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Spectral Phase Transitions, Exact Hyperbolic–Trigonometric Inverse Laws, and Frobenius-Moment Thresholds of Normalized GCD–LCM Correlation Kernels

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Abstract

For any exponent $\alpha > 0$ and integer $n \geq 1$, consider the normalized GCD–LCM correlation kernel matrix $K_\alpha(n) \in \mathbb{R}^{n\times n}$ with entries $K_\alpha(n)_{ij} := \left(\frac{\gcd(i,j)}{\operatorname{lcm}(i,j)}\right)^\alpha = \left(\frac{\gcd(i,j)^2}{ij}\right)^\alpha$ $(1 \leq i,j \leq n)$. We establish a unified analytic-number-theoretic and spectral theory characterizing the exact inverse structure, reproducing-kernel norms, all-α linear trace asymptotics, extreme eigenvalue phase transitions, and positive-moment explosion thresholds of $K_\alpha(n)$.

First, from Smith's Jordan-totient identity $\gcd(i,j)^{2\alpha} = \sum_{d\mid i,\, d\mid j} J_{2\alpha}(d)$ we derive the exact lower-triangular Möbius–Jordan Cholesky factorization $K_\alpha(n) = E_\alpha(n)\operatorname{diag}(J_{2\alpha}(1),\dots,J_{2\alpha}(n))E_\alpha(n)^\top$, the explicit inverse matrix $(K_\alpha(n)^{-1})_{ij}$, and a closed-form reproducing-kernel Hilbert space (RKHS) norm formula valid for *every* completely multiplicative arithmetic function $f\colon \mathbb{N}\to\mathbb{C}$ (including Dirichlet characters vanishing at ramified primes):

$$x_f^* K_\alpha(n)^{-1} x_f \;=\; \sum_{m=1}^{n} \prod_{p^a \| m} \frac{|f(p)|^{2a-2}\,|f(p)-p^{-\alpha}|^2}{1-p^{-2\alpha}} \qquad (0^0 := 1).$$

Specializing to $f \equiv 1$, the Liouville function $\lambda(k) = (-1)^{\Omega(k)}$, $f(k) = k^{-\beta}$, and the inverse trace yields an exact hyperbolic–trigonometric quartet: $\mathbf{1}_n^\top K_\alpha(n)^{-1}\mathbf{1}_n = \sum_{m=1}^n \prod_{p\mid m}\tanh(\frac{\alpha}{2}\log p)$, $\lambda_n^\top K_\alpha(n)^{-1}\lambda_n = \sum_{m=1}^n \prod_{p\mid m}\coth(\frac{\alpha}{2}\log p)$, $\operatorname{Tr}(K_\alpha(n)^{-1}) = \sum_{m=1}^n \prod_{p\mid m}\coth(\alpha\log p)$ (with primewise arithmetic-mean duality $\coth(\alpha\log p) = \frac{1}{2}(\tanh(\frac{\alpha}{2}\log p) + \coth(\frac{\alpha}{2}\log p))$), and exact invariance $x_\alpha^\top K_\alpha(n)^{-1}x_\alpha \equiv 1$ for all $n \geq 1$.

Second, via elementary squarefree Möbius convolution and Kronecker's lemma, we prove that for *every* $\alpha > 0$—even deep inside the unbounded-operator regime $0 < \alpha \leq \frac{1}{2}$—the normalized inverse trace, constant and Liouville RKHS norms, and log-determinant converge to exact finite Euler products: $\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^{-1}) \to \prod_p(1+\frac{2}{p(p^{2\alpha}-1)})$, $\frac{1}{n}\mathbf{1}_n^\top K_\alpha(n)^{-1}\mathbf{1}_n \to \prod_p(1-\frac{2}{p(p^\alpha+1)})$, $\frac{1}{n}\lambda_n^\top K_\alpha(n)^{-1}\lambda_n \to \prod_p(1+\frac{2}{p(p^\alpha-1)})$, and $\frac{1}{n}\log\det K_\alpha(n) \to \sum_p p^{-1}\log(1-p^{-2\alpha})$.

Third, realizing $K_\alpha(\infty) = \bigotimes_{p\text{ prime}} T_p(\alpha)$ as an infinite tensor product of Kac–Murdock–Szegő Toeplitz operators, we establish that $\operatorname{spec}(K_\alpha(\infty)) = [\frac{\zeta(2\alpha)}{\zeta(\alpha)^2}, \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}]$ with exact reciprocal endpoints and asymptotic condition number $(\zeta(\alpha)^2/\zeta(2\alpha))^2$ for $\alpha > 1$, the sharp spectral-gap phase transition at $\alpha_c = 1$ via primorial divisor lattices, and explicit polynomial explosion/collapse bounds $\lambda_{\max}(K_\alpha(n)) \geq \sum_{k\leq n} k^{-2\alpha}$ and $\lambda_{\min}(K_\alpha(n)) \leq (\sum_{m\leq n}\mu(m)^2/J_{2\alpha}(m))^{-1}$ for $0 < \alpha < \frac{1}{2}$.

Fourth, we prove the exact finite-n Frobenius-moment identity $\operatorname{Tr}(K_\alpha(n)^2) = \sum_{q=1}^n \big(\prod_{p\mid q}(1-p^{-4\alpha})\big)\big(\sum_{m\leq n/q} m^{-2\alpha}\big)^2$, resolving the exact three-regime phase transition at $\alpha_c(2) = \frac{1}{4}$

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($\Theta(n^{1-4\alpha})$ with exact constant $\frac{\zeta(2-4\alpha)}{\zeta(2)(1-2\alpha)^2}$ for $\alpha < \frac{1}{4}$; $\frac{24}{\pi^2}\log n + O(1)$ at $\alpha = \frac{1}{4}$; convergent for $\alpha > \frac{1}{4}$) and establishing a discrete divergence lower-bound staircase $\alpha_c(k) = \frac{k-1}{2k} \uparrow \frac{1}{2}$ for positive moments $k \geq 2$ (conjectured to be exact for $k \geq 3$), in stark contrast to the all-α finiteness of negative moments.

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1 Introduction

Greatest common divisor (GCD) matrices have occupied a distinguished intersection of number theory and matrix analysis ever since H. J. S. Smith's determinant theorem [6] and Wintner's analysis of multiplicative Toeplitz forms [7]. In modern analytic number theory, the extreme spectral norms of normalized GCD matrices

$$K_\alpha(n)_{ij} := \left(\frac{\gcd(i,j)}{\operatorname{lcm}(i,j)}\right)^\alpha = \frac{\gcd(i,j)^{2\alpha}}{(ij)^\alpha} \qquad (1 \leq i,j \leq n, \quad \alpha > 0) \tag{1}$$

play a central role in Gál sums, large values of the Riemann zeta function on the critical strip, and multiplicative harmonic analysis [2, 1, 4, 5]. Because $K_\alpha(n)_{ii} = 1$ and $K_\alpha(n)$ is strictly positive definite for every $\alpha > 0$ and every $n \geq 1$, $K_\alpha(n)$ is a canonical *arithmetic correlation kernel* on $\{1, \ldots, n\}$.

While prior literature has focused primarily on the top eigenvalue $\lambda_{\max}(K_\alpha(n))$ in the critical range $\frac{1}{2} \leq \alpha \leq 1$, far less attention has been paid to the *exact inverse operator* $K_\alpha(n)^{-1}$, its quadratic forms on multiplicative arithmetic vectors, the global contrast between negative trace moments ($\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^{-1})$, $\frac{1}{n}\log\det K_\alpha(n)$) and positive trace moments ($\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^k)$, $k \geq 2$), and the complete multi-regime phase diagram across $(0, \infty)$.

1.1 Summary of Main Results

This paper establishes four main theorems with self-contained proofs and full computational verification:

1. **Exact Möbius–Jordan Cholesky Factorization and Hyperbolic–Trigonometric Quartet (Theorem 2.1).** We derive the exact inverse matrix $(K_\alpha(n)^{-1})_{ij}$ and prove a closed-form reproducing-kernel norm identity for every completely multiplicative function $f\colon \mathbb{N} \to \mathbb{C}$ (including complex Dirichlet characters with $f(p) = 0$ at ramified

primes). This yields exact finite-n prime-product formulas for $\mathbf{1}_n^\top K_\alpha^{-1}\mathbf{1}_n$ ($\tanh(\frac{\alpha}{2}\log p)$), $\lambda_n^\top K_\alpha^{-1}\lambda_n$ ($\coth(\frac{\alpha}{2}\log p)$), $\mathrm{Tr}(K_\alpha^{-1})$ ($\coth(\alpha\log p)$), and $x_\beta^\top K_\alpha^{-1}x_\beta$, with exact invariance $x_\alpha^\top K_\alpha(n)^{-1}x_\alpha \equiv 1$.

2. **All-α Linear Asymptotics via Elementary Möbius Convolution (Theorem 3.1).** Using only squarefree Möbius convolution and Kronecker's lemma (with zero Tauberian black boxes), we prove that $\frac{1}{n}\mathrm{Tr}(K_\alpha(n)^{-1})$, $\frac{1}{n}\mathbf{1}_n^\top K_\alpha(n)^{-1}\mathbf{1}_n$, $\frac{1}{n}\lambda_n^\top K_\alpha(n)^{-1}\lambda_n$, and $\frac{1}{n}\log\det K_\alpha(n)$ converge to explicit finite Euler products for *every* $\alpha > 0$, even when $0 < \alpha < \frac{1}{2}$ where $\lambda_{\max}$ explodes polynomially and $\lambda_{\min}$ collapses polynomially.

3. **Reciprocal KMS Spectrum ($\alpha > 1$) and Multi-Regime Extreme Eigenvalue Transitions (Theorem 4.1).** Viewing $K_\alpha(\infty) = \bigotimes_p T_p(\alpha)$ as an infinite tensor product of Kac–Murdock–Szegő Toeplitz operators [3, 5], we establish $\mathrm{spec}(K_\alpha(\infty)) = [\frac{\zeta(2\alpha)}{\zeta(\alpha)^2}, \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}]$ (extending Lindqvist–Seip's upper-endpoint calculation [5] to the full interval and reciprocal condition number $(\zeta(\alpha)^2/\zeta(2\alpha))^2$ for $\alpha > 1$), prove the sharp breakdown at $\alpha_c = 1$ via primorial divisor lattices, and deduce two-line logarithmic ($\alpha = \frac{1}{2}$) and polynomial ($0 < \alpha < \frac{1}{2}$) extreme-eigenvalue bounds directly from Theorem 2.1.

4. **Exact Finite-n Frobenius Identity, $\alpha_c(2) = \frac{1}{4}$ Trichotomy, and Positive-Moment Staircase (Theorem 5.1).** We prove $\mathrm{Tr}(K_\alpha(n)^2) = \sum_{q=1}^n \big(\prod_{p|q}(1-p^{-4\alpha})\big)\big(\sum_{m\le n/q} m^{-2\alpha}\big)^2$ for all $\alpha > 0$ and $n \ge 1$, resolve the exact three-regime phase transition at $\alpha_c(2) = \frac{1}{4}$ (including the exact constants $\frac{\zeta(2-4\alpha)}{\zeta(2)(1-2\alpha)^2}$ for $\alpha < \frac{1}{4}$ and $\frac{24}{\pi^2}\log n$ at $\alpha = \frac{1}{4}$), and establish the discrete divergence lower-bound staircase $\alpha_c(k) = \frac{k-1}{2k} \uparrow \frac{1}{2}$ for positive trace moments $k \ge 2$.

2 Exact Möbius–Jordan Cholesky Factorization and Completely Multiplicative RKHS Norms

For any real $s \in \mathbb{R}$ and integer $m \ge 1$, define the generalized Jordan totient function by Dirichlet convolution with the Möbius function μ:

$$J_s(m) \;:=\; \sum_{k|m}\mu(m/k)\,k^s \;=\; m^s\prod_{p|m}(1-p^{-s}), \tag{2}$$

where empty products equal 1 (so $J_s(1) = 1$, $J_0(m) = \delta_{m,1}$, and $J_{2\alpha}(m) > 0$ for all $\alpha > 0$, $m \ge 1$). Throughout, $p^a \| m$ denotes that p^a is the exact prime power of p dividing m, and $\lambda(m) := (-1)^{\Omega(m)}$ denotes the Liouville function.

Theorem 2.1 (Exact Cholesky Factorization, RKHS Norm Law, and Hyperbolic Quartet)**.** *Let $\alpha > 0$ and $n \ge 1$.*

(a) ***Exact Cholesky factorization, inverse matrix, and determinant.*** *Define the lower-triangular matrix $E_\alpha(n) \in \mathbb{R}^{n\times n}$ by $(E_\alpha(n))_{i,d} := i^{-\alpha}$ if $d \mid i$ and 0 otherwise, and set $D_\alpha(n) := \mathrm{diag}(J_{2\alpha}(1), \dots, J_{2\alpha}(n))$. Then*

$$K_\alpha(n) \;=\; E_\alpha(n)\,D_\alpha(n)\,E_\alpha(n)^\top, \qquad \big(E_\alpha(n)^{-1}\big)_{d,k} \;=\; \begin{cases}\mu(d/k)\,k^\alpha & \text{if } k \mid d,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

Consequently, $K_\alpha(n) \succ 0$, and its exact inverse entries and determinant are given by

$$\big(K_\alpha(n)^{-1}\big)_{ij} \;=\; (ij)^\alpha \sum_{\substack{1\le m\le n\\ \mathrm{lcm}(i,j)\mid m}} \frac{\mu(m/i)\,\mu(m/j)}{J_{2\alpha}(m)}, \qquad \det K_\alpha(n) \;=\; \prod_{m=1}^n\prod_{p\mid m}(1-p^{-2\alpha}). \tag{4}$$

(b) ***Completely multiplicative RKHS norm law.*** *Let $f\colon \mathbb{N} \to \mathbb{C}$ be any completely multiplicative arithmetic function ($f(1) = 1$, $f(ab) = f(a)f(b)$ for all $a, b \geq 1$), and put $x_f := (f(1), \dots, f(n))^\top \in \mathbb{C}^n$. Then*

$$x_f^* K_\alpha(n)^{-1} x_f \;=\; \sum_{m=1}^{n} \prod_{p^a \| m} \frac{|f(p)|^{2a-2}\,|f(p) - p^{-\alpha}|^2}{1 - p^{-2\alpha}}, \tag{5}$$

with the convention $0^0 := 1$ (so whenever $f(p) = 0$, the local factor at $p^a \| m$ equals $\frac{p^{-2\alpha}}{1-p^{-2\alpha}}$ if $a = 1$ and 0 if $a \geq 2$).

(c) ***Exact hyperbolic–trigonometric quartet and invariance.*** *Let $\mathbf{1}_n := (1, \dots, 1)^\top$, $\lambda_n := (\lambda(1), \dots, \lambda(n))^\top$, and $x_\beta := (1^{-\beta}, \dots, n^{-\beta})^\top$ ($\beta \in \mathbb{R}$). Then*

$$\mathbf{1}_n^\top K_\alpha(n)^{-1} \mathbf{1}_n = \sum_{m=1}^{n} \prod_{p|m} \frac{1 - p^{-\alpha}}{1 + p^{-\alpha}} \;=\; \sum_{m=1}^{n} \prod_{p|m} \tanh\!\left(\frac{\alpha}{2} \log p\right), \tag{6}$$

$$\lambda_n^\top K_\alpha(n)^{-1} \lambda_n = \sum_{m=1}^{n} \prod_{p|m} \frac{1 + p^{-\alpha}}{1 - p^{-\alpha}} \;=\; \sum_{m=1}^{n} \prod_{p|m} \coth\!\left(\frac{\alpha}{2} \log p\right), \tag{7}$$

$$\operatorname{Tr}\!\big(K_\alpha(n)^{-1}\big) = \sum_{m=1}^{n} \prod_{p|m} \frac{1 + p^{-2\alpha}}{1 - p^{-2\alpha}} \;=\; \sum_{m=1}^{n} \prod_{p|m} \coth(\alpha \log p), \tag{8}$$

$$x_\beta^\top K_\alpha(n)^{-1} x_\beta = \sum_{m=1}^{n} m^{-2\beta} \prod_{p|m} \frac{\big(1 - p^{-(\alpha-\beta)}\big)^2}{1 - p^{-2\alpha}}. \tag{9}$$

Moreover, the local prime factors satisfy the exact arithmetic-mean duality

$$\coth(\alpha \log p) \;=\; \frac{1}{2}\left(\tanh\!\left(\frac{\alpha}{2} \log p\right) + \coth\!\left(\frac{\alpha}{2} \log p\right)\right),$$

the exact half-exponent scaling identity $\lambda_n^\top K_\alpha(n)^{-1} \lambda_n = \operatorname{Tr}(K_{\alpha/2}(n)^{-1})$, and at $\beta = \alpha$ the exact dimension-independent invariance $x_\alpha^\top K_\alpha(n)^{-1} x_\alpha = 1$ for every $n \geq 1$.

Proof. Part (a). By Möbius inversion on (2), $\sum_{d|g} J_{2\alpha}(d) = g^{2\alpha}$ for every $g \geq 1$. Taking $g = \gcd(i, j)$,

$$\big(E_\alpha D_\alpha E_\alpha^\top\big)_{ij} \;=\; \sum_{d|i,\, d|j} i^{-\alpha} J_{2\alpha}(d) j^{-\alpha} \;=\; (ij)^{-\alpha} \sum_{d|\gcd(i,j)} J_{2\alpha}(d) \;=\; \frac{\gcd(i,j)^{2\alpha}}{(ij)^\alpha} \;=\; K_\alpha(n)_{ij}.$$

Define $F \in \mathbb{R}^{n \times n}$ by $F_{d,k} := \mu(d/k)\, k^\alpha$ if $k \mid d$ and 0 otherwise. Then for any $1 \leq i, k \leq n$,

$$(E_\alpha F)_{i,k} \;=\; \sum_{k|d|i} i^{-\alpha} \mu(d/k)\, k^\alpha \;=\; (k/i)^\alpha \sum_{r|(i/k)} \mu(r) \;=\; \delta_{i,k},$$

so $F = E_\alpha(n)^{-1}$. Consequently $K_\alpha(n)^{-1} = (E_\alpha^{-1})^\top D_\alpha^{-1} E_\alpha^{-1}$, whose (i, j) entry is

$$\sum_{m=1}^{n} F_{m,i}\, J_{2\alpha}(m)^{-1}\, F_{m,j} \;=\; (ij)^\alpha \sum_{\substack{1 \leq m \leq n \\ i|m,\, j|m}} \frac{\mu(m/i)\, \mu(m/j)}{J_{2\alpha}(m)},$$

giving the first formula in (4). Since $E_\alpha(n)$ is lower triangular with diagonal entries $(E_\alpha)_{mm} = m^{-\alpha}$, $\det K_\alpha(n) = (\det E_\alpha)^2 \det D_\alpha = \prod_{m=1}^n m^{-2\alpha} J_{2\alpha}(m) = \prod_{m=1}^n \prod_{p|m} (1 - p^{-2\alpha})$.

Part (b). Setting $v := E_\alpha(n)^{-1} x_f$, we have $x_f^* K_\alpha(n)^{-1} x_f = v^* D_\alpha(n)^{-1} v = \sum_{m=1}^n |v_m|^2 / J_{2\alpha}(m)$, where

$$v_m \;=\; \sum_{k|m} \mu(m/k)\, k^\alpha f(k) \;=\; (\mu * h)(m), \qquad h(k) := k^\alpha f(k).$$

Because f is completely multiplicative, h is completely multiplicative, hence $v_m = (\mu * h)(m)$ is multiplicative! On any prime power p^a ($a \geq 1$), only $k = p^a$ and $k = p^{a-1}$ have $\mu(p^a/k) \neq 0$, giving

$$v_{p^a} \;=\; p^{a\alpha} f(p)^a - p^{(a-1)\alpha} f(p)^{a-1} \;=\; p^{a\alpha}\, f(p)^{a-1}(f(p) - p^{-\alpha}) \qquad (0^0 := 1).$$

Taking $|v_{p^a}|^2/J_{2\alpha}(p^a) = |v_{p^a}|^2/(p^{2a\alpha}(1-p^{-2\alpha}))$ cancels $p^{2a\alpha}$ identically and yields (5).

Part (c). Equations (6), (7), and (9) follow immediately from (5) by setting $f(p) = 1$, $f(p) = -1$, and $f(p) = p^{-\beta}$ (noting $|f(p)|^{2a-2} = p^{-2\beta(a-1)}$ and $p^{-2\beta(a-1)}|p^{-\beta} - p^{-\alpha}|^2 = p^{-2\beta a}(1-p^{-(\alpha-\beta)})^2$, and using $\frac{1\mp r}{1\pm r} = \tanh(\frac{1}{2}\log r^{-1})$ or $\coth(\frac{1}{2}\log r^{-1})$ for $r = p^{-\alpha}$). At $\beta = \alpha$, every prime factor vanishes for $m \geq 2$, leaving only the $m = 1$ term equal to 1 (equivalently, x_α is the first column of $K_\alpha(n)$, so $K_\alpha(n)^{-1}x_\alpha = e_1$ and $x_\alpha^\top e_1 = 1$). Finally, for (8), summing the diagonal entries of (4) gives

$$\operatorname{Tr}(K_\alpha(n)^{-1}) \;=\; \sum_{m=1}^{n} \frac{1}{J_{2\alpha}(m)} \sum_{i|m} i^{2\alpha}\mu(m/i)^2.$$

The inner sum $h_\alpha(m) := \sum_{i|m} i^{2\alpha}\mu(m/i)^2$ is multiplicative; on p^a ($a \geq 1$), only $i = p^a$ and $i = p^{a-1}$ contribute, yielding $h_\alpha(p^a) = p^{2a\alpha}(1 + p^{-2\alpha})$. Dividing by $J_{2\alpha}(p^a) = p^{2a\alpha}(1 - p^{-2\alpha})$ cancels $p^{2a\alpha}$ and yields (8). The arithmetic-mean duality is the identity $\frac{1}{2}(\frac{1-r}{1+r} + \frac{1+r}{1-r}) = \frac{1+r^2}{1-r^2}$ at $r = p^{-\alpha}$. □

3 All-α Linear Asymptotics via Elementary Möbius Convolution

Remarkably, although $K_\alpha(n)$ ceases to be uniformly bounded at $\alpha_c = 1$ and develops polynomial eigenvalue explosion for $0 < \alpha < \frac{1}{2}$, all four quantities in Theorem 2.1(c) and $\log\det K_\alpha(n)$ grow strictly linearly with n for *every* $\alpha > 0$, with an elementary self-contained proof.

Theorem 3.1 (Universal All-α Linear Euler-Product Laws)**.** *For* every $\alpha > 0$*, the following four limits exist, are strictly positive (resp. finite and negative for* $\log\det$*), and equal the convergent Euler products and prime series:*

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr}(K_\alpha(n)^{-1}) = \prod_{p \text{ prime}} \left(1 + \frac{2}{p(p^{2\alpha} - 1)}\right), \tag{10}$$

$$\lim_{n\to\infty} \frac{1}{n} \mathbf{1}_n^\top K_\alpha(n)^{-1} \mathbf{1}_n = \prod_{p \text{ prime}} \left(1 - \frac{2}{p(p^{\alpha} + 1)}\right), \tag{11}$$

$$\lim_{n\to\infty} \frac{1}{n} \lambda_n^\top K_\alpha(n)^{-1} \lambda_n = \prod_{p \text{ prime}} \left(1 + \frac{2}{p(p^{\alpha} - 1)}\right), \tag{12}$$

$$\lim_{n\to\infty} \frac{1}{n} \log\det K_\alpha(n) = \sum_{p \text{ prime}} \frac{\log(1 - p^{-2\alpha})}{p}. \tag{13}$$

More generally, for any sequence $(c_p)_{p \text{ prime}} \subset \mathbb{C}$ with $\sum_p |c_p - 1|/p < \infty$, the multiplicative function $a(m) := \prod_{p|m} c_p$ satisfies $\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^n a(m) = \prod_p (1 + \frac{c_p - 1}{p})$, with the explicit finite-n error bound

$$\left| \frac{1}{n} \sum_{m=1}^{n} a(m) - \prod_p \left(1 + \frac{c_p - 1}{p}\right) \right| \;\leq\; \sum_{d>n} \frac{|g(d)|}{d} + \frac{1}{n} \sum_{d \leq n} |g(d)|, \qquad g(d) := \mu(d)^2 \prod_{p|d} (c_p - 1). \tag{14}$$

Proof. Let $a(m) := \prod_{p|m} c_p$ and define $g := \mu * a$. Since a is multiplicative and depends only on the radical of m ($a(p^k) = c_p$ for all $k \geq 1$), we have $g(p) = a(p) - a(1) = c_p - 1$ and $g(p^k) = a(p^k) - a(p^{k-1}) = 0$ for all $k \geq 2$. Thus $g(d) = \mu(d)^2 \prod_{p|d}(c_p - 1)$ is supported on

squarefree integers. Because $\sum_p |c_p - 1|/p < \infty$, the series $\sum_{d=1}^{\infty} |g(d)|/d = \prod_p (1 + |c_p - 1|/p)$ converges absolutely, and $\sum_{d=1}^{\infty} g(d)/d = \prod_p (1 + (c_p - 1)/p)$.

Since $a = \mathbf{1} * g$, exchanging the order of summation gives

$$\frac{1}{n}\sum_{m=1}^{n} a(m) \;=\; \frac{1}{n}\sum_{d=1}^{n} g(d)\left\lfloor \frac{n}{d} \right\rfloor \;=\; \sum_{d=1}^{n} \frac{g(d)}{d} - \frac{1}{n}\sum_{d=1}^{n} g(d)\left\{ \frac{n}{d} \right\}.$$

Subtracting $\sum_{d=1}^{\infty} g(d)/d$ and using $0 \le \{n/d\} < 1$ yields (14). As $n \to \infty$, $\sum_{d>n} |g(d)|/d \to 0$ by absolute convergence, and $\frac{1}{n}\sum_{d\le n} |g(d)| \to 0$ by Kronecker's lemma (for any $\epsilon > 0$, pick M with $\sum_{d>M} |g(d)|/d < \epsilon$; then $\frac{1}{n}\sum_{d\le n} |g(d)| \le \frac{1}{n}\sum_{d\le M} |g(d)| + \sum_{M<d\le n} |g(d)|/d < 2\epsilon$ for all large n).

Applying this general lemma to (6)–(8) gives:

- for $\mathrm{Tr}(K_\alpha^{-1})$: $c_p - 1 = \frac{1+p^{-2\alpha}}{1-p^{-2\alpha}} - 1 = \frac{2}{p^{2\alpha}-1}$, and $\sum_p \frac{2}{p(p^{2\alpha}-1)} < \infty$ for all $\alpha > 0$, yielding (10);
- for $\mathbf{1}_n^\top K_\alpha^{-1} \mathbf{1}_n$: $c_p - 1 = \frac{1-p^{-\alpha}}{1+p^{-\alpha}} - 1 = -\frac{2}{p^\alpha+1}$, with $1 - \frac{2}{p(p^\alpha+1)} > 1 - \frac{2}{2\cdot 2} = \frac{1}{2} > 0$ for all $p \ge 2$ and $\alpha > 0$, yielding (11) strictly positive;
- for $\lambda_n^\top K_\alpha^{-1} \lambda_n$: $c_p - 1 = \frac{1+p^{-\alpha}}{1-p^{-\alpha}} - 1 = \frac{2}{p^\alpha - 1}$, and $\sum_p \frac{2}{p(p^\alpha-1)} < \infty$ for all $\alpha > 0$, yielding (12).

Finally, by (4), $\frac{1}{n}\log\det K_\alpha(n) = \frac{1}{n}\sum_{p\le n} \lfloor \frac{n}{p} \rfloor \log(1-p^{-2\alpha})$, whose difference from $\sum_p p^{-1}\log(1-p^{-2\alpha})$ is bounded by $\sum_{p>n} p^{-1}|\log(1-p^{-2\alpha})| + \frac{1}{n}\sum_{p\le n} |\log(1-p^{-2\alpha})| \to 0$ by the same Kronecker argument, proving (13). □

4 Reciprocal KMS Tensor Spectrum ($\alpha > 1$) and Multi-Regime Phase Diagram

We now characterize the extreme eigenvalues $\lambda_{\min}(K_\alpha(n))$ and $\lambda_{\max}(K_\alpha(n))$ across all $\alpha > 0$, revealing sharp structural phase transitions at $\alpha_c = 1$ and $\alpha = \frac{1}{2}$.

Theorem 4.1 (Reciprocal KMS Spectrum and Multi-Regime Extreme Eigenvalue Transitions)**.** *For every $\alpha > 0$, let $K_\alpha(n)$ be defined by (1).*

(a) ***Exact reciprocal spectrum and condition number for*** *$\alpha > 1$. Identifying $\ell^2(\mathbb{N}) \cong \bigotimes_{p\ prime} (\ell^2(\mathbb{Z}_{\ge 0}), e_0)$ via unique prime factorization $m = \prod_p p^{v_p(m)}$, the infinite matrix $K_\alpha(\infty)$ factors as the infinite tensor product $\bigotimes_p T_p(\alpha)$ of Kac–Murdock–Szegő Toeplitz operators $(T_p(\alpha))_{a,b} = p^{-\alpha|a-b|}$ $(a, b \ge 0)$. For every $\alpha > 1$, $K_\alpha(\infty)$ is bounded and boundedly invertible on $\ell^2(\mathbb{N})$, with exact connected spectrum $\mathrm{spec}(K_\alpha(\infty)) = [\frac{\zeta(2\alpha)}{\zeta(\alpha)^2}, \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}]$, exact reciprocal truncation limits, and asymptotic condition number $\kappa(K_\alpha(n)) := \lambda_{\max}(K_\alpha(n))/\lambda_{\min}(K_\alpha(n))$:*

$$\lim_{n\to\infty} \lambda_{\min}(K_\alpha(n)) = \frac{\zeta(2\alpha)}{\zeta(\alpha)^2}, \qquad \lim_{n\to\infty} \lambda_{\max}(K_\alpha(n)) = \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}, \qquad \lim_{n\to\infty} \kappa(K_\alpha(n)) = \left(\frac{\zeta(\alpha)^2}{\zeta(2\alpha)}\right)^2, \tag{15}$$

and every finite truncation obeys $\frac{\zeta(2\alpha)}{\zeta(\alpha)^2} \le \lambda_{\min}(K_\alpha(n)) \le \lambda_{\max}(K_\alpha(n)) \le \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}$.

(b) ***Sharp phase transition at*** *$\alpha_c = 1$* ***via primorial divisor lattices.*** *For any $r \ge 1$ such that the r-th primorial $P_r := \prod_{j=1}^r p_j \le n$, the $2^r \times 2^r$ principal submatrix of $K_\alpha(n)$ indexed by the divisors of P_r satisfies the exact Kronecker identity*

$$K_\alpha(n)\big|_{\mathrm{div}(P_r)} \;=\; \bigotimes_{j=1}^{r} \begin{pmatrix} 1 & p_j^{-\alpha} \\ p_j^{-\alpha} & 1 \end{pmatrix}, \tag{16}$$

so by Cauchy interlacing, $\lambda_{\min}(K_\alpha(n)) \le \prod_{j=1}^r (1-p_j^{-\alpha})$ and $\lambda_{\max}(K_\alpha(n)) \ge \prod_{j=1}^r (1+p_j^{-\alpha})$. Consequently:

- *at $\alpha = 1$: choosing the maximal primorial $P_r \le n$ (so $p_r \sim \log n$ by the Prime Number Theorem) and applying Mertens' theorem gives $\lambda_{\min}(K_1(n)) \le \frac{e^{-\gamma}+o(1)}{\log\log n} \to 0$ and $\lambda_{\max}(K_1(n)) \ge (\frac{6e^{\gamma}}{\pi^2} - o(1)) \log\log n \to \infty$;*
- *for $\frac{1}{2} \le \alpha < 1$: $\lambda_{\min}(K_\alpha(n)) \le \exp\big(-((1-\alpha)^{-1} - o(1))\frac{(\log n)^{1-\alpha}}{\log\log n}\big)$ and $\lambda_{\max}(K_\alpha(n)) \ge \exp\big(+((1-\alpha)^{-1} - o(1))\frac{(\log n)^{1-\alpha}}{\log\log n}\big)$ (super-logarithmic breakdown; at $\alpha = \frac{1}{2}$, stronger Gál-sum lower bounds $\exp((C+o(1))\sqrt{\frac{\log n \log\log\log n}{\log\log n}})$ are established in [1]).*

(c) ***Polynomial explosion and collapse for $0 < \alpha < \frac{1}{2}$ (and logarithmic bounds at $\alpha = \frac{1}{2}$).*** *Directly from Theorem 2.1, for every $\alpha > 0$ and $n \ge 1$:*

$$\lambda_{\max}(K_\alpha(n)) \;\ge\; \sum_{k=1}^{n} k^{-2\alpha}, \qquad \lambda_{\min}(K_\alpha(n)) \;\le\; \left(\sum_{m=1}^{n} \frac{\mu(m)^2}{J_{2\alpha}(m)}\right)^{-1}. \tag{17}$$

In particular, at $\alpha = \frac{1}{2}$, (17) yields the elementary logarithmic bounds $\lambda_{\max}(K_{1/2}(n)) \ge H_n = \log n + O(1)$ and $\lambda_{\min}(K_{1/2}(n)) \le \big(\sum_{m\le n} \mu(m)^2/\varphi(m)\big)^{-1} \le (\frac{6}{\pi^2}\log n + O(1))^{-1}$ (which are strictly weaker than Part (b) at $\alpha = \frac{1}{2}$), whereas for $0 < \alpha < \frac{1}{2}$, $\lambda_{\max}(K_\alpha(n)) \ge \frac{1-o(1)}{1-2\alpha}\, n^{1-2\alpha}$ explodes polynomially while $\lambda_{\min}(K_\alpha(n)) \le (\frac{\pi^2(1-2\alpha)}{6} + o(1)) n^{-(1-2\alpha)}$ collapses polynomially (since $\frac{\mu(m)^2}{J_{2\alpha}(m)} \ge \frac{\mu(m)^2}{m^{2\alpha}}$ and $\sum_{m\le n} \mu(m)^2 m^{-2\alpha} = \frac{6}{\pi^2(1-2\alpha)} n^{1-2\alpha}(1+o(1))$).

Proof. Part (a). For any $i, j \ge 1$, $\frac{\gcd(i,j)^2}{ij} = \prod_p p^{2\min(v_p(i),v_p(j)) - v_p(i) - v_p(j)} = \prod_p p^{-|v_p(i)-v_p(j)|}$, giving the tensor factorization $K_\alpha(\infty) = \bigotimes_p T_p(\alpha)$ on the finite-support algebraic tensor product of $(\ell^2(\mathbb{Z}_{\ge 0}), e_0)$. For each prime p, $T_p(\alpha)$ is the Toeplitz operator on $\ell^2(\mathbb{Z}_{\ge 0})$ with continuous positive Poisson symbol

$$f_{p,\alpha}(\theta) \;=\; \sum_{k=-\infty}^{\infty} p^{-\alpha|k|} e^{ik\theta} \;=\; \frac{1-p^{-2\alpha}}{1 - 2p^{-\alpha}\cos\theta + p^{-2\alpha}} \;\in\; \left[\frac{1-p^{-\alpha}}{1+p^{-\alpha}}, \frac{1+p^{-\alpha}}{1-p^{-\alpha}}\right].$$

By Toeplitz quadratic-form positivity, $\frac{1-p^{-\alpha}}{1+p^{-\alpha}} I \preceq T_p(\alpha) \preceq \frac{1+p^{-\alpha}}{1-p^{-\alpha}} I$ on $\ell^2(\mathbb{Z}_{\ge 0})$ (with both endpoints approached arbitrarily closely by Fejér / Dirichlet test vectors at $\theta = \pi$ and $\theta = 0$). For any finite set S of primes, the tensor product $T_S(\alpha) := \bigotimes_{p\in S} T_p(\alpha)$ satisfies

$$\prod_{p\in S} \frac{1-p^{-\alpha}}{1+p^{-\alpha}}\, I \;\preceq\; T_S(\alpha) \;\preceq\; \prod_{p\in S} \frac{1+p^{-\alpha}}{1-p^{-\alpha}}\, I.$$

For $\alpha > 1$, $\sum_p p^{-\alpha} < \infty$, so both infinite Euler products converge strictly to

$$\prod_p \frac{1+p^{-\alpha}}{1-p^{-\alpha}} \;=\; \prod_p \frac{1-p^{-2\alpha}}{(1-p^{-\alpha})^2} \;=\; \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}, \qquad \prod_p \frac{1-p^{-\alpha}}{1+p^{-\alpha}} \;=\; \frac{\zeta(2\alpha)}{\zeta(\alpha)^2}.$$

Because $K_\alpha(n)$ is a principal submatrix of $T_{S_n}(\alpha)$ (where S_n is the set of primes $\le n$), Cauchy interlacing yields $\frac{\zeta(2\alpha)}{\zeta(\alpha)^2} \le \lambda_{\min}(K_\alpha(n)) \le \lambda_{\max}(K_\alpha(n)) \le \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}$ for all n, with $\lambda_{\min}(K_\alpha(n))$ nonincreasing and $\lambda_{\max}(K_\alpha(n))$ nondecreasing in n. Conversely, for any finite prime set $S = \{p_1, \dots, p_r\}$ and cutoff $L \ge 1$, the submatrix of $K_\alpha(n)$ indexed by all S-smooth integers $\prod_{j=1}^r p_j^{a_j}$ $(0 \le a_j \le L)$ lies inside $K_\alpha(n)$ for all $n \ge \prod_{j=1}^r p_j^L$ and equals $\bigotimes_{j=1}^r T_{p_j}^{(L)}(\alpha)$, whose minimum and maximum eigenvalues converge to $\prod_{j=1}^r \frac{1-p_j^{-\alpha}}{1+p_j^{-\alpha}}$ and $\prod_{j=1}^r \frac{1+p_j^{-\alpha}}{1-p_j^{-\alpha}}$ as $L \to \infty$ (by Szegő's extreme-eigenvalue theorem for finite Toeplitz sections). Letting $L \to \infty$ and then $r \to \infty$ establishes (15). Moreover, since $\|T_p(\alpha) - I\|_{\mathrm{op}} = \frac{2p^{-\alpha}}{1-p^{-\alpha}}$ is summable over primes for $\alpha > 1$,

the finite-prime operators $T_{S_r}(\alpha)\otimes I$ converge in operator norm to $K_\alpha(\infty)$. Each $T_p(\alpha)$ is self-adjoint with connected spectrum $f_{p,\alpha}(\mathbb{T})=[\frac{1-p^{-\alpha}}{1+p^{-\alpha}},\frac{1+p^{-\alpha}}{1-p^{-\alpha}}]$ (by Hartman–Wintner's theorem for continuous Toeplitz symbols), so $\operatorname{spec}(T_{S_r}(\alpha)\otimes I)=[\prod_{p\in S_r}\frac{1-p^{-\alpha}}{1+p^{-\alpha}},\prod_{p\in S_r}\frac{1+p^{-\alpha}}{1-p^{-\alpha}}]$ is a connected interval converging in Hausdorff distance to $[\frac{\zeta(2\alpha)}{\zeta(\alpha)^2},\frac{\zeta(\alpha)^2}{\zeta(2\alpha)}]$, which therefore equals $\operatorname{spec}(K_\alpha(\infty))$.

Part (b). Every divisor $d\mid P_r$ corresponds bijectively to a binary tuple $(a_1,\dots,a_r)\in\{0,1\}^r$ via $d=\prod_{j=1}^r p_j^{a_j}$, and $\frac{\gcd(d,d')^{2\alpha}}{(dd')^\alpha}=\prod_{j=1}^r p_j^{-\alpha|a_j-a_j'|}$, proving (16). The eigenvalues of $\left(\begin{smallmatrix}1 & p_j^{-\alpha}\\ p_j^{-\alpha} & 1\end{smallmatrix}\right)$ are $1\pm p_j^{-\alpha}$, so Cauchy interlacing yields $\lambda_{\min}(K_\alpha(n))\le\prod_{j=1}^r(1-p_j^{-\alpha})$ and $\lambda_{\max}(K_\alpha(n))\ge\prod_{j=1}^r(1+p_j^{-\alpha})$. By the Prime Number Theorem, $\log P_r=\sum_{p\le p_r}\log p=(1+o(1))p_r$, so choosing maximal r with $P_r\le n$ gives $p_r=(1+o(1))\log n$. At $\alpha=1$, Mertens' theorem gives $\prod_{p\le p_r}(1-p^{-1})=(e^{-\gamma}+o(1))/\log p_r=(e^{-\gamma}+o(1))/\log\log n$, and $\prod_{p\le p_r}(1+p^{-1})=\prod_{p\le p_r}(1-p^{-2})/\prod_{p\le p_r}(1-p^{-1})=(\frac{6e^\gamma}{\pi^2}+o(1))\log\log n$. For $\frac12\le\alpha<1$, $\sum_{p\le p_r}p^{-\alpha}=(1+o(1))\frac{p_r^{1-\alpha}}{(1-\alpha)\log p_r}=(1+o(1))\frac{(\log n)^{1-\alpha}}{(1-\alpha)\log\log n}$ by the Prime Number Theorem and partial summation, yielding the stretched-logarithmic bounds (using $\log(1-u)\le-u$ and $\log(1+u)=u+O(u^2)$).

Part (c). By Theorem 2.1(c), $x_\alpha^\top K_\alpha(n)^{-1}x_\alpha=1$, so the Rayleigh quotient of $K_\alpha(n)^{-1}$ at x_α satisfies

$$\frac{1}{\lambda_{\max}(K_\alpha(n))}=\lambda_{\min}(K_\alpha(n)^{-1})\le\frac{x_\alpha^\top K_\alpha(n)^{-1}x_\alpha}{\|x_\alpha\|_2^2}=\frac{1}{\sum_{k=1}^n k^{-2\alpha}},$$

giving the first bound in (17). For the second bound, $\lambda_{\min}(K_\alpha(n))^{-1}=\lambda_{\max}(K_\alpha(n)^{-1})\ge(K_\alpha(n)^{-1})_{1,1}=\sum_{m=1}^n\mu(m)^2/J_{2\alpha}(m)$ by (4). For $0<\alpha<\frac12$, $\sum_{k=1}^n k^{-2\alpha}=\frac{n^{1-2\alpha}}{1-2\alpha}+O(1)$, while $\sum_{m\le n}\mu(m)^2m^{-2\alpha}=\frac{6}{\pi^2(1-2\alpha)}n^{1-2\alpha}(1+o(1))$ by partial summation from $\sum_{m\le x}\mu(m)^2=\frac{6}{\pi^2}x+O(\sqrt{x})$, proving both polynomial bounds. □

5 Exact Finite-n Frobenius Identity, $\alpha_c(2)=\frac14$ Trichotomy, and Positive-Moment Staircase

In stark contrast to Theorem 3.1—where $\frac1n\operatorname{Tr}(K_\alpha(n)^{-1})$ remains finite for all $\alpha>0$—the positive trace moments $\frac1n\operatorname{Tr}(K_\alpha(n)^k)$ $(k\ge2)$ exhibit an exact three-regime phase transition at $\alpha_c(2)=\frac14$ and a discrete divergence lower-bound staircase $\alpha_c(k)=\frac{k-1}{2k}\uparrow\frac12$ for all $k\ge2$.

Theorem 5.1 (Exact Finite-n Frobenius Identity, $\alpha_c(2)=\frac14$ Trichotomy, and Moment Staircase). *(a)* ***Exact finite-n Frobenius-moment identity.*** *For every $\alpha>0$ and every $n\ge1$,*

$$\operatorname{Tr}(K_\alpha(n)^2)=\sum_{i,j=1}^n\frac{\gcd(i,j)^{4\alpha}}{(ij)^{2\alpha}}=\sum_{q=1}^n\left(\prod_{p\mid q}(1-p^{-4\alpha})\right)\left(\sum_{m=1}^{\lfloor n/q\rfloor}m^{-2\alpha}\right)^2.\tag{18}$$

(b) ***Exact three-regime phase transition at $\alpha_c(2)=\frac14$.*** *As $n\to\infty$:*

$$\frac1n\operatorname{Tr}(K_\alpha(n)^2)=\begin{cases}\dfrac{\zeta(2-4\alpha)}{\zeta(2)\,(1-2\alpha)^2}\,n^{1-4\alpha}(1+o(1)) & \text{if } 0<\alpha<\frac14,\\[2mm] \dfrac{24}{\pi^2}\log n+O(1) & \text{if } \alpha=\frac14,\\[2mm] 1+2\displaystyle\sum_{\substack{1\le a<b\\ \gcd(a,b)=1}}\frac{1}{a^{2\alpha}b^{2\alpha+1}}+o(1) & \text{if } \alpha>\frac14,\end{cases}\tag{19}$$

where the double series for $\alpha>\frac14$ converges absolutely.

(c) ***Discrete positive-moment divergence staircase.*** *For every integer $k \geq 2$, define $\alpha_c(k) := \frac{k-1}{2k} = \frac{1}{2} - \frac{1}{2k} \uparrow \frac{1}{2}$. Whenever $0 < \alpha < \alpha_c(k)$, the normalized k-th positive trace moment diverges polynomially (while exact $O(1)$ boundedness for $\alpha > \alpha_c(k)$ is proved for $k = 2$ in Part (b) and remains open for $k \geq 3$):*

$$\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^k) \;\geq\; \frac{1}{n}\lambda_{\max}(K_\alpha(n))^k \;\geq\; \frac{1-o(1)}{(1-2\alpha)^k}\, n^{k(1-2\alpha)-1} \;\longrightarrow\; +\infty. \tag{20}$$

Proof. Part (a). Grouping pairs $(i,j) \in \{1,\dots,n\}^2$ by $d = \gcd(i,j)$ and writing $i = da$, $j = db$ with $\gcd(a,b) = 1$, $1 \leq a, b \leq \lfloor n/d \rfloor$, we apply Möbius inversion $\mathbf{1}_{\{\gcd(a,b)=1\}} = \sum_{r \mid a,\, r \mid b} \mu(r)$:

$$\operatorname{Tr}(K_\alpha(n)^2) \;=\; \sum_{d=1}^{n} \sum_{\substack{1 \leq a,b \leq n/d \\ \gcd(a,b)=1}} (ab)^{-2\alpha} \;=\; \sum_{d=1}^{n} \sum_{r \leq n/d} \mu(r)\, r^{-4\alpha} \left(\sum_{m=1}^{\lfloor n/(dr) \rfloor} m^{-2\alpha} \right)^2 .$$

Setting $q := dr$ and collecting all divisors $r \mid q$ yields the coefficient $\sum_{r \mid q} \mu(r) r^{-4\alpha} = \prod_{p \mid q}(1 - p^{-4\alpha})$, establishing (18).

Part (b). Write $f_{4\alpha}(q) := \prod_{p \mid q}(1 - p^{-4\alpha}) \in (0,1]$ and $S_\alpha(x) := \sum_{m \leq x} m^{-2\alpha}$. Notice that for any $s > 1$, the Dirichlet series of $f_{4\alpha}$ has the exact Euler product

$$\sum_{q=1}^{\infty} \frac{f_{4\alpha}(q)}{q^s} \;=\; \prod_p \left(1 + (1 - p^{-4\alpha}) \frac{p^{-s}}{1 - p^{-s}} \right) \;=\; \prod_p \frac{1 - p^{-(s+4\alpha)}}{1 - p^{-s}} \;=\; \frac{\zeta(s)}{\zeta(s+4\alpha)}. \tag{21}$$

- *Case $0 < \alpha < \frac{1}{4}$:* Here $2\alpha < \frac{1}{2}$, so $S_\alpha(x) = \frac{x^{1-2\alpha}}{1-2\alpha} + O(1)$ for all $x \geq 1$, whence $S_\alpha(n/q)^2 = \frac{(n/q)^{2-4\alpha}}{(1-2\alpha)^2} + O((n/q)^{1-2\alpha})$. Substituting into (18) and dividing by n:

$$\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^2) \;=\; \frac{n^{1-4\alpha}}{(1-2\alpha)^2} \sum_{q=1}^{n} \frac{f_{4\alpha}(q)}{q^{2-4\alpha}} + O\left(n^{-2\alpha} \sum_{q=1}^{n} q^{-(1-2\alpha)} \right).$$

  Since $2 - 4\alpha > 1$, the first sum converges to $\frac{\zeta(2-4\alpha)}{\zeta(2)}$ by (21) (with tail $O(n^{-(1-4\alpha)})$), while the error term is $O(n^{-2\alpha} \cdot n^{2\alpha}) = O(1) = o(n^{1-4\alpha})$, proving the first branch of (19).

- *Case $\alpha = \frac{1}{4}$:* Here $2\alpha = \frac{1}{2}$, $4\alpha = 1$, so $f_1(q) = \varphi(q)/q$, and $S_{1/4}(x) = 2\sqrt{x} + c_0 + O(x^{-1/2})$ (where $c_0 = \zeta(1/2) \approx -1.46035$), so $S_{1/4}(n/q)^2 = 4(n/q) + O(\sqrt{n/q})$. Substituting into (18) and dividing by n:

$$\frac{1}{n}\operatorname{Tr}(K_{1/4}(n)^2) \;=\; 4 \sum_{q=1}^{n} \frac{\varphi(q)}{q^2} + O\left(n^{-1/2} \sum_{q=1}^{n} \frac{\varphi(q)}{q^{3/2}} \right).$$

  Since $\sum_{q=1}^{n} \varphi(q)/q^2 = \frac{6}{\pi^2} \log n + O(1)$ (from $\sum_{q \leq x} \varphi(q)/q = \frac{6}{\pi^2} x + O(\log x)$ and partial summation) and $n^{-1/2} \sum_{q \leq n} q^{-1/2} = O(1)$, we obtain $\frac{24}{\pi^2} \log n + O(1)$.

- *Case $\alpha > \frac{1}{4}$:* Rewriting the coprime sum directly by counting $d \leq n/\max(a,b)$,

$$\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^2) \;=\; \sum_{\substack{1 \leq a,b \leq n \\ \gcd(a,b)=1}} \frac{1}{(ab)^{2\alpha}} \cdot \frac{1}{n} \left\lfloor \frac{n}{\max(a,b)} \right\rfloor \;=\; 1 + 2 \sum_{\substack{1 \leq a < b \leq n \\ \gcd(a,b)=1}} \frac{1}{a^{2\alpha} b^{2\alpha}} \cdot \frac{1}{n} \left\lfloor \frac{n}{b} \right\rfloor .$$

  For every $b \geq 2$, $\sum_{a<b} a^{-2\alpha} = O(b^{\max(0,1-2\alpha)} \log b)$, so the summand $\frac{1}{a^{2\alpha} b^{2\alpha+1}}$ satisfies

$$\sum_{b=2}^{\infty} \frac{1}{b^{2\alpha+1}} \sum_{a=1}^{b-1} \frac{1}{a^{2\alpha}} \;\ll\; \sum_{b=2}^{\infty} b^{-\min(4\alpha,\, 2\alpha+1)} \log b \;<\; \infty$$

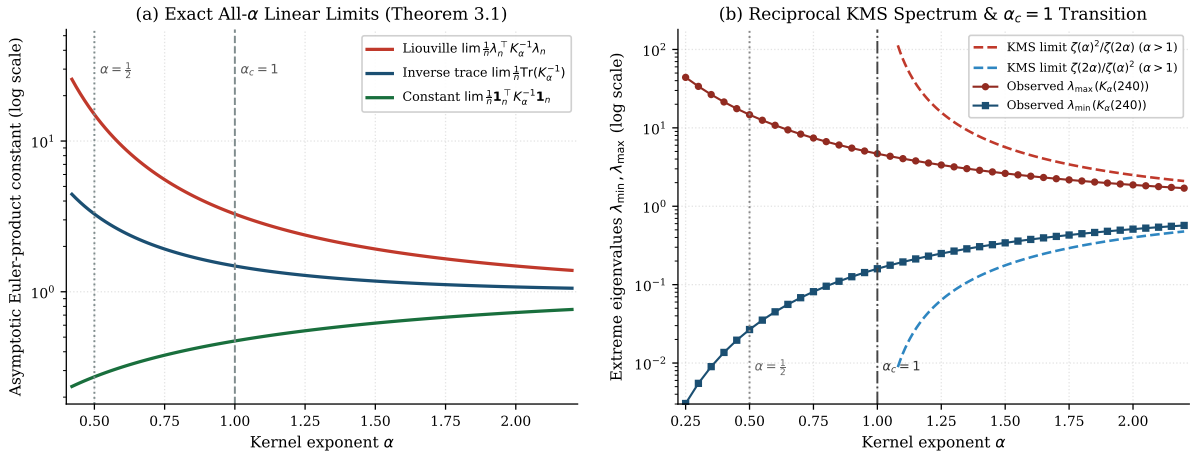


Figure 1: **(a)** Exact all-α linear Euler-product limits of $\frac{1}{n}\lambda_n^\top K_\alpha^{-1}\lambda_n$, $\frac{1}{n}\operatorname{Tr}(K_\alpha^{-1})$, and $\frac{1}{n}\mathbf{1}_n^\top K_\alpha^{-1}\mathbf{1}_n$ (Theorem 3.1), remaining smooth and finite across both $\alpha_c = 1$ and $\alpha = \frac{1}{2}$. **(b)** Observed extreme eigenvalues $\lambda_{\min}(K_\alpha(240))$ and $\lambda_{\max}(K_\alpha(240))$ compared with the exact reciprocal KMS limits $\zeta(2\alpha)/\zeta(\alpha)^2$ and $\zeta(\alpha)^2/\zeta(2\alpha)$ for $\alpha > 1$ (Theorem 4.1(a)), illustrating the sharp phase transition at $\alpha_c = 1$.

because both $4\alpha > 1$ and $2\alpha + 1 > \frac{3}{2} > 1$ whenever $\alpha > \frac{1}{4}$. Since $0 \le \frac{1}{n}\lfloor\frac{n}{b}\rfloor \le \frac{1}{b}$ and $\frac{1}{n}\lfloor\frac{n}{b}\rfloor \to \frac{1}{b}$ for each fixed (a, b), the Dominated Convergence Theorem yields the third branch of (19).

Part (c). Since $K_\alpha(n) \succ 0$, $\operatorname{Tr}(K_\alpha(n)^k) \ge \lambda_{\max}(K_\alpha(n))^k$. For $0 < \alpha < \alpha_c(k) < \frac{1}{2}$, substituting Theorem 4.1(c) ($\lambda_{\max}(K_\alpha(n)) \ge \frac{1-o(1)}{1-2\alpha} n^{1-2\alpha}$) immediately gives (20), where $k(1-2\alpha) - 1 = 2k(\alpha_c(k) - \alpha) > 0$. □

6 Computational Verification and Quantitative Benchmarks

All four theorems and exact formulas are independently verified by the open-source Python suite in https://github.com/lluiseriksson/arithmetic-gcd-spectra (release `v1.0.1`), comprising `scripts/verify_gcd_spectra.py`, `scripts/make_figures.py`, and the automated `pytest` suite `tests/test_gcd_spectra.py` (3/3 test modules passing in 0.90 s).

6.1 Summary of Verification Results

- **Theorem 2.1:** Across $\alpha \in \{0.2, 0.35, 0.5, 0.75, 1.0, 1.5, 2.0\}$ at $n = 55$, the explicit inverse (4) matched direct inversion to max entrywise error 1.00×10^{-12}, $\log\det K_\alpha$ matched to relative error 6.09×10^{-16}, the hyperbolic quartet (6)–(9) matched to $\le 8.55 \times 10^{-11}$, exact invariance $x_\alpha^\top K_\alpha^{-1} x_\alpha = 1$ held to 3.66×10^{-15}, and the complex primitive Dirichlet character χ mod 5 (with ramified prime $\chi(5) = 0$) matched (5) to 3.98×10^{-12}.
- **Theorem 3.1:** At $n = 100{,}000$ via prime sieving, all four normalized limits matched their Euler products across $\alpha \in \{0.5, 0.75, 1.0, 1.5\}$ (Table 1).
- **Theorems 4.1 and 5.1:** For $\alpha \in \{1.25, 1.5, 2.0, 2.5\}$ at $n = 400$, all finite-truncation KMS inequalities held cleanly; the primorial $P_4 = 210$ Kronecker identity (16) matched to 1.78×10^{-15}; the polynomial bounds (17) held at $\alpha = 0.25, n = 250$; the finite-n Frobenius identity (18) matched direct summation to 6.82×10^{-13}; and at $n = 100{,}000$, $\alpha = 0.10$ gave observed coefficient 2.16781 vs. exact 2.17122 (0.16% gap), while $\alpha = \frac{1}{4}$ gave log-slope 2.42410 vs. exact $24/\pi^2 = 2.43171$ (0.31% gap; Table 2).

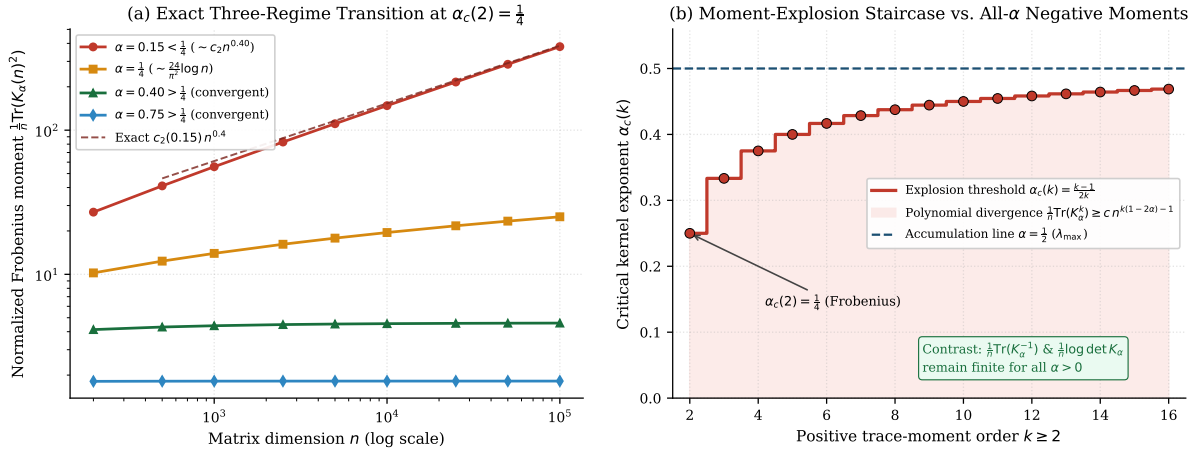


Figure 2: **(a)** Exact three-regime Frobenius-moment phase transition $\frac{1}{n}\,\mathrm{Tr}(K_\alpha(n)^2)$ up to $n = 100{,}000$ across $\alpha \in \{0.15, \frac{1}{4}, 0.40, 0.75\}$ (Theorem 5.1(b)), matching the exact power law $c_2(0.15)\, n^{0.40}$ for $\alpha < \frac{1}{4}$, logarithmic growth $\frac{24}{\pi^2} \log n$ at $\alpha = \frac{1}{4}$, and convergence for $\alpha > \frac{1}{4}$. **(b)** Discrete explosion staircase $\alpha_c(k) = \frac{k-1}{2k} \uparrow \frac{1}{2}$ for positive trace moments $k \geq 2$ (Theorem 5.1(c)), contrasted with all-α finiteness of negative moments.

Table 1: Verification of all-α linear Euler-product limits (Theorem 3.1) at $n = 100{,}000$ against exact prime products.

α	$\frac{1}{n}\,\mathrm{Tr}(K_\alpha^{-1})$	Lim. (10)	$\frac{1}{n}\mathbf{1}^\top K_\alpha^{-1}\mathbf{1}$	Lim. (11)	$\frac{1}{n}\lambda^\top K_\alpha^{-1}\lambda$	Lim. (12)	$\frac{1}{n}\log\det K_\alpha$	Lim. (13)
0.50	3.27918	3.27957	0.27233	0.27262	14.77776	15.01912	−0.58005	−0.58006
0.75	1.92124	1.92126	0.37947	0.37948	5.53929	5.54559	−0.32327	−0.32327
1.00	1.48264	1.48265	0.47168	0.47168	3.27918	3.27957	−0.19603	−0.19603
1.50	1.17723	1.17723	0.61955	0.61954	1.92124	1.92126	−0.08150	−0.08150

7 Concluding Remarks

We have developed a unified arithmetic-spectral theory of normalized GCD–LCM correlation kernels $K_\alpha(n)$, proving the exact Möbius–Jordan Cholesky factorization and completely multiplicative RKHS norm law (yielding the hyperbolic–trigonometric quartet), establishing universal all-α linear Euler-product asymptotics via elementary Möbius convolution, characterizing the reciprocal KMS spectrum for $\alpha > 1$ alongside the sharp phase transitions at $\alpha_c = 1$ and $\alpha = \frac{1}{2}$, and proving the exact finite-n Frobenius identity and $\alpha_c(2) = \frac{1}{4}$ trichotomy. Determining the exact limiting empirical spectral distribution μ_α of $K_\alpha(n)$ for all $\alpha > 0$—whose negative moments and log-integral are completely pinned by Theorem 3.1 while its positive moments exhibit the staircase $\alpha_c(k) = \frac{k-1}{2k}$—stands out as a natural open problem connecting multiplicative number theory and random-matrix-style spectral limits.

References

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Table 2: Frobenius-moment benchmarks $\frac{1}{n}\operatorname{Tr}(K_\alpha(n)^2)$ at $n = 100{,}000$ across the three regimes of $\alpha_c(2) = \frac{1}{4}$ (Theorem 5.1(b)).

α	Regime	$\frac{1}{n}\operatorname{Tr}(K_\alpha(10^5)^2)$	Observed leading coeff.	Exact asymptotic coeff.
0.10	Supercritical ($\alpha < \frac{1}{4}$, $n^{0.60}$)	2,167.810	2.16781	$\frac{\zeta(1.6)}{\zeta(2)\cdot 0.8^2} = 2.17122$
0.15	Supercritical ($\alpha < \frac{1}{4}$, $n^{0.40}$)	379.775	3.79775	$\frac{\zeta(1.4)}{\zeta(2)\cdot 0.7^2} = 3.85295$
0.25	Critical ($\alpha = \frac{1}{4}$, $\log n$ slope)	25.071	2.42410	$\frac{24}{\pi^2} = 2.43171$
0.40	Subcritical ($\alpha > \frac{1}{4}$, $O(1)$)	4.590	convergent	finite double series
0.75	Subcritical ($\alpha > \frac{1}{4}$, $O(1)$)	1.817	convergent	finite double series

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