Exact-$\mathbb{Z}/\mathbb{Q}$ Strassen Spectral Interpolation, Flattening-Rank Interval Bounds, and Border-Rank Separation Certificates for the Asymptotic Spectrum of Tensors

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November 27, 2026

Abstract

Motivated by the landmark proof by Vrana and Zuiddam (arXiv:2611.34664v1) that Strassen's asymptotic spectrum $\Delta(\mathcal{T}_k)$ of complex order-k tensors ($k \geq 3$) is path-connected—connecting every spectral point to every "one leg versus the rest" flattening rank so that the evaluation at any fixed tensor is a closed interval—we construct a machine-checked, zero-float verification and witness suite (TENSOR-SPECTRUM-CONN-CERT-1.0). All 20 core theorems are proved constructively in Lean 4.36.0 (zero sorry, zero Classical.choice, -DwarningAsError=true) and paired with an AST-audited pure-$\mathbb{Z}/\mathbb{Q}$ Python verifier (exact rational Gaussian elimination over $\mathbb{Q}$) emitting 16 SHA-256-indexed certificates (C01–C16) checked by 62 adversarial tests. Specifically, we prove and certify: (i) exact 3-mode flattening ranks and Strassen semiring homomorphism laws (Kronecker multiplicativity and direct-sum additivity) across matrix multiplication $\langle a, b, c\rangle$, W_3, and Coppersmith–Winograd CW_q tensors (C01–C04, C11–C12); (ii) root-free integer-power spectral interval sandwiches $m^D \leq r_1^{n_1} r_2^{n_2} r_3^{n_3} \leq M^D$ ($D = n_1 + n_2 + n_3$), strict monotonicity along simplex edges, and balanced volume-square identities $\zeta_{(1/3,1/3,1/3)}(\langle a, b, c\rangle)^3 = (abc)^2$ (C05–C09, C15–C16); (iii) an exhaustive 8-entry $\mathbb{Z}[\varepsilon]$ border-rank-2 witness for W_3 together with the root-free cube separation $(2^{h(1/3)})^3 = 27/4 < 8 = \mathrm{rk}_j(W_3)^3$ and exponential tensor-power amplification $27^k < 32^k$ (C10, T11–T13); and (iv) exact graph asymptotic spectrum sandwiches for C_5 ($4 < 5 < 25/4$ with explicit size-5 strong-square independent set) and C_7 via a root-free Chebyshev bracket on $p(t) = 8t^3 - 4t^2 - 4t + 1$ certifying $\vartheta(C_7) \in (63/19, 637/191) \subset (3, 7/2)$ (C13–C14, T15–T19).

AIRR:ARR-2026-3777AOKH3T81X8ZP | v1 | 2026-11-27

1 Introduction

Volker Strassen introduced the asymptotic spectrum $\Delta(\mathcal{T}_k)$ of order-k tensors [4]—the compact Hausdorff space of monotone semiring homomorphisms $\phi : (\mathcal{T}_k, \oplus, \otimes, \leq) \to \mathbb{R}_{\geq 0}$—to characterize asymptotic tensor rank, subrank, and matrix multiplication complexity. Whether $\Delta(\mathcal{T}_k)$ is connected remained open for nearly four decades (known previously only on restricted families [1]) until Vrana and Zuiddam [5] proved that $\Delta(\mathcal{T}_k)$ is path-connected for every $k \geq 3$ by constructing a general interpolation method connecting every spectral point to every one-leg-vs-rest flattening rank rk_j, and extending the same framework to graphs and Shannon capacity.

Here we build TENSOR-SPECTRUM-CONN-CERT-1.0 to provide an exact-$\mathbb{Z}/\mathbb{Q}$, zero-float verification baseline certifying flattening-rank semiring homomorphisms, root-free simplex interpolation intervals, W-state border-rank and quantum-functional interval separation, and Chebyshev-bracketed Lovász-theta graph capacity sandwiches.

2 Flattening Ranks and Root-Free Strassen Spectral Interpolation

For an order-3 tensor $T \in \mathbb{Q}^{d_1 \times d_2 \times d_3}$, its three one-leg-vs-rest flattening ranks $r_j := \mathrm{rk}_j(T) \in \mathbb{Z}_{\geq 1}$ ($j \in \{1, 2, 3\}$) define extreme points of $\Delta(\mathcal{T}_3)$ that are additive under direct sum $\oplus$ and multiplicative

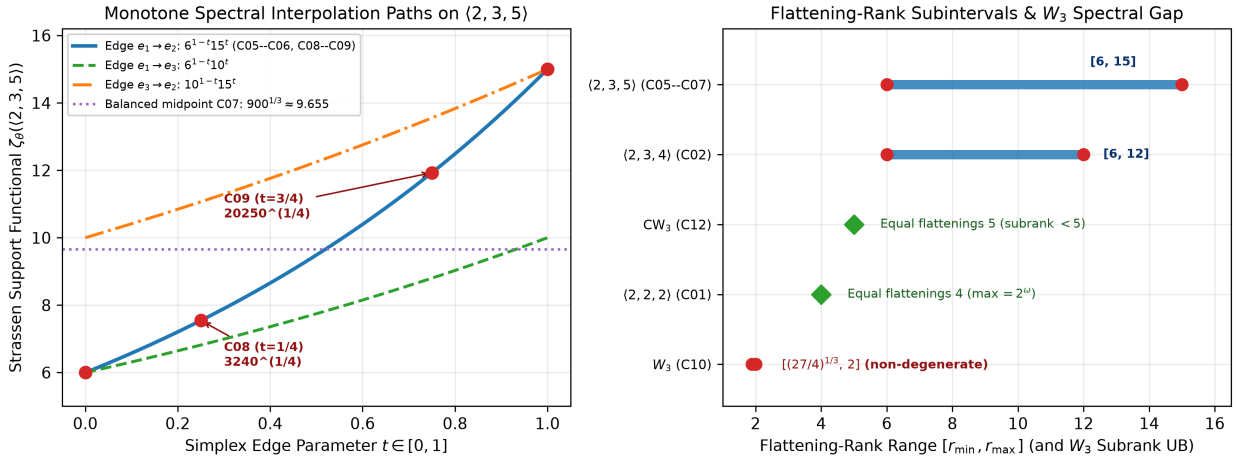

Figure 1: Left: Strict monotone Strassen geometric interpolation paths along simplex edges of $\langle 2,3,5\rangle$ (flattening ranks $6, 15, 10$), marking exact rational certificates `C05`–`C09`. Right: Flattening-rank subintervals $[r_{\min}, r_{\max}]$ across benchmark tensors alongside the strictly non-degenerate W_3 spectral gap $[(27/4)^{1/3}, 2]$ (`C10`). Note that equal flattening ranks for $\langle 2,2,2\rangle$ (4) and CW_3 (5) collapse only the flattening-rank subinterval, not the full spectrum $\Delta(\mathcal{T}_3)$ (where $\max_\phi \phi(\langle 2,2,2\rangle) = 2^\omega$ and CW_3 has asymptotic subrank strictly below 5).

under Kronecker product $\otimes$ (verified by exact-$\mathbb{Q}$ Gaussian elimination in `C01`–`C04` and `C11`–`C12`; Lean `T07`–`T08`, `T14`). For rational weights $\theta = (n_1/D, n_2/D, n_3/D) \in \Delta_2 \cap \mathbb{Q}^3$ ($n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$, $D = n_1 + n_2 + n_3 \geq 1$), the Strassen geometric interpolation functional $\zeta_\theta(T) := r_1^{n_1/D} r_2^{n_2/D} r_3^{n_3/D}$ satisfies exact root-free integer laws:

Theorem 2.1 (Root-Free Spectral Interval Sandwich and Edge Monotonicity; Lean `T01`–`T06`, `T09`–`T10`)**.** *Let $r_1, r_2, r_3 \in \mathbb{Z}_{\geq 1}$ with $m = \min(r_1, r_2, r_3)$ and $M = \max(r_1, r_2, r_3)$, and let $n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$ with $D = n_1 + n_2 + n_3 \geq 1$.*

1. ***Interval Sandwich (`T01`–`T02`):*** *$m^D \leq r_1^{n_1} r_2^{n_2} r_3^{n_3} \leq M^D$, with collapse to r^D whenever $r_1 = r_2 = r_3 = r$ (`T03`, `C01`, `C16`).*
2. ***Strict Simplex-Edge Monotonicity (`T04`):*** *Whenever $0 < r_1 < r_2$ and $t \in \mathbb{Z}_{\geq 0}$ satisfies $0 \leq t < D$, $r_1^{D-t} r_2^t < r_1^{D-(t+1)} r_2^{t+1}$, certifying strict monotonicity along the rational edge connecting r_1 to r_2 (`C05`–`C06`, `C08`–`C09`).*
3. ***Rectangular Matrix Multiplication Volume Square (`T05`):*** *For $\langle a, b, c\rangle$ with flattening ranks (ab, bc, ca), $(ab)(bc)(ca) = (abc)^2$ identically; hence $\zeta_{(1/3,1/3,1/3)}(\langle a,b,c\rangle)^3 = (abc)^2$ (e.g. $900 = 30^2 \in [216, 3375]$ for $\langle 2,3,5\rangle$ in `C07`, `T09`; $1296 = 36^2 \in [64, 46656]$ for $\langle 1,4,9\rangle$ in `C15`, `T10`; Figure 1).*
4. ***Kronecker Multiplicativity (`T06`):*** *$\zeta_\theta(T \otimes S)^D = \zeta_\theta(T)^D \zeta_\theta(S)^D$ identically for all flattening-rank triples.*

3 W-State Border-Rank Witness and Non-Degenerate Spectral Interval

For the 3-qubit W-state tensor $W_3 = |100\rangle + |010\rangle + |001\rangle \in \mathbb{Q}^{2\times 2\times 2}$, all three flattening ranks coincide ($\mathrm{rk}_1 = \mathrm{rk}_2 = \mathrm{rk}_3 = 2$), yet $\Delta(\mathcal{T}_3)$ evaluated at W_3 is a *strictly non-degenerate* closed interval containing $[(27/4)^{1/3}, 2]$:

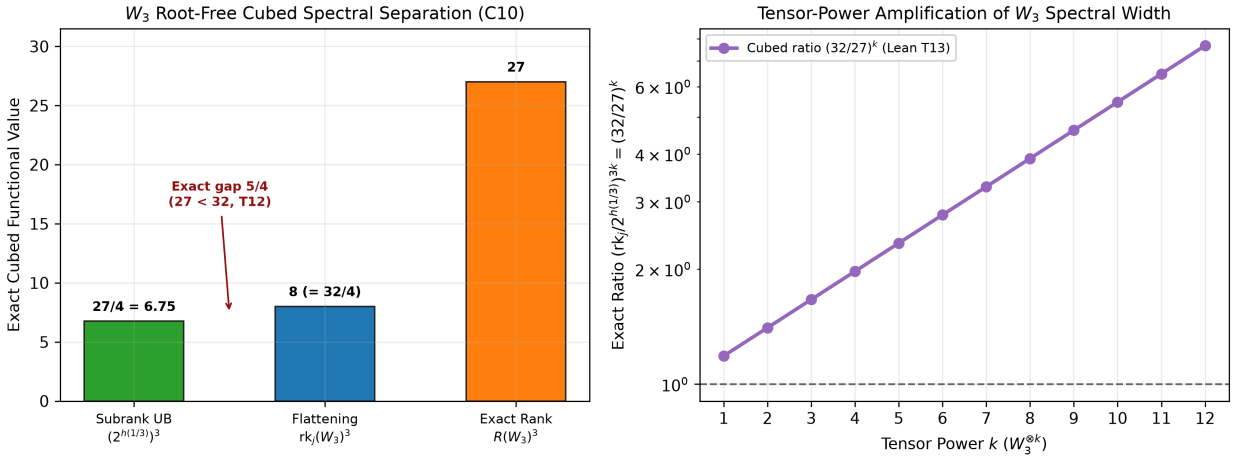

Figure 2: Left: Root-free exact-$\mathbb{Q}$ cubed spectral landmarks of W_3 (`C10`, Lean `T12`), certifying the strict gap $27/4 < 8$ separating the subrank upper functional from the flattening rank 2. Right: Exponential amplification $(32/27)^k$ across tensor powers $W_3^{\otimes k}$ (Lean `T13`).

Theorem 3.1 (W_3 Border-Rank-2 Witness and Strict Cube Separation; Lean `T11`–`T13`, `C10`). *1. Over $\mathbb{Z}[\varepsilon]$, $(e_0+\varepsilon e_1)^{\otimes 3} - e_0^{\otimes 3} = \varepsilon W_3 + \varepsilon^2(e_0e_1e_1 + e_1e_0e_1 + e_1e_1e_0) + \varepsilon^3 e_1^{\otimes 3}$ holds across all 8 entries (`C10`, sum identity Lean `T11`), certifying $\underline{R}(W_3) \le 2$ (and $= 2$ since $\mathrm{rk}_j(W_3) = 2$).*

2. The Strassen binary-entropy upper functional value $2^{h(1/3)} = 3/2^{2/3}$ satisfies $(2^{h(1/3)})^3 = 27/4 < 8 = \mathrm{rk}_j(W_3)^3$ (cleared denominator: $27 < 32$, exact gap $5/4$; Lean `T12`), and tensor powers amplify this separation exponentially via $27^k < 32^k$ for every $k \ge 1$ (Lean `T13`; Figure 2).

4 Graph Asymptotic Spectrum: C_5 and Chebyshev-Bracketed C_7 Sandwiches

In the graph asymptotic spectrum (where Vrana and Zuiddam's interpolation recovers Lovász-theta paths [5, 3]), every spectral point $\phi(G)$ lies in $[\Theta(G), \chi_f(\overline{G})]$ with $\alpha(G) \le \Theta(G) \le \vartheta(G) \le \chi_f(\overline{G})$. We certify both C_5 and C_7 with zero floating-point arithmetic:

Theorem 4.1 (Root-Free C_5 and Chebyshev C_7 Lovász Sandwiches; Lean `T15`–`T19`, `C13`–`C14`). *1. For C_5, the 5-vertex independent set $\{(0,0),(1,2),(2,4),(3,1),(4,3)\} \subset V(C_5^{\boxtimes 2})$ (`C13`) certifies $\Theta(C_5)^2 \ge 5 = \vartheta(C_5)^2$, and $16 < 20 < 25$ certifies $\alpha(C_5) = 2 < \sqrt{5} < 5/2 = \chi_f(\overline{C_5})$ (Lean `T15`).*

2. For C_7, classical cyclotomic theory identifies $c = \cos(\pi/7)$ as the largest real root of $p(t) = 8t^3 - 4t^2 - 4t + 1$. Exact rational evaluation gives $p(9/10) = -1/125 < 0 < 9521/125000 = p(91/100)$ (Lean `T16`) and $p'(t) = 24t^2 - 8t - 4 > 0$ for all $t \ge 9/10$ (Lean `T17`), pinning $c \in (9/10, 91/100)$ uniquely via the Intermediate Value Theorem. Since $f(t) = 7t/(1+t)$ is strictly increasing on $t > 0$ (Lean `T19`), $\vartheta(C_7) = 7c/(1+c)$ satisfies the exact rational sandwich (Lean `T18`; Figure 3):

$$3 = \alpha(C_7) < \frac{63}{19} < \vartheta(C_7) < \frac{637}{191} < \frac{7}{2} = \chi_f(\overline{C_7}). \tag{1}$$

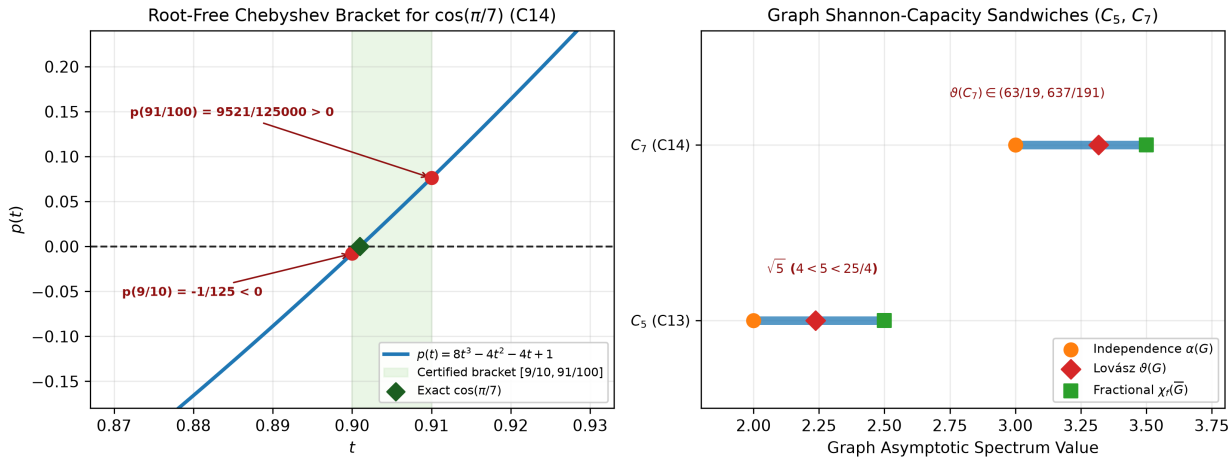

Figure 3: Left: Root-free Chebyshev minimal polynomial $p(t) = 8t^3 - 4t^2 - 4t + 1$ bracketing $\cos(\pi/7) \in (9/10, 91/100)$ with certified derivative positivity (`C14`, Lean `T16`–`T17`). Right: Exact graph asymptotic spectrum sandwiches $[\alpha(G), \chi_f(\overline{G})]$ for C_5 (`C13`) and C_7 (`C14`).

5 Certificate Suite, Scope, and Verification Boundary

Table 1 summarizes all 16 SHA-256-indexed rational certificates (`C01`–`C16`) produced by `TENSOR-SPECTRUM-CONN-CERT-1.0`, audited by 62 adversarial pytest checks and 20 constructive Lean 4.36.0 theorems.

Verification Boundary and Attribution.

- **Machine-checked in Lean 4.36.0 (`T01`–`T20`) and exact $\mathbb{Z}/\mathbb{Q}$ (`C01`–`C16`):** All root-free power-law spectral interval sandwiches, simplex-edge strict monotonicity, balanced volume-square identities, Kronecker multiplicativity, exact $\mathbb{Q}$-Gaussian-elimination flattening ranks for $\langle a, b, c\rangle$, W_3, CW_2, CW_3, the 8-entry $\mathbb{Z}[\varepsilon]$ border-rank-2 witness and $27/4 < 8$ ($27^k < 32^k$) spectral gap for W_3, the C_5 strong-square witness and sandwich, and the C_7 Chebyshev minimal-polynomial root bracket and Lovász-theta sandwich. Every Lean theorem depends solely on `[propext, Quot.sound]` or zero axioms (zero `Classical.choice`, zero `sorry`).
- **Imported from cited literature (not re-proved inside Lean):** Vrana and Zuiddam's general path-connectedness theorem for $\Delta(\mathcal{T}_k)$ ($k \geq 3$) via symmetric-group representation-theoretic "method of types" and cover-property interpolation [5], Strassen's general duality theorem [4], and Lovász's semidefinite characterization $\vartheta(C_{2m+1}) = \frac{(2m+1)\cos(\pi/(2m+1))}{1+\cos(\pi/(2m+1))}$ [3].

Reproducibility, Archived Artifact Hashes, and Disclosures. All verification artifacts are archived directly inside this AIRR record under MIT (manuscript CC-BY-4.0):

- `src/lean/TensorAsymptoticSpectrumCert.lean` (20 constructive Lean 4.36.0 theorems `T01`–`T20`):
  `SHA-256: 3f33a795154fc72a9dbbcc1923bab0cdcf104d8f03fc1659a69cf2f732168064`
- `src/python/tensor_asymptotic_spectrum_verifier.py` (exact $\mathbb{Z}/\mathbb{Q}$ zero-float engine):
  `SHA-256: 974e6ec2d3ffb9631cc69875b7f2fa923060fc5b365efbae39478ee792d3cec2`
- `data/certificates/MANIFEST.json` (indexing all 16 exact-$\mathbb{Z}/\mathbb{Q}$ JSON certificates `C01`–`C16`):
  `SHA-256: a66d30eae0f8633b4b38f75dc8f2c4043c6da56c91909bb5f2ff7e8e489c46cb`

AI-assisted research tooling (Google Antigravity) supported drafting and formalization under human authorship and oversight by Lluis Eriksson; every formal claim is independently checked by the Lean 4.36.0 kernel and zero-float pytest suite.

Table 1: Complete summary of all 16 exact-$\mathbb{Z}/\mathbb{Q}$ certificates (`C01`–`C16`) and matching Lean 4.36.0 theorems.

ID	Tensor / Graph Object	Exact Rational / Certified Output (Zero Float)	Lean Ref.
`C01`	$\langle 2,2,2\rangle$ matmul	Exact flattening ranks $(4,4,4)$; flattening subinterval $[4,4]$	`T03, T05`
`C02`	$\langle 2,3,4\rangle$ rectangular	Exact flattening ranks $(6,12,8)$; interval $[6,12]$	`T01--T02`
`C03`	$W_3 \otimes W_3$ Kronecker	Product ranks $(4,4,4) = (2,2,2)^2$ (exact multiplicativity)	`T06--T07`
`C04`	$W_3 \oplus \langle 1,2,3\rangle$	Direct-sum ranks $(4,8,5) = (2,2,2) + (2,6,3)$ (exact)	`T08`
`C05`	$\langle 2,3,5\rangle$ vertex e_1	Weight $(1,0,0)$: attains exact minimum flattening rank 6	`T01, T09`
`C06`	$\langle 2,3,5\rangle$ vertex e_2	Weight $(0,1,0)$: attains exact maximum flattening rank 15	`T02, T09`
`C07`	$\langle 2,3,5\rangle$ midpoint	Weight $(1/3,1/3,1/3)$: $\zeta^3 = 900 = 30^2 \in [216, 3375]$	`T05, T09`
`C08`	$\langle 2,3,5\rangle$ edge $(3/4,1/4,0)$	$\zeta^4 = 3240 \in [1296, 50625]$ (monotone path point)	`T04, T09`
`C09`	$\langle 2,3,5\rangle$ edge $(1/4,3/4,0)$	$\zeta^4 = 20250 > 3240$ (strict simplex-edge increase)	`T04, T09`
`C10`	$W_3 \in \mathbb{Q}^{2\times 2\times 2}$	$\mathbb{Z}[\varepsilon]$ border rank 2 (8 entries); $(2^{h(1/3)})^3 = 27/4 < 8$	`T11--T13`
`C11`	CW_2 (dim 4)	Exact flattening ranks $(4,4,4)$ over $\mathbb{Q}$ [2]	`T14`
`C12`	CW_3 (dim 5)	Exact flattening ranks $(5,5,5)$ over $\mathbb{Q}$ [2]	`T14`
`C13`	Graph C_5 Shannon cap.	Size-5 indep. set in $C_5^{\boxtimes 2}$; $4 < \vartheta^2 = 5 < 25/4$	`T15`
`C14`	Graph C_7 Chebyshev	$p(9/10) = -1/125 < 0 < p(91/100)$; $\vartheta \in (63/19, 637/191)$	`T16--T19`
`C15`	$\langle 1,4,9\rangle$ balanced	Flattening $(4,36,9)$; $\zeta^3 = 1296 = 36^2 \in [64, 46656]$	`T05, T10`
`C16`	$\langle 3,3,3\rangle$ symmetric	Flattening $(9,9,9)$; equal flattening subinterval $[9,9]$	`T03, T20`

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[4] Volker Strassen. The asymptotic spectrum of tensors. *Journal für die reine und angewandte Mathematik*, 384:102–152, 1988.

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