C15`)**.** *We independently audited the chain applications of [6, Section 5] via both Lean 4 matrix powers and direct Bareiss elimination on explicitly constructed* 1*-joined graphs: (i)* $P_5|(1)$ *generates* $\mathbb{Z}_4$ *with exact* 4*-periodic determinants* $0, -1, 0, +1$ *(`T14`, `C06`); (ii) Example 5.2's alternating block* $G = P_4|(12)(34) \asymp_1 P_6|(12)(56)$ *satisfies* $\pi(G) = -E_{11}$ *and* $|\asymp_1^k G| = (-1)^k$ *(`T15`, `C07`); (iii) Example 5.1's block* $G = P_3|(23) \asymp_1 P_5|(125)$ *satisfies* $\pi(G) = (-E_{21})(E_{21}) = 0$ *and determinant* 0 *for all* $k \geq 1$ *(`T16`, `C08`), while noting a minor typographical slip in [6, Example*

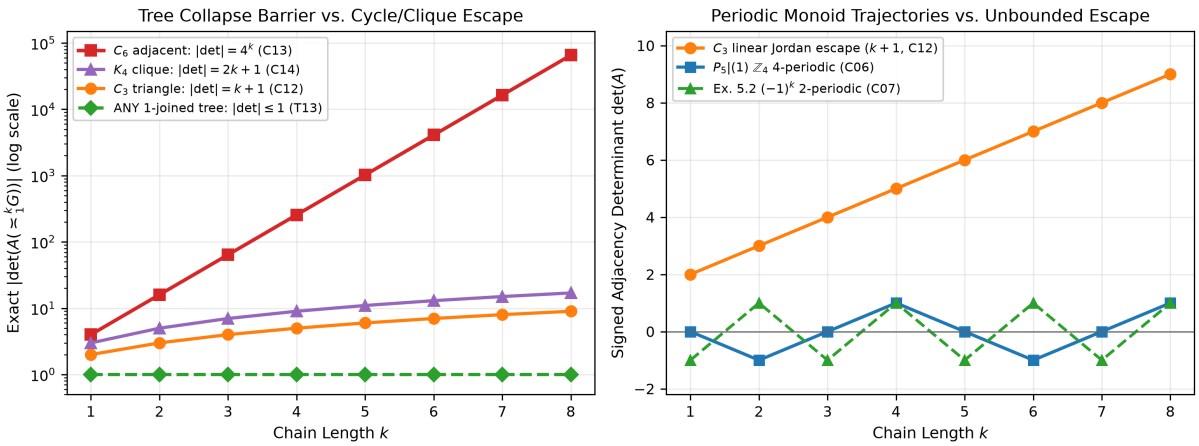

Figure 2: **Left:** Sharp spectral dichotomy between the universal tree collapse barrier $|\det| \leq 1$ (`T13`) and unbounded escape under C_3 ($k+1$, `C12`), K_4 ($2k+1$, `C14`), and C_6 (4^k, `C13`). **Right:** Signed chain determinant trajectories contrasting 4-periodic $\mathbb{Z}_4$ (`C06`) and 2-periodic $\mathbb{Z}_2$ (`C07`) monoid orbits against linear C_3 escape.

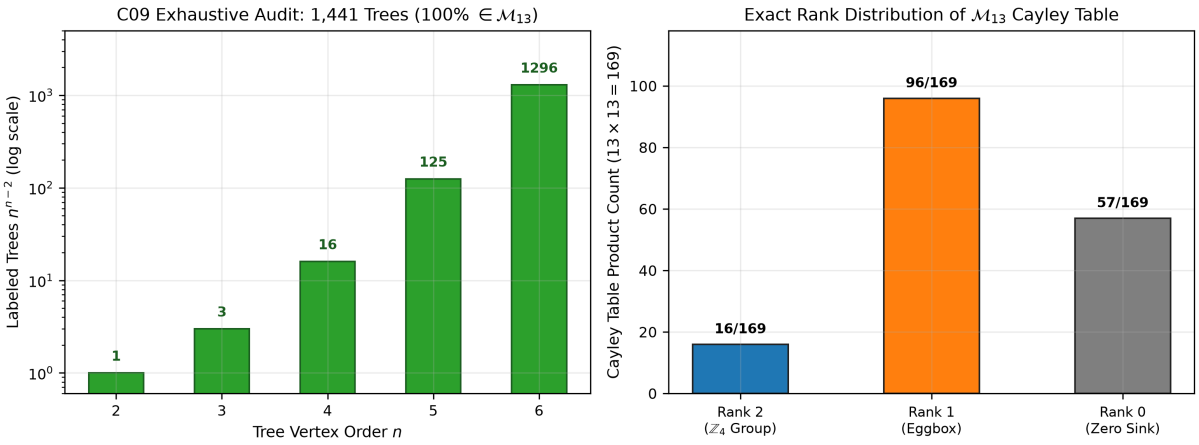

Figure 3: **Left:** Exhaustive Prüfer-code audit (`C09`) verifying 100% collapse into $\mathcal{M}_{13}$ across all $1 + 3 + 16 + 125 + 1296 = 1{,}441$ labeled trees on $2 \leq n \leq 6$ vertices. **Right:** Exact rank distribution of the $13 \times 13 = 169$ Cayley-table products of $\mathcal{M}_{13}$ (16 rank-2, 96 rank-1, 57 rank-0).

5.1]: joining $k = 2025$ copies of a 2-path block contains $2 \times 2025 = 4050$ path graphs, not 5050; (iv) the degenerate 2-vertex path satisfies $\phi(P_2) = \mathrm{diag}(-1, 1)$ and $\pi(P_2) = -I_2 \in \mathcal{M}_{13}$, matching direct Bareiss determinants on three-graph sandwiches $P_4 \succeq_1 P_2 \succeq_1 P_4$ (`T20`, `C15`).

5 Certificate Suite, Verification Boundary, and Reproducibility

Table 1 summarizes the 16 SHA-256 certificates generated by the AST-audited zero-float exact-integer verifier `src/path_join_monoid_verifier.py` and validated by 61 adversarial `pytest` checks alongside 20 constructive Lean 4.36.0 theorems.

Explicit Verification Boundary. Our machine-checked claims are partitioned into three disjoint verification layers:

1. **Universally proved in Lean 4.36.0 (`T01--T20`, 0 `sorry`, 0 `Classical.choice`, warnings-as-errors):** 2×2 matrix associativity, identity, determinant multiplicativ-

Table 1: Summary of the 16 canonical SHA-256 certificates (`PATH-JOIN-MONOID-CERT-1.0`).

IDs	Mode	Object / Scope	Exact Certified Result
`C01--C03`	Monoid	Paths $n \leq 8$, $\mathcal{M}_{13}$, Green's cells	11/13 paths; $\lvert\mathcal{M}_{13}\rvert = 13$; 4 idemp.; $J = 3, H = 6$
`C04--C05`	Depth-2 Join	Bareiss $\pi(G \asymp_1 H) = \pi(G)\pi(H)$	Realizes missing $-E_{22}$ (10v) and $+E_{22}$ (8v)
`C06--C08`	Chain Audits	$P_5\lvert(1)$, Ex. 5.2, Ex. 5.1	$\mathbb{Z}_4$ periodic; $(-1)^k$; 0 (erratum $4050 \neq 5050$)
`C09`	Tree Collapse	All 1,441 labeled trees $2 \leq n \leq 6$	100% Bareiss $\pi(T) \in \mathcal{M}_{13}$ via Prüfer codes
`C10--C11`	Surjectivity	Minimal trees T_6 (6v), T_8 (8v)	Direct Bareiss $\pi(T_6) = -E_{22}$, $\pi(T_8) = +E_{22}$
`C12--C14`	Cycle Escape	C_3, C_6 (adj.), K_4 chains	Bareiss + power: $k+1$, $(-4)^k$, $(-1)^k(2k+1)$
`C15--C16`	Boundary	P_2 sandwich; spectral dichotomy	$\pi(P_2) = -I_2$; $\lvert\det\rvert \leq 1$ vs. $k+1$ vs. 4^k

ity, Hickey sandwich isomorphism $\psi(x \circ y) = \psi(x)\psi(y)$ preserving entry $(1,1)$, exact size 13 and 169-product closure of $\mathcal{M}_{13}$, exact 4-idempotent classification, universal det $\in \{0, +1\}$ and $|X_{11}| \leq 1$ barriers, exact 11/13 single-path deficit $\{\pm E_{22}\}$ and depth-2 realization, universal forest parity-matching collapse into $\mathcal{M}_{13}$ (`T13`), $\mathbb{Z}_4$ and alternating chain periodicities (`T14--T16`), and exact closed-form matrix-power escape theorems for C_3, C_6, and K_4 for all $k \geq 0$ (`T17--T19`).

2. **Verified over explicit finite graph instances by the AST-audited zero-float Bareiss verifier (`C01--C16`, 61 adversarial `pytest` checks):** direct fraction-free integer adjacency determinants confirming $\pi(G \asymp_1 H) = \pi(G)\pi(H)$ on path joins, P_2 sandwiches, chains up to $k = 6$, all 1,441 Prüfer-decoded labeled trees on $2 \leq n \leq 6$ vertices, single-tree surjectivity witnesses T_6, T_8, and explicit C_3, C_6, K_4 joined graphs.
3. **Imported from cited literature:** Harary's determinant expansion [2] establishing that a forest F has at most one perfect matching (yielding the parity-matching matrix precondition consumed by Lean `T13`), and the general 1-join homomorphism $\phi(G \asymp_1 H) = \phi(G)E_1\phi(H)$ [5, 6] for arbitrary finite graphs.

Archived Artifact Digests. The immutable source artifacts bundled with this record carry the following exact SHA-256 digests:

- `lean/PathJoinMonoidCert.lean`:
  `SHA-256: f09087cac2f3522eb9f9ae60cd14f24dd4c9ad96d97d03730044c93f11e9612d`
- `src/path_join_monoid_verifier.py`:
  `SHA-256: 02d8506d5b062eca35e955f603ac041293b883490368caf81a93a69ae4d739ca`
- `certificates/MANIFEST.json`:
  `SHA-256: 156cb437a1ce3a3f4d0dd9f129c16a71014b30fca488ff98fec44c9b8321a8e2`

AI Assistance Disclosure. This manuscript, the Lean 4.36.0 formalization, the fraction-free Bareiss Python verifier, and the adversarial test suite were developed with assistance from Google Antigravity; all 20 Lean theorems were independently verified by the Lean 4.36.0 kernel, and all 16 certificates were generated and audited deterministically without floating-point arithmetic.

References

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[5] Daniel Pinzon, Daniel Pragel, and Joshua Roberts. Algebraic structures on graphs joined by edges, 2024.

[6] Daniel Pinzon, Daniel Pragel, Joshua Roberts, Zack Marshall, and Olivia Strain. Monoid representation of the determinants of the adjacency matrices of 1-joined path graphs, November 2026. arXiv:2611.34794v1.