AIRR.SCIENCE · ARR-2026-352ESGYHXQ9Y0VDJ v1 [math.DS] 29 Nov 2026

Endemic Markov Factorization, Sharp Thompson Contraction, Volterra–KL Lyapunov Laws, and Green–Kubo Perron-Centrality Control for Multigroup SIS Epidemic Networks

Lluis Eriksson
Independent Researcher, Östersund, Sweden
editor@airr.science

November 29, 2026

Abstract

We establish four self-contained, non-asymptotic structural theorems governing equilibrium geometry, spectral stability, global nonlinear Lyapunov contraction, and convex optimal vaccination control for multigroup Susceptible–Infected–Susceptible (SIS) epidemic networks $\dot{x} = (\mathbf{1}_n - x) \circ (Bx) - D_\gamma x$ on $[0,1]^n$ ($n \geq 2$) with irreducible nonnegative transmission matrix $B \in \mathbb{R}^{n\times n}_{\geq 0}$, recovery rates $\gamma \in \mathbb{R}^n_{>0}$, next-generation matrix $K = D_\gamma^{-1}B$, and basic reproduction number $R_0 = \rho(K)$. First, we prove the global threshold dichotomy with an explicit exponential extinction rate $(1 - R_0)\gamma_{\min}$ for $R_0 < 1$, a short irreducibility proof of critical extinction at $R_0 = 1$, a closed-form supercritical prevalence envelope $\mathcal{B} = [x^-, x^+]$ for $R_0 > 1$, and the sharp nonlinear Thompson-metric contraction law $d_T(T(x), T(y)) \leq \ln\big(e^{d_T(x,y)}/(1 - \alpha + \alpha e^{d_T(x,y)})\big) \leq (1 - \alpha)d_T(x, y)$ on $\mathcal{B}$ for the endemic fixed-point operator $T(x)_i = (Kx)_i/(1 + (Kx)_i)$. Second, introducing the row-stochastic endemic Markov matrix $P^* = D_{x^*}^{-1} D_{1-x^*} K D_{x^*}$ with stationary distribution π^*, we prove the exact Jacobian similarity $D_{x^*}^{-1} J(x^*) D_{x^*} = -D_{\gamma/(1-x^*)}(I_n - D_{1-x^*}P^*)$, derive the exact two-sided spectral abscissa bracket $-\max_i(Bx^*)_i \leq \alpha(J(x^*)) \leq -\min_i(Bx^*)_i < 0$, and establish positive M-matrix intervention sensitivities, an exact π^*-weighted conservation identity, and universal uniform log-elasticity bounds. Third, we prove that the endemic Volterra–KL Lyapunov functional $V(x) = \sum_{i=1}^n \frac{\pi_i^*}{\gamma_i}(\frac{x_i}{x_i^*} - 1 - \ln\frac{x_i}{x_i^*})$ satisfies an exact nonnegative mixing-plus-saturation dissipation identity and decays globally exponentially as $V(x(t)) \leq \exp(-c_0 \min_i(Bx^*)_i\, t)\, V(x(0))$ from every $x(0) \in (0,1]^n$. Fourth, under log-recovery control $K(u) = D_{e^{-u}}K$, we prove that the exact Hessian of $\ln R_0(u)$ is the symmetric part of the Green–Kubo asymptotic variance matrix $D_p(2Z - I_n + \mathbf{1}_n p^\top)$ of the next-generation Markov chain $Q(u)$, establish its positive semidefiniteness in one line via $L^2(p)$-contractivity ($\|y\|_p^2 - \|Qy\|_p^2 \geq 0$), and derive the equalized Perron-centrality-per-cost KKT optimality law and closed-form R_0 lower bound. All four theorems are verified across 270 randomized network instances with zero bound violations.

Keywords: multigroup SIS epidemic models, basic reproduction number, Thompson metric, endemic Markov chain, Volterra–KL Lyapunov function, Green–Kubo variance, Perron centrality, optimal vaccination control.
MSC 2020: 34D23, 92D30, 15B48, 60J10, 90C25.

1 Introduction

Heterogeneous multigroup Susceptible–Infected–Susceptible (SIS) epidemic networks provide a foundational model of persistent contagion on weighted directed graphs [1, 4, 5, 8]. Given $n \geq 2$ coupled subpopulations with infection prevalences $x_i(t) \in [0,1]$, nonnegative transmission

rates $B = (B_{ij}) \in \mathbb{R}^{n\times n}_{\geq 0}$, and strictly positive recovery rates $\gamma = (\gamma_1, \ldots, \gamma_n)^\top \in \mathbb{R}^n_{>0}$, the deterministic mean-field dynamics evolve according to

$$\dot{x}_i = (1 - x_i) \sum_{j=1}^{n} B_{ij} x_j - \gamma_i x_i, \qquad i \in [n] := \{1, \ldots, n\}, \tag{1}$$

or in vector form $\dot{x} = (\mathbf{1}_n - x) \circ (Bx) - D_\gamma x$, where $\circ$ denotes the Hadamard product and $D_z := \operatorname{diag}(z_1, \ldots, z_n)$. Defining the *next-generation matrix* $K := D_\gamma^{-1} B \in \mathbb{R}^{n\times n}_{\geq 0}$ and the *basic reproduction number* $R_0 := \rho(K)$ [2, 3], the classical Lajmanovich–Yorke theorem [1] shows that when B is irreducible, the disease-free equilibrium 0 is globally asymptotically stable for $R_0 \leq 1$, whereas a unique endemic equilibrium $x^* \in (0, 1)^n$ is globally asymptotically stable on $(0, 1]^n$ for $R_0 > 1$.

Despite extensive qualitative theory [1, 4, 7, 8], four quantitative non-asymptotic questions have lacked a unified, elementary, closed-form treatment:

1. *Sharp nonlinear contraction and endemic localization:* What explicit closed-form order box $\mathcal{B} = [x^-, x^+] \subset (0, 1)^n$ traps x^* in terms of (R_0, w), and what is the sharp nonlinear Thompson-metric contraction constant of the endemic fixed-point operator $T(x)_i = (Kx)_i/(1 + (Kx)_i)$ on $\mathcal{B}$?

2. *Endemic spectral abscissa and policy sensitivity:* How does the Jacobian $J(x^*)$ factorize through an intrinsic row-stochastic Markov chain P^* at the endemic equilibrium, what sharp two-sided bounds trap the local exponential convergence rate $\alpha(J(x^*))$, and what exact conservation and log-elasticity laws govern equilibrium responses to recovery or transmission interventions?

3. *Global non-asymptotic exponential decay of the full nonlinear flow:* Does the nonlinear ODE (1) admit an explicit global Lyapunov functional $V(x)$ whose dissipation decomposes into exact nonnegative Markov mixing and logistic saturation terms and yields an explicit initial-condition-dependent exponential decay rate from every initial state $x(0) \in (0, 1]^n$?

4. *Second-order geometry of R_0 under resource-constrained control:* When log-recovery interventions shift $\gamma_i \mapsto \gamma_i e^{u_i}$ (so $K(u) = D_{e^{-u}} K$) [6, 9], what exact spectral object equals the Hessian $\nabla^2 \ln R_0(u)$, and what closed-form optimality condition and universal lower bound govern budget-constrained R_0 minimization?

Standing Assumptions and Notation

Throughout, $n \geq 2$, $B \in \mathbb{R}^{n\times n}_{\geq 0}$ is irreducible, $\gamma \in \mathbb{R}^n_{>0}$, and $K := D_\gamma^{-1} B$ is therefore nonnegative and irreducible with $R_0 := \rho(K) > 0$. By the Perron–Frobenius theorem [10], R_0 is a simple eigenvalue of K with strictly positive left and right eigenvectors $v, w \in \mathbb{R}^n_{>0}$ ($v^\top K = R_0 v^\top$, $Kw = R_0 w$), normalized uniquely by $\|w\|_\infty = \max_i w_i = 1$ and $\langle v, w \rangle = 1$. We write $w_{\min} := \min_i w_i \in (0, 1]$ and $\gamma_{\min} := \min_i \gamma_i > 0$. On $\mathbb{R}^n_{>0}$, Thompson's part metric [11, 12] is $d_T(x, y) := \max_{i\in[n]} |\ln(x_i/y_i)|$. We use $D_{1-x} := \operatorname{diag}(1 - x_1, \ldots, 1 - x_n)$ and write $\phi(z) := z - 1 - \ln z \geq 0$ for $z > 0$. Notice that $[0, 1]^n$ is positively invariant under (1): at $x_i = 0$, $\dot{x}_i = (Bx)_i \geq 0$, while at $x_i = 1$, $\dot{x}_i = -\gamma_i < 0$.

2 Global Threshold Dichotomy, Endemic Envelope, and Sharp Thompson Contraction

Positive equilibria $x \in (0, 1)^n$ of (1) satisfy $(1 - x_i)(Kx)_i = x_i$, which is algebraically equivalent to the fixed-point equation $x = T(x)$ for the monotone operator $T : \mathbb{R}^n_{\geq 0} \to [0, 1)^n$ (mapping

$\mathbb{R}^n_{>0}$ into $(0,1)^n$) defined by

$$T(x)_i := \frac{(Kx)_i}{1+(Kx)_i}, \qquad i \in [n]. \tag{2}$$

Theorem 2.1 (Global Dichotomy, Endemic Envelope, and Sharp Thompson Contraction)**.** *Let $B \in \mathbb{R}^{n\times n}_{\geq 0}$ be irreducible, $\gamma \in \mathbb{R}^n_{>0}$, $K = D_\gamma^{-1}B$, and $R_0 = \rho(K)$ with normalized Perron eigenvectors $v, w \in \mathbb{R}^n_{>0}$ ($\|w\|_\infty = 1$, $\langle v, w\rangle = 1$).*

(a) ***Subcritical exponential extinction and critical extinction.*** *Define $L_v(x) := \sum_{i=1}^n \frac{v_i}{\gamma_i} x_i$. Along every trajectory $x(t) \in [0,1]^n$ of (1),*

$$\dot{L}_v(x) = -(1-R_0)\langle v, x\rangle - \sum_{i=1}^n v_i x_i (Kx)_i. \tag{3}$$

In particular, if $R_0 < 1$, then $L_v(x(t)) \leq e^{-(1-R_0)\gamma_{\min} t}\, L_v(x(0))$ for all $t \geq 0$. If $R_0 = 1$, then $x(t) \to 0$ as $t \to \infty$ for every $x(0) \in [0,1]^n$.

(b) ***Supercritical endemic envelope and nonlinear Thompson contraction.*** *Assume $R_0 > 1$, and define the order box $\mathcal{B} := \{x \in \mathbb{R}^n : x^- \leq x \leq x^+\} \subset (0,1)^n$ by*

$$x_i^- := \left(1 - \frac{1}{R_0}\right) w_i, \qquad x_i^+ := \min\left\{ \left(1 - \frac{1}{R_0}\right) \frac{w_i}{w_{\min}},\ \frac{(K\mathbf{1}_n)_i}{1+(K\mathbf{1}_n)_i} \right\}, \qquad i \in [n]. \tag{4}$$

Set $\alpha := \min_{i\in[n]} x_i^- = (1 - R_0^{-1}) w_{\min} \in (0,1)$. Then $T(\mathcal{B}) \subseteq \mathcal{B}$, and for all $x, y \in \mathcal{B}$,

$$d_T(T(x), T(y)) \leq \ln \frac{e^{d_T(x,y)}}{1 - \alpha + \alpha e^{d_T(x,y)}} \leq (1-\alpha)\, d_T(x,y). \tag{5}$$

Consequently, T has a unique fixed point $x^* \in \mathbb{R}^n_{>0}$, which satisfies $x^- \leq x^* \leq x^+$ and is the unique nonzero equilibrium of (1) in $[0,1]^n$.

Proof. (a) Differentiating $L_v(x)$ along (1) and using $v^\top K = R_0 v^\top$ gives

$$\begin{aligned}\dot{L}_v(x) &= \sum_{i=1}^n \frac{v_i}{\gamma_i}\big((1-x_i)(Bx)_i - \gamma_i x_i\big) \\ &= v^\top K x - v^\top x - \sum_{i=1}^n v_i x_i (Kx)_i = -(1-R_0)\langle v, x\rangle - \sum_{i=1}^n v_i x_i (Kx)_i.\end{aligned}$$

When $R_0 < 1$, $\langle v, x\rangle = \sum_i \gamma_i \frac{v_i}{\gamma_i} x_i \geq \gamma_{\min} L_v(x)$ yields $\dot{L}_v(x) \leq -(1-R_0)\gamma_{\min} L_v(x)$, and Grönwall's inequality gives exponential decay. When $R_0 = 1$, $\dot{L}_v(x) = -\sum_i v_i x_i (Kx)_i \leq 0$ on the compact positively invariant set $[0,1]^n$. By LaSalle's invariance principle, the ω-limit set Ω of any trajectory is a nonempty positively invariant subset of $E := \{x \in [0,1]^n : x_i (Kx)_i = 0\ \forall i\}$. Suppose for contradiction that some $x(0) \in \Omega$ is nonzero, and let $S := \operatorname{supp}(x(0)) \neq \emptyset$. Since K is irreducible and $n \geq 2$, every row of K is nonzero; hence $S \neq [n]$ (otherwise $x_i(0)(Kx(0))_i > 0$ for all i). By irreducibility of B, there exist $k \notin S$ and $j \in S$ with $B_{kj} > 0$ (thus $K_{kj} > 0$). Then $x_j(0) > 0$ and $\dot{x}_k(0) = (Bx(0))_k \geq B_{kj} x_j(0) > 0$, so for sufficiently small $t > 0$ both $x_k(t) > 0$ and $x_j(t) > 0$, whence $x_k(t)(Kx(t))_k \geq K_{kj} x_k(t) x_j(t) > 0$, contradicting positive invariance of $\Omega \subseteq E$. Hence $\Omega = \{0\}$ and $x(t) \to 0$.

(b) Note that $x_i^- \leq x_i^+$ because $w_{\min} \leq 1$ and $Kx^- = (R_0 - 1)w \leq (R_0 - 1)\mathbf{1}_n$, while more directly $T(x^-)_i = \frac{(R_0-1)w_i}{1+(R_0-1)w_i} \geq \frac{(R_0-1)w_i}{1+(R_0-1)} = x_i^-$ (as $w_i \leq 1$) and $T(x^-)_i \leq T(\mathbf{1}_n)_i = \frac{(K\mathbf{1}_n)_i}{1+(K\mathbf{1}_n)_i}$ (as $x^- \leq \mathbf{1}_n$), confirming $x^- \leq x^+$. Since T is entrywise nondecreasing on $\mathbb{R}^n_{\geq 0}$, for every $x \in \mathcal{B}$

we have $T(x) \geq T(x^-) \geq x^-$. Likewise, letting $\tilde{x}^+ := (1 - R_0^{-1})(w/w_{\min})$, we have $x \leq \tilde{x}^+$ and $x \leq \mathbf{1}_n$, so

$$T(x)_i \leq T(\tilde{x}^+)_i = \frac{(R_0 - 1)w_i/w_{\min}}{1 + (R_0 - 1)w_i/w_{\min}} \leq \frac{(R_0 - 1)w_i/w_{\min}}{1 + (R_0 - 1)} = \tilde{x}_i^+$$

(using $w_i/w_{\min} \geq 1$) and $T(x)_i \leq T(\mathbf{1}_n)_i = \frac{(K\mathbf{1}_n)_i}{1+(K\mathbf{1}_n)_i}$. Thus $T(\mathcal{B}) \subseteq \mathcal{B}$.

Now let $x, y \in \mathcal{B}$ and set $d := d_T(x, y) \geq 0$. Then $x \leq e^d y$, so $(Kx)_i \leq e^d (Ky)_i$ for every $i \in [n]$. Because $u \mapsto u/(1 + u)$ is increasing on $(0, \infty)$,

$$\frac{T(x)_i}{T(y)_i} \leq \frac{e^d (Ky)_i/(1 + e^d (Ky)_i)}{(Ky)_i/(1 + (Ky)_i)} = \frac{e^d(1 + (Ky)_i)}{1 + e^d (Ky)_i} = \frac{e^d}{1 - T(y)_i + T(y)_i e^d}.$$

Since $e^d \geq 1$, the denominator $1 - t + te^d$ is nondecreasing in $t \in [0, 1]$. As $T(y)_i \geq x_i^- \geq \alpha$, we obtain $T(x)_i/T(y)_i \leq e^d/(1 - \alpha + \alpha e^d)$. Swapping x and y and taking logarithms establishes the first inequality in (5). By convexity of the exponential function, $1 - \alpha + \alpha e^d \geq e^{\alpha d}$, so $\ln\big(e^d/(1 - \alpha + \alpha e^d)\big) \leq (1 - \alpha)d$.

By the Banach fixed-point theorem on the complete metric space $(\mathcal{B}, d_T)$, T admits a unique fixed point $x^* \in \mathcal{B}$. To prove uniqueness among all nonzero equilibria in $[0, 1]^n$, note first that any nonzero equilibrium $x \in [0, 1]^n \setminus \{0\}$ satisfies $x \in (0, 1)^n$: if $x_i = 1$ then $0 = -\gamma_i < 0$, and if $x_i = 0$ then $(Bx)_i = 0$, propagating $x_j = 0$ to all in-neighbors and hence to all of $[n]$ by irreducibility. For any two fixed points $x, y \in (0, 1)^n$, let $\lambda := \min_i(x_i/y_i) > 0$. If $\lambda < 1$, then $x \geq \lambda y$ gives $x_i = T(x)_i \geq T(\lambda y)_i = \frac{\lambda(Ky)_i}{1+\lambda(Ky)_i} > \frac{\lambda(Ky)_i}{1+(Ky)_i} = \lambda y_i$ for every $i \in [n]$, contradicting the minimality of λ. Hence $\lambda \geq 1$, so $x \geq y$, and by symmetry $x = y$. □

Remark 2.2 (Limiting contraction factor for uniform-row-sum networks)**.** When $K\mathbf{1}_n = R_0\mathbf{1}_n$ ($w = \mathbf{1}_n$, $w_{\min} = 1$), the equilibrium is uniform, $x^* = (1 - R_0^{-1})\mathbf{1}_n$, and the exact local contraction rate at x^* is $\rho(DT(x^*)) = 1 - x_i^* = R_0^{-1}$ (Theorem 3.1(a)), matching the limit $1 - \alpha = 1 - (1 - R_0^{-1})w_{\min} \to R_0^{-1}$ as $w_{\min} \to 1^-$. Note that for any $w_{\min} < 1$, $T(y)_i \geq \frac{(R_0-1)w_{\min}}{1+(R_0-1)w_{\min}} > \alpha$ strictly on $\mathcal{B}$, so substituting $\tilde{\alpha} := \frac{(R_0-1)w_{\min}}{1+(R_0-1)w_{\min}}$ tightens (5) slightly while sharing the same $w_{\min} \to 1^-$ limit $1 - \tilde{\alpha} \to R_0^{-1}$.

3 Endemic Markov Factorization, Spectral Abscissa Bracket, and Intervention Laws

For $R_0 > 1$ with unique endemic equilibrium $x^* \in (0, 1)^n$, define the *endemic transition matrix* $P^* \in \mathbb{R}_{\geq 0}^{n \times n}$ by

$$P^* := D_{x^*}^{-1} D_{1-x^*} K D_{x^*}, \qquad P_{ij}^* = \frac{(1 - x_i^*)K_{ij}x_j^*}{x_i^*} = \frac{(1 - x_i^*)B_{ij}x_j^*}{\gamma_i x_i^*}. \tag{6}$$

Because $x_i^* = (1 - x_i^*)(Kx^*)_i$, we have $(P^*\mathbf{1}_n)_i = \frac{(1-x_i^*)(Kx^*)_i}{x_i^*} = 1$, so P^* is an irreducible *row-stochastic* Markov matrix with unique strictly positive stationary distribution $\pi^* \in \Delta_n^\circ := \{p \in \mathbb{R}_{>0}^n : \sum_i p_i = 1\}$ satisfying $(\pi^*)^\top P^* = (\pi^*)^\top$. Let

$$J(x^*) := D_{1-x^*}B - \operatorname{diag}(Bx^* + \gamma)$$

denote the Jacobian of the SIS vector field (1) at x^*, and let $\alpha(J(x^*)) := \max_{\lambda \in \sigma(J(x^*))} \operatorname{Re}(\lambda)$ denote its spectral abscissa.

Theorem 3.1 (Endemic Markov Factorization, Spectral Bracket, and Intervention Laws)**.** *Assume $R_0 > 1$ with unique endemic equilibrium $x^* \in (0, 1)^n$ and endemic Markov chain (P^*, π^*).*

(a) ***Exact Markov factorization and two-sided spectral abscissa bracket.*** *The discrete endemic-map Jacobian $DT(x^*)$ and continuous-time SIS Jacobian $J(x^*)$ satisfy*

$$D_{x^*}^{-1} DT(x^*) D_{x^*} = D_{1-x^*} P^*, \qquad 1 - x^*_{\max} \le \rho\big(DT(x^*)\big) \le 1 - x^*_{\min} < 1, \tag{7}$$

$$D_{x^*}^{-1} J(x^*) D_{x^*} = -D_{\gamma/(1-x^*)}\big(I_n - D_{1-x^*} P^*\big). \tag{8}$$

Moreover, the spectral abscissa of $J(x^*)$ obeys the exact hazard-rate bracket

$$-\max_{i\in[n]} (Bx^*)_i \;\le\; \alpha\big(J(x^*)\big) \;\le\; -\min_{i\in[n]} (Bx^*)_i < 0, \tag{9}$$

where $(Bx^*)_i = \gamma_i x_i^*/(1-x_i^*) > 0$.

(b) ***Positive M-matrix intervention law, π^*-conservation, and log-elasticity bounds.*** *On the supercritical irreducible domain $\mathcal{D}_+ := \{(B,\gamma) \in \mathbb{R}^{n\times n}_{\ge 0} \times \mathbb{R}^n_{>0} : B \text{ irreducible},\ R_0(D_\gamma^{-1}B) > 1\}$ (relatively open in $\mathbb{R}^{n\times n}_{\ge 0} \times \mathbb{R}^n_{>0}$), the equilibrium map $(B,\gamma) \mapsto x^*$ admits a local C^∞ extension to an open neighborhood in $\mathbb{R}^{n\times n} \times \mathbb{R}^n_{>0}$ around every point of $\mathcal{D}_+$, with entrywise strictly positive inverse $(-J(x^*))^{-1} > 0$. Under any infinitesimal perturbation $(\delta B, \delta\gamma)$ (setting $\delta \ln x^* := D_{x^*}^{-1}\delta x^*$ and $\delta \ln \gamma := D_\gamma^{-1}\delta\gamma$), the first-order log-prevalence response satisfies*

$$\delta \ln x^* = \big(I_n - D_{1-x^*}P^*\big)^{-1} D_{1-x^*}\big(D_{Bx^*}^{-1}(\delta B)x^* - \delta \ln \gamma\big), \tag{10}$$

together with the exact π^*-weighted conservation identity

$$\sum_{i=1}^n \pi_i^* x_i^* (P^* \delta \ln x^*)_i = \sum_{i=1}^n \pi_i^* (1-x_i^*)\left(\frac{(\delta B x^*)_i}{(Bx^*)_i} - \delta \ln \gamma_i\right). \tag{11}$$

In particular, for a uniform proportional recovery increase ($\delta \ln \gamma = \mathbf{1}_n$, $\delta B = 0$) or uniform proportional transmission scaling ($\delta B = B$, $\delta \ln \gamma = 0$; equivalently, the negative response to a uniform transmission cut $\delta B = -B$), every patch satisfies the exact log-elasticity bounds

$$\frac{1-x^*_{\max}}{x^*_{\max}} \;\le\; -\frac{\partial \ln x_i^*}{\partial \ln \gamma_{\text{unif}}} = \frac{\partial \ln x_i^*}{\partial \ln B_{\text{unif}}} \;\le\; \frac{1-x^*_{\min}}{x^*_{\min}} \qquad \forall i \in [n]. \tag{12}$$

Proof. (a) Since $\frac{\partial}{\partial u}\big(\frac{u}{1+u}\big) = \frac{1}{(1+u)^2}$ and $1 + (Kx^*)_i = \frac{1}{1-x_i^*}$, we have $DT(x^*) = D^2_{1-x^*}K$, so $D_{x^*}^{-1}DT(x^*)D_{x^*} = D_{1-x^*}(D_{x^*}^{-1}D_{1-x^*}KD_{x^*}) = D_{1-x^*}P^*$. Because $D_{1-x^*}P^*$ is nonnegative irreducible with row sums $(D_{1-x^*}P^*\mathbf{1}_n)_i = 1 - x_i^*$, Collatz–Wielandt bounds on $\rho(D_{1-x^*}P^*)$ immediately prove (7).

For $J(x^*)$, the equilibrium identity $(Bx^*)_i = \frac{\gamma_i x_i^*}{1-x_i^*}$ implies $(Bx^*)_i + \gamma_i = \frac{\gamma_i}{1-x_i^*}$. Hence

$$\begin{aligned} D_{x^*}^{-1} J(x^*) D_{x^*} &= D_{x^*}^{-1} D_{1-x^*} D_\gamma K D_{x^*} - D_{\gamma/(1-x^*)} \\ &= D_\gamma P^* - D_{\gamma/(1-x^*)} = -D_{\gamma/(1-x^*)}\big(I_n - D_{1-x^*}P^*\big), \end{aligned}$$

proving (8). Let $M := D_{x^*}^{-1}J(x^*)D_{x^*}$. Its off-diagonal entries are $\gamma_i P^*_{ij} \ge 0$, so M is an irreducible Metzler matrix, and choosing $c > \max_i \frac{\gamma_i}{1-x_i^*}$ makes $M + cI_n$ nonnegative and irreducible with row sums

$$\big((M+cI_n)\mathbf{1}_n\big)_i = c - \frac{\gamma_i}{1-x_i^*}\big(1 - (1-x_i^*)\big) = c - \frac{\gamma_i x_i^*}{1-x_i^*} = c - (Bx^*)_i.$$

By Perron–Frobenius / Collatz–Wielandt bounds on the nonnegative irreducible matrix $M + cI_n$, its spectral radius $\rho(M+cI_n) = \alpha(M) + c = \alpha(J(x^*)) + c$ lies in $[c - \max_i (Bx^*)_i,\ c - \min_i (Bx^*)_i]$. Subtracting c establishes (9).

(b) The defining map $G(x,B,\gamma) := (\mathbf{1}_n - x) \circ (Bx) - D_\gamma x$ is C^∞ on the open ambient space $\mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathbb{R}^n_{>0}$. At any $(B,\gamma) \in \mathcal{D}_+$ with unique positive equilibrium $x^* \in (0,1)^n$,

$\partial_x G(x^*, B, \gamma) = J(x^*)$ is invertible by (9), so the implicit function theorem yields open neighborhoods $U \subset \mathbb{R}^{n\times n} \times \mathbb{R}^n_{>0}$ of (B, γ) and $W \subset (0,1)^n$ of x^* together with a unique C^∞ solution branch $\tilde{x}^* : U \to W$ solving $G(\tilde{x}^*(B', \gamma'), B', \gamma') = 0$. To verify that $\tilde{x}^*$ coincides with the global endemic equilibrium x^* on $U \cap \mathcal{D}_+$ near (B, γ) (including along boundary faces where some $B_{ij} = 0$), observe that x^* is continuous on $\mathcal{D}_+$: for any sequence $(B^{(k)}, \gamma^{(k)}) \to (B, \gamma)$ in $\mathcal{D}_+$, compactness of $[0,1]^n$ ensures every subsequence of $x^*(B^{(k)}, \gamma^{(k)})$ has a further subsequence converging to some $\bar{x} \in [0,1]^n$. Since $R_0(K) > 1$ is a simple eigenvalue of the irreducible matrix $K = D_\gamma^{-1} B$, both $R_0(K^{(k)}) \to R_0(K) > 1$ and the normalized Perron eigenvector $w(K^{(k)}) \to w(K) \in \mathbb{R}^n_{>0}$ vary continuously, so the lower bound (4) passes to the limit $\bar{x} \geq (1 - R_0(K)^{-1}) w(K) > 0$. By continuity of G, $G(\bar{x}, B, \gamma) = 0$, hence $\bar{x} = x^*(B, \gamma)$ by uniqueness in Theorem 2.1(b). Thus $x^*(B', \gamma') \in W$ for all $(B', \gamma') \in \mathcal{D}_+$ sufficiently close to (B, γ), forcing $x^*(B', \gamma') = \tilde{x}^*(B', \gamma')$ there. Because $\rho(D_{1-x^*} P^*) \leq 1 - x^*_{\min} < 1$ and P^* is irreducible, the Neumann series $(I_n - D_{1-x^*} P^*)^{-1} = \sum_{k=0}^\infty (D_{1-x^*} P^*)^k$ converges and is entrywise strictly positive, whence $(-J(x^*))^{-1} = D_{x^*} (I_n - D_{1-x^*} P^*)^{-1} D_{(1-x^*)/\gamma} D_{x^*}^{-1} > 0$. Differentiating $G(x^*, B, \gamma) = 0$ gives $-J(x^*)\delta x^* = D_{1-x^*}(\delta B) x^* - D_{x^*} \delta\gamma$. Substituting $-J(x^*)\delta x^* = D_{x^*} D_{\gamma/(1-x^*)} (I_n - D_{1-x^*} P^*) \delta \ln x^*$ and left-multiplying by $D_{(1-x^*)/\gamma} D_{x^*}^{-1}$ (using $\frac{(1-x_i^*)^2}{\gamma_i x_i^*} = \frac{1-x_i^*}{(Bx^*)_i}$) yields

$$\big(I_n - D_{1-x^*} P^*\big) \delta \ln x^* = D_{1-x^*} \big(D_{Bx^*}^{-1} (\delta B) x^* - \delta \ln \gamma \big), \tag{13}$$

which inverts to (10). Left-multiplying (13) by $(\pi^*)^\top$ and using $(\pi^*)^\top (I_n - D_{1-x^*} P^*) = (\pi^*)^\top (I_n - P^*) + (\pi^*)^\top D_{x^*} P^* = (\pi^*)^\top D_{x^*} P^*$ proves (11). Finally, since $(I_n - D_{1-x^*} P^*) \mathbf{1}_n = \mathbf{1}_n - (\mathbf{1}_n - x^*) = x^*$, we have $(I_n - D_{1-x^*} P^*)^{-1} x^* = \mathbf{1}_n$. Because $x^*_{\min} \leq x^*_j \leq x^*_{\max}$ implies

$$\frac{1 - x^*_{\max}}{x^*_{\max}} \, x^* \;\leq\; \mathbf{1}_n - x^* \;\leq\; \frac{1 - x^*_{\min}}{x^*_{\min}} \, x^*$$

entrywise, multiplying by the entrywise positive matrix $(I_n - D_{1-x^*} P^*)^{-1}$ proves (12). □

4 Global Endemic Volterra–KL Lyapunov Contraction

For $R_0 > 1$ with endemic Markov chain (P^*, π^*), define the *endemic Volterra–KL Lyapunov functional* on $(0,1]^n$ by

$$V(x) := \sum_{i=1}^n \frac{\pi_i^*}{\gamma_i} \phi\!\left(\frac{x_i}{x_i^*}\right) = \sum_{i=1}^n \frac{\pi_i^*}{\gamma_i} \left(\frac{x_i}{x_i^*} - 1 - \ln \frac{x_i}{x_i^*} \right) \geq 0. \tag{14}$$

Theorem 4.1 (Volterra–KL Dissipation Decomposition and Global Exponential Decay)**.** *Assume $R_0 > 1$ with unique endemic equilibrium $x^* \in (0,1)^n$ and endemic Markov chain (P^*, π^*).*

(a) ***Exact mixing-plus-saturation dissipation identity.*** *For every $x \in (0,1]^n$, setting $z_i := x_i / x_i^* > 0$, the time derivative of V along (1) satisfies*

$$\dot{V}(x) = -\sum_{i,j=1}^n \pi_i^* P_{ij}^* \, \phi\!\left(\frac{z_j}{z_i}\right) - \sum_{i=1}^n \pi_i^* \frac{x_i^*}{1 - x_i^*} \frac{(z_i - 1)^2}{z_i} \, (P^* z)_i \;\leq\; 0, \tag{15}$$

with $\dot{V}(x) = 0$ on $(0,1]^n$ if and only if $x = x^*$.

(b) ***Global non-asymptotic exponential contraction.*** *For every $c \in (0,1]$, the compact order box $\mathcal{C}_c := \{x \in [0,1]^n : c x^* \leq x \leq \mathbf{1}_n\} \subset (0,1]^n$ is positively invariant under (1). Consequently, for every initial condition $x(0) \in (0,1]^n$, setting $c_0 := \min\{1, \min_{j\in[n]} (x_j(0)/x_j^*)\} \in (0,1]$, the trajectory $x(t)$ satisfies*

$$V(x(t)) \;\leq\; \exp\!\left(-c_0 \min_{i\in[n]} (Bx^*)_i \cdot t \right) V(x(0)) \qquad \forall t \geq 0. \tag{16}$$

Proof. (a) Using $\gamma_i x_i^* = (1 - x_i^*)(Bx^*)_i$ and $(Bx)_i/(Bx^*)_i = (P^*z)_i$, we rewrite the normalized vector field as

$$\frac{\dot{x}_i}{\gamma_i x_i^*} = \frac{1 - x_i}{1 - x_i^*}(P^*z)_i - z_i = \big((P^*z)_i - z_i\big) - \frac{x_i^*(z_i - 1)}{1 - x_i^*}(P^*z)_i.$$

Multiplying by $\pi_i^*(1 - z_i^{-1}) = \pi_i^* \frac{z_i - 1}{z_i}$ and summing over $i \in [n]$ gives

$$\dot{V}(x) = \sum_{i,j=1}^{n} \pi_i^* P_{ij}^* \left(z_j - z_i - \frac{z_j}{z_i} + 1 \right) - \sum_{i=1}^{n} \pi_i^* \frac{x_i^*}{1 - x_i^*} \frac{(z_i - 1)^2}{z_i} (P^*z)_i.$$

Because $(\pi^*)^\top P^* = (\pi^*)^\top$ and $P^* \mathbf{1}_n = \mathbf{1}_n$, both $\sum_{i,j} \pi_i^* P_{ij}^* (z_j - z_i) = 0$ and $\sum_{i,j} \pi_i^* P_{ij}^* \ln(z_j/z_i) = 0$ vanish identically. Adding the latter vanishing sum transforms $1 - z_j/z_i$ into $1 - z_j/z_i + \ln(z_j/z_i) = -\phi(z_j/z_i)$, establishing (15). Both sums in (15) are nonnegative; if $\dot{V}(x) = 0$, then $(z_i - 1)^2 (P^*z)_i = 0$ for all $i \in [n]$, and since $(P^*z)_i > 0$ (as $z > 0$ and P^* is row-stochastic), this forces $z_i = 1$ for all i, i.e., $x = x^*$.

(b) Fix $c \in (0, 1]$ and consider any boundary point $x \in \mathcal{C}_c$ where $x_i = cx_i^*$ for some $i \in [n]$. Because $x \geq cx^*$ entrywise and $B \geq 0$, we have $(Bx)_i \geq c(Bx^*)_i = c\frac{\gamma_i x_i^*}{1 - x_i^*}$, so

$$\dot{x}_i = (1 - cx_i^*)(Bx)_i - \gamma_i c x_i^* \geq c\big((1 - cx_i^*) - (1 - x_i^*)\big)(Bx^*)_i = c(1 - c)x_i^*(Bx^*)_i \geq 0.$$

At the upper boundary $x_i = 1$, $\dot{x}_i = -\gamma_i < 0$. By standard flow invariance for locally Lipschitz ODEs (Nagumo's condition on the compact box $[cx^*, \mathbf{1}_n]$), $\mathcal{C}_c$ is positively invariant. Hence for any $x(0) \in (0, 1]^n$, we have $z_j(t) \geq c_0$ for all $j \in [n]$ and $t \geq 0$, so $(P^*z(t))_i \geq c_0$ for all $i \in [n]$. Moreover, for every $z > 0$,

$$\frac{(z - 1)^2}{z} - \phi(z) = \left(z - 2 + \frac{1}{z} \right) - (z - 1 - \ln z) = \frac{1}{z} - 1 - \ln\left(\frac{1}{z}\right) = \phi\left(\frac{1}{z}\right) \geq 0,$$

so $\frac{(z_i - 1)^2}{z_i} \geq \phi(z_i)$ universally. Dropping the nonnegative mixing term in (15) yields

$$\dot{V}(x(t)) \leq -c_0 \sum_{i=1}^{n} \pi_i^* \frac{x_i^*}{1 - x_i^*} \phi(z_i(t)) = -c_0 \sum_{i=1}^{n} (Bx^*)_i \frac{\pi_i^*}{\gamma_i} \phi(z_i(t)) \leq -\left(c_0 \min_{i \in [n]} (Bx^*)_i \right) V(x(t)).$$

Integrating by Grönwall's inequality proves (16). □

5 Green–Kubo Hessian of $\ln R_0(u)$ and Convex Perron-Centrality Control

Consider log-recovery (or vaccination) control $u \in \mathbb{R}^n$ acting diagonally on recovery rates via $\gamma_i(u) := \gamma_i e^{u_i}$, which yields the controlled next-generation matrix $K(u) := D_{e^{-u}} K$ and controlled reproduction number $R_0(u) := \rho(K(u)) > 0$. Let $v(u), w(u) \in \mathbb{R}^n_{>0}$ be the positive left and right Perron eigenvectors of $K(u)$ normalized by $\langle v(u), w(u) \rangle = 1$, define the *Perron centrality distribution*

$$p_i(u) := v_i(u) w_i(u) > 0, \qquad \sum_{i=1}^{n} p_i(u) = 1,$$

and define the *next-generation Markov chain*

$$Q(u) := \frac{1}{R_0(u)} D_{w(u)}^{-1} K(u) D_{w(u)}, \tag{17}$$

which is irreducible and row-stochastic with unique stationary distribution $p(u)^\top Q(u) = p(u)^\top$. Because 1 is a simple eigenvalue of $Q(u)$, the matrix $I_n - Q(u) + \mathbf{1}_n p(u)^\top$ is invertible (even

when $Q(u)$ is periodic), and the *group inverse* $Z(u) := (I_n - Q(u))^{\#}$ (obtained by subtracting the rank-one projection $\mathbf{1}_n p(u)^\top$ from the Kemeny–Snell fundamental matrix) is well defined by

$$Z(u) := \big(I_n - Q(u) + \mathbf{1}_n p(u)^\top\big)^{-1} - \mathbf{1}_n p(u)^\top. \tag{18}$$

Theorem 5.1 (Green–Kubo Hessian of $\ln R_0(u)$ and Perron-Centrality Optimal Control)**.** *Let $B \in \mathbb{R}^{n\times n}_{\geq 0}$ be irreducible, $\gamma \in \mathbb{R}^n_{>0}$, and $K = D_\gamma^{-1} B$.*

(a) ***Exact gradient, Green–Kubo Hessian, and convexity.*** *The log-reproduction functional $\psi(u) := \ln R_0(u)$ is C^∞ and convex on $\mathbb{R}^n$, with exact first and second derivatives*

$$\nabla\psi(u) = -p(u), \tag{19}$$

$$h^\top \nabla^2\psi(u)\, h = \tilde{h}^\top D_{p(u)}\big(2Z(u) - I_n\big)\tilde{h} = \|y\|^2_{L^2(p(u))} - \|Q(u)y\|^2_{L^2(p(u))} \geq 0, \tag{20}$$

for every $h \in \mathbb{R}^n$, where $\tilde{h} := h - \langle p(u), h\rangle \mathbf{1}_n$, $y := Z(u)\tilde{h}$, and $\|z\|^2_{L^2(p)} := \sum_{i=1}^n p_i z_i^2$. In particular, $\nabla^2\psi(u)$ is the symmetric part of the Green–Kubo asymptotic variance matrix $D_{p(u)}(2Z(u) - I_n + \mathbf{1}_n p(u)^\top)$ of $Q(u)$, has nullspace containing $\mathbb{R}\mathbf{1}_n$, and is strictly positive definite on $(\mathbb{R}\mathbf{1}_n)^\perp$ whenever $B \in \mathbb{R}^{n\times n}_{>0}$.

(b) ***Equalized Perron-centrality KKT law and universal R_0 floor.*** *For any positive patch costs $c \in \mathbb{R}^n_{>0}$ and budget $B_{\mathrm{bud}} > 0$, minimizing $R_0(u)$ over $\mathcal{U} := \{u \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n c_i u_i \leq B_{\mathrm{bud}}\}$ is a convex program admitting an optimal allocation $u^* \in \mathcal{U}$ characterized by the necessary and sufficient KKT conditions: there exists $\lambda^* > 0$ such that*

$$\frac{p_i(u^*)}{c_i} = \lambda^* \quad (\forall i \text{ with } u_i^* > 0), \qquad \frac{p_i(u^*)}{c_i} \leq \lambda^* \quad (\forall i \text{ with } u_i^* = 0), \qquad \sum_{i=1}^n c_i u_i^* = B_{\mathrm{bud}}. \tag{21}$$

Moreover, every feasible allocation $u \in \mathcal{U}$ satisfies the universal lower bound

$$R_0(u) \;\geq\; R_0(0)\exp\!\left(-\sum_{i=1}^n p_i(0)\, u_i\right) \;\geq\; R_0(0)\exp\!\left(-\left(\max_{i\in[n]} \frac{p_i(0)}{c_i}\right) B_{\mathrm{bud}}\right). \tag{22}$$

Proof. (a) Fix $u_0 \in \mathbb{R}^n$ and direction $h \in \mathbb{R}^n$. Because $R_0(u_0)$ is a simple eigenvalue of $K(u_0) = D_{e^{-u_0}} K$, the curve $t \mapsto (R_0(u_0 + th), v(u_0 + th), w(u_0 + th))$ is C^∞ near $t = 0$ under the local scalar gauge $\langle v(u_0), w(u_0 + th)\rangle = 1$ and $\langle v(u_0 + th), w(u_0 + th)\rangle = 1$ (which uniquely fixes the eigenvector scaling along $u_0 + th$ and yields $v(u_0)^\top \dot{w}(u_0) = 0$ at $t = 0$). Writing derivatives with respect to t at $t = 0$ with overdots and suppressing the argument u_0, differentiating $K(u_0 + th)w(u_0 + th) = e^{\psi(u_0+th)} w(u_0 + th)$ once yields

$$-D_h K w + K\dot{w} = \dot{\psi} e^{\psi} w + e^{\psi}\dot{w}. \tag{23}$$

Left-multiplying by $v^\top$ and using $v^\top K = e^{\psi} v^\top$ and $v^\top w = 1$ gives $-e^{\psi} v^\top D_h w = \dot{\psi} e^{\psi}$, hence $\dot{\psi} = -\langle p, h\rangle$, proving (19). Imposing the gauge normalization $v^\top \dot{w} = 0$ (which fixes the scaling of $w(u + th)$) and setting $\eta := D_w^{-1}\dot{w}$ (so $p^\top \eta = 0$), multiplying (23) on the left by $e^{-\psi} D_w^{-1}$ yields

$$(I_n - Q)\eta = -(h - \langle p, h\rangle \mathbf{1}_n) = -\tilde{h}. \tag{24}$$

Check that Z in (18) satisfies $(I_n - Q)Z = I_n - \mathbf{1}_n p^\top$ and $p^\top Z = 0$: indeed, $(I_n - Q + \mathbf{1}_n p^\top)\mathbf{1}_n = \mathbf{1}_n$ and $p^\top(I_n - Q + \mathbf{1}_n p^\top) = p^\top$, so $(I_n - Q + \mathbf{1}_n p^\top)^{-1}\mathbf{1}_n = \mathbf{1}_n$ and $p^\top(I_n - Q + \mathbf{1}_n p^\top)^{-1} = p^\top$, whence $Z\mathbf{1}_n = 0$, $p^\top Z = 0$, and $(I_n - Q)Z = (I_n - Q + \mathbf{1}_n p^\top)(I_n - Q + \mathbf{1}_n p^\top)^{-1} - \mathbf{1}_n p^\top = I_n - \mathbf{1}_n p^\top$. Since $p^\top \eta = 0$ and $p^\top \tilde{h} = 0$, the unique solution of (24) is $\eta = -Z\tilde{h} = -y$, where $y := Z\tilde{h}$ as in (20).

Differentiating $K(u+th)w(u+th) = e^{\psi(u+th)}w(u+th)$ a second time at $t=0$ gives

$$D_h^2 K w - 2D_h K\dot{w} + K\ddot{w} = (\ddot{\psi} + \dot{\psi}^2)e^{\psi}w + 2\dot{\psi}e^{\psi}\dot{w} + e^{\psi}\ddot{w}.$$

Left-multiplying by $v^\top$ cancels $v^\top K\ddot{w} = e^{\psi}v^\top\ddot{w}$ and $v^\top\dot{w} = 0$, leaving

$$\ddot{\psi} = v^\top D_h^2 w - \dot{\psi}^2 - 2e^{-\psi}v^\top D_h K\dot{w} = \sum_{i=1}^n p_i h_i^2 - \langle p, h\rangle^2 - 2p^\top D_h Q\eta.$$

Using $\sum_i p_i h_i^2 - \langle p,h\rangle^2 = \tilde{h}^\top D_p \tilde{h}$, $p^\top Q\eta = p^\top\eta = 0$ (so $p^\top D_h Q\eta = \tilde{h}^\top D_p Q\eta$), and $Q\eta = \eta + \tilde{h}$ from (24), we obtain

$$\ddot{\psi} = \tilde{h}^\top D_p\tilde{h} - 2\tilde{h}^\top D_p(\eta+\tilde{h}) = 2\tilde{h}^\top D_p y - \tilde{h}^\top D_p\tilde{h} = \tilde{h}^\top D_p(2Z - I_n)\tilde{h}.$$

Since $\tilde{h} = (I_n - Q)y$, this rewrites identically as

$$\ddot{\psi} = \langle y - Qy,\ 2y - (y-Qy)\rangle_{L^2(p)} = \langle y-Qy,\ y+Qy\rangle_{L^2(p)} = \|y\|_{L^2(p)}^2 - \|Qy\|_{L^2(p)}^2.$$

By Jensen's inequality on each row of the stochastic matrix Q and $p^\top Q = p^\top$,

$$\|Qy\|_{L^2(p)}^2 = \sum_{i=1}^n p_i\left(\sum_{j=1}^n Q_{ij}y_j\right)^2 \le \sum_{i=1}^n p_i\sum_{j=1}^n Q_{ij}y_j^2 = \sum_{j=1}^n p_j y_j^2 = \|y\|_{L^2(p)}^2,$$

proving $\ddot{\psi}\ge 0$ and hence convexity of ψ on $\mathbb{R}^n$. Note also that $\tilde{h} = (I_n - \mathbf{1}_n p^\top)h$, and since both $p^\top Z = 0$ and $Z\mathbf{1}_n = 0$, we have $\tilde{h}^\top D_p(2Z-I_n)\tilde{h} = h^\top D_p(2Z - I_n + \mathbf{1}_n p^\top)h$, so $\nabla^2\psi(u_0)$ is the symmetric part of $D_p(2Z - I_n + \mathbf{1}_n p^\top)$. If $B>0$, then $Q_{ij}>0$ for all i,j; equality in Jensen's inequality forces y to be constant across $[n]$, whence $p^\top y = 0$ forces $y=0$, $\tilde{h} = (I_n - Q)y = 0$, and $h\in\mathbb{R}\mathbf{1}_n$.

(b) Minimizing $R_0(u)$ over the compact polytope $\mathcal{U}$ is equivalent to minimizing the smooth convex function $\psi(u) = \ln R_0(u)$ over $\mathcal{U}$. Slater's condition holds trivially ($B_{\text{bud}} > 0$), so the KKT conditions are necessary and sufficient for global optimality: there exist $\lambda^*\ge 0$ and $\mu_i \ge 0$ such that $-p_i(u^*) + \lambda^* c_i - \mu_i = 0$ and $\mu_i u_i^* = 0$. Since $p_i(u^*) > 0$ and $c_i > 0$, $\lambda^* = (p_i(u^*) + \mu_i)/c_i > 0$ strictly, forcing the budget constraint to bind ($\sum_i c_i u_i^* = B_{\text{bud}}$) and giving (21). Finally, convexity of ψ with $\nabla\psi(0) = -p(0)$ yields $\psi(u)\ge\psi(0) - \langle p(0), u\rangle$ for all $u\in\mathbb{R}^n$; for $u\in\mathcal{U}$, $\sum_i p_i(0)u_i = \sum_i \frac{p_i(0)}{c_i}(c_i u_i) \le (\max_i \frac{p_i(0)}{c_i})B_{\text{bud}}$, and exponentiating proves (22). □

6 Computational Verification and Explicit Benchmarks

All four theorems were verified programmatically via a self-contained Python suite (`sis_network.py`, `verify_all.py`, and `test_sis_network.py`; four automated unit tests passing in 1.52 s) across 270 deterministic randomized network instances using NumPy `default_rng` streams.

- **Theorem 2.1 suite (seed `202631`, 75 networks):** Across $n\in\{3,5,8\}$ (25 networks each; $B_{\text{raw}}\sim\text{Exp}(1)+0.05$, $\gamma\sim\text{Unif}(0.5,2.0)$), each network was rescaled once to $R_0\in(0.3,0.85)$ (integrated by RK4 with $\Delta t = 0.01$ to $t=2.0$ from $x_0\sim\text{Unif}(0.05,0.95)$) and once to $R_0\in(1.3,4.0)$ (testing $x^-\le x^*\le x^+$ and (5) on uniform pairs in $\mathcal{B}$). Maximum violations were 0.0 across all four inequalities.

- **Theorem 3.1 suite (seed `202632`, 75 networks):** Across $n\in\{3,5,7\}$ (25 networks each; $B_{\text{raw}}\sim\text{Exp}(1)+0.05$, $\gamma\sim\text{Unif}(0.4,2.2)$, rescaled to $R_0\in(1.4,3.8)$, perturbations $\delta B\sim\mathcal{N}(0,0.3^2)$, $\delta\ln\gamma\sim\mathcal{N}(0,0.3^2)$), the similarity identity (8) matched to 6.75×10^{-14}, the spectral abscissa bracket (9) and log-elasticity bounds (12) had 0.0 violation, central-difference sensitivities ($\varepsilon = 10^{-6}$) matched (10) to 8.92×10^{-9}, and the π^*-conservation identity (11) held to 2.00×10^{-14}.

Table 1: Quantitative summary on trial 0 of each dimension block $n \in \{3, 4, 5, 6\}$ (seed `202634`, strictly positive $B > 0$ rescaled to $R_0 = 2.5$). In every network, $\alpha(J(x^*)) \le -\min_i(Bx^*)_i < 0$ (Theorem 3.1(a)) and $\mathrm{tr}(\nabla^2 \ln R_0) > 0$ (Theorem 5.1(a), strictly positive since $B > 0$).

Network	R_0	$x^*_{\min}$	$x^*_{\max}$	$\alpha(J(x^*))$	Bound $-\min_i(Bx^*)_i$	$\max_i p_i$	$\mathrm{tr}(\nabla^2 \ln R_0)$
`Net_n3`	2.5000	0.5422	0.6591	−1.3515	−1.0259	0.3615	0.4577
`Net_n4`	2.5000	0.4320	0.6130	−1.8405	−1.2858	0.8173	0.5867
`Net_n5`	2.5000	0.4016	0.6657	−1.5335	−1.0658	0.3264	0.5812
`Net_n6`	2.5000	0.3247	0.7168	−1.4564	−0.7028	0.3599	0.6926

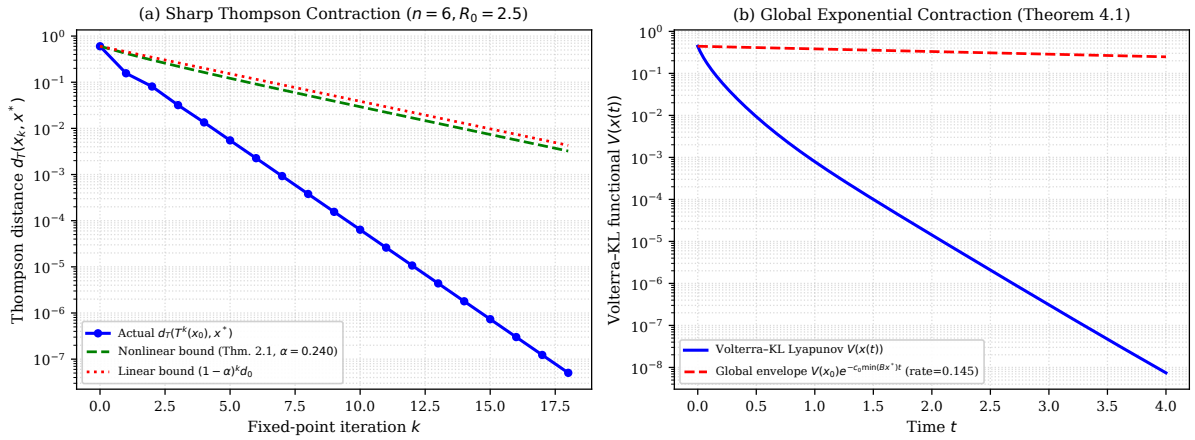


Figure 1: **(a)** Sharp nonlinear Thompson contraction (5) of the endemic operator T vs. the linear bound $(1-\alpha)^k d_0$ on a standalone $n = 6, R_0 = 2.5$ network (fresh seed `202631`, $\alpha = 0.240$, initialized at x^-). **(b)** Global exponential contraction of the Volterra–KL functional $V(x(t))$ along the full nonlinear SIS ODE (1) from $x_0 = (0.12, 0.85, 0.20, 0.90, 0.15, 0.78)$ bounded by the Theorem 4.1(b) envelope $V(x_0)\exp(-c_0 \min_i(Bx^*)_i t)$ (rate 0.145).

- **Theorem 4.1 suite (seed `202633`, 60 networks):** Across $n \in \{3, 5, 7\}$ (20 networks each; $B_{\mathrm{raw}} \sim \mathrm{Exp}(1) + 0.05$, $\gamma \sim \mathrm{Unif}(0.5, 2.0)$, rescaled to $R_0 \in (1.5, 3.5)$, $x_0 \sim \mathrm{Unif}(0.05, 0.95)$, RK4 with $\Delta t = 0.005$ to $t = 2.0$), the instantaneous dissipation identity (15) held to 1.67×10^{-14} and the global exponential envelope (16) exhibited 0.0 violation.
- **Theorem 5.1 suite (seed `202634`, 60 networks):** Looping in order over $n \in \{3, 4, 5, 6\}$ (15 networks each; $B_{\mathrm{raw}} \sim \mathrm{Exp}(1) + 0.05$, $\gamma \sim \mathrm{Unif}(0.5, 2.0)$, directions $h \sim \mathcal{N}(0, 0.5^2)$, controls $u \sim \mathrm{Unif}(0, 1)$), central-difference directional derivatives ($\varepsilon = 10^{-5}$) matched (19) to 1.68×10^{-10} and the Green–Kubo Hessian (20) to 3.15×10^{-5} (IEEE `float64` finite-difference cancellation floor $O(\varepsilon_{\mathrm{mach}}/\varepsilon^2)$), minimum eigenvalues of $\nabla^2\psi(u)$ satisfied $\ge -6.80 \times 10^{-16}$, and the convex lower bound (22) had 0.0 violation.

7 Discussion and Open Directions

The four theorems above show that two canonical Markov chains—the endemic chain (P^*, π^*) at x^* and the next-generation chain $(Q(u), p(u))$—encode the exact local stability, policy sensitivity, global nonlinear Lyapunov contraction, and second-order convex control geometry of multigroup SIS networks. Three natural extensions remain open:

1. **Tightening the global Volterra–KL rate via mixing conductance:** Theorem 4.1(b) drops the nonnegative mixing dissipation $-\sum_{i,j} \pi^*_i P^*_{ij} \phi(z_j/z_i)$ to obtain the explicit rate

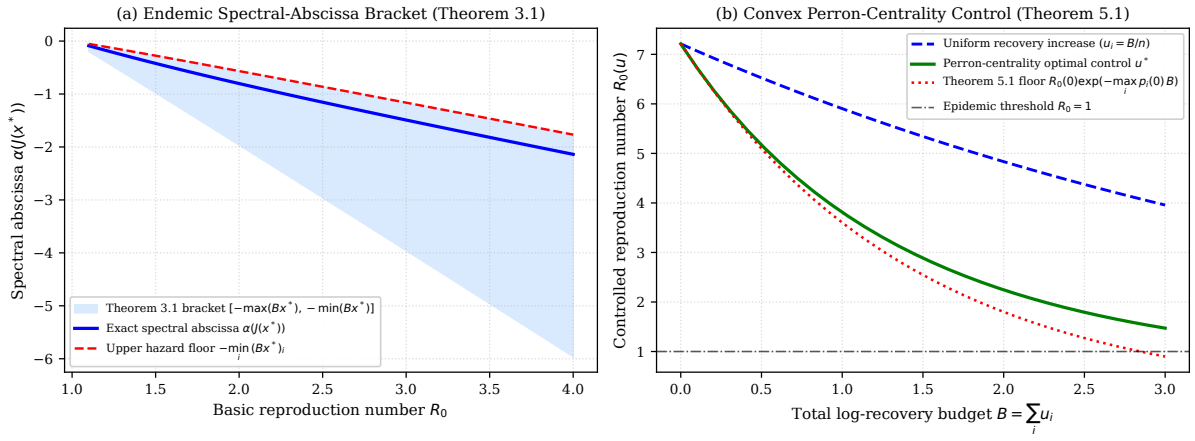


Figure 2: **(a)** Exact spectral abscissa $\alpha(J(x^*))$ across $R_0 \in [1.1, 4.0]$ for a standalone $n = 6$ network (fresh seed `202632`) tightly bracketed by $[-\max_i(Bx^*)_i, -\min_i(Bx^*)_i]$ (Theorem 3.1(a)). **(b)** Budget-constrained $R_0(u)$ minimization on a heterogeneous $n = 5$ hub-spoke network (fresh seed `202634`, unit costs $c_i = 1$): Perron-centrality optimal control (Theorem 5.1(b)) outperforms uniform allocation and stays above the universal exponential floor (22).

$c_0 \min_i(Bx^*)_i$; incorporating a Cheeger or modified log-Sobolev constant of P^* could improve the global rate on heterogeneous networks where $\min_i(Bx^*)_i \ll \max_i(Bx^*)_i$.

2. **Joint transmission-and-recovery optimal control:** Extending the Green–Kubo Hessian representation (20) from diagonal recovery controls $D_{e^{-u}}K$ to rank-constrained edge-weight interventions $B_{ij}e^{-w_{ij}}$ yields a coupled edge-space convex program whose explicit second-order curvature remains to be characterized.

3. **Higher-order simplicial and hypergraph SIS dynamics:** Combining the multilinear Birkhoff–Hopf contraction tools of nonlinear Perron–Frobenius theory with Volterra–KL Lyapunov functionals for simplicial contagion models (where bistability occurs below $R_0 = 1$) is a promising direction.

Reproducibility and Data Availability

All analytical verifications, unit tests, and figure generation scripts are openly archived under the MIT license (version `v1.0.3`) at:
`https://github.com/lluiseriksson/sis-epidemic-markov-geometry`.

References

[1] A. Lajmanovich and J. A. Yorke, "A deterministic model for gonorrhea in a nonhomogeneous population," *Mathematical Biosciences*, vol. 28, no. 3–4, pp. 221–236, 1976.

[2] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, "On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations," *Journal of Mathematical Biology*, vol. 28, no. 4, pp. 365–382, 1990.

[3] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," *Mathematical Biosciences*, vol. 180, no. 1–2, pp. 29–48, 2002.

[4] A. Fall, A. Iggidr, G. Sallet, and J. J. Tewa, "Epidemiological models and Lyapunov functions," *Mathematical Modelling of Natural Phenomena*, vol. 2, no. 1, pp. 62–83, 2007.

[5] P. Van Mieghem, J. Omić, and R. Kooij, "Virus spread in networks," *IEEE/ACM Transactions on Networking*, vol. 17, no. 1, pp. 1–14, 2009.

[6] V. M. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, and G. J. Pappas, "Optimal resource allocation for network protection against spreading processes," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 1, pp. 99–108, 2014.

[7] A. Khanafer, T. Başar, and B. Gharesifard, "Stability of epidemic models over directed graphs: a positive systems approach," *Automatica*, vol. 74, pp. 126–134, 2016.

[8] W. Mei, S. Mohagheghi, S. Zampieri, and F. Bullo, "On the dynamics of deterministic epidemic propagation over networks," *Annual Reviews in Control*, vol. 44, pp. 116–128, 2017.

[9] S. Friedland and S. Karlin, "Some inequalities for the spectral radius of non-negative matrices and applications," *Duke Mathematical Journal*, vol. 42, no. 3, pp. 459–490, 1975.

[10] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, vol. 9. SIAM, 1994.

[11] A. C. Thompson, "On certain contraction mappings in a partially ordered vector space," *Proceedings of the American Mathematical Society*, vol. 14, no. 3, pp. 438–443, 1963.

[12] B. Lemmens and R. Nussbaum, *Nonlinear Perron–Frobenius Theory*, Cambridge Tracts in Mathematics, vol. 189. Cambridge University Press, 2012.