Tetrahedral Echoes on the Fuzzy Sphere

Exact finite frames, spin towers, a sharp link-reflection-positive boundary, and a connected Hilbert–Schmidt nuclear cutoff

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Abstract

Four balanced spin kicks along the vertices of a regular tetrahedron generate an order-dependent eight-pulse echo. We prove an exact fact stronger than the qubit depolarizing calculation: all 24 echo orders have one conjugacy class in $SU(2)$. Randomizing only the global orientation therefore produces an order-law-independent, self-adjoint random-unitary channel on every spin-j matrix algebra. Its complete Hilbert–Schmidt spectrum is

$$\lambda_\ell(y) = \frac{U_{2\ell}(q(y))}{2\ell+1} = \frac{\sin((2\ell+1)\alpha(y))}{(2\ell+1)\sin\alpha(y)}, \qquad 0 \le \ell \le 2j,$$

with multiplicity $2\ell+1$, where $q = \cos\alpha = 1-A/2$, $A = 512z^4-256z^3+64z^2$, and $z = \sin^2(\sqrt{3}y)/3$. The maximal link-reflection-positive interval connected to zero is exactly $\alpha(y) \le \pi/(4j+1)$. Its endpoint is the unique small-pulse solution $A(y_j) = 4\sin^2[\pi/(2(4j+1))]$. Within every spin-$j$ strict positivity window, the first nonzero mass is the same function $m_1(y)$, independent of j.

The Haar frame is not an implementation idealization. Applying the classical positive Gauss–Legendre/trapezoidal product cubature at the architecture bandlimit $4j$ realizes the channel exactly with $(4j+1)(2j+1)$ weighted orientations when the frame may depend on the echo order, or with $(4j+1)^2(2j+1)$ order-oblivious orientations. No sampling limit is taken.

The full tower has a sharp large-spin law. If $N = 4j+1$, $y = \xi/\sqrt{N}$, and $(2\ell+1)/N \to u$, then

$$\lambda_\ell \longrightarrow \operatorname{sinc}(8\xi^2 u), \qquad \frac{m_\ell}{N^2} \longrightarrow -\frac{\log \operatorname{sinc}(8\xi^2 u)}{4\xi^4}$$

in the positive phase. Thus $\xi_c = \sqrt{\pi/8}$ is the exact boundary for positivity of the *entire* asymptotic tower: above it some rank sector is negative, although the top sector alternates sign in later sinc bands. The fixed-rank limit recovers the fuzzy-sphere Laplacian, while a simultaneous ultraviolet limit yields an explicit heat-trace integral. Finally, Poissonizing the transfer gives a genuine CPTP semigroup and an M-body connected nuclear profile $\exp(3e^{-c})-1$ at time $(\log M + c)/\Gamma_1$. This is a connected Hilbert–Schmidt nuclear cutoff, not a trace-distance mixing theorem. Exact symbolic and deterministic matrix certificates accompany the manuscript.

1 Question, answer, and novelty boundary

The fuzzy sphere replaces scalar functions on S^2 by the matrix algebra $\mathrm{End}(V_j)$, with rotations acting by conjugation in the spin-j irreducible representation. Its standard Laplacian is diagonal on $\mathrm{End}(V_j) = \bigoplus_{\ell=0}^{2j} V_\ell$, with eigenvalues $\ell(\ell+1)$. This finite geometry and its large-j limit are classical [3, 4, 5]. Covariant $SU(2)$ channels and their semiclassical behavior are also established subjects [6, 7].

Here the channel is generated by a specific finite pulse architecture. Four balanced noncommuting kicks are applied in an arbitrary order and completed by a physical reverse echo. The order is not assumed uniform. Instead, the tetrahedron receives a randomized global orientation. We ask whether this architecture produces an exactly solvable positive transfer on every fuzzy sphere, whether its frame average has an exact finite realization, and what remains of positivity when pulse size and spin grow together.

The answer has four levels. First, every echo order has the same fundamental trace, hence the same $SU(2)$ conjugacy class. Haar framing therefore erases the order law exactly, including deterministic laws. Second, the degree-$4j$ conjugacy orbit admits explicit positive finite cubatures, either order conditioned with $O(j^2)$ frames or order oblivious with $O(j^3)$. Third, central harmonic analysis turns the common class into a normalized-character spectrum, giving an exact link-reflection-positive window and finite-pulse mass tower. Fourth, scaling the pulse as $j^{-1/2}$ reveals a nontrivial ultraviolet sinc profile and the sharp boundary $\sqrt{\pi/8}$ for positivity of the full tower.

The novelty statement is deliberately scoped. We do not claim the Clebsch–Gordan decomposition, the $SU(2)$ character formula, fuzzy-sphere convergence, general theory of equivariant channels, reflection positivity, or generic tensor-product cutoff theory [8, 9, 10, 11]. To our knowledge, the architecture-level chain

$$\text{common tetrahedral echo class} \Longrightarrow \text{exact spin tower} \Longrightarrow \text{sharp link-positive scaling boundary}$$

and its double-scaling heat profile have not previously been written down. This is a scoped search statement, not proof of absolute priority.

The Gauss–Legendre/trapezoidal product rule itself is classical [1, 2]. We claim neither a new cubature nor minimal cardinality. The applied contribution is the exact bandlimit $4j$, the two finite frame counts, and their specialization to this echo channel.

Relation to the preceding tetrahedral-qubit papers

Two submitted predecessors are direct inputs, and neither is replaced here. The first proves that the regular tetrahedron uniquely maximizes the equal-norm four-kick qubit curvature gap and gives an exactly calibrated effective depolarizing twirl [12]. The second constructs a general finite-dimensional reflection-positive transfer framework and solves the uniformly order-averaged tetrahedral qubit transfer, its first zero, mass, and Hausdorff gap certificates [13]. Their relation to the present results is summarized in Table 1.

Manuscript	Earlier tetrahedral result	Result added here
Sharp costs and exact semigroups	Sharp qubit curvature extremality; effective-holonomy depolarizing schedule	Complete physical echo class for every spin; no qubit action-optimality claim is repeated
Reflection positivity and exact gap certificates	Uniform-order qubit transfer, link positivity, finite transfer mass, Hausdorff certificates	Arbitrary order law after global framing; all-spin character tower and exact spin-dependent positive window
Present paper	Uses the same tetrahedral physical word	Finite positive frames, fuzzy-sphere limit, uniform sinc double scaling, critical boundary layer, and connected HS nuclear profile

Table 1: Inherited and new parts of the tetrahedral line. The present paper is a sequel, not a replacement.

The text remains autonomous: the echo identity, spectrum, positivity theorem, limits, robustness bounds, and certificates are all proved here. Conversely, it does not reproduce the predecessors' action-optimality, general transfer window, Hausdorff moment certificates, or finite-sample statistical layer.

2 Tetrahedral complete echoes

Let $V_j \simeq \mathbb{C}^{2j+1}$ carry the spin-j irreducible representation π_j of $SU(2)$, where $j \in \frac{1}{2}\mathbb{N}$. Its Hermitian generators obey $[J_a, J_b] = i\epsilon_{abc}J_c$. Choose

$$\boldsymbol{v}_1 = (1,1,1), \quad \boldsymbol{v}_2 = (1,-1,-1), \quad \boldsymbol{v}_3 = (-1,1,-1), \quad \boldsymbol{v}_4 = (-1,-1,1) \tag{1}$$

and define

$$h_a^{(j)} = 2\boldsymbol{v}_a \cdot \boldsymbol{J}. \tag{2}$$

For $j = \frac{1}{2}$, $2\boldsymbol{J} = \boldsymbol{\sigma}$. The kicks are balanced and $\boldsymbol{v}_a \cdot \boldsymbol{v}_b = -1$ for $a \neq b$.

For an order $\sigma = (\sigma_1, \ldots, \sigma_4)$, with σ_1 acting first, put

$$U_{\sigma,j}(y) = e^{-iyh_{\sigma_4}^{(j)}} \cdots e^{-iyh_{\sigma_1}^{(j)}}, \tag{3}$$

$$W_{\sigma,j}(y) = U_{\sigma,j}(y) U_{\sigma^R,j}(y)^*, \qquad \sigma^R = (\sigma_4, \ldots, \sigma_1). \tag{4}$$

This is an eight-pulse word. Since π_j is a representation, there is a fundamental element $w_\sigma(y) \in SU(2)$ such that $W_{\sigma,j}(y) = \pi_j(w_\sigma(y))$.

Define

$$z(y) = \frac{\sin^2(\sqrt{3}y)}{3}, \qquad A(y) = 512z^4 - 256z^3 + 64z^2. \tag{5}$$

Lemma 2.1 (Common conjugacy invariant)**.** *For every $\sigma \in S_4$,*

$$\frac{1}{2} \operatorname{Tr}_{V_{1/2}} w_\sigma(y) = q(y) := 1 - \frac{A(y)}{2}. \tag{6}$$

Consequently all 24 echoes belong to one conjugacy class in $SU(2)$.

Proof. Put $c = \cos(\sqrt{3}y)$ and $s = \sin(\sqrt{3}y)/\sqrt{3}$. A fundamental pulse is $cI - is\boldsymbol{v}_a \cdot \boldsymbol{\sigma}$. If a partial product is $aI - i\boldsymbol{b} \cdot \boldsymbol{\sigma}$, left multiplication gives

$$(a, \boldsymbol{b}) \mapsto (ca - s\boldsymbol{v} \cdot \boldsymbol{b},\, c\boldsymbol{b} + sa\boldsymbol{v} + s\boldsymbol{v} \times \boldsymbol{b}). \tag{7}$$

The proper rotation group of the tetrahedron acts on the four labels as A_4. For $\tau \in A_4$, define the left action on an ordering by $(\tau\sigma)_r := \tau(\sigma_r)$, and choose its spin lift u_τ. Then

$$u_\tau(\boldsymbol{v}_a \cdot \boldsymbol{\sigma})u_\tau^* = \boldsymbol{v}_{\tau(a)} \cdot \boldsymbol{\sigma}, \qquad w_{\tau\sigma} = u_\tau w_\sigma u_\tau^*.$$

Hence the trace is constant on the two left A_4-orbits in S_4, which are exactly the even and odd orderings. It is therefore enough to evaluate one representative of each parity. Eight applications of (7), using only $\boldsymbol{v}_a^2 = 3$, $\boldsymbol{v}_a \cdot \boldsymbol{v}_b = -1$ for $a \neq b$, and the oriented tetrahedral cross products, give

Parity	Representative	$\frac{1}{2}\operatorname{Tr} w_\sigma(y)$
even	$(1, 2, 3, 4)$	$(c^2 - s^2)(c^6 + 13c^4s^2 + 35c^2s^4 + 79s^6)$
odd	$(2, 1, 3, 4)$	$(c^2 - s^2)(c^6 + 13c^4s^2 + 35c^2s^4 + 79s^6)$

Thus both orbits give

$$\operatorname{Tr} w_\sigma(y) = 2(c^2 - s^2)(c^6 + 13c^4s^2 + 35c^2s^4 + 79s^6). \tag{8}$$

With $z = s^2$ and $c^2 = 1 - 3z$, the right side divided by two is

$$(c^2 - s^2)(c^6 + 13c^4s^2 + 35c^2s^4 + 79s^6) = 1 - 32z^2 + 128z^3 - 256z^4 = 1 - A/2.$$

This reduces the analytic check from 24 words to the two displayed representatives; the ancillary script independently expands all 24 words. An element of $SU(2)$ has eigenvalues $e^{\pm i\alpha}$, $0 \leq \alpha \leq \pi$, and its conjugacy class is determined by its fundamental trace $2\cos\alpha$. □

Write

$$q(y) = \cos\alpha(y), \qquad 0 \leq \alpha(y) \leq \pi. \tag{9}$$

Since $A = 4\sin^2(\alpha/2)$, A is an exact finite-pulse class-angle invariant.

3 Haar framing and the exact spin tower

For any probability law μ on S_4, define

$$\mathcal{T}^{\mu}_{j,y}(X) = \sum_{\sigma\in S_4} \mu_\sigma \int_{SU(2)} \mathrm{Ad}_{\pi_j(g w_\sigma(y) g^{-1})}(X)\,\mathrm{d}g, \tag{10}$$

where $\mathrm{d}g$ is normalized Haar measure and $\mathrm{Ad}_U(X) = UXU^*$.

Theorem 3.1 (Order-law universality and exact spectrum)**.** *The channel (10) is independent of μ; denote it by $\mathcal{T}_{j,y}$. It is unital, trace preserving, completely positive, $SU(2)$-equivariant, and Hilbert–Schmidt self-adjoint. Under*

$$\mathrm{End}(V_j) = \bigoplus_{\ell=0}^{2j} V_\ell, \tag{11}$$

its eigenvalue on V_ℓ is

$$\boxed{\lambda_\ell(y) = \frac{\chi_\ell(w(y))}{2\ell+1} = \frac{U_{2\ell}(q(y))}{2\ell+1} = \frac{\sin((2\ell+1)\alpha(y))}{(2\ell+1)\sin\alpha(y)}} \tag{12}$$

with multiplicity $2\ell+1$.

Proof. Lemma 2.1 gives $w_\sigma = k_\sigma w k_\sigma^{-1}$. Haar invariance makes every summand in (10) equal, proving order-law independence. Each integrand is a unitary conjugation. In $SU(2)$, w^{-1} lies in the same class as w, so the conjugacy average equals its Hilbert–Schmidt adjoint.

The decomposition (11) is multiplicity free. The average commutes with $SU(2)$, so Schur's lemma makes it scalar on each V_ℓ. Its trace there is $\chi_\ell(w)$, giving the normalized character. The Weyl formula $\chi_\ell(w) = \sin((2\ell+1)\alpha)/\sin\alpha$ proves (12). □

Once the global frame is Haar random, averaging the order is redundant. Conversely, averaging only the 24 orders gives tetrahedral, not full $SU(2)$, symmetry and does not generally make every higher-rank sector scalar.

4 Exact finite-frame realizations

The Haar integral in (10) can be replaced exactly by a finite positive average. Two versions separate whether the controller may use the realized order. Let

$$L = 4j, \qquad N = L+1 = 4j+1, \qquad r = 2j+1. \tag{13}$$

Set $\phi_a = 2\pi a/N$ and $\psi_b = 2\pi b/N$, and let $(x_c, \omega_c)_{c=1}^r$ be the r-point Gauss–Legendre nodes and weights on $[-1,1]$. Thus $\omega_c > 0$ and $\sum_c \omega_c = 2$. Write $\beta_c = \arccos x_c$ and choose arbitrary $SU(2)$ lifts of

$$g_{ac} = R_z(\phi_a)R_y(\beta_c), \qquad\qquad g_{abc} = R_z(\phi_a)R_y(\beta_c)R_z(\psi_b). \tag{14}$$

Theorem 4.1 (Exact finite global frames)**.** *Choose $k_\sigma \in SU(2)$ so that $w_\sigma = k_\sigma w k_\sigma^{-1}$. For a known realized order, put $h_{ac}^{(\sigma)} = g_{ac}k_\sigma^{-1}$. Then*

$$\mathcal{T}_{j,y}(X) = \sum_{a=0}^{N-1}\sum_{c=1}^{r} \frac{\omega_c}{2N}\,\mathrm{Ad}_{\pi_j(h_{ac}^{(\sigma)} w_\sigma h_{ac}^{(\sigma)-1})}(X). \tag{15}$$

This uses exactly $Nr = (4j+1)(2j+1)$ positive-weight orientations.

If the same frame law must work without knowing the order, then for every σ

$$\mathcal{T}_{j,y}(X) = \sum_{a,b=0}^{N-1}\sum_{c=1}^{r} \frac{\omega_c}{2N^2}\,\mathrm{Ad}_{\pi_j(g_{abc} w_\sigma g_{abc}^{-1})}(X). \tag{16}$$

This order-oblivious realization uses $N^2 r = (4j+1)^2(2j+1)$ orientations. Both formulas are exact, not asymptotic sampling statements.

Proof. Let $W = \mathrm{End}(V_j)$, $\rho_j(g) = \mathrm{Ad}_{\pi_j(g)}$ on W, and $R = \rho_j(w)$. From (11), R lies in $\bigoplus_{\ell=0}^{2j} \mathrm{End}(V_\ell)$. Under $A \mapsto \rho_j(g) A \rho_j(g)^*$, the cyclic submodule generated by R therefore contains only integer spins $J \le 2\ell \le 4j = L$. This is the architecture bandlimit.

Conjugate w into the z-torus. Then R commutes with the axial stabilizer, so every scalar coefficient of its orbit is a combination of

$$D^J_{m0}(\phi, \beta, 0) = e^{-im\phi} d^J_{m0}(\beta), \qquad 0 \le J \le L.$$

The $N = L + 1$ equally spaced azimuths annihilate every nonzero $|m| \le L$, without aliasing. The remaining coefficient is $d^J_{00}(\beta) = P_J(\cos\beta)$. Gauss–Legendre quadrature with $r = 2j + 1$ nodes is exact through degree $2r - 1 = 4j + 1 > L$. Since normalized orbit measure is $\mathrm{d}\phi/(2\pi)\ \mathrm{d}x/2$, this proves (15). The conjugator k_σ^{-1} reduces every realized order to the same reference class element.

For an order-independent list, retain the third Euler angle. General matrix coefficients are $D^J_{mn} = e^{-im\phi} d^J_{mn}(\beta) e^{-in\psi}$. The two N-point trapezoidal rules kill all terms except $m = n = 0$, and the same polar rule finishes the Haar integral, proving (16). The weights in the two formulas sum respectively to $N(2)/(2N) = 1$ and $N^2(2)/(2N^2) = 1$. Finally, either lift gives the same adjoint action because the center acts as a scalar in V_j. □

The product cubatures used here are classical spherical-harmonic quadrature rules [1, 2]. Theorem 4.1 identifies the finite bandlimit and applies those rules exactly to the present pulse architecture. It makes no minimality or spherical-design claim.

5 Exact link reflection positivity

For self-adjoint T, define site and link forms

$$\mathfrak{S} = \sum_{p,q=0}^{r} \left\langle X_p, T^{p+q} X_q \right\rangle_{\mathrm{HS}}, \qquad\qquad \mathfrak{L} = \sum_{p,q=0}^{r} \left\langle X_p, T^{p+q+1} X_q \right\rangle_{\mathrm{HS}}. \tag{17}$$

The site form is a squared norm. The link form is nonnegative for all data exactly when $T \ge 0$.

On $0 \le y \le \pi/(2\sqrt{3})$, z, A, α increase away from zero because

$$\frac{\mathrm{d}A}{\mathrm{d}z} = 128z(16z^2 - 6z + 1) > 0. \tag{18}$$

Theorem 5.1 (Maximal connected link-positive window)**.** *On the pulse component attached to $y = 0$,*

$$\mathcal{T}_{j,y} \ge 0 \quad \Longleftrightarrow \quad 0 \le \alpha(y) \le \frac{\pi}{4j+1}. \tag{19}$$

The transfer is strictly positive when the last inequality is strict. Its endpoint y_j is the unique first-branch solution

$$A(y_j) = 4\sin^2\left(\frac{\pi}{2(4j+1)}\right). \tag{20}$$

At y_j, only the top eigenvalue first reaches zero; immediately after it is negative.

Proof. For $0 < \alpha < \pi/(4j+1)$, every $(2\ell+1)\alpha$ lies in $(0, \pi)$, so all eigenvalues in (12) are positive. At equality, the top mode $\ell = 2j$ vanishes and all lower modes remain positive. Immediately above, its sine is negative. Equation (20) follows from $A = 4\sin^2(\alpha/2)$; uniqueness follows from (18). □

Disconnected positive islands can occur at larger class angle. The theorem therefore concerns the maximal component containing the identity, not the global positivity set.

In the strict window define

$$H_{j,y} = -\frac{1}{4y^4} \log \mathcal{T}_{j,y}, \qquad m_\ell(y) = -\frac{1}{4y^4} \log \lambda_\ell(y). \tag{21}$$

Since $\sin x/x$ decreases strictly on $(0, \pi)$,

$$1 = \lambda_0 > \lambda_1 > \cdots > \lambda_{2j} > 0, \qquad 0 < m_1 < m_2 < \cdots < m_{2j}. \tag{22}$$

Thus $m_1(y)$ is independent of $j \ge \frac{1}{2}$.

Table 2: Exact connected link-positive endpoints from (20).

j	y_j	j	y_j
1/2	0.455698535295322	3	0.182254064679147
1	0.320178428422916	5	0.140751807450614
3/2	0.259660967397034	10	0.099304268084188
2	0.224049291863730	20	0.070140459670326

6 Fixed-rank fuzzy-sphere heat flow

Let $L = \ell(\ell+1)$. Exact expansion gives

$$\lambda_\ell(y) = 1 - \frac{128}{3}Ly^4 + 256Ly^6 + \left(\frac{8192}{15}L^2 - \frac{20096}{15}L\right)y^8 + O(y^{10}), \tag{23}$$

$$m_\ell(y) = \frac{32}{3}L - 64Ly^2 + \left(\frac{4096}{45}L^2 + \frac{5024}{15}L\right)y^4 + O(y^6). \tag{24}$$

Hence, for fixed j,

$$H_{j,y} \longrightarrow \frac{32}{3}\Delta_j, \qquad \Delta_j|_{V_\ell} = \ell(\ell+1)I. \tag{25}$$

The limiting heat trace is

$$Z_j(t) = \sum_{\ell=0}^{2j}(2\ell+1)e^{-(32/3)t\ell(\ell+1)}. \tag{26}$$

It converges to the scalar sphere heat trace. With $b = 32t/3$, $K = 2j+1$, and $r = e^{-b(2K+1)}$,

$$0 \le Z_\infty(t) - Z_j(t) \le e^{-bK(K+1)}\left(\frac{2K+1}{1-r} + \frac{2r}{(1-r)^2}\right). \tag{27}$$

This follows by writing $\ell = K + k$, using $\ell(\ell+1) \ge K(K+1) + k(2K+1)$, and summing two geometric series.

7 Double scaling and the sinc phase

Set

$$N = 4j+1, \qquad y_N = \frac{\xi}{\sqrt{N}}, \qquad \xi > 0. \tag{28}$$

The architecture polynomial has expansion

$$A(y) = 64y^4 - 384y^6 + \frac{6976}{5}y^8 + O(y^{10}). \tag{29}$$

Since $A = 4\sin^2(\alpha/2)$,

$$N\alpha(y_N) = 8\xi^2 + O(N^{-1}). \tag{30}$$

Theorem 7.1 (Uniform sinc tower and full-tower phase boundary)**.** *Put $u_{N,\ell} = (2\ell+1)/N$. Uniformly over all $0 \le \ell \le 2j$,*

$$\lambda_\ell(y_N) = \operatorname{sinc}(8\xi^2 u_{N,\ell}) + O_\xi(N^{-1}), \qquad \operatorname{sinc} x = \frac{\sin x}{x}. \tag{31}$$

Consequently

$$\xi_c = \sqrt{\frac{\pi}{8}} \tag{32}$$

is the sharp boundary for positivity of the entire asymptotic tower. If $\xi < \xi_c$, all ranks are strictly positive for sufficiently large j. If $\xi > \xi_c$, some rank is negative for sufficiently large j. At equality the top eigenvalue tends to zero.

For $0 < \xi < \xi_c$, uniformly over rank,

$$\frac{m_\ell(y_N)}{N^2} = \Phi_\xi(u_{N,\ell}) + O_\xi(N^{-1}), \qquad \Phi_\xi(u) = -\frac{\log \operatorname{sinc}(8\xi^2 u)}{4\xi^4}. \tag{33}$$

Proof. Rewrite

$$\lambda_\ell = \frac{\sin((2\ell+1)\alpha)}{(2\ell+1)\alpha}\frac{\alpha}{\sin\alpha}.$$

Equation (30) and $0 < u_{N,\ell} \le 1$ give (31) uniformly. Below criticality, all limiting arguments lie in $[0,\pi)$, so the full tower is positive.

If $\xi > \xi_c$, then $B = 8\xi^2 > \pi$. Choose

$$u \in \left(\frac{\pi}{B}, \min\left\{\frac{2\pi}{B}, 1\right\}\right).$$

This interval is nonempty and $\sin(Bu) < 0$. Admissible rank fractions become dense in $[0,1]$, so a nearby sector is negative for all sufficiently large N. Notice that the top value $\operatorname{sinc}(B)$ itself alternates sign in successive sinc bands; top-mode negativity is not asserted beyond the first negative band. At criticality its argument tends to π.

Below criticality, sinc is uniformly bounded away from zero on the rank interval, so the logarithm is uniformly Lipschitz. Since $4y_N^4 = 4\xi^4/N^2$, taking logarithms proves (33). □

The profile recovers the fuzzy Laplacian:

$$\Phi_\xi(u) = \frac{8}{3}u^2 + O(\xi^4 u^4) \qquad (\xi \to 0). \tag{34}$$

Corollary 7.2 (Ultraviolet heat-trace law)**.** *For $0 < \xi < \xi_c$ and $\tau > 0$,*

$$\lim_{j\to\infty} \frac{1}{N^2} \operatorname{Tr}_{\operatorname{End}(V_j)} \exp\left[-\frac{\tau}{N^2} H_{j,\xi/\sqrt{N}}\right] = \frac{1}{2}\int_0^1 u e^{-\tau\Phi_\xi(u)}\,du. \tag{35}$$

Proof. The left side is $N^{-2}\sum_{\ell=0}^{2j}(2\ell+1)e^{-\tau m_\ell/N^2}$. The mesh $u_{N,\ell}$ has spacing $2/N$, and the multiplicity is $Nu_{N,\ell}$. Uniform convergence in (33) gives the Riemann integral. □

Series reversion of (20) yields

$$y_j = \sqrt{\frac{\pi}{8N}}\left[1 + \frac{3\pi}{16N} + \frac{401\pi^2}{7680N^2} + O(N^{-3})\right]. \tag{36}$$

The N^{-1} correction in (36) resolves the critical edge itself.

Proposition 7.3 (Critical boundary layer)**.** *Let $\xi_c = \sqrt{\pi/8}$ and*

$$y_N = \frac{\xi_c}{\sqrt{N}}\left(1 + \frac{s}{N}\right), \qquad s \in \mathbb{R}. \tag{37}$$

Then the top-rank eigenvalue satisfies

$$\boxed{\lim_{N\to\infty} N\lambda_{2j}(y_N) = \frac{3\pi}{8} - 2s.} \tag{38}$$

Thus the finite-size sign change occurs at $s = 3\pi/16$, exactly the first correction in (36).

Proof. From (29) and $A = 4\sin^2(\alpha/2)$,

$$\alpha(y) = 8y^2 - 24y^4 + O(y^6).$$

Substituting (37) and $8\xi_c^2 = \pi$ gives

$$N\alpha(y_N) = \pi + \frac{2\pi(s - 3\pi/16)}{N} + O(N^{-2}).$$

Since $\lambda_{2j} = \sin(N\alpha)/(N\sin\alpha)$ and $N\sin\alpha \to \pi$, multiplication by N gives (38). □

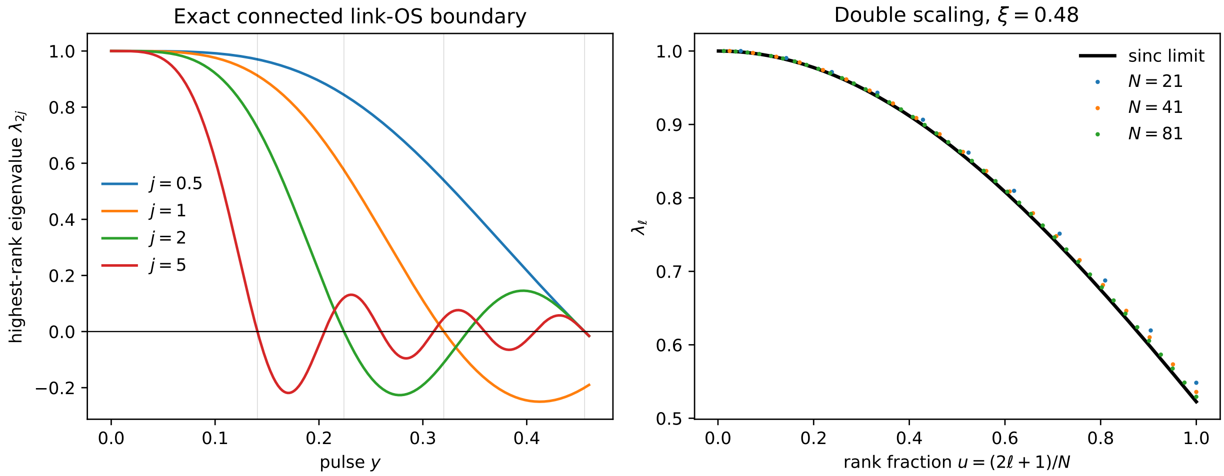

Figure 1: Left: highest-rank eigenvalues and their first positivity endpoints. Oscillations beyond the first zero explain why Theorem 5.1 concerns the component attached to zero. Right: convergence to the sinc tower at $\xi = 0.48 < \sqrt{\pi/8}$.

8 Poissonized CPTP dynamics and a connected Hilbert–Schmidt nuclear cutoff

Integer powers of $\mathcal{T}_{j,y}$ are CPTP. Fractional spectral powers need not be, so the OS heat operator must be separated from continuous quantum Markov dynamics. Define

$$\mathcal{P}_s = \exp\left[\frac{s}{4y^4}(\mathcal{T}_{j,y} - I)\right] = e^{-s/(4y^4)} \sum_{n=0}^{\infty} \frac{(s/(4y^4))^n}{n!} \mathcal{T}_{j,y}^n. \tag{39}$$

This is CPTP, being a Poisson mixture. Its rates are

$$\Gamma_\ell(y) = \frac{1 - \lambda_\ell(y)}{4y^4}, \qquad 0 = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{2j}, \tag{40}$$

and $\Gamma_\ell \to (32/3)\ell(\ell+1)$ as $y \to 0$.

For M independent sites, define the connected Hilbert–Schmidt nuclear trace

$$\nu_M(s) = \mathrm{Tr}_{\mathrm{HS}}(\mathcal{P}_s^{\otimes M}) - 1 = \left[1 + \sum_{\ell=1}^{2j}(2\ell+1)e^{-s\Gamma_\ell}\right]^M - 1. \tag{41}$$

Theorem 8.1 (Exact connected Hilbert–Schmidt nuclear profile)**.** *For fixed $j \geq \frac{1}{2}$ and y in the strict connected window, let*

$$s_M(c) = \frac{\log M + c}{\Gamma_1(y)}.$$

Then

$$\lim_{M\to\infty} \nu_M(s_M(c)) = \exp(3e^{-c}) - 1. \tag{42}$$

Proof. The rank-one term is $3e^{-c}/M$. For $\ell \geq 2$,

$$Me^{-s_M(c)\Gamma_\ell} = e^{-c\Gamma_\ell/\Gamma_1} M^{1-\Gamma_\ell/\Gamma_1} \to 0.$$

Thus the bracket in (41) is $1 + 3e^{-c}/M + o(M^{-1})$, proving (42). □

Generic product-channel cutoffs are known [11]. The architecture-specific content is the exact rate tower, leading degeneracy 3, and closed profile. We call this a connected Hilbert–Schmidt nuclear cutoff; without an extra norm comparison it is not a trace-distance mixing theorem.

9 Robustness and implementation hypotheses

Proposition 9.1 (Finite-frame weights, orientations, and pulses)**.** *Use either exact finite rule in Theorem 4.1, with ideal weights p_ν, ideal effective frames g_ν, and implemented values $\widehat{p}_\nu, \widehat{g}_\nu$, where $p_\nu, \widehat{p}_\nu \geq 0$ and $\sum_\nu p_\nu = \sum_\nu \widehat{p}_\nu = 1$. Put*

$$\eta_\nu = \|\pi_j(\widehat{g}_\nu) - \pi_j(g_\nu)\|_\infty, \qquad \varepsilon_{\mathrm{fr}} = \sum_\nu |\widehat{p}_\nu - p_\nu| + 4\sum_\nu \widehat{p}_\nu \eta_\nu.$$

Then the frame-implemented channel differs from $\mathcal{T}_{j,y}$ by at most $\varepsilon_{\mathrm{fr}}$ in induced Hilbert–Schmidt norm. If the relative physical frame rotation has angle $\theta_\nu \in [0,\pi]$, choose the lift along the shortest path so that $g_\nu^{-1}\widehat{g}_\nu = \exp[-i(\theta_\nu/2)\boldsymbol{n}_\nu \cdot \boldsymbol{\sigma}]$. Then $\eta_\nu \leq j\theta_\nu$.

If h_a is perturbed to $h_a + E_a$, the eight-pulse channel changes by at most

$$4|y| \sum_{a=1}^{4} \|E_a\|_\infty . \tag{43}$$

For a common bound $\|E_a\|_\infty \leq \epsilon$, this is $16|y|\epsilon$.

Proof. Write $G_\nu = \pi_j(g_\nu)$ and let $V_\nu = G_\nu \pi_j(w_\sigma) G_\nu^*$; the conditioned rule is included by absorbing k_σ^{-1} into g_ν. Since

$$\|\mathrm{Ad}_U - \mathrm{Ad}_V\|_{2\to 2} \leq 2\|U - V\|_\infty, \qquad \left\|\widehat{V}_\nu - V_\nu\right\|_\infty \leq 2\eta_\nu,$$

separating weight and frame errors gives

$$\left\|\widehat{\mathcal{T}} - \mathcal{T}\right\|_{2\to 2} \leq \sum_\nu |\widehat{p}_\nu - p_\nu| + 4\sum_\nu \widehat{p}_\nu \eta_\nu.$$

The angular estimate follows by integrating the spin-j generator along a shortest rotation path. Duhamel's formula gives $\left\|e^{-iy(H+E)} - e^{-iyH}\right\|_\infty \leq |y|\,\|E\|_\infty$. Each kick appears twice, so telescoping bounds the word error by $2|y|\sum_a \|E_a\|_\infty$. Finally $\|\mathrm{Ad}_U - \mathrm{Ad}_V\|_{2\to 2} \leq 2\|U - V\|_\infty$. □

If self-adjointness is retained by pairing every implemented word with its adjoint, a sufficient link-positive margin is

$$\varepsilon_{\mathrm{fr}} + 4|y| \sum_a \|E_a\|_\infty < \lambda_{2j}(y). \tag{44}$$

Norm closeness alone does not make a non-self-adjoint transfer link positive; adjoint pairing is a separate requirement. Unlike total variation against Haar—which equals one for every finite discrete law—the bound above is adapted to perturbations of the exact finite cubatures themselves.

10 Reproducibility and limitations

The accompanying archive contains the producer `certify_spin_j.py` and the independent finite-frame verifier `verify_finite_frames.py`. From the archive root, run `python certify_spin_j.py` and `python verify_finite_frames.py`; dependencies are listed in `requirements.txt`.

The ancillary scripts perform:

1. exact enumeration of all 24 fundamental echo traces;
2. exact checks of (23)–(24) for six ranks;
3. the explicit conditioned finite conjugacy-sphere cubature in Theorem 4.1 through $j = 2$, agreeing with (12) to at worst 1.3×10^{-15}, together with an independently oversampled matrix quadrature;
4. endpoint, asymptotic, and double-scaling convergence tables.

Two clean replays produced identical bytes. The frozen SHA-256 digests are

`spin_j_certificate.json:`	`94a53240f38bb3a111d2dd1dccfb2eba` `34dba4fe8bbf2f5303004b09121db021`,
`spin_j_os_and_scaling.pdf:`	`871ed8e6c93957bee5a970d9b12313cb` `72801fadb6938723ed3c8fef6ec6de8f`,
`finite_frame_verification.json:`	`cebbbbb2cddeba684b53d3b171d175b3` `9fd8b5de8d70ebdc9e37d1cad150a29f`.

The result has explicit boundaries:

- Link-OS is finite-dimensional transfer reflection positivity, not an infinite-volume field theory or Yang–Mills mass gap.
- The endpoint is semidefinite; the logarithmic Hamiltonian needs strict positivity.
- Full higher-spin scalarity requires global $SU(2)$ framing. Tetrahedral symmetry alone is insufficient; exact finite framing is supplied by Theorem 4.1.
- The fixed-rank and double-scaling limits have different orders of limits.
- Above ξ_c, some rank is negative. The top rank alternates sign in later sinc bands.
- The Poissonized semigroup is CPTP; the OS fractional powers are not claimed CPTP.
- The cutoff observable is exactly the connected Hilbert–Schmidt nuclear trace (41).

11 Conclusion

Every complete tetrahedral echo order lies in one exact $SU(2)$ conjugacy class. Global-frame randomization therefore removes the order law and normalized characters solve the channel on every fuzzy sphere. The required average has explicit exact finite realizations: quadratic in j when frames may be conditioned on the realized order, and cubic when one frame list must be order oblivious.

The solution identifies the complete spin tower, maximal connected link-positive window, exact endpoint, and finite-pulse mass ordering. At low rank the generator tends to the fuzzy Laplacian. When pulse and spin scale together, the entire tower converges to a sinc law with sharp full-tower boundary $\xi_c = \sqrt{\pi/8}$ and an explicit ultraviolet heat trace. Poissonization adds a genuine quantum Markov dynamics with an exact connected Hilbert–Schmidt many-body nuclear profile.

Concrete next questions are to minimize the exact frame cardinality, classify disconnected finite-spin positive islands, and identify other balanced spherical architectures with solvable class invariants.

AI-assistance disclosure

The author used an AI system for literature triage, symbolic-algebra scripting, numerical cross-checks, manuscript organization, and language editing. The claims were rederived into explicit proofs and deterministic certificates. The author retains responsibility for all definitions, calculations, citations, claims, and submission.

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