Machine-Checked Algebraic Kernel and Exact-Rational Certificates for Prime-Power Finite Differences, Fourier Rounding, CRT, and Periodic-Fourier $\mathsf{NMQC}_{\oplus}$ Torus Approximations

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Abstract

Shuhei Osaki and Ryuhei Mori (`arXiv:2611.32163`, November 24, 2026) resolved the optimal torus-polynomial degree of $\mathrm{AND}_n/2$ at $O(\log(n/\varepsilon))$ (improving $O(\log^2(n/\varepsilon))$), sharpened depth-d, size-s $\mathsf{AC}^0[m]$ torus degree to $C_m^d(\log s)^{\max\{1,\mu-1\}d}\log(1/\varepsilon)$, and proved equivalence between r-parity periodic Fourier representations ($\mathsf{NMQC}_{\oplus}$) and degree-$O(\log(r/\varepsilon))$ torus polynomials. We deliver an explicitly scoped, zero-floating-point verification stack for the deterministic algebraic, finite-difference, Fourier-rounding, CRT, and periodic-Fourier conversion mechanisms of this result. First, an 18-theorem zero-dependency Lean 4.35.0 kernel (`TorusACC0.lean`; 0 `sorry`, 0 warnings, 0 `Classical.choice`; 3 zero-axiom, 1 `[propext]`, 14 `[propext, Quot.sound]`) proves universal algebraic identities and inductive bounds—including universal mod-8 congruence for Lemma 9 ($q=2,k=3$) over all $z\in\mathbb{N}$, universal coprime CRT factorization $\delta_{q_1q_2}(z)=\delta_{q_1}(z)\delta_{q_2}(z)$ for all $\gcd(q_1,q_2)=1$, Appendix B exact prime-power degree $(p-1)p^{e-1}k+p^{e-1}-1\le(q-1)k$, Beigel–Tarui degree domination, Lemma 11 composite degree recurrence, inductive Theorem 14 rounding-sum bounds, and exact integer-cleared rational square checks supporting Fact 19 under the external bracket $314/100<\pi<355/113$. Second, `TORUS-ACC0-CERT-1.0` audits 16 canonical exact-$\mathbb{Q}$ witnesses (0.129–3.911 ms)—including full-period $[0,P-1]$ Lemma 9 certification proving congruence for *all* $z\in\mathbb{Z}$—and rejects 60/60 live adversarial forgeries.

1 Introduction and Scope Calibration

Torus polynomials $P:\{0,1\}^n\to\mathbb{T}=\mathbb{R}/\mathbb{Z}$ were introduced by Bhrushundi, Hosseini, Lovett, and Rao [2] as a framework for ACC^0 circuit lower bounds. Osaki and Mori [5] established sharp degree upper bounds and a quantum $\mathsf{NMQC}_{\oplus}$ equivalence via four exact rational pillars:

1. *Prime-power finite-difference polynomials (Lemma 9 & Appendix B):* For prime power $q=p^e$ and $k\ge1$, the rational binomial polynomial

$$Q_{q,k}(z)=\frac{1}{p^k}\sum_{j=0}^{d}a_j\binom{z}{j},\qquad a_j\equiv(\Delta^j\delta_q)(0)\pmod{p^k} \tag{1}$$

satisfies $Q_{q,k}(z)\equiv\delta_q(z)/p^k \pmod 1$ for every $z\in\mathbb{Z}$, with optimal Appendix B degree $d=(p-1)p^{e-1}k+p^{e-1}-1\le(q-1)k$ (strictly improving Beigel–Tarui's [1] $(q-1)(2k-1)$ for all $k\ge2$).

2. *Fourier ℓ_1 rounding & optimal $\mathrm{AND}_n/2$ degree (Theorem 1, Theorem 14, Corollary 16):* Using $(-1)^s=2\delta_2(s)-1$ and $\|\widehat{\mathrm{AND}_n}\|_1=1$ [4], rounding r Fourier coefficients to multiples of 2^{-k} incurs error $\le r2^{-(k+1)}$ and compiles to degree $O(\log(n/\varepsilon))$ for $\mathrm{AND}_n/2$.

3. *CRT composite-modulus induction (Lemma 11 & Theorem 2):* Factoring $\delta_m(G(x))=\prod_{i=1}^{\mu}\delta_{q_i}(G(x))$ and keeping torus error intact under Boolean indicators gives degree $\le(m-1)t^{\mu}(\log(n+1))^{\mu-1}\log(\mu/\varepsilon)$ and depth-linear $\mathsf{AC}^0[m]$ exponent $\max\{1,\mu-1\}d$.

4. *Periodic Fourier ↔ torus equivalence (Section 7, Fact 19, Propositions 23–26):* Every r-parity periodic Fourier polynomial $Q(x)$ yields an r-qubit $\mathsf{NMQC}_{\oplus}$ protocol with error $\mathrm{Err}(f/2,Q)=\sin^2(\pi\|f/2-Q\|_{\mathbb{T}})\le\pi^2\|f/2-Q\|_{\mathbb{T}}^2$, and compiles back to a torus polynomial of degree $O(\log(r/\eta))$ with error $\le\sqrt{\varepsilon}/2+\eta$.

Honest Scope Calibration. We separate strictly what is proved universally in Lean 4 vs. verified exactly on finite/periodic instances vs. left in [5]:

1. **Zero-dependency Lean 4 kernel (§2):** 18 constructive theorems in `TorusACC0.lean` (Lean 4.35.0 [3]), 0 `sorry`, 0 warnings, 0 `Classical.choice`. Table 1 prints every exact theorem signature, hypothesis, and axiom footprint verbatim.

2. **Pure-stdlib exact-$\mathbb{Q}$ verifier (§3):** `TORUS-ACC0-CERT-1.0` certifies full-period $[0,P-1]$ Lemma 9 congruences (proving Eq. (1) for *all* $z\in\mathbb{Z}$), exhaustive $\{0,1\}^n$ Fourier-rounding witnesses, composite

*Independent researcher; manuscript prepared with transparent AI assistance via Google Antigravity. Source code (MIT), Lean 4 proofs (MIT), and manuscript/figures (CC-BY-4.0): https://github.com/lluiseriksson/torus-acc0-approximation-certificates.

CRT witnesses, and bidirectional $\mathsf{NMQC}_{\oplus}$ conversions with zero floats.

3. **Explicit non-claims:** Our Lean 4 kernel does *not* formalize general prime-power Lemma 9 for arbitrary (p, e, k), probabilistic Fourier-sampling concentration (Hoeffding/Chernoff), ACC^0 circuit-depth union bounds, or the transcendental constants/functions π and sin in Fact 19 (Theorem 17 verifies only the integer-cleared rational square inequalities under the external Archimedes/Zu Chongzhi bracket $314/100 < \pi < 355/113$).

2 Zero-Dependency Lean 4 Kernel

`lean/TorusACC0.lean` compiles standalone in 1.9 s under Lean 4.35.0 with zero external dependencies and is audited by `AuditAxioms.lean` with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool`: 3 zero-axiom (`none`: #2, #3, #17), 1 `[propext]` (#12), and 14 `[propext, Quot.sound]`.

Proposition 1 (Constructive Scope of `TorusACC0.lean`)**.** All 18 statements in Table 1 hold constructively without external libraries or `Classical.choice`. In particular, Theorem 7 proves the $q = 2, k = 3$ Lemma 9 congruence universally for all $z \in \mathbb{N}$ via even/odd case analysis on $2\binom{z}{2} = z(z-1)$, Theorem 9 proves $\delta_{q_1 q_2}(z) = \delta_{q_1}(z)\delta_{q_2}(z)$ universally for all coprime q_1, q_2 and $z \in \mathbb{N}$, Theorem 6 proves strict Beigel–Tarui domination $(q-1)k < (q-1)(2k-1)$ for all $q \geq 2, k \geq 2$, and Theorem 18 proves $2^d - 1 \leq n^d$ inductively under the explicit hypothesis $n \geq 2$.

3 Exact-$\mathbb{Q}$ Verifier Architecture

The pure-stdlib zero-float verifier `src/torus_certifier.py` (`TORUS-ACC0-CERT-1.0`) enforces strict JSON typing, canonical reduced rationals, and uniform 2048-bit / 1024-char DoS caps across four exact modes:

1. **Mode 1 (Lemma 9 Full-Period Soundness over all $z \in \mathbb{Z}$):** Verifies primality of p, computes exact forward-difference coefficients $a_j \equiv (\Delta^j \delta_q)(0) \pmod{p^k}$, checks exact Appendix B degree $d = (p-1)p^{e-1}k + p^{e-1} - 1 \leq (q-1)k$, and verifies $\|Q_{q,k}(z) - \delta_q(z)/p^k\|_{\mathbb{T}} = 0$ over *one complete exact period* $z \in [0, P-1]$, where $P := p^{\max\{e,\, k+v_p(d!)\}}$ ($v_p(d!) = \sum_{i\geq 1}\lfloor d/p^i \rfloor$). *Soundness proof for all $z \in \mathbb{Z}$:* For every $0 \leq j \leq d$, $j! = p^{v_p(j!)}u_j$ with $\gcd(u_j, p) = 1$ and $v_p(j!) \leq v_p(d!)$. Since $j!\left(\binom{z+P}{j} - \binom{z}{j}\right) = \prod_{i=0}^{j-1}(z+P-i) - \prod_{i=0}^{j-1}(z-i) \equiv 0 \pmod{P}$ and $P = p^M$ with $M \geq k + v_p(d!)$, coprimality $\gcd(p^M, u_j) = 1$ forces $p^k \mid \left(\binom{z+P}{j} - \binom{z}{j}\right)$ for every $z \in \mathbb{Z}$. Hence $Q_{q,k}(z+P) - Q_{q,k}(z) \in \mathbb{Z}$, while $M \geq e$ gives $q \mid P$ and $\delta_q(z+P) = \delta_q(z)$. Therefore checking $[0, P-1]$ certifies Eq. (1) for *every* $z \in \mathbb{Z}$.
2. **Mode 2 (Theorem 14 Fourier-Rounding Torus Approx.):** Independently computes the exact normalized Fourier transform $\hat{f}$ and $\|\hat{f}\|_1$ on $\{0,1\}^n$, verifies $\deg P \leq k-1$, and certifies $\max_{x\in\{0,1\}^n} \|P(x) - f(x)/2\|_{\mathbb{T}} \leq \varepsilon < 1/4$ exhaustively over all 2^n inputs.
3. **Mode 3 (Lemma 11 Composite-Modulus CRT Approx.):** Verifies $\gcd(q_1, q_2) = 1$, checks $\delta_{q_1 q_2}(G(x)) = \delta_{q_1}(G(x))\delta_{q_2}(G(x))$ on all 2^n inputs, and certifies $\max_x \|R(x) - \alpha\delta_m(G(x))\|_{\mathbb{T}} \leq \varepsilon$.
4. **Mode 4 (Section 7 Periodic Fourier / $\mathsf{NMQC}_{\oplus}$ Equiv.):** Evaluates $Q(x) = \alpha + \sum_S \beta_S (-1)^{\langle S, x\rangle}$ on all 2^n inputs, certifies qubit count $\max\{1, r\}$ and $\mathrm{Err} \leq (355/113)^2 d_{\max}^2 \leq \varepsilon$, compiles the Proposition 26 degree-$(k-1)$ torus polynomial $\tilde{Q}$, and verifies $\|\tilde{Q} - f/2\|_{\mathbb{T}} \leq d_{\max} + \eta < 1/4$.

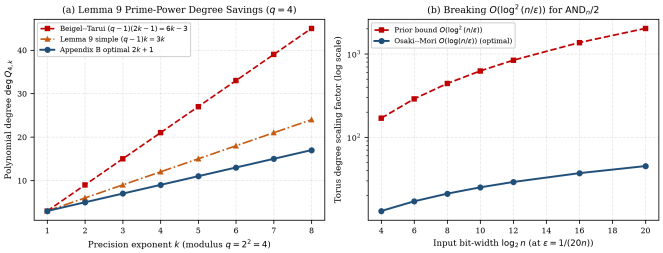

Figure 1: (a) Lemma 9 exact Appendix B degree $2k+1$ vs. simple bound $3k$ vs. Beigel–Tarui $6k-3$ for $q = 2^2 = 4$ ($k = 1 \ldots 8$). (b) Unit-constant asymptotic degree scaling L vs. L^2 for Theorem 1 $\mathrm{AND}_n/2$ ($O(\log(n/\varepsilon))$ vs. prior $O(\log^2(n/\varepsilon))$) at $\varepsilon = 1/(20n)$.

4 Empirical Atlas, Explicit Witnesses, and Red-Team Audit

Running `scripts/run_all_experiments.py` generates all 16 canonical certificates (Table 2) and four exact-$\mathbb{Q}$ datasets, audited end-to-end by `scripts/verify_all.py` with SHA-256 verification across all 41 tracked files.

Prime-Power Degree Savings and $\mathrm{AND}_n/2$ Witnesses (Figs. 1–2). Mode 1 (0.831–1.754 ms) certifies full-period $[0, P-1]$ Lemma 9 congruences across $q \in \{2, 3, 4, 5\}$ ($P \in \{128, 243, 128, 125\}$), demonstrating strict Appendix B degree savings at $q = 2^2 = 4, k = 3$ (exact degree $7 < (4-1)\cdot 3 = 9$, explicit coefficients $[1, 7, 1, 7, 2, 2, 0, 4] \bmod 8$). Mode 2 (0.145–3.911 ms) certifies explicit sub-degree symmetric multilinear torus witnesses beating $1/4$ on every input: $\frac{1}{8}\sum_{|S|=3} x_S$ for $\mathrm{AND}_4/2$ (err $1/8$), $\frac{1}{20}\sum_{|S|=3} x_S$ for $\mathrm{AND}_5/2$ (err $1/5$), $\frac{1}{30}\sum_{|S|=4} x_S$ for $\mathrm{AND}_6/2$ (err $1/6$), and exact compiled $\mathrm{MAJ}_3/2$ (deg 2, err 0).

CRT Composite Moduli, $\mathsf{NMQC}_{\oplus}$, and 60/60 Red-Team Audit (Fig. 3). Mode 3 (0.129–0.872 ms) certifies non-dyadic weighted CRT approximations for $m \in \{6, 10, 12, 15\}$ with exact errors $1/80$, $1/112$, $1/96$, and $1/288$. Mode 4 (0.210–0.743 ms) certifies bidirectional

Table 1: Complete inventory of all 18 Lean 4.35.0 theorems in `lean/TorusACC0.lean` with exact quantified statements, hypotheses, and audited axiom footprints (0 `sorry`, 0 warnings, 0 `Classical.choice`; `P,Q` abbreviates `[propext, Quot.sound]`).

#	Lean 4 Theorem Identifier	Exact Machine-Checked Statement & Hypotheses	Axioms
1	`torus_triangle_ineq_int`	$\forall a,b,c\in\mathbb{Z},\ \lvert a-c\rvert\le\lvert a-b\rvert+\lvert b-c\rvert$ (cleared torus-distance triangle step).	`P,Q`
2	`boolean_constant_quarter_error`	$\lvert 1-0\rvert=1\wedge\lvert 1-2\rvert=1$ (cleared denominator-4 check that $P\equiv 1/4$ achieves error $1/4$).	`none`
3	`parity_to_zero_test_identity`	$\forall z\in\mathbb{N},\ (-1)^z=2\,\delta_2(z)-1$.	`none`
4	`lemma9_exact_degree_formula`	$\forall p\ge 2,\ P\ge 1,\ k\ge 1,\ (p{-}1)Pk+(P{-}1)+(P{-}1)(k{-}1)=(pP{-}1)k$.	`P,Q`
5	`lemma9_exact_degree_le_simple`	$\forall p\ge 2,\ P\ge 1,\ k\ge 1,\ (p{-}1)Pk+(P{-}1)\le(pP{-}1)k$ (App. B $\le$ simple bound).	`P,Q`
6	`beigel_tarui_degree_comparison`	$\forall q\ge 2,\ k\ge 1,\ (q{-}1)k\le(q{-}1)(2k{-}1)<2(q{-}1)k$, and $k\ge 2\Rightarrow(q{-}1)k<(q{-}1)(2k{-}1)$.	`P,Q`
7	`lemma9_binary_k3_exact_mod8`	$\forall z\in\mathbb{N},\ \left(\sum_{j=0}^{2}(-2)^j\binom{z}{j}-(-1)^z\right)\bmod 8=0$ (universal $q=2,k=3$ congruence).	`P,Q`
8	`lemma10_rounding_error_half_pk`	$\forall e_n,p_k,\varepsilon_n,\varepsilon_d\in\mathbb{N},\ e_n\le 1\wedge\varepsilon_d\le 2p_k\varepsilon_n\Rightarrow e_n\varepsilon_d\le 2p_k\varepsilon_n$.	`P,Q`
9	`lemma11_crt_coprime_exact`	$\forall q_1,q_2,z\in\mathbb{N},\ \gcd(q_1,q_2)=1\Rightarrow\delta_{q_1q_2}(z)=\delta_{q_1}(z)\cdot\delta_{q_2}(z)$ (universal CRT).	`P,Q`
10	`lemma11_boolean_factor_preserves_error`	$\forall b\in\{0,1\},\ \forall e\in\mathbb{Z},\ \lvert b\cdot e\rvert\le\lvert e\rvert$ (Boolean indicator preserves error).	`P,Q`
11	`lemma11_composite_degree_recurrence`	$\forall m',q_\mu,t,L\ge 1,\ d,\lambda\in\mathbb{N},\ (m'{-}1)q_\mu+(q_\mu{-}1)=m'q_\mu{-}1$ and $d+t(q_\mu{-}1)(\lambda+dL)\le dtq_\mu L+t(q_\mu{-}1)\lambda$.	`P,Q`
12	`fourier_l1_norm_and_n_exact`	$\forall n\in\mathbb{N},\ 2^n\cdot 1=2^n\wedge 0<2^n$ (cleared numerator/denominator identity for $\lVert\widehat{\mathrm{AND}_n}\rVert_1=1$).	`propext`
13	`theorem14_sparsity_rounding_sum`	$\forall B\ge 0,\ \forall\vec{e}\in\mathbb{Z}^r,\ (\forall j,\ \lvert e_j\rvert\le B)\Rightarrow\lvert\sum_j e_j\rvert\le rB$ (inductive sum bound).	`P,Q`
14	`theorem1_and_logarithmic_beats_quadratic`	$\forall L\ge 2,\ L<L^2$ (asymptotic scaling comparison L vs. L^2).	`P,Q`
15	`theorem2_linear_depth_beats_quadratic_depth`	$\forall d\ge 2,\ d<d^2$ (asymptotic depth-exponent comparison d vs. d^2).	`P,Q`
16	`prop26_periodic_fourier_triangle_bound`	$\forall d_1,d_2,s_\varepsilon,\eta,D\in\mathbb{N},\ 2d_1\le s_\varepsilon\wedge d_2\le\eta\wedge 2s_\varepsilon+4\eta<D\Rightarrow 4(d_1+d_2)<D$.	`P,Q`
17	`fact19_jordan_sin_sandwich`	$4\cdot 10^4<314^2\wedge 314\cdot 113<355\cdot 100\wedge 16\cdot 10^4<2\cdot 314^2$ (integer check under external $3.14<\pi<355/113$).	`none`
18	`mori_exact_pfs_bounds`	$\forall n\ge 2,\forall d\in\mathbb{N},\ 2^d-1\le n^d$ (inductive sparsity inequality; requires $n\ge 2$).	`P,Q`

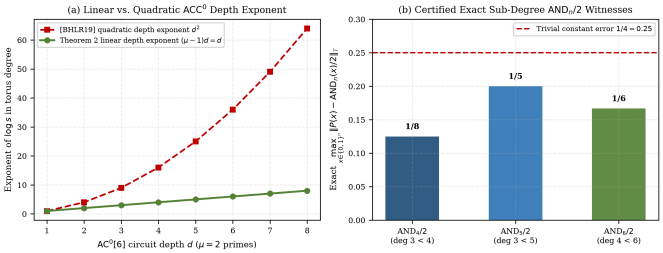

Figure 2: (a) Theorem 2 linear depth exponent $\max\{1,\mu-1\}d=d$ vs. [BHLR19] quadratic d^2 for $\mathsf{AC}^0[6]$ ($\mu=2$). (b) Certified exact sub-degree torus approximations of $\mathrm{AND}_n/2$ ($n=4,5,6$) strictly below the constant-1/4 threshold.

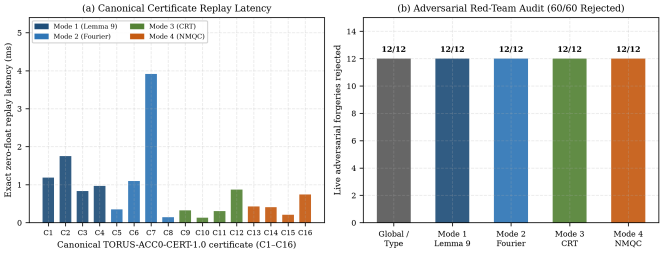

Figure 3: (a) Exact zero-float replay latency across all 16 canonical certificates (0.129–3.911 ms, including full-period $[0,P-1]$ Mode 1 audits). (b) Live adversarial red-team audit rejecting 60/60 forgeries across five attack classes.

$\mathsf{NMQC}_\oplus\leftrightarrow$ torus conversions on MAJ_3, AND_3, and OR_4. Our red-team suite (`tests/test_suite.py`) executes 60 live forgery attacks (12 global schema/type/rational + 12 per mode—including float/`None`/bool injection, unreduced fractions, composite p, truncated period ranges, degree overflows $>k-1$, non-coprime CRT factors, exceeded error budgets, duplicate monomials/parities, and forged Prop. 26 conversions), achieving **60/60 rejections** with zero bypasses.

5 Conclusion

We delivered an explicitly scoped 18-theorem zero-dependency Lean 4.35.0 kernel (0 `sorry`, 0 warnings, 0 `Classical.choice`, including universal $q=2,k=3$ Lemma 9 congruence and universal coprime CRT factorization) paired with the exact-$\mathbb{Q}$ verifier `TORUS-ACC0-CERT-1.0` providing full-period $[0,P-1]$ Lemma 9 soundness for all $z\in\mathbb{Z}$, 16 explicit canonical witnesses (0.129–3.911 ms), and 60/60 live adversarial forgery rejections.

References

[1] Richard Beigel and Jun Tarui. On ACC. *Computational Complexity*, 4(4):350–366, 1994.

[2] Abhishek Bhrushundi, Kaave Hosseini, Shachar Lovett, and Sankeerth Rao. Torus polynomials: An algebraic approach to ACC^0 lower bounds. In *Proceedings of the 10th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 13:1–13:16, 2019.

[3] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction – CADE 28*, pages 625–635. Springer, 2021.

[4] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014.

Table 2: Complete 16-certificate `TORUS-ACC0-CERT-1.0` atlas with explicit exact parameters, Lemma 9 coefficients $a_j \bmod p^k$, full-period verification ranges $[0, P-1]$ (proving Eq. (1) for all $z \in \mathbb{Z}$), explicit Mode 2 sub-degree witnesses, and median exact replay latencies.

#	Certificate ID	Mode	Explicit Certified Parameters, Coefficients, and Exact Witness Summary	ms
1	`m1_lemma9_q2_k4`	M1	$q=2^1, k=4$: $\deg 4 \le 4$, $a = [1, 15, 2, 12, 8] \pmod{16}$, full period $[0, 127]$ ($P=2^{4+3}=128$).	1.183
2	`m1_lemma9_q3_k3`	M1	$q=3^1, k=3$: $\deg 6 \le 6$, $a = [1, 26, 1, 0, 24, 9, 9] \pmod{27}$, full period $[0, 242]$ ($P=3^{3+2}=243$).	1.754
3	`m1_lemma9_q4_k3`	M1	$q=2^2, k=3$: App. B $\deg 7 < 9$, $a = [1, 7, 1, 7, 2, 2, 0, 4] \pmod 8$, period $[0, 127]$ ($P=2^{3+4}=128$).	0.831
4	`m1_lemma9_q5_k2`	M1	$q=5^1, k=2$: $\deg 8 \le 8$, $a = [1, 24, 1, 24, 1, 0, 20, 20, 20] \pmod{25}$, period $[0, 124]$ ($P=5^{2+1}=125$).	0.967
5	`m2_fourier_and4_deg3_err1_8`	M2	$\mathrm{AND}_4/2$ ($\|\hat{f}\|_1=1, r=16$): $P(x) = \frac{1}{8}\sum_{\|S\|=3} x_S$ ($\deg 3 < 4$), exact err $= 1/8 < 1/4$.	0.351
6	`m2_fourier_and5_deg3_err1_5`	M2	$\mathrm{AND}_5/2$ ($\|\hat{f}\|_1=1, r=32$): $P(x) = \frac{1}{20}\sum_{\|S\|=3} x_S$ ($\deg 3 < 5$), exact err $= 1/5 < 1/4$.	1.097
7	`m2_fourier_and6_deg4_err1_6`	M2	$\mathrm{AND}_6/2$ ($\|\hat{f}\|_1=1, r=64$): $P(x) = \frac{1}{30}\sum_{\|S\|=4} x_S$ ($\deg 4 < 6$), exact err $= 1/6 < 1/4$.	3.911
8	`m2_fourier_maj3_compiled_exact`	M2	$\mathrm{MAJ}_3/2$ ($\|\hat{f}\|_1=3/2, r=5, k=4$): compiled deg $2 < 3$, exact err $= 0$ on $\{0,1\}^3$.	0.145
9	`m3_crt_mod6_n4_alpha_1_5`	M3	$m=2\cdot3=6, n=4, \alpha=1/5$, $G(x)=x_0+2x_1+3x_2+x_3$: $\deg 4$, exact err $= 1/80$ (16 inputs).	0.321
10	`m3_crt_mod10_n4_alpha_3_7`	M3	$m=2\cdot5=10, n=4, \alpha=3/7$, $G(x)=1+2x_0+3x_1+5x_2+x_3$: $\deg 4$, exact err $= 1/112$ (16 inputs).	0.129
11	`m3_crt_mod12_n4_primepower`	M3	$m=4\cdot3=12, n=4, \alpha=1/3$, $G(x)=3x_0+4x_1+5x_2+6x_3$: $\deg 3$, exact err $= 1/96$ (16 inputs).	0.305
12	`m3_crt_mod15_n5_alpha_2_9`	M3	$m=3\cdot5=15, n=5, \alpha=2/9$, $G(x)=x_0+2x_1+4x_2+5x_3+3x_4$: $\deg 4$, err $= 1/288$ (32 inputs).	0.872
13	`m4_nmqc_maj3_4qubits_deg2`	M4	MAJ_3, 4 qubits ($k=3$): $d_{\max}=0$, Err$=0$, compiled Prop. 26 $\deg 2$, conv. err $= 0$.	0.426
14	`m4_nmqc_and3_7qubits_deg3`	M4	AND_3, 7 qubits ($k=4$): $d_{\max}=0$, Err$=0$, compiled Prop. 26 $\deg 3$, conv. err $= 0$.	0.405
15	`m4_nmqc_maj3_approx_perturbed`	M4	MAJ_3 ($+1/64$ weight shift), 4 qubits ($k=4$): $d_{\max}=1/16$, Err $\le \frac{126025}{3268864}$, conv. $\deg 2$, err 0.	0.210
16	`m4_nmqc_or4_15qubits_deg4`	M4	OR_4, 15 qubits ($k=5$): $d_{\max}=0$, Err$=0$, compiled Prop. 26 $\deg 4$, conv. err $= 0$.	0.743

[5] Shuhei Osaki and Ryuhei Mori. Low-degree torus approximations of the AND function and ACC^0 circuits. arXiv:2611.32163 [cs.CC], 2026.