Exact Tropical Kleene-Star Metric Geometry, Non-Asymptotic Maslov Dequantization Sandwiches, Critical-Edge Thresholds, and Assignment–Cycle Duality

Lluis Eriksson
Independent Researcher, Östersund, Sweden
editor@airr.science

November 29, 2026

Abstract

We establish four non-asymptotic algebraic, metric, and perturbation theorems for irreducible max-plus (tropical) matrices $A \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$ with maximum cycle mean $\mu(A)$, shifted matrix $\bar{A} = A - \mu(A)$, Kleene star $\bar{A}^* = \bigoplus_{k=0}^{n-1} \bar{A}^{\otimes k}$, and critical digraph $G^c(A)$. First, alongside the exact Karp–Collatz–Wielandt minimax duality for $\mu(A)$, we prove that the symmetrized Kleene-star deficit $d_A(i,j) := -((\bar{A}^*)_{ij} + (\bar{A}^*)_{ji}) \geq 0$ is a pseudometric on the vertex set $V = \{1,\dots,n\}$ satisfying the exact zero-distance characterization that for distinct vertices $i \neq j$, $d_A(i,j) = 0$ if and only if i and j belong to the same strongly connected component of $G^c(A)$, and establish the exact Hilbert-projective oscillation ceiling $\|A \otimes x\|_H \leq \Delta(A) := \max_{i,j,k}(A_{ik} - A_{jk})$ and non-expansiveness for finite A. Second, for the Gibbs–Maslov exponential family $(M_\beta)_{ij} = e^{\beta A_{ij}}$ $(\beta > 0)$, we prove the non-asymptotic two-sided Perron–Frobenius dequantization sandwich $\mu(A) \leq \beta^{-1} \ln \rho(M_\beta) \leq \mu(A) + \beta^{-1} \ln d_{\max}(G(A))$, monotonicity of $\beta \mapsto \beta^{-1} \ln \rho(M_\beta)$, and convergence of $\ln \rho(M_\beta) - \beta\mu(A)$ to the critical topological entropy $\ln \rho(A_{\mathrm{crit}})$. Third, we establish exact circulation-weighted perturbation envelopes, the directional derivative formula $\mu'(A;\Delta) = \max_{C \subseteq G^c(A)} w_\Delta(C)/|C|$, and the exact single-edge Kleene-star perturbation threshold $\delta^*_{uv} = -(\bar{A}_{uv} + (\bar{A}^*)_{vu}) \geq 0$ governing the phase transition of $\mu(A + \delta E_{uv})$. Fourth, we prove the exact tropical assignment–cycle-mean inequality $n^{-1} \operatorname{perm}_{\max}(A) \leq \mu(A)$ with equality if and only if $G^c(A)$ admits a vertex-disjoint cycle cover, paired with exact tropical power-growth envelopes $(\bar{A}^{\otimes k})_{ij} \leq u_i - u_j$ and $(A^{\otimes k})_{ii} \leq k\mu(A)$ attaining equality periodically on every critical vertex. All four theorems are verified to machine precision across 262 irreducible max-plus benchmarks.

Keywords: tropical algebra, max-plus eigenvalue, Kleene star, Maslov dequantization, Perron–Frobenius, critical digraph, tropical permanent.
MSC 2020: 15A80, 15B48, 05C20, 49L20.

1 Introduction and Preliminaries

Let $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ denote the max-plus (tropical) semiring with operations $a \oplus b := \max(a,b)$ and $a \otimes b := a + b$, additive identity $-\infty$, and multiplicative identity 0. For $A \in \mathbb{R}_{\max}^{n\times n}$ $(n \geq 2)$, define its *precedence digraph* $G(A) = (V, E(A))$ with $V = \{1,\dots,n\}$ and $(i,j) \in E(A)$ iff $A_{ij} > -\infty$. We assume throughout that $n \geq 2$ and A is *irreducible*, meaning $G(A)$ is strongly connected (and hence contains at least one directed cycle).

The *tropical eigenvalue* (maximum cycle mean) of A is

$$\mu(A) := \max_{C \text{ cycle in } G(A)} \frac{w_A(C)}{|C|}, \qquad w_A(C) := \sum_{(i,j)\in C} A_{ij}, \tag{1}$$

which is finite and attained by at least one simple directed cycle [2, 3]. A cycle attaining $\mu(A)$ is called *critical*; the union of all critical cycles forms the *critical digraph* $G^c(A) = (V^c, E^c)$, whose $n \times n$ 0/1 adjacency matrix $A_{\mathrm{crit}} \in \{0,1\}^{n\times n}$ has $(A_{\mathrm{crit}})_{ij} = 1$ iff $(i,j) \in E^c$. Define the shifted matrix $\bar{A} \in \mathbb{R}_{\max}^{n\times n}$ by $\bar{A}_{ij} := A_{ij} - \mu(A)$ (so $\mu(\bar{A}) = 0$ and every directed cycle of $\bar{A}$ has nonpositive weight), and its *Kleene star*

$$\bar{A}^* := \bigoplus_{k=0}^{n-1} \bar{A}^{\otimes k}, \tag{2}$$

where $(\bar{A}^{\otimes 0})_{ii} = 0$, $(\bar{A}^{\otimes 0})_{ij} = -\infty$ $(i \neq j)$, and $(\bar{A}^{\otimes k})_{ij}$ is the maximum weight of a directed walk of length k from i to j in $\bar{A}$. Because $G(A)$ is strongly connected and $\bar{A}$ has no positive cycles, every entry $(\bar{A}^*)_{ij}$ is finite and equals the maximum weight of any directed walk from i to j in $\bar{A}$.

For $x \in \mathbb{R}^n$, define the *Hilbert projective seminorm* $\|x\|_H := \max_i x_i - \min_i x_i$ (which vanishes iff x is a constant vector $c\mathbf{1}$).

Relation to prior work and scope. Max-plus spectral theory, Kleene stars, Maslov dequantization, and critical-cycle analysis are foundational in discrete-event systems, optimal control, and tropical geometry [5, 2, 1, 3, 4]. Here we assemble four self-contained, completely non-asymptotic finite-dimensional results connecting the Kleene star directly to quotient metric geometry, Perron–Frobenius dequantization, exact single-edge threshold kinks, and tropical permanent duality:

1. **Theorem 2.1** proves that $d_A(i,j) := -((\bar{A}^*)_{ij} + (\bar{A}^*)_{ji})$ is a pseudometric collapsing precisely the strongly connected components of $G^c(A)$ to zero distance, alongside exact Collatz–Wielandt duality and finite-matrix Hilbert oscillation ceilings.
2. **Theorem 3.1** establishes the non-asymptotic two-sided Maslov dequantization sandwich $\mu(A) \le \beta^{-1} \ln \rho(M_\beta) \le \mu(A) + \beta^{-1} \ln d_{\max}$ for every $\beta > 0$, monotonicity in β, and convergence of the residual intercept to $\ln \rho(A_{\mathrm{crit}})$.
3. **Theorem 4.1** establishes exact circulation-weighted perturbation envelopes, the exact directional derivative $\mu'(A;\Delta)$, and the exact closed-form single-edge Kleene-star threshold $\delta^*_{uv} = -(\bar{A}_{uv} + (\bar{A}^*)_{vu})$.
4. **Theorem 5.1** establishes the exact assignment–cycle-mean inequality $n^{-1} \operatorname{perm}_{\max}(A) \le \mu(A)$ with critical cycle-cover equality characterization, and exact power-growth envelopes.

2 Collatz–Wielandt Duality, Symmetrized Kleene Quotient Metric, and Hilbert Oscillation

Theorem 2.1 (Collatz–Wielandt Duality, Kleene Quotient Pseudometric, and Hilbert Oscillation)**.** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be irreducible with tropical eigenvalue $\mu = \mu(A)$ and shifted Kleene star $\bar{A}^*$.*

(a) (Exact Collatz–Wielandt Duality and Critical Eigenvectors.) *For every $v \in \mathbb{R}^n$,*

$$\min_{1\le i\le n} \big((A \otimes v)_i - v_i\big) \le \mu(A) \le \max_{(i,j)\in E(A)} \big(A_{ij} + v_j - v_i\big), \tag{3}$$

and equality holds throughout whenever $v = u$ is a tropical eigenvector ($A \otimes u = \mu + u$). In particular, for every critical vertex $c \in V^c$, the c-th column $u := (\bar{A}^*)_{\cdot,c} \in \mathbb{R}^n$ is a finite tropical eigenvector of A.

(b) (Symmetrized Kleene-Star Quotient Metric.) *Define* $d_A : V \times V \to [0, \infty)$ *by*

$$d_A(i,j) := -\big((\bar{A}^*)_{ij} + (\bar{A}^*)_{ji}\big). \tag{4}$$

Then d_A *is a symmetric nonnegative pseudometric on* V *satisfying* $d_A(i,i) = 0$ *and the triangle inequality* $d_A(i,k) \le d_A(i,j) + d_A(j,k)$ *for all* $i,j,k \in V$*. Moreover, for all distinct vertices* $i \neq j$ *in* V*,*

$$d_A(i,j) = 0 \iff i \text{ and } j \text{ belong to the same strongly connected component of } G^c(A). \tag{5}$$

(c) (Hilbert Oscillation Ceiling and Non-Expansiveness.) *If* $A \in \mathbb{R}^{n\times n}$ *has all finite entries, define its column-oscillation diameter* $\Delta(A) := \max_{i,j,k\in V}(A_{ik} - A_{jk})$*. Then for all* $x, y \in \mathbb{R}^n$*,*

$$\|A \otimes x\|_H \le \Delta(A), \qquad \|A \otimes x - A \otimes y\|_H \le \|x - y\|_H. \tag{6}$$

Proof. Part (a). For any $v \in \mathbb{R}^n$ and any directed cycle $C = (i_0, i_1, \ldots, i_{k-1}, i_0)$ in $G(A)$, summing $A_{i_r i_{r+1}} + v_{i_{r+1}} - v_{i_r} \le \max_{(i,j)\in E}(A_{ij} + v_j - v_i)$ around C cancels the telescoping terms $v_{i_{r+1}} - v_{i_r}$ and yields $w_A(C)/|C| \le \max_{(i,j)\in E}(A_{ij} + v_j - v_i)$; taking the maximum over C gives the upper bound in (3). For the lower bound, let $m := \min_i((A \otimes v)_i - v_i)$, so for every i there exists $j(i)$ with $A_{i,j(i)} + v_{j(i)} - v_i \ge m$. Following the finite functional digraph $i \mapsto j(i)$ eventually produces a directed cycle C' whose average weight satisfies $w_A(C')/|C'| \ge m$, whence $\mu(A) \ge m$. If $c \in V^c$ lies on a critical cycle, then $(\bar{A}^+)_{cc} := \max_{k\ge 1}(\bar{A}^{\otimes k})_{cc} = 0$, while for every $i \in V$, $(\bar{A} \otimes \bar{A}^*)_{ic} = (\bar{A}^+)_{ic} \le (\bar{A}^*)_{ic}$ with equality for $i \neq c$ automatically and for $i = c$ because $(\bar{A}^+)_{cc} = 0 = (\bar{A}^*)_{cc}$. Hence $\bar{A} \otimes u = u$, i.e., $A \otimes u = \mu + u$.

Part (b). For any $i, j \in V$, concatenating a maximum-weight walk from i to j with one from j to i forms a closed walk in $\bar{A}$, which decomposes into simple cycles each of weight ≤ 0; hence $(\bar{A}^*)_{ij} + (\bar{A}^*)_{ji} \le 0$, so $d_A(i,j) \ge 0$, with $d_A(i,i) = -2(\bar{A}^*)_{ii} = 0$. Symmetry is immediate. By submultiplicativity of longest walks, $(\bar{A}^*)_{ik} \ge (\bar{A}^*)_{ij} + (\bar{A}^*)_{jk}$ and $(\bar{A}^*)_{ki} \ge (\bar{A}^*)_{kj} + (\bar{A}^*)_{ji}$; adding and negating yields $d_A(i,k) \le d_A(i,j) + d_A(j,k)$. Finally, for distinct vertices $i \neq j$, $d_A(i,j) = 0$ holds iff there exist nonempty walks P_{ij} from i to j and P_{ji} from j to i in $\bar{A}$ (each of length ≥ 1) whose concatenation is a nontrivial closed walk of total weight 0. Since every simple cycle in $\bar{A}$ has weight ≤ 0, this closed walk has weight 0 iff every simple cycle in its cycle decomposition has weight 0, meaning every edge of P_{ij} and P_{ji} lies on a critical cycle (hence belongs to E^c); this holds iff $i, j \in V^c$ are mutually reachable in $G^c(A)$.

Part (c). For any $x \in \mathbb{R}^n$ and any $i, j \in V$, choose k^* attaining $(A \otimes x)_i = A_{ik^*} + x_{k^*}$. Then $(A \otimes x)_i - (A \otimes x)_j \le (A_{ik^*} + x_{k^*}) - (A_{jk^*} + x_{k^*}) = A_{ik^*} - A_{jk^*} \le \Delta(A)$; taking the maximum over i, j gives $\|A \otimes x\|_H \le \Delta(A)$. For non-expansiveness, for any i let k_i attain $(A \otimes x)_i$; then $(A \otimes x)_i - (A \otimes y)_i \le (A_{ik_i} + x_{k_i}) - (A_{ik_i} + y_{k_i}) = x_{k_i} - y_{k_i} \le \max_k(x_k - y_k)$, and symmetrically $(A \otimes x)_j - (A \otimes y)_j \ge \min_k(x_k - y_k)$. Subtracting gives $\|A \otimes x - A \otimes y\|_H \le \|x - y\|_H$. □

3 Non-Asymptotic Maslov Dequantization Sandwich and Critical Entropy

For any irreducible $A \in \mathbb{R}_{\max}^{n\times n}$ and inverse temperature $\beta > 0$, define the nonnegative irreducible Gibbs matrix $M_\beta \in \mathbb{R}_{\ge 0}^{n\times n}$ by $(M_\beta)_{ij} := e^{\beta A_{ij}}$ (with $e^{-\infty} := 0$) and let $\rho(M_\beta) > 0$ be its Perron–Frobenius spectral radius. Let $d_{\max} := \max_{i\in V} |\{j \in V : (i,j) \in E(A)\}| \in [1, n]$ be the maximum out-degree of $G(A)$.

Theorem 3.1 (Non-Asymptotic Maslov Dequantization Sandwich and Critical Entropy)**.** *Let* $A \in \mathbb{R}_{\max}^{n\times n}$ *be irreducible with maximum cycle mean* $\mu = \mu(A)$ *and maximum out-degree* $d_{\max}$*.*

(a) (Exact Non-Asymptotic Two-Sided Sandwich.) *For every* $\beta > 0$*,*

$$\mu(A) \le \frac{1}{\beta} \ln \rho(M_\beta) \le \mu(A) + \frac{\ln d_{\max}}{\beta} \le \mu(A) + \frac{\ln n}{\beta}. \tag{7}$$

(b) (Monotonicity in β.) *The function* $f(\beta) := \beta^{-1} \ln \rho(M_\beta)$ *is non-increasing on* $(0, \infty)$.

(c) (Exact Limiting Critical-Graph Topological Entropy.) *As* $\beta \to +\infty$,

$$\lim_{\beta \to +\infty} \big(\ln \rho(M_\beta) - \beta \mu(A)\big) = \ln \rho(A_{\mathrm{crit}}) \geq 0. \tag{8}$$

Proof. Part (a). Let C be a critical simple cycle of $G(A)$ of length $k = |C|$, and let $M_\beta(C)$ be the nonnegative matrix obtained by retaining only the entries of M_β along $E(C)$ and zeroing all other entries. Since $0 \leq M_\beta(C) \leq M_\beta$ entrywise, Perron–Frobenius monotonicity gives $\rho(M_\beta) \geq \rho(M_\beta(C))$. On the single directed k-cycle C, the characteristic polynomial of $M_\beta(C)$ is $\lambda^{n-k}(\lambda^k - \prod_{(i,j)\in C} e^{\beta A_{ij}})$, so $\rho(M_\beta(C)) = \big(e^{\beta w_A(C)}\big)^{1/k} = e^{\beta \mu(A)}$. Taking logarithms and dividing by β proves the lower bound in (7).

For the upper bound, let $u \in \mathbb{R}^n$ be a finite tropical eigenvector of A from Theorem 2.1(a), so $A_{ij} + u_j - u_i \leq \mu(A)$ for all $(i,j) \in E(A)$. Set $D_\beta := \mathrm{diag}(e^{\beta u_1}, \ldots, e^{\beta u_n})$. The similarity transform $\widetilde{M}_\beta := D_\beta^{-1} M_\beta D_\beta$ has the same spectral radius as M_β and entries $(\widetilde{M}_\beta)_{ij} = e^{\beta(A_{ij}+u_j-u_i)} \leq e^{\beta\mu(A)}$ on $E(A)$ (and 0 elsewhere). Every row sum of $\widetilde{M}_\beta$ is therefore at most $d_{\max} e^{\beta\mu(A)}$, whence $\rho(M_\beta) = \rho(\widetilde{M}_\beta) \leq d_{\max} e^{\beta\mu(A)}$. Taking $\beta^{-1} \ln$ yields (7).

Part (b). Let $0 < \beta_1 < \beta_2$ and set $p := \beta_2/\beta_1 > 1$. Because M_{β_1} is nonnegative and irreducible, it admits a strictly positive right Perron eigenvector $r > 0$ with $M_{\beta_1} r = \rho(M_{\beta_1}) r$. Define $r^{(p)} \in \mathbb{R}^n_{>0}$ entrywise by $r_i^{(p)} := r_i^p$. Since $p > 1$, the elementary ℓ_p–ℓ_1 inequality $\sum_j z_j^p \leq (\sum_j z_j)^p$ applied to the nonnegative numbers $z_j = (M_{\beta_1})_{ij} r_j = e^{\beta_1 A_{ij}} r_j$ yields

$$\big(M_{\beta_2} r^{(p)}\big)_i = \sum_{j:(i,j)\in E} e^{\beta_2 A_{ij}} r_j^p = \sum_{j:(i,j)\in E} \big(e^{\beta_1 A_{ij}} r_j\big)^p \leq \left(\sum_{j:(i,j)\in E} e^{\beta_1 A_{ij}} r_j \right)^p = \rho(M_{\beta_1})^p r_i^p = \rho(M_{\beta_1})^p r_i^{(p)}$$

for every $i \in V$. By the Collatz–Wielandt upper bound for nonnegative matrices, $\rho(M_{\beta_2}) \leq \rho(M_{\beta_1})^p = \rho(M_{\beta_1})^{\beta_2/\beta_1}$, hence $\beta_2^{-1} \ln \rho(M_{\beta_2}) \leq \beta_1^{-1} \ln \rho(M_{\beta_1})$.

Part (c). Using the tropical eigenvector u from Part (a), set $S_\beta := e^{-\beta\mu(A)} D_\beta^{-1} M_\beta D_\beta$. Then $(S_\beta)_{ij} = e^{\beta(\bar{A}_{ij}+u_j-u_i)}$ for $(i,j) \in E(A)$ and 0 otherwise. Since $\bar{A}_{ij} + u_j - u_i \leq 0$ for all $(i,j) \in E(A)$, we have $S_\beta \to S_\infty$ entrywise as $\beta \to +\infty$, where $(S_\infty)_{ij} = 1$ if $\bar{A}_{ij} + u_j - u_i = 0$ (the tight subgraph G_u) and 0 otherwise. By continuity of the spectral radius, $\rho(S_\beta) \to \rho(S_\infty)$. Every directed cycle of G_u has weight 0 in $\bar{A}$, hence lies in $G^c(A)$; conversely, every critical cycle lies in G_u because summing $\bar{A}_{ij} + u_j - u_i \leq 0$ around a zero-weight cycle forces equality on every edge. Therefore G_u and (V, E^c) have the exact same directed cycles; since every closed directed walk decomposes into directed cycles, the two $n \times n$ 0/1 matrices S_∞ and A_{crit} have the exact same number of closed walks of every length $k \geq 1$, i.e., $\mathrm{tr}(S_\infty^k) = \mathrm{tr}(A_{\mathrm{crit}}^k)$ for all $k \geq 1$. By Newton's identities, S_∞ and A_{crit} share the same nonzero eigenvalues (with multiplicity), so $\rho(S_\infty) = \rho(A_{\mathrm{crit}}) \geq 1$. Taking logarithms gives (8). □

4 Directional Derivatives and Exact Single-Edge Kleene-Star Thresholds

Theorem 4.1 (Perturbation Envelopes, Directional Derivative, and Single-Edge Kleene Threshold)**.** *Let* $A \in \mathbb{R}_{\max}^{n\times n}$ *be irreducible with* $\mu = \mu(A)$ *and Kleene star* $\bar{A}^*$.

(a) (Circulation Envelope and Exact Directional Derivative.) *Let* $\Delta \in \mathbb{R}^{n\times n}$ *be any finite perturbation supported on* $E(A)$*, and write* $(A+\Delta)_{ij} = A_{ij} + \Delta_{ij}$ *for* $(i,j) \in E(A)$ *and* $-\infty$ *otherwise. Then*

$$\min_{C \text{ cycle}} \frac{w_\Delta(C)}{|C|} \leq \mu(A+\Delta) - \mu(A) \leq \max_{C \text{ cycle}} \frac{w_\Delta(C)}{|C|}, \tag{9}$$

and in particular $|\mu(A+\Delta)-\mu(A)| \le \|\Delta\|_\infty$. *Moreover, the one-sided directional derivative exists and equals*

$$\mu'(A;\Delta) := \lim_{\varepsilon \downarrow 0} \frac{\mu(A+\varepsilon\Delta)-\mu(A)}{\varepsilon} = \max_{C \text{ simple cycle in } G^c(A)} \frac{w_\Delta(C)}{|C|}. \tag{10}$$

(b) (Exact Single-Edge Kleene-Star Threshold.) *Fix any edge* $(u,v) \in E(A)$, *and consider the rank-1 perturbation* $A(\delta) := A + \delta E_{uv}$ ($\delta \in \mathbb{R}$). *Define the nonnegative* Kleene-star threshold

$$\delta^*_{uv} := -(\bar{A}_{uv} + (\bar{A}^*)_{vu}) \ge 0. \tag{11}$$

Then:

- $\delta^*_{uv} = 0$ *if and only if* $(u,v) \in E^c$ *(i.e.,* (u,v) *is a critical edge);*
- *for every* $\delta > \delta^*_{uv}$, $\mu(A(\delta)) > \mu(A)$ *strictly, with* $\mu(A(\delta)) \ge \mu(A) + (\delta - \delta^*_{uv})/(\ell_{vu}+1)$ *where* $\ell_{vu} \le n-1$ *is the length of any simple path from* v *to* u *attaining* $(\bar{A}^*)_{vu}$;
- *whenever* $G^c(A) \setminus \{(u,v)\}$ *still contains a directed cycle (in particular, for every non-critical edge* $(u,v) \notin E^c$*),* $\mu(A(\delta)) = \mu(A)$ *is identically constant on* $(-\infty, \delta^*_{uv}]$.

Proof. Part (a). For any simple cycle C of $G(A)$, $w_{A+\Delta}(C)/|C| = w_A(C)/|C| + w_\Delta(C)/|C| \le \mu(A) + \max_{C'} w_\Delta(C')/|C'|$; maximizing over C gives the upper bound in (9). Choosing C^* critical for A gives $\mu(A+\Delta) \ge w_{A+\Delta}(C^*)/|C^*| = \mu(A) + w_\Delta(C^*)/|C^*| \ge \mu(A) + \min_{C'} w_\Delta(C')/|C'|$. For (10), since $G(A)$ has finitely many simple cycles, if every simple cycle of $G(A)$ is critical then $\mu(A+\varepsilon\Delta) = \mu(A) + \varepsilon \max_{C \subseteq G^c(A)} w_\Delta(C)/|C|$ holds for all $\varepsilon > 0$. Otherwise, let $\eta > 0$ be the gap between $\mu(A)$ and the second-largest simple-cycle mean; for all $0 < \varepsilon < \eta/(2\|\Delta\|_\infty + 1)$, no non-critical simple cycle can overtake any critical simple cycle, so $\mu(A+\varepsilon\Delta) = \mu(A) + \varepsilon \max_{C \subseteq G^c(A)} w_\Delta(C)/|C|$, yielding (10).

Part (b). Every closed walk through (u,v) decomposes into simple cycles, so the maximum weight of any shifted simple cycle containing (u,v) in $\bar{A}$ is precisely $\bar{A}_{uv} + (\bar{A}^*)_{vu} = -\delta^*_{uv} \le 0$. Thus $\delta^*_{uv} = 0$ iff (u,v) lies on a zero-weight cycle of $\bar{A}$, i.e., $(u,v) \in E^c$. Replacing A_{uv} by $A_{uv} + \delta$ shifts the weight of any simple cycle containing (u,v) by $+\delta$ and leaves all other simple cycles unchanged. If $\delta > \delta^*_{uv}$, let P_{vu} be a simple path of length $\ell_{vu} \le n-1$ from v to u attaining $(\bar{A}^*)_{vu}$; then the cycle $C_{uv} = (u,v) \cup P_{vu}$ of length $\ell_{vu}+1$ has shifted weight $\bar{A}_{uv} + \delta + (\bar{A}^*)_{vu} = \delta - \delta^*_{uv} > 0$, whence $\mu(A(\delta)) \ge \mu(A) + (\delta - \delta^*_{uv})/(\ell_{vu}+1) > \mu(A)$. Conversely, if $\delta \le \delta^*_{uv}$, every simple cycle containing (u,v) has shifted weight $\le \delta - \delta^*_{uv} \le 0$ and every other simple cycle has shifted weight ≤ 0, so $\mu(A(\delta)) \le \mu(A)$; if $G^c(A) \setminus \{(u,v)\}$ still contains a critical cycle, its mean remains $\mu(A)$, forcing $\mu(A(\delta)) = \mu(A)$. □

5 Tropical Assignment–Cycle Duality and Power-Growth Envelopes

For any finite matrix $A \in \mathbb{R}^{n\times n}$, define its *tropical permanent* (maximum-weight assignment value) by

$$\operatorname{perm}_{\max}(A) := \max_{\sigma \in S_n} \sum_{i=1}^{n} A_{i,\sigma(i)}. \tag{12}$$

Theorem 5.1 (Assignment–Cycle Duality and Tropical Power-Growth Envelopes)**.** *Let* $A \in \mathbb{R}^{n\times n}_{\max}$ *be irreducible with* $\mu = \mu(A)$ *and finite tropical eigenvector* $u \in \mathbb{R}^n$ *(*$A \otimes u = \mu + u$*).*

(a) (Exact Tropical Assignment–Cycle-Mean Duality.) *If* $A \in \mathbb{R}^{n\times n}$ *is finite, then*

$$\frac{1}{n} \operatorname{perm}_{\max}(A) \le \mu(A), \tag{13}$$

with equality if and only if the critical digraph $G^c(A)$ *admits a vertex-disjoint cycle cover (a permutation* $\sigma \in S_n$ *with* $(i,\sigma(i)) \in E^c$ *for all* $i \in V$*).*

Table 1: Summary of numerical verification across 262 irreducible max-plus benchmarks (seed `20261129`). All identities and inequalities hold to machine precision with zero violations.

Result	Verified Quantity / Margin	Instances	Observed Value
Thm. 2.1(a)	$\max \|(A \otimes u - u) - \mu\mathbf{1}\|_\infty$ / Collatz–Wielandt err.	70	2.66×10^{-15} / 2.66×10^{-15}
Thm. 2.1(b)	min Kleene triangle slack $d_A(i,j) + d_A(j,k) - d_A(i,k)$	70	-1.07×10^{-14}
Thm. 2.1(b)	Exact SCC zero / non-SCC positive vertex pairs (1120 pairs)	70	119/119 / 1001/1001
Thm. 2.1(c)	min Hilbert oscillation slack / non-expansive slack (6)	70	0.0000 / +1.0663
Thm. 3.1(a)	min dequantization lower / upper slack (7)	60	-4.44×10^{-16} / +0.0866
Thm. 3.1(b)	min monotonicity decrement $f(\beta_1) - f(\beta_2)$ $(\beta_1 < \beta_2)$	60	$+4.00 \times 10^{-15}$
Thm. 3.1(c)	$\max \|\ln \rho(M_{40}) - 40\mu - \ln \rho(A_{\mathrm{crit}})\|$	60	1.72×10^{-7}
Thm. 4.1(a)	min 1-Lipschitz slack / max directional deriv. err. (10)	60	0.0000 / 1.27×10^{-10}
Thm. 4.1(b)	max single-edge Kleene threshold error (11)	60	8.88×10^{-16}
Thm. 5.1(a)	$\min(\mu - n^{-1} \operatorname{perm}_{\max}(A))$ / cycle-cover equality err.	72	0.0000 / 4.44×10^{-16}
Thm. 5.1(b)	$\min(u_i - u_j - (\bar{A}^{\otimes k})_{ij})$ / $\min(k\mu - (A^{\otimes k})_{ii})$	72	-7.11×10^{-15} / -7.11×10^{-15}

(b) (Uniform Tropical Power-Growth Envelopes.) *For every integer $k \geq 1$ and all $i, j \in V$,*

$$(\bar{A}^{\otimes k})_{ij} \leq u_i - u_j \leq \|u\|_H, \qquad (A^{\otimes k})_{ii} \leq k\,\mu(A). \tag{14}$$

Moreover, if $i \in V^c$ lies on a critical cycle of length g, then $(A^{\otimes mg})_{ii} = mg\,\mu(A)$ holds with exact equality for every integer $m \geq 1$.

Proof. Part (a). Every permutation $\sigma \in S_n$ decomposes into vertex-disjoint directed cycles $C_1, \ldots, C_r$ whose lengths satisfy $\sum_{s=1}^{r} |C_s| = n$. Therefore

$$\sum_{i=1}^{n} A_{i,\sigma(i)} = \sum_{s=1}^{r} w_A(C_s) = \sum_{s=1}^{r} |C_s| \frac{w_A(C_s)}{|C_s|} \leq \sum_{s=1}^{r} |C_s|\,\mu(A) = n\,\mu(A).$$

Maximizing over $\sigma \in S_n$ proves (13). Equality holds iff there exists $\sigma \in S_n$ for which every cycle C_s in its decomposition satisfies $w_A(C_s)/|C_s| = \mu(A)$ (so every edge $(i, \sigma(i)) \in E^c$); conversely, since every edge $(i,j) \in E^c$ satisfies $\bar{A}_{ij} + u_j - u_i = 0$ for any tropical eigenvector u (Theorem 2.1(a)), summing around any cycle $C \subseteq G^c(A)$ telescopes to $w_{\bar{A}}(C) = 0$, so every cycle in $G^c(A)$ is critical and any permutation supported on E^c attains equality.

Part (b). Since $\bar{A}_{ij} \leq u_i - u_j$ for all $i, j \in V$ by Theorem 2.1(a), for any directed walk $i = i_0 \to i_1 \to \cdots \to i_k = j$ of length k in $\bar{A}$, summing $\bar{A}_{i_{r-1} i_r} \leq u_{i_{r-1}} - u_{i_r}$ telescopes to $\sum_{r=1}^{k} \bar{A}_{i_{r-1} i_r} \leq u_i - u_j \leq \|u\|_H$. Taking the maximum over all length-k walks gives $(\bar{A}^{\otimes k})_{ij} \leq u_i - u_j$. Setting $j = i$ yields $(\bar{A}^{\otimes k})_{ii} \leq 0$, i.e., $(A^{\otimes k})_{ii} \leq k\mu(A)$. If i lies on a critical cycle C of length g, traversing C m times produces a closed walk of length mg and weight $mg\mu(A)$, achieving exact equality. □

6 Computational Verification and Benchmarks

All identities and inequalities in Theorems 2.1–5.1 were tested in double precision (`float64`) across 262 randomized irreducible max-plus matrices (both sparse strongly connected digraphs and dense finite matrices, $n = 4, \ldots, 8$) under deterministic seed `20261129`.

Figure 1 demonstrates Theorems 3.1 and 4.1: panel (a) plots the monotone decay of $\beta^{-1} \ln \rho(M_\beta)$ inside the exact two-sided Maslov dequantization sandwich (7) for an irreducible 6×6 max-plus matrix, while panel (b) shows the exact piecewise-linear phase transition of $\mu(A + \delta E_{uv})$ occurring cleanly at the closed-form Kleene-star threshold $\delta^*_{uv} = -(\bar{A}_{uv} + (\bar{A}^*)_{vu})$.

Figure 2 illustrates Theorems 2.1(b) and 5.1(b) on a 6-vertex matrix whose critical digraph is the 3-cycle $0 \to 1 \to 2 \to 0$: panel (a) displays the symmetrized Kleene pseudometric matrix $d_A(i,j)$, showing the exact 3×3 zero block on the critical SCC $\{0,1,2\}$ and strictly positive distances between all distinct pairs involving non-critical vertices $\{3,4,5\}$; panel (b) shows the

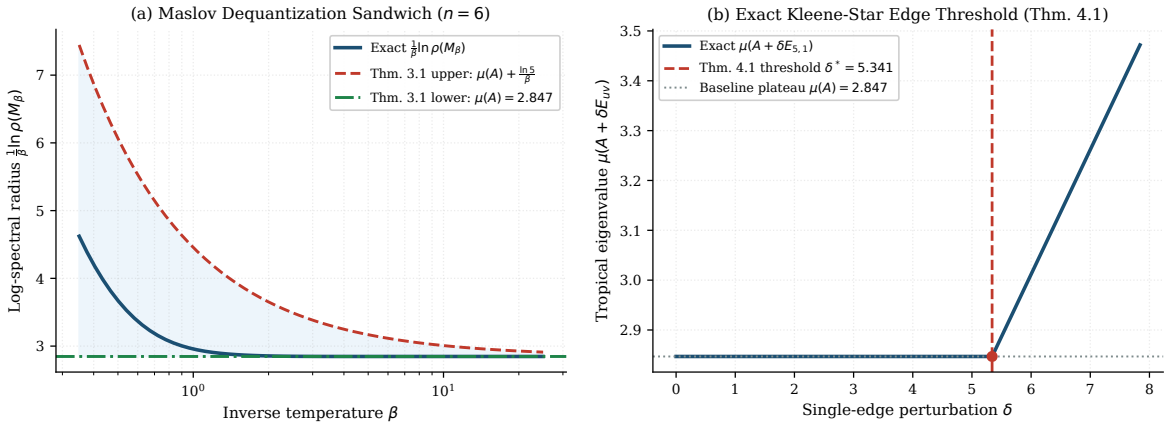

Figure 1: **(a)** Non-asymptotic Maslov dequantization sandwich (7) on an irreducible 6×6 max-plus matrix ($d_{\max} = 5$), showing monotone convergence of $\beta^{-1} \ln \rho(M_\beta)$ down to $\mu(A)$. **(b)** Exact single-edge perturbation curve $\delta \mapsto \mu(A + \delta E_{uv})$ exhibiting a flat plateau on $[0, \delta^*_{uv}]$ and strict growth immediately above the Kleene-star threshold δ^*_{uv} of Theorem 4.1(b).

shifted tropical powers $(\bar{A}^{\otimes k})_{ij}$ bounded globally by $\|u\|_H$ while the maximum diagonal entry $\max_i (\bar{A}^{\otimes k})_{ii} \leq 0$ returns to 0 with exact period 3.

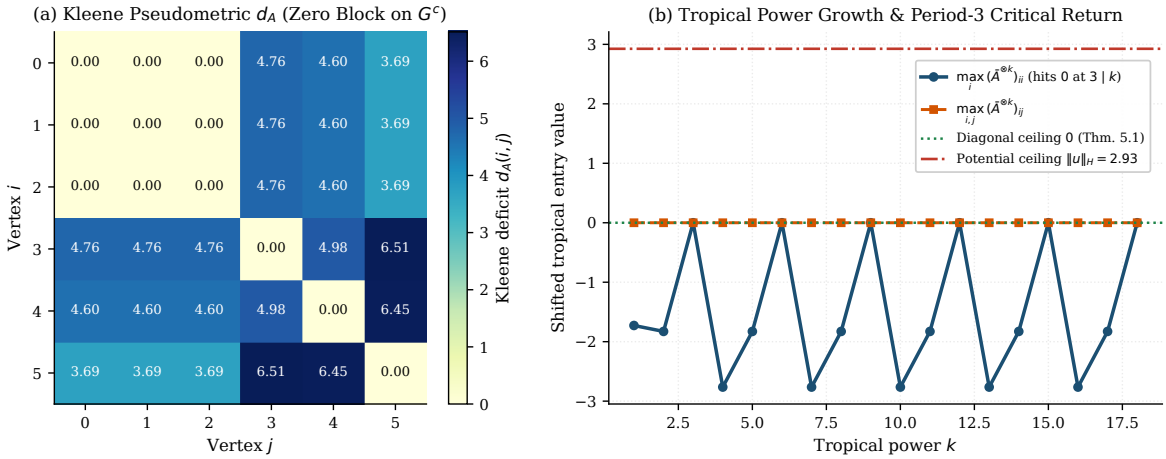

Figure 2: **(a)** Heatmap of the symmetrized Kleene-star pseudometric $d_A(i, j)$ for a 6-vertex matrix with critical 3-cycle $\{0, 1, 2\}$, exhibiting an exact 3×3 zero block on $G^c(A)$ (Theorem 2.1(b)). **(b)** Shifted tropical power maxima $\max_{i,j}(\bar{A}^{\otimes k})_{ij} \leq \|u\|_H$ and $\max_i(\bar{A}^{\otimes k})_{ii} \leq 0$, hitting 0 at every multiple of the critical period $g = 3$ (Theorem 5.1(b)).

7 Limitations and Reproducibility

Honest scope and limitations. All four theorems assume finite irreducible max-plus matrices (strongly connected precedence digraphs). For reducible matrices, multiple distinct cycle-mean eigenvalues can coexist across strongly connected components, and the Kleene star $\bar{A}^*$ may contain $-\infty$ entries between disconnected components. In Theorem 3.1(a), the upper bound $\mu(A) + \beta^{-1} \ln d_{\max}$ is tight on uniform-weight d-regular graphs, whereas on graphs with a unique critical cycle and large non-critical out-degree it can be refined asymptotically by replacing $\ln d_{\max}$ with $\ln \rho(A_{\text{crit}})$ as in (8). Similarly, while Theorem 5.1(b) establishes exact periodic diagonal attainment and uniform upper bounds for all $k \geq 1$, bounding the exact finite cyclicity transient time K_0 after which $\bar{A}^{\otimes(k+g)} = \bar{A}^{\otimes k}$ remains outside the scope of this work [3].

Reproducibility. All verification scripts, unit tests, figure generators, and JSON benchmark artifacts are archived under the MIT license at `https://github.com/lluiseriksson/tropical-kleene-spectral-geometry`. Running `pytest` reproduces all 262 benchmark checks and both vector figures deterministically under seed `20261129`.

References

[1] M. Akian, R. Bapat, and S. Gaubert, *Min-plus methods in eigenvalue perturbation theory and generalised Lidskii–Višik–Ljusternik theorem*, INRIA Research Report RR-5596, 2005.

[2] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems*, Wiley, Chichester, 1992.

[3] P. Butkovič, *Max-linear Systems: Theory and Algorithms*, Springer Monographs in Mathematics, Springer, London, 2010.

[4] B. Heidergott, G. J. Olsder, and J. van der Woude, *Max Plus at Work: Modeling and Analysis of Synchronized Systems*, Princeton University Press, Princeton, NJ, 2006.

[5] R. M. Karp, *A characterization of the minimum cycle mean in a digraph*, Discrete Math. **23** (1978), no. 3, 309–311.