

Finite convex interpolation for polynomial Lyapunov existence

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Abstract

We characterize polynomial Lyapunov existence over all degrees for planar homogeneous polynomial fields with nonvanishing angular velocity. A moment threshold separates strict polynomial certificates from the absence of local analytic weak certificates in the stable case. Classical confluent Nevanlinna–Pick interpolation gives a finite convex semialgebraic formulation, with exact boundary decision for algebraic coefficients. For each odd field degree, the certificate region is open and semialgebraic, and compact admissible families have a computable common certificate degree. For rational fields in dense binary encoding, we prove polynomial-time additive approximation of the threshold and polynomial-time decision under a supplied positive gap promise. Explicit inner and outer radii permit a classical weak convex optimization reduction; a uniform functional perturbation argument handles algebraic poles without constructing their common number field. This approximates a value and does not bound the output size of a Lyapunov polynomial polynomially. Jackson bounds and a constructive sufficient-degree estimate complement the finite formulation. An exact cubic example has an algebraic threshold and minimum Lyapunov degree eight at one rational parameter. Beyond the rotating class, a boundary-factor obstruction produces a full twelve-dimensional rational box of stable quintic fields with sign-changing angular velocity and no local analytic weak certificate. One explicit rational C^1 Lyapunov function works for the entire box. The latter result is a sufficient obstruction, not a general classification with angular zeros. General certified solver implementations are not supplied.

1 The certificate problem

Consider a real planar homogeneous polynomial vector field X of degree $d \geq 1$. Write its polar equations as

$$\dot{r} = r^d F(\theta), \quad \dot{\theta} = r^{d-1} G(\theta). \quad (1)$$

We assume that G never vanishes. Its sign is constant; reflecting the plane if necessary makes $G > 0$. Antipodal parity then forces d to be odd, and F, G are π -periodic trigonometric polynomials. Define

$$a = -\frac{1}{2\pi} \int_0^{2\pi} \frac{F(\theta)}{G(\theta)} d\theta, \quad q(\phi) = \frac{F(\phi/2)}{G(\phi/2)} + a. \quad (2)$$

The profile q is real analytic and has zero mean.

A *weak Lyapunov function* here satisfies $V(0) = 0$, is positive away from the origin in its domain, and has $L_X V \leq 0$. A *strict function* has $L_X V < 0$. Homogeneous polynomial certificates are required globally; analytic certificates may be defined only near the origin. We study the stable rotating case $a > 0$.

Theorem 1.1 (Finite decision in the rotating class). *For a planar homogeneous polynomial field with real algebraic coefficients and G of one strict sign, existence of a strict polynomial Lyapunov function is decidable by a finite algorithm. In the case $a > 0$, the following are equivalent:*

- (i) a strict homogeneous polynomial Lyapunov function exists;
- (ii) a weak polynomial Lyapunov function exists;
- (iii) a positive local real-analytic weak Lyapunov function exists.

The algorithm also decides inputs on the boundary between existence and nonexistence. It uses finite semialgebraic interpolation data, without prescribing a Lyapunov degree. Exact boundary decision uses quantifier elimination. Polynomial-time approximation and promised-gap decision for dense rational inputs are established separately in Theorem 5.3.

Theorem 1.1 concerns existence of a particular kind of certificate. Stability in this class already follows from the sign of the classical polar average a . The distinction matters: Liu and Fitzsimmons [1] exhibit a stable homogeneous cubic field with no polynomial or local analytic weak Lyapunov function. Their family, stability analysis and original counterexample are prior results. We use the same factorization and leading-term viewpoint, extending the certificate problem to every profile in (2).

Earlier results show that polynomial Lyapunov degree cannot be bounded solely by dimension and vector-field degree [2, 3]; this does not rule out an input-dependent decision procedure. Fixed-degree semialgebraic searches and converse sums-of-squares results address different questions. Gram and Toeplitz methods also provide sufficient certificates for systems with angular variables [8]. Our finite interpolation formulation decides existence over all polynomial degrees for the rotating homogeneous class.

Korda's recent stability decision procedure covers planar homogeneous fields, including angular zeros [9]; Part III of its supplement explicitly distinguishes stability from polynomial-certificate existence. Its Theorem 20.3 already computes the normalized polar mean as an algebraic residue sum. We use that established calculation in our normalization. The additional problem here is the existence of a polynomial certificate of some degree.

The analytic tools used below are classical: Fejér–Riesz factorization [7], positive-real representations, Jackson kernels [4], semidefinite duality, confluent interpolation [5], and real quantifier elimination [6]. The contribution is the reduction connecting them to this all-degree certificate problem.

2 A moment threshold and finite error bounds

For a real 2π -periodic function use the conventions

$$\langle v \rangle = \frac{1}{2\pi} \int_0^{2\pi} v(t) dt, \quad q(t) = \sum_{k \in \mathbb{Z}} q_k e^{ikt}, \quad H \cos kt = \sin kt, \quad H \sin kt = -\cos kt.$$

Assume for now that $q \in C^2$, $q_0 = 0$. For $N \geq 1$, set

$$T_N(m) = (m_{k-j})_{j,k=0}^N, \quad m_{-k} = \overline{m_k}.$$

Define the finite optimum

$$a_N(q) = \min\{b : m_0 = 1, \quad T_N(m) \succeq 0, \quad T_N(w) \succeq 0, \\ w_0 = b, \quad w_k = im_k - q_k \quad (1 \leq k \leq N)\}. \quad (3)$$

The diagonal of $T_N(w)$ imposes $b \geq 0$. The choice $m_k = 0$, $b \geq \max q$, is feasible, since $b - q \geq 0$. Principal minors give $|m_k| \leq 1$, so the minimum is attained after restricting $b \leq \max q$. Principal submatrices also show

$$0 \leq a_1(q) \leq a_2(q) \leq \cdots \leq \max q.$$

Thus

$$A(q) = \sup_{N \geq 1} a_N(q) \quad (4)$$

is well defined. The integer N is a moment order, not a Lyapunov polynomial degree.

Theorem 2.1 (Finite brackets). *Let $\vartheta_N = \pi/(N+2)$. Then*

$$0 \leq A(q) - a_N(q) \leq E_N := \frac{\pi^2}{4}(1 - \cos \vartheta_N) \|q''\|_\infty \leq \frac{\pi^4 \|q''\|_\infty}{8(N+2)^2}. \quad (5)$$

Independently, if the weighted Fourier sum converges,

$$0 \leq A(q) - a_N(q) \leq \frac{\pi^2}{(N+2)^2} \sum_{k \geq 1} k^2 |q_k|. \quad (6)$$

The two right-hand majorants need not be ordered.

Proof. Put

$$v_j = \sqrt{\frac{2}{N+2}} \sin((j+1)\vartheta_N), \quad 0 \leq j \leq N, \quad K_N(t) = \left| \sum_{j=0}^N v_j e^{ijt} \right|^2.$$

The kernel is even, nonnegative and has mean one. Its real Fourier coefficients are

$$\kappa_k = \sum_{j=0}^{N-k} v_j v_{j+k} \quad (0 \leq k \leq N), \quad \kappa_k = 0 \quad (k > N),$$

with $\kappa_0 = 1$ and $\kappa_1 = \cos \vartheta_N$. These are the classical Jackson coefficients:

$$\kappa_k = \frac{(N+2-k) \cos(k\vartheta_N) + \cot(\vartheta_N) \sin(k\vartheta_N)}{N+2} \quad (0 \leq k \leq N); \quad (7)$$

see [4, eq. (2.11)]. Extending v by zero to $\ell^2(\mathbb{Z})$ and using the bilateral shift gives

$$1 - \kappa_k = \frac{1}{2} \|v - S^k v\|^2 \leq k^2(1 - \kappa_1), \quad k \geq 1. \quad (8)$$

Take an optimizer in (3) and set

$$u_N(t) = 1 + 2 \operatorname{Re} \sum_{k=1}^N \kappa_k m_k e^{ikt}, \quad w_N(t) = a_N + 2 \operatorname{Re} \sum_{k=1}^N \kappa_k w_k e^{ikt}.$$

For $z_j(t) = v_j e^{ijt}$, these are respectively $z(t)^* T_N(m) z(t)$ and $z(t)^* T_N(w) z(t)$. Both are non-negative, and

$$w_N = a_N - q_N - H u_N, \quad q_N = K_N * q.$$

With $E = \|q - q_N\|_\infty$, the two nonnegative densities u_N and $w_N + E - (q - q_N)$ give feasible moments at every order at damping $a_N + E$. Hence $A \leq a_N + E$. This argument constructs compatible full moment sequences; it does not assume that arbitrary simultaneous truncations extend.

Taylor's theorem, evenness of the kernel, and $t^2 \leq (\pi^2/2)(1 - \cos t)$ on $[-\pi, \pi]$ yield

$$E \leq \frac{\|q''\|_\infty}{2} \langle t^2 K_N(t) \rangle \leq \frac{\pi^2}{4} \|q''\|_\infty (1 - \kappa_1).$$

Finally $1 - \cos \vartheta_N \leq \vartheta_N^2/2$ gives (5). Applying (8) directly to the Fourier series gives $E \leq 2(1 - \kappa_1) \sum_{k \geq 1} k^2 |q_k|$, proving (6). $\square$

3 All degrees of polynomial certificates

For $|\alpha| < 1$, define the positive quadratic form

$$Q_\alpha(x, y) = (1 + |\alpha|^2)(x^2 + y^2) - 2\operatorname{Re}(\alpha)(x^2 - y^2) + 4\operatorname{Im}(\alpha)xy. \quad (9)$$

Its circle profile is $|1 - \alpha e^{2i\theta}|^2$.

Theorem 3.1 (Strict threshold). *For $a > 0$ and a real mean-zero C^2 profile q , the following are equivalent:*

- (i) $a > A(q)$;
- (ii) a finite product $P = \prod_{j=1}^M Q_{\alpha_j}$, $|\alpha_j| < 1$, satisfies

$$-a + q(2\theta) + \frac{p'(\theta)}{2Mp(\theta)} < 0, \quad p(\theta) = P(\cos \theta, \sin \theta);$$

- (iii) some positive homogeneous polynomial of even degree satisfies the corresponding weak inequality with ≤ 0 .

In particular equality $a = A(q) > 0$ admits no such weak certificate.

Proof. For a positive homogeneous polynomial of degree $2M$, Fejér–Riesz factorization of its circle profile, choosing the factor with all zeros outside the closed disk, gives

$$p(\theta) = C \prod_{j=1}^M |1 - \alpha_j e^{2i\theta}|^2, \quad |\alpha_j| < 1.$$

Factors with $\alpha_j = 0$ pad the actual Fourier degree to M . Differentiation gives

$$\frac{p'(\theta)}{2Mp(\theta)} = \operatorname{Im} h(e^{2i\theta}), \quad h(z) = \frac{1}{M} \sum_{j=1}^M \frac{1 + \alpha_j z}{1 - \alpha_j z}, \quad h(0) = 1. \quad (10)$$

Thus $u = \operatorname{Re} h$ has a strictly positive minimum on the circle. If $w = a - q - Hu \geq 0$, its coefficients satisfy $w_k = im_k - q_k$, where m_k are the coefficients of u . Both moment matrices are positive, so $A \leq a$.

To prove the necessary *strict* inequality, let $Q(z) = 2 \sum_{k \geq 1} q_k z^k$. The C^2 hypothesis implies absolute convergence on the closed disk. For small $\varepsilon > 0$,

$$h_\varepsilon = (1 - \varepsilon)h + \varepsilon(1 - iQ)$$

still has positive real part and $h_\varepsilon(0) = 1$. Since $\operatorname{Im}(1 - iQ) = -q$ on the circle, its slack is

$$a - q - \operatorname{Im} h_\varepsilon = (1 - \varepsilon)w + \varepsilon a.$$

It remains positive after replacing a by $a - \varepsilon a/2$. This gives $A < a$, using $a > 0$.

Conversely suppose $a > A$. Theorem 2.1 supplies N with $a_N + E_N < a$. Form the nonnegative densities u_N, w_N in its proof, and let $h_N(z) = 1 + 2 \sum \kappa_k m_k z^k$. For $r < 1$ close enough to one,

$$\operatorname{Re} h_N(rz) > 0 \quad (|z| \leq 1), \quad a - q - \operatorname{Im} h_N(re^{it}) = (w_N)_r + (a - a_N) - [q - (q_N)_r] > 0.$$

Here a subscript r means Poisson smoothing. The Herglotz representation of h_N has a probability measure on the circle. Approximate it by equal-weight empirical measures. For this fixed $r < 1$, the kernel is uniformly continuous on the closed disk, so the resulting functions

$$h_M(z) = \frac{1}{M} \sum_{j=1}^M \frac{1 + re^{-it_j} z}{1 - re^{-it_j} z}$$

approximate $h_N(rz)$ uniformly. They retain the strict slack when M is large. Equation (10) now gives the required product of positive quadratic forms. $\square$

Proposition 3.2 (Local analytic and global polynomial certificates). *For the field (1) with $G > 0$ and $a > 0$, the three conditions in Theorem 1.1 are equivalent to $a > A(q)$.*

Proof. For $P = r^{2M}p(\theta)$,

$$\frac{L_X P}{2Mr^{d-1}GP} = -a + q(2\theta) + \frac{p'(\theta)}{2Mp(\theta)}. \quad (11)$$

Thus Theorem 3.1 handles homogeneous certificates. If V is local analytic, take its first nonzero homogeneous Taylor term P_m . Positivity of V and homogeneity of X give $P_m \geq 0$ and $L_X P_m \leq 0$. For $p_m = P_m|_{S^1}$,

$$p'_m + m(F/G)p_m \leq 0.$$

If $p_m(\theta_0) = 0$, multiplication by $\exp(m \int_{\theta_0}^{\theta} F/G)$ yields a nonnegative nonincreasing function starting at zero, over a full following period. This forces $p_m \equiv 0$, a contradiction. Hence P_m is positive definite, m is even, and Theorem 3.1 gives $a > A(q)$. A polynomial is analytic; a strict homogeneous polynomial is itself a local analytic weak certificate. $\square$

The local leading-term mechanism follows the argument used for the specific cubic counterexample in [1]. For completeness, the classical stability statement is also direct. A periodic primitive $B'(\theta) = F/G + a$ gives

$$V_{\text{sm}}(r, \theta) = r^2 e^{-2B(\theta)}, \quad L_X V_{\text{sm}} = -2ar^{d-1}GV_{\text{sm}}.$$

This function is coercive and C^1 at the origin; for $a > 0$ it proves global asymptotic stability. If $a = 0$ the polar trajectories are closed, and if $a < 0$ they spiral outward. Neither case admits a strict Lyapunov function. This does not exclude weak first integrals when $a = 0$.

4 A finite interpolation representation

We now assume

$$q(t) = R(e^{it}), \quad (12)$$

where R is rational, has no pole on the unit circle, is real there and has zero circle mean. Algebraic coefficients will be needed only for the effective decision conclusion.

4.1 The quadrant dual

Lemma 4.1 (Finite dual). *The number $a_N(q)$ equals the maximum of*

$$\langle q \operatorname{Im} f \rangle - c \quad (13)$$

over analytic polynomials f of degree at most N such that $f(0) = c+i$, $c \geq 0$, and $\operatorname{Re} f, \operatorname{Im} f \geq 0$ on the circle.

Proof. Positive semidefinite Gram multipliers for the two Toeplitz constraints correspond to nonnegative trigonometric polynomials U, V of degree at most N . Every such polynomial has a Gram representation by Fejér–Riesz. With formal moments of u , $\langle u \rangle = 1$, the Lagrangian is

$$b - \langle uU \rangle - \langle (b - q - Hu)V \rangle = b(1 - \langle V \rangle) + \langle qV \rangle - \langle u(U + HV) \rangle.$$

We used skew-adjointness of H under the real circle pairing. Stationarity requires $\langle V \rangle = 1$ and $U + HV = c$ constant, with $c = \langle U \rangle \geq 0$. This says exactly that $f = U + iV$ is analytic, $f(0) = c + i$. The dual objective is (13). The primal is strictly feasible at $m_k = 0$, $b > \max q$; its finite optimum is attained. Finite-dimensional semidefinite strong duality proves the assertion. $\square$

Let ζ_j be the interior poles of $R(z)/z$, of orders n_j . For any analytic f on $\mathbb{D}$, define the finite jet functional

$$\Lambda_R(f) = \sum_{|\zeta_j| < 1} \operatorname{Res}_{z=\zeta_j} \frac{R(z)f(z)}{z}. \quad (14)$$

Only derivatives through order $n_j - 1$ at ζ_j occur. For polynomials, the residue theorem gives $\Lambda_R(f) = \langle qf \rangle$. For a general analytic f , (14) is the definition; a boundary integral is not required.

Proposition 4.2 (Quadrant formulation). *Let $\mathcal{Q}_c$ be the analytic maps from $\mathbb{D}$ to the open first quadrant with $f(0) = c + i$, where $c > 0$. Then*

$$A(q) = \sup_{c>0, f \in \mathcal{Q}_c} (\operatorname{Im} \Lambda_R(f) - c). \quad (15)$$

Proof. Any dual polynomial with $c > 0$ belongs to $\mathcal{Q}_c$ by the strong minimum principle for its two harmonic parts. A dual polynomial with $c = 0$ is identically i and contributes zero. Constants $c + i$ already have objectives tending to zero as $c \downarrow 0$.

For $f \in \mathcal{Q}_c$ and $r < 1$, both parts of $f(rz)$ have positive minima on the closed disk. Its Taylor polynomials converge uniformly there and eventually preserve both positive parts, with the same constant term. Their objectives approach $\operatorname{Im} \Lambda_R(f(r \cdot)) - c$. Every interior jet in (14) converges to the jet of f as $r \uparrow 1$. The union of the finite duals therefore has the same supremum as (15). $\square$

4.2 Interior jets and a Pick matrix

Include 0 among the nodes, adding multiplicity one if it is not already a pole of $R(z)/z$. Let the distinct nodes be $\zeta_1, \dots, \zeta_s$, with multiplicities n_j , and put $m = \sum n_j$. Introduce normalized jet variables

$$u_{jp} = \frac{f^{(p)}(\zeta_j)}{p!}, \quad 0 \leq p < n_j.$$

At node 0, impose the value $c + i$. Require

$$c > 0, \quad \operatorname{Re} u_{j0} > 0, \quad \operatorname{Im} u_{j0} > 0. \quad (16)$$

Define the jets of the proposed square $g = f^2$ by

$$v_{jp} = \sum_{\ell=0}^p u_{j\ell} u_{j,p-\ell}.$$

Form the Hermitian $m \times m$ matrix

$$\mathcal{P}_{(j,p),(k,\ell)} = \frac{\partial_z^p \partial_{\bar{w}}^\ell g(z) - \overline{g(w)}}{p! \ell! i(1 - z\bar{w})} \Big|_{z=\zeta_j, w=\zeta_k}. \quad (17)$$

The right-hand side uses only the specified jets v . In particular for values alone its entry is $(v_{j0} - \bar{v}_{k0})/(i(1 - \zeta_j \bar{\zeta}_k))$.

Lemma 4.3 (Classical confluent interpolation). *The tuples satisfying (16), the value condition at 0, and $\mathcal{P} \succeq 0$, are exactly the jet tuples of the functions in $\bigcup_{c>0} \mathcal{Q}_c$.*

Proof. If f maps to the first quadrant, $g = f^2$ maps to the upper half-plane. Its positive Pick kernel and its derivatives give $\mathcal{P} \succeq 0$.

Conversely, classical interior confluent Nevanlinna–Pick interpolation provides an analytic g with $\operatorname{Im} g \geq 0$ and the prescribed jets when $\mathcal{P} \succeq 0$, including singular semidefinite data.

This is the half-plane theorem; see [5, Section 3, pp. 246–247], after multiplication of g by $-i$. The normalized derivative kernels in (17) and the repeated-node basis there are related by an invertible change of basis. All interpolation nodes are interior.

At zero, $g(0) = (c+i)^2$ has imaginary part $2c > 0$; the minimum principle implies $\text{Im } g > 0$ throughout $\mathbb{D}$. The unique analytic square root taking values in the first quadrant exists on the upper half-plane. The inequalities in (16) select this root at each node. Successive derivatives of $f^2 = g$ recover all higher prescribed jets, since the coefficient of the next jet is $2u_{j0} \neq 0$. $\square$

Theorem 4.4 (Finite algebraic threshold). *For a rational profile (12), $A(q)$ is the supremum of the finite problem*

$$\sup\{J(c, u) : (c, u) \in \mathcal{S}_R\}, \quad J(c, u) = \text{Im } \Lambda_R(u) - c, \quad (18)$$

where $\mathcal{S}_R$ is defined by (16), the value at 0, and $\mathcal{P} \succeq 0$. If R has algebraic coefficients, $\mathcal{S}_R$ is semialgebraic over real algebraic numbers, $A(q)$ is algebraic, and its comparison with any algebraic number is decidable. The supremum in (18) need not be attained.

Proof. Proposition 4.2 and Lemma 4.3 give (18). Interior poles and residue coefficients of an algebraic rational function are algebraic and effectively computable. Each matrix entry in (17) is a polynomial of degree at most two in the real jet variables, with algebraic coefficients. The denominators $1 - \zeta_j \bar{\zeta}_k$ are nonzero constants. Hermitian positivity can be expressed using principal minors, or a real symmetric block matrix. Thus $\mathcal{S}_R$ is semialgebraic and J is affine.

The set is nonempty, and its supremum is finite and nonnegative by (4). For algebraic $a > 0$, the condition $a > A(q)$ is the first-order real sentence

$$\exists \varepsilon > 0 \forall (c, u) \quad [(c, u) \in \mathcal{S}_R \implies J(c, u) \leq a - \varepsilon]. \quad (19)$$

Real quantifier elimination [6] decides this sentence. The supremum itself is defined by the conditions that B is an upper bound and that, for each $\varepsilon > 0$, some feasible J exceeds $B - \varepsilon$. Eliminating quantifiers describes the singleton $\{A(q)\}$ over the real algebraic numbers. Such a singleton is algebraic, and real root isolation gives an effective specification. No compactness or maximizer of (18) is assumed. $\square$

Proof of Theorem 1.1. The no-zero condition on the angular component is decidable on the unit circle by real quantifier elimination. Reflect if needed. The functions F, G have only even angular frequencies, so F/G is rational in $z = e^{2i\theta}$ with algebraic coefficients and no circle poles. Its mean is a sum of interior residues of $(F/G)(z)/z$, and hence a in (2) is algebraic and computable; see also [9, Supplement, Theorem 20.3]. For $a \leq 0$ there is no strict certificate, as noted after Proposition 3.2. For $a > 0$, form $R = F/G + a$, its interior nodes and the finite problem (18). Decide (19) and apply Proposition 3.2. Equality $a = A(q) > 0$ yields a negative answer by the strict threshold theorem. $\square$

Remark 4.5. The matrix $\mathcal{P}$ has size determined by the poles of the input profile, rather than a candidate Lyapunov degree. Its entries are quadratic in the jets, so this representation is not a linear matrix inequality in the displayed variables. Nevertheless, $\mathcal{S}_R$ is convex: convex combinations of quadrant maps with $\text{Im } f(0) = 1$ remain in that class, and taking jets is linear. Lemma 4.3 identifies its image with $\mathcal{S}_R$. Thus (18) is a convex semialgebraic optimization problem. This observation alone supplies neither a finite lifted semidefinite representation nor an efficiency bound. The finite moment programs (3) are also convex, but their approximation alone does not decide equality.

5 Polynomial-time additive approximation

The convexity of the jet set has a quantitative consequence. The input model throughout this section is dense binary rational data: all coefficients or jets up to the indicated degree are explicit. The statements do not apply to compressed exponent encodings.

Theorem 5.1 (Rational jet data). *Let $\zeta_j \in \mathbb{Q}[i] \cap \mathbb{D}$ be distinct nodes including zero, with positive multiplicities n_j , and let*

$$\Lambda(f) = \sum_{j,p < n_j} b_{jp} \frac{f^{(p)}(\zeta_j)}{p!}, \quad b_{jp} \in \mathbb{Q}[i], \quad \Lambda(1) = 0.$$

Over quadrant maps $f(0) = c + i$, $c > 0$, define $A_\Lambda = \sup(\operatorname{Im} \Lambda(f) - c)$. For rational $0 < \varepsilon \leq 1$, a rational approximation to A_Λ with absolute error at most ε can be computed in time polynomial in the binary encoding of the data and the tolerance. In particular, the cost for $\varepsilon = 2^{-p}$ is polynomial in p . This is a value approximation, not a polynomial-time construction of a Lyapunov polynomial.

Proof. Write $f = c + iH$. Then $H(0) = 1$, $\operatorname{Re} H > 0$, and Herglotz representation gives, for $|z| \leq r < 1$,

$$|H(z)| \leq U_0(r) := \frac{1+r}{1-r}, \quad \left| \frac{H^{(p)}(z)}{p!} \right| \leq U_p(r) := \frac{2}{(1-r)^{p+1}} \quad (p \geq 1). \quad (20)$$

Take the rational radii $r_j = (1 + |\zeta_j|^2)/2$, which dominate $|\zeta_j|$, and put

$$B = 1 + \sum_{j,p} (|\operatorname{Re} b_{jp}| + |\operatorname{Im} b_{jp}|) U_p(r_j).$$

Since $\Lambda(1) = 0$, the affine objective is $J = \operatorname{Re} \Lambda(H) - c \leq B - c$. Constants $c + i$ show $0 \leq A_\Lambda \leq B$, and near-maximizers may be chosen with $c \leq B + 1$.

For $0 < \delta \leq 1/2$, let K_δ be the feasible jet set with $\delta \leq c \leq C := 2B + 2$. It is convex. Harnack gives the positive closed value bounds

$$\operatorname{Re} f(\zeta_j) \geq \delta \frac{1 - r_j}{1 + r_j}, \quad \operatorname{Im} f(\zeta_j) \geq \frac{1 - r_j}{1 + r_j}.$$

Add the interval for c and the Pick PSD condition. Lemma 4.3 shows that these closed conditions characterize K_δ exactly. Equation (20) bounds all remaining coordinates, so this body is compact. Take a nearly optimal map. Its convex combination with the constant $1 + i$, using weights $1 - \delta, \delta$, shows

$$0 \leq A_\Lambda - \max_{K_\delta} J \leq \delta(B + 1). \quad (21)$$

This does not assume attainment of the original supremum.

Put $m = \sum n_j$ and remove the fixed coordinate $\operatorname{Im} f(0) = 1$, leaving $n = 2m - 1$ real coordinates. Let x_0 be the jets of $1 + i$, and let the confluent Vandermonde matrix be

$$V_{(j,p),\ell} = \binom{\ell}{p} \zeta_j^{\ell-p} \quad (0 \leq \ell < m),$$

with zero entries if $\ell < p$. It is invertible. Define

$$T = \max \left(1, \sum_{a,b} (|\operatorname{Re}(V^{-1})_{ab}| + |\operatorname{Im}(V^{-1})_{ab}|) \right), \quad r_0 = (4T)^{-1}.$$

Any real jet perturbation of Euclidean norm at most r_0 has a Hermite interpolating polynomial p with $\sup_{\mathbb{D}} |p| \leq Tr_0 \leq 1/4$. Thus $1 + i + p$ stays in the quadrant and its anchor has real part in $[3/4, 5/4]$. Consequently $B(x_0, r_0) \subseteq K_\delta$. With $U = \max_{j,p} U_p(r_j)$, the rational radius $R_0 = 2n(C + U + 2)$ contains K_δ about both zero and x_0 .

Membership of rational tuples in K_δ is polynomial-time: form the quadratic squared-map jets and the rational Hermitian Pick matrix, test the scalar bounds, and test PSD by exact Schur elimination. A negative diagonal rejects; a zero diagonal requires its row to be zero; a positive pivot allows a Schur complement. Rational pivots are ratios of minors, so determinant bounds control their bit lengths polynomially. No enumeration of all principal minors is needed. This exact test is a valid weak-membership oracle.

Apply the classical centered-convex-body oracle theorem of Grötschel–Lovász–Schrijver [10, Corollary 4.3.12]. Its hypotheses are the rational center, inner and outer radii just exhibited and weak membership. For tolerance $\eta < r_0$, it returns $\hat{x}$ within η of K_δ , whose objective is at least the maximum on its η -interior minus η . The empty-interior alternative is impossible because of the inner ball. Write $J(x) = w \cdot x + j_0$, $W = \max(1, \sum |w_i|)$. Contracting a maximizer toward x_0 by η/r_0 puts it in that interior and loses at most $\eta W R_0/r_0$. Proximity bounds any excess objective by ηW . Therefore

$$|J(\hat{x}) - \max_{K_\delta} J| \leq \eta[1 + W(1 + R_0/r_0)].$$

Choose

$$\delta = \frac{\varepsilon}{4(B+1)}, \quad \eta = \min\left(\frac{r_0}{2}, \frac{\varepsilon}{4[1 + W(1 + R_0/r_0)]}\right).$$

Together with (21), this gives error at most $\varepsilon/2$. All displayed rational data have polynomial binary length: powers have exponents $O(m)$, and rational matrix inversion and determinant bounds control the inverse Vandermonde. Tiny geometric radii affect their logarithms, not the number of bits exponentially. The oracle theorem bounds both the number and size of its queries, completing the ordinary Turing polynomial-time reduction. $\square$

The theorem uses classical weak optimization, not a new general convex optimization algorithm or an asserted finite LMI lift. It already applies to trigonometric polynomials with rational Fourier coefficients, since their functionals use only jets at zero. Equality and the size of an explicit certificate remain separate issues.

5.1 Rational profiles with algebraic poles

Exact arithmetic in the compositum of all pole fields need not have polynomial degree. The next argument avoids that arithmetic by approximating the objective on a fixed analytic function class.

Lemma 5.2 (Uniform perturbation of the functional). *Let $f_p = f^{(p)}/p!$, and suppose that the functionals*

$$\Lambda(f) = \sum b_{jp} f_p(\zeta_j), \quad \tilde{\Lambda}(f) = \sum \tilde{b}_{jp} f_p(\tilde{\zeta}_j)$$

annihilate constants. Suppose corresponding nodes have the same jet orders and all nodes have modulus at most $\rho < 1$. Then

$$|A_\Lambda - A_{\tilde{\Lambda}}| \leq \sum_{j,p} |b_{jp} - \tilde{b}_{jp}| U_p(\rho) + \sum_{j,p} |\tilde{b}_{jp}| (p+1) U_{p+1}(\rho) |\zeta_j - \tilde{\zeta}_j|.$$

Proof. Write $f = c + iH$ as before. Constants cancel exactly in both objectives. Since $H'_p = (p+1)H_{p+1}$, integrate the derivative bound along the chord between corresponding nodes. Equation (20) bounds the coefficient error. Taking suprema over the same class of quadrant maps proves the assertion. No continuity claim about perturbed Pick feasible sets is used. $\square$

Theorem 5.3 (Polynomial-time value approximation for rational fields). *Let $S = N/D$, $N, D \in \mathbb{Q}[i][z]$, be given densely, real on the unit circle, with D having no zeros there. Both $\mu = \langle S \rangle$ and $A(S - \mu)$ admit rational additive ε -approximations in time polynomial in the dense binary input length and the tolerance encoding.*

Consequently, given a rational planar homogeneous field with nonvanishing angular speed, and a rational promised separation $\gamma > 0$, existence of a strict polynomial Lyapunov function of some degree can be decided in polynomial time on inputs satisfying $|a - A(q)| \geq \gamma$. For stable inputs $a > 0$, this also decides existence of a local analytic weak certificate under the same promise. The latter equivalence is not asserted for centers.

Proof. Clear a common denominator of N, D , scaling both together, and put $Q = zD$. At an interior root ζ_j of Q of multiplicity n_j , set

$$E_j(t) = \sum_{k \geq 0} \frac{Q^{(n_j+k)}(\zeta_j)}{(n_j+k)!} t^k, \quad c_{jp} = [t^{n_j-1-p}] \frac{N(\zeta_j + t)}{E_j(t)}.$$

Then $\Lambda_S(f) = \sum c_{jp} f_p(\zeta_j)$ is the residue functional, and $\mu = \sum_j c_{j0}$. Include the anchor zero with multiplicity at least one. The normalized functional $\Lambda = \Lambda_S - \mu f(0)$ has the same derivative coefficients, and its zeroth anchor coefficient is

$$b_{00} = - \sum_{\zeta_j \neq 0} c_{j0}.$$

Thus it annihilates constants without first constructing the algebraic mean in a common number field. Removable poles may be cancelled by rational polynomial gcd, or retained with zero residues.

We record the bit bounds needed to approximate these data. Let n, τ bound degrees and cleared integer coefficient lengths; both are polynomial in the dense input. Define $Q^\#(z) = z^{\deg Q} \overline{Q(1/\bar{z})}$. The primitive squarefree part P of

$$zQQ^\#\overline{Q}Q^\#$$

is an integer polynomial of polynomial degree and coefficient length; bars on polynomial factors here conjugate coefficients. Its roots include zero, all roots of Q , and the reciprocal conjugates of their nonzero roots. Squarefree extraction uses polynomial gcd and preserves polynomial bit bounds.

For P of degree $d_0 \geq 2$ and height $H_0 \geq 1$, Cauchy's root bound and its nonzero integral discriminant give the conservative separation $\sigma = (4H_0)^{-d_0^2}$ between distinct roots. Indeed all roots have modulus at most $2H_0$, and isolating one squared distance in the discriminant product, while bounding every other distance by $4H_0$, proves this lower bound. When $d_0 = 1$ only zero is relevant and we set $\sigma = 1$. For an interior nonzero root with $|\zeta| > 1/2$,

$$\sigma \leq |\zeta - 1/\bar{\zeta}| = \frac{1 - |\zeta|^2}{|\zeta|}, \quad 1 - |\zeta| \geq \sigma/4.$$

Hence $\beta = \min(1/2, \sigma/4)$ is a lower bound for the distance of every interior node from the unit circle. The reciprocal argument likewise separates exterior roots from it. Both σ and β have polynomial binary length.

The multiplicities and number of distinct roots of Q are computed by gcd over $\mathbb{Q}[i]$. Certified complex root isolation and refinement have polynomial bit complexity here; for example, the geometric bounds of Mehlhorn–Sagraloff–Wang [11, Theorems 3–4] apply. Root magnitudes, inverse separations and inverse products of root differences in those bounds have polynomial logarithms by the preceding estimates. The rational coefficient oracle is exact. Refining to 2^{-k} accuracy therefore costs a polynomial in the original input and k .

Residue coefficients have the same approximation property, even at multiple poles. At multiplicity r ,

$$\left| \frac{Q^{(r)}(\zeta)}{r!} \right| = |\text{lead}(Q)| \prod_{w \neq \zeta} |\zeta - w|^{\text{mult}(w)} \geq \sigma^{\deg Q}.$$

The leading coefficient is a nonzero Gaussian integer. For an explicit bound let $n \geq 1$ dominate both polynomial degrees, let 2^τ bound real and imaginary coefficient parts, and put

$$C_0 = (n+1)^3 2^{\tau+2n+4}, \quad h = \sigma^{\deg Q}, \quad B_0 = 2C_0/h.$$

On $|z| \leq 2$, C_0 bounds all factorial-normalized derivatives of N, Q and their first derivatives, by the binomial theorem. Within distance $\rho_0 = \min(1/4, h/(2C_0))$ of the root, the series denominator's constant term has modulus at least $h/2$. Write its coefficients as $e_j(w)$, and those of the translated numerator as $a_j(w)$. Series division obeys

$$t_\ell = \frac{a_\ell - \sum_{j=1}^{\ell} e_j t_{\ell-j}}{e_0}.$$

Induction gives $|t_\ell| \leq B_0(1 + B_0)^\ell$. Differentiating the recurrence gives

$$|t'_\ell| \leq B_0 \left(1 + \sum_{k < \ell} |t_k| + \sum_{k < \ell} |t'_k| + |t_\ell| \right).$$

A second induction bounds every derivative for $\ell \leq n$ by $L_0 = (n+2)B_0^2(1+B_0)^{2n}$, whose logarithm is polynomial in the input. Root accuracy $\min(\rho_0, \delta/(2L_0))$, followed by rounding within $\delta/2$, gives residue error at most δ . Exact rational series evaluation has polynomial bit growth: at order ℓ its denominator introduces at most $\ell + 1$ powers of the common constant denominator, in addition to polynomial coefficient and evaluation denominators. Thus the computation is polynomial in the input and requested precision bits, treating each pole separately.

Let m be the total interior multiplicity including the anchor. Coarse certified coefficient enclosures supply rational $M \geq 1 + \max |b_{jp}|$ with polynomial length. Set

$$\rho = 1 - \beta/2, \quad U = \max_{0 \leq p \leq m} U_p(\rho), \quad \delta = \min \left(\frac{\beta}{8}, \frac{\sigma}{8}, \frac{\varepsilon}{8mU(1+2mM)} \right).$$

Keep zero exact, approximate other nodes within δ , and approximate all coefficients except b_{00} within $\delta/(m+1)$. Set the new anchor coefficient to minus the sum of the other zeroth coefficients. Every coefficient error is at most δ ; constants cancel exactly. Nodes remain distinct inside radius ρ and $|\tilde{b}_{jp}| \leq 2M$. Lemma 5.2 bounds the threshold error by $m\delta U + 2m^2 M \delta U \leq \varepsilon/8$. Theorem 5.1, with tolerance $\varepsilon/2$, approximates the new rational functional. It need not itself arise from a real circle profile. The total error is less than ε , with polynomial encoding throughout. Approximating and summing the zeroth residues separately gives the asserted approximation of the real mean μ .

Finally, a dense rational homogeneous field gives Laurent F, G of degree at most $(d+1)/2$, with polynomial bit cost. Multiplication by that power of z represents F/G with polynomial numerator and denominator degrees at most $d+1$. After reflection if necessary, $G > 0$; its denominator has no unit-circle root. Approximate $-\mu$ and A to $\gamma/4$ to distinguish the promised cases. For $\gamma \geq 1$, fixed smaller tolerances suffice. The strict-certificate criterion also rejects $a \leq 0$; the equivalence with weak analytic certificates is used only for $a > 0$, as in Theorem 1.1. $\square$

Neither theorem makes exact equality polynomial-time, bounds the output size of a polynomial certificate polynomially, or implements the classical oracle algorithm. The gain is precision dependence: increasing the Jackson hierarchy alone gives no corresponding polynomial bound in $\log(1/\varepsilon)$. The primary ingredients remain classical interpolation, convex-body optimization and root refinement; the reduction establishes their hypotheses for this certificate threshold.

6 A semialgebraic region of certificate existence

The preceding construction is uniform in the coefficients, even when interior poles collide or cancel. This gives a statement about families without fixing a certificate degree.

Theorem 6.1 (Uniform coefficient region). *Fix an odd integer $d \geq 1$. In the coefficient space $\mathbb{R}^{2(d+1)}$ of planar homogeneous fields of degree d , let $\mathcal{R}_d$ denote the fields whose angular component has one strict sign, and let $\mathcal{C}_d \subset \mathcal{R}_d$ denote those admitting a strict polynomial Lyapunov function of some degree. Then $\mathcal{C}_d$ is open and semialgebraic, with an effectively constructible first-order real definition over $\mathbb{Q}$. The interpolation matrix in this definition has size at most $d + 2$, including the normalization node at zero.*

Proof. The condition defining $\mathcal{R}_d$ is a strict sign condition on the unit circle, hence semialgebraic and open. Treat $G > 0$; a fixed reflection treats $G < 0$. Put $k = (d + 1)/2$. The Laurent polynomials F, G in $z = e^{2i\theta}$ have degrees at most k , so

$$N(z) = z^k F(z), \quad D(z) = z^k G(z)$$

are polynomials of degree at most $d + 1$. Their coefficients depend $\mathbb{Q}[i]$ -linearly on the field coefficients. The polynomial D is nonzero and has no root on the unit circle.

There are finitely many degree and root-multiplicity patterns for D . For each pattern introduce its leading coefficient and distinct complex roots, impose the coefficient identities for their product, and specify which roots satisfy $|\zeta| < 1$ and which satisfy $|\zeta| > 1$. All these conditions are real semialgebraic. Taking the finite union of patterns handles collisions and degree drops without a continuous choice of root labels.

The mean $-a$ is the sum of interior residues of $N/(zD)$. On a fixed pattern these residues are rational functions of coefficients and root variables: repeated poles require only finitely many derivatives. All denominators are nonzero on the pattern, and auxiliary inverse variables turn each division into a polynomial equality. This classical mean computation, also explicit in [9, Supplement, Theorem 20.3], therefore defines a semialgebraically in the parameters.

Set $R = N/D + a$. In (14) and (17), use the interior roots of zD with their full denominator multiplicities. This includes zero with multiplicity at least one. Removable poles may be retained: their unused residue coefficients vanish. The extra jets do not change the feasible objectives, because every quadrant map supplies them and Lemma 4.3 still characterizes the full jet tuples. The total multiplicity is at most $\deg D + 1 \leq d + 2$. If $D(0) = 0$, the node zero is counted once with its multiplicity increased by one.

Both the residue objective and the Pick matrix entries are rational functions of the root and coefficient variables, polynomial of degree at most two in the jets. Their denominators, including $1 - \zeta_j \bar{\zeta}_\ell$, do not vanish. Thus the anchor condition, quadrant inequalities and positive semidefiniteness are semialgebraic jointly in all parameters. Existentially quantify the root and mean variables, require $a > 0$, and then impose (19), with its quantifiers over the feasible jets. Any admissible root listing describes the same profile and threshold. The finite union of these formulas defines exactly $\mathcal{C}_d$; real quantifier elimination gives the asserted effective semialgebraic description over $\mathbb{Q}$.

Finally a strict homogeneous certificate P for X has positive minima of P and $-L_X P$ on the circle. The same P remains strict under small coefficient changes in X , while the sign of G persists. Proposition 3.2 supplies a homogeneous certificate at every point of $\mathcal{C}_d$, proving openness. $\square$

Corollary 6.2 (Finitely many parameter transitions). *For a polynomial or rational one-real-parameter family of fixed odd degree, with real algebraic coefficients and a semialgebraic parameter domain over the real algebraic numbers, assume its rational denominators do not vanish on that domain. The parameters where the field belongs to $\mathcal{C}_d$ form a finite union of intervals and points with algebraic finite endpoints. If the domain is open, this set is relatively open and has no isolated point.*

Proof. Take the inverse image of $\mathcal{C}_d$ under the semialgebraic coefficient map. One-dimensional semialgebraic sets have the stated form; continuity gives the additional openness assertion. $\square$

Corollary 6.3 (A computable degree for compact families). *Given a compact semialgebraic coefficient set K , defined over real algebraic numbers, inclusion $K \subset \mathcal{C}_d$ is decidable. If it holds, an algorithm finds an even degree $2M$ such that every $X \in K$ admits a strict homogeneous certificate of degree $2M$. The certificate may depend on X .*

Proof. Decide inclusion using Theorem 6.1. A strict homogeneous certificate at each $X \in K$ works on a neighborhood. Compactness gives a finite cover with finitely many certificate degrees. Taking positive integer powers raises these certificates to their least common multiple degree and preserves strictness. Hence some even degree works throughout K .

For $M = 1, 2, \dots$, quantifier elimination decides

$$\forall X \in K \exists P \in \mathbb{R}[x, y]_{2M} \forall (x, y) : \quad x^2 + y^2 = 1 \implies P(x, y) > 0 \text{ and } -L_X P(x, y) > 0,$$

where $\mathbb{R}[x, y]_{2M}$ denotes homogeneous forms of degree $2M$ and the existential variables are their coefficients. The search terminates by the finite-cover argument. For empty K it may return degree two. This is a common degree, not necessarily a common polynomial, and no efficiency bound is asserted. $\square$

These are consequences of the finite interpolation representation; they do not give a new decision procedure for stability itself.

7 Effective separation and a sufficient degree

The finite brackets also have a quantitative use independent of quantifier elimination. If $a < A(q)$, some $a_N > a$ excludes certificates of every degree. If $a > A(q)$, some $a_N + E_N < a$ proves existence. For nonzero separation, it suffices to use

$$N = O\left(\max\left\{1, \sqrt{\frac{\|q''\|_\infty}{|a - A(q)|}}\right\}\right).$$

This statement concerns exact bounds on the finite optimum. A numerical implementation needs certified errors smaller than the separation; floating solver statuses do not supply them.

Theorem 7.1 (Constructive sufficient degree). *Let $\Delta = a - A(q) > 0$ and $L = 2 \sum_{k \geq 1} k |q_k| < \infty$. If $L = 0$, $P = x^2 + y^2$ is a certificate. Otherwise let*

$$\tau = \min\{1/2, \Delta/(4L)\}, \quad r = 1 - \tau, \quad M = \max\left\{1, \left\lceil \frac{8\pi r}{\Delta(1 - r^2)} \right\rceil\right\}.$$

There is a product of M positive quadratic forms with normalized slack at least $\Delta/2$. When $\Delta \leq 2L$, one may use $M \leq \lceil 32\pi L/\Delta^2 \rceil$. The Lyapunov polynomial degree is $2M$.

Proof. Take a diagonal subsequence of finite optimizers. The bounds $|m_k| \leq 1$, $a_N \leq \max q$, and closedness of every finite positive cone give compatible full moments. The classical positive trigonometric moment representation yields a probability measure μ and a positive measure ν of mass A , with $\nu_k = i\mu_k - q_k$, $k \geq 1$. Poisson smoothing at radius r loses at most

$$\|q - q_r\|_\infty \leq L(1 - r) \leq \Delta/4.$$

The smoothed Herglotz function thus has slack at least $3\Delta/4$.

Choose a cut of the circle with no atom of μ , and use its M midpoint quantiles with equal weights. Their cumulative discrepancy is at most $1/(2M)$, also in the presence of atoms. Integration by parts and the total variation of the Herglotz kernel give, uniformly on the closed disk,

$$\|h_M - h_r\|_\infty \leq \frac{1}{2M} \int_0^{2\pi} \frac{2r}{|1 - re^{it}|^2} dt = \frac{2\pi r}{M(1 - r^2)} \leq \Delta/4.$$

Use (10). The last bound on M follows from $\tau = \Delta/(4L)$ and $r/(1 - r^2) \leq 1/\tau$. $\square$

No optimality of the exponent or constants is claimed. Strict binary forms are sums of two squares of real forms, so both the constructed P and $-L_X P$ admit such decompositions. This is the classical binary positivity fact, not a new general SOS theorem. A small rational perturbation of the coefficients of P preserves both strict inequalities on the unit circle. Thus for a rational input field a rational-coefficient certificate exists whenever the strict criterion holds. This does not assert a two-square decomposition over the rationals.

8 Two explicit profiles

8.1 One harmonic

If q is a trigonometric polynomial of degree s , the functional only uses Taylor coefficients at zero:

$$\Lambda_R(f) = \sum_{k=1}^s \overline{q_k} f_k, \quad f_0 = c + i.$$

Put $g_k = \sum_{\ell=0}^k f_\ell f_{k-\ell}$. The finite interpolation condition reduces to

$$T_s(\mu) \succeq 0, \quad \mu_0 = 2c, \quad \mu_k = g_k/(2i) \quad (k \geq 1).$$

This is the classical Carathéodory–Fejér condition for $-ig$ to have nonnegative real part; its imaginary constant is unrestricted.

For $q = b \cos \phi$, $b \geq 0$, the 2×2 condition is

$$|f_1| \leq \frac{2c}{\sqrt{1 + c^2}}.$$

The objective is $(b/2) \operatorname{Im} f_1 - c$, so

$$A(b \cos \phi) = \sup_{c > 0} \left(\frac{bc}{\sqrt{1 + c^2}} - c \right) = (b^{2/3} - 1)_+^{3/2}. \quad (22)$$

The maximum for $b > 1$ occurs at $c = \sqrt{b^{2/3} - 1}$; for $b \leq 1$ the supremum is zero. Changing the phase replaces b by $|b|$. Consequently the stable cubic family with $G = 1$ and $F = -a + b \cos 2\theta$ has a polynomial or analytic weak certificate exactly when $|b| < (1 + a^{2/3})^{3/2}$. The family itself is from [1].

8.2 A rational profile with nonconstant angular speed

Consider

$$q(\phi) = \frac{3 \cos \phi}{1 + (3/5) \cos \phi} + \frac{5}{4}. \quad (23)$$

Writing $C = 25/4$, $r = 1/3$, one has $q = C(1 - P_{-r})$, where $P_{-r}(\phi) = (1 - r^2)/|1 + re^{i\phi}|^2$. Thus

$$\Lambda_R(f) = -C[f(-r) - f(0)].$$

Set $f(0) = c + i$, $f(-r) = x + iy$, with $c, x, y > 0$. The two-node Pick condition reduces exactly to

$$D(c, x, y) := 18cxy - (c^2 - 1 - x^2 + y^2)^2 - 4(c + xy)^2 \geq 0. \quad (24)$$

Therefore

$$A(q) = \frac{25}{4} - \inf_{\substack{c, x, y > 0 \\ D(c, x, y) \geq 0}} \left(c + \frac{25}{4}y \right). \quad (25)$$

Proposition 8.1 (An exact algebraic instance). *Let*

$$\begin{aligned} \mathcal{R}(t) = & 16777216t^{12} - 8141930496t^{10} + 1537904087040t^8 \\ & - 104707265987520t^6 + 2126206939995165t^4 \\ & - 15011745808074246t^2 + 81397225546445341. \end{aligned}$$

It has a unique root t_* in $(0, 19/4)$, and the profile (23) satisfies $A(q) = 25/4 - t_*$. In particular

$$1.657217310707 < A(q) < 1.657217310708. \quad (26)$$

The endpoints in (26) are exact decimal rationals.

Proof. The feasible rational triple $(c, x, y) = (1, 1, 3/5)$ has $D = 94/625 > 0$ and objective $19/4 < 25/4$. Restrict a minimizing sequence to objective at most $19/4$. The equivalent disk condition

$$\left| (x + iy)^2 - \left(c^2 - 1 + \frac{5}{2}ic \right) \right| \leq \frac{3}{2}c \quad (27)$$

bounds x , as c, y are already bounded. If $c \rightarrow 0$, the squared value tends to -1 , forcing $x \rightarrow 0$, $y \rightarrow 1$; the objective would tend to $25/4$, a contradiction. Hence c is bounded below. The imaginary part of (27) gives $2xy \geq c$, so x, y also stay bounded away from zero. A minimum is attained in the positive octant and lies on $D = 0$.

At a regular minimizing boundary point, the multiplier equations, and at a singular point the vanishing gradient, imply

$$D = 0, \quad D_x = 0, \quad 4D_y - 25D_c = 0, \quad 4t - 4c - 25y = 0. \quad (28)$$

Exact rational elimination in the variable order x, c, y, t gives the univariate eliminant

$$t^2(4t - 25)^2(4t + 25)^2(t^2 + 1)^2 \mathcal{R}(t). \quad (29)$$

The supplementary program recomputes a graded reverse lexicographic Gröbner basis over $\mathbb{Q}$, converts it to lexicographic order, verifies (29) and its zero remainder, and performs exact Sturm root counts. These give one root of $\mathcal{R}$ in $(0, 19/4)$, no endpoint root, and one root in

$$\left(\frac{4592782689292}{10^{12}}, \frac{4592782689293}{10^{12}} \right).$$

Since every minimizer satisfies (28) and $0 < t \leq 19/4$, none of the other factors in (29) vanishes. Existence of the minimum identifies it with the unique root t_* , giving (26). No numerical optimizer is used for global optimality. $\square$

This is realized by a concrete cubic vector field. With

$$\rho = x^2 + y^2, \quad H = \rho + \frac{3}{5}(x^2 - y^2), \quad K = (-a + \frac{5}{4})H + 3(x^2 - y^2),$$

take $\dot{x} = Kx - Hy$, $\dot{y} = Ky + Hx$. Then $G = 1 + (3/5) \cos 2\theta > 0$ and $F = (-a + 5/4)G + 3 \cos 2\theta$. It is globally asymptotically stable for every $a > 0$; polynomial and local analytic weak certificates exist precisely for $a > A(q)$. The algebraic boundary value is therefore an explicit equality input to the finite decision theorem.

8.3 An exact minimum degree

At $a = 2$, the preceding cubic has a strict integer-coefficient certificate, while finite rational identities exclude every smaller degree.

Proposition 8.2. *For the cubic above at $a = 2$, the minimum degree of a polynomial weak Lyapunov function is eight, even allowing nonhomogeneous polynomials and inequalities only near the origin, with $V(0) = 0$. A homogeneous degree-eight strict certificate exists globally.*

Proof. Define

$$\begin{aligned} P(x, y) = & 216x^8 - 2260x^7y + 12820x^6y^2 - 45066x^5y^3 + 99726x^4y^4 \\ & - 137242x^3y^5 + 113341x^2y^6 - 51276xy^7 + 9784y^8. \end{aligned}$$

The descending x -coefficients of $5(-L_X P)$, of degree ten, are

$$\begin{aligned} & 2528, -28576, 65016, 322736, -1562236, 1039608, \\ & 5502688, -13799076, 13740768, -6471612, 1188936. \end{aligned}$$

Exact signed Euclidean Sturm chains for $P(t, 1)$ and $-L_X P(t, 1)$ have variations $(4, 4)$ and $(5, 5)$ at minus and plus infinity, respectively. Neither has a real root. Their values at $(0, 1)$ are 9784 and $1188936/5$, and at the omitted projective point $(1, 0)$ they are 216 and $2528/5$. Homogeneity proves both strict positivities.

For the lower bound write, at degree $2M$,

$$p(\phi) = p_0 + 2 \operatorname{Re} \sum_{k=1}^M p_k e^{ik\phi}, \quad d(\phi) = -MF(\phi)p(\phi) - G(\phi)p'(\phi),$$

where $\phi = 2\theta$, $F = -3/4 + (51/20) \cos \phi$, and $G = 1 + (3/5) \cos \phi$. On the circle $-L_X P = 2d$. Let $v_M = (1, 2 \cos \phi, -2 \sin \phi, \dots, 2 \cos M\phi, -2 \sin M\phi)^T$. For a real rational t , set $\cos \phi = (1 - t^2)/(1 + t^2)$, $\sin \phi = 2t/(1 + t^2)$, and

$$b_M(t) = -MF(\phi)v_M(\phi) - G(\phi)v'_M(\phi).$$

Thus $b_M(t)$ is a rational column, and its pairing with the real Fourier coefficients gives $d(\phi)$. For the nodes in the following table form $B_M = [b_M(t_1) \cdots b_M(t_{2M+1})]$, and define $w = -B_M^{-1}(1, 0, \dots, 0)^T$. These formulas specify the weights exactly over $\mathbb{Q}$.

M	Nodes t_j	$\operatorname{sign} \det B_M$	$\min_j w_j >$
1	$-3/10, 17/20, 9/10$	1	1
2	$-7/20, 9/20, 1/2, 23/20, 6/5$	-1	2
3	$-7/20, 1/4, 3/10, 7/10, 3/4, 13/10, 27/20$	1	6

Direct rational arithmetic gives the nonzero determinants and positive weight bounds displayed; the supplementary verifier supplies the full rational weights and checks every coefficient of

$$\sum_j w_j d(\phi_j) = -p_0.$$

For a nonzero nonnegative p , its mean p_0 is positive. The identity is therefore incompatible with $d \geq 0$, excluding degrees two, four and six over arbitrary real coefficients.

Finally, the first nonzero homogeneous term of any local polynomial weak certificate with $V(0) = 0$ is nonnegative, nonzero, of positive even degree, and has nonpositive Lie derivative. If its total degree were below eight, one of the three excluded cases would occur. The exact verifier uses no optimizer; numerical searches only discovered the fixed coefficients and support nodes. $\square$

This is a minimum-degree result for this instance. It does not give a general minimum-degree law or a new positivity technique.

9 A robust obstruction when angular velocity changes sign

The preceding decision and complexity results require nonvanishing angular velocity. A sufficient obstruction extends to angular zeros and yields an explicit open set of stable systems outside that class. This is not a complete characterization with angular zeros.

Lemma 9.1 (A mean obstruction with boundary factors). *For a homogeneous field of odd degree, let F, G be its real trigonometric polar profiles in $\phi = 2\theta$, with $G \not\equiv 0$, and write*

$$G(\phi) = \sum_{k=-s}^s \gamma_k e^{ik\phi}, \quad T_G(\alpha) = 2 \operatorname{Im} \sum_{k=1}^s \overline{\gamma_k} \alpha^k.$$

If

$$\langle F \rangle + \min_{|\alpha| \leq 1} T_G(\alpha) > 0, \tag{30}$$

then the homogeneous field has no nonzero nonnegative homogeneous polynomial of positive degree with nonpositive Lie derivative. It consequently has no positive local real-analytic weak Lyapunov function.

Proof. A nonzero nonnegative homogeneous polynomial has degree $2M > 0$ and circle restriction

$$p(\phi) = C \prod_{j=1}^M |1 - \alpha_j e^{i\phi}|^2, \quad C > 0, \quad |\alpha_j| \leq 1.$$

Zero parameters may complete the factor list. Where $p > 0$, the Lie inequality is

$$F + G \operatorname{Im} h \leq 0, \quad h(z) = \frac{1}{M} \sum_{j=1}^M \frac{1 + \alpha_j z}{1 - \alpha_j z}.$$

Indeed $\dot{\phi} = 2r^{d-1}G$ and $p'/(Mp) = \operatorname{Im} h$. A zero of p at which $G \neq 0$ is impossible: on a short interval with constant sign of G , the inequality $Gp' + MFp \leq 0$, an integrating factor, and $p \geq 0$ propagate that zero to a one-sided interval. The analytic trigonometric polynomial would vanish identically.

For $|\alpha| < 1$, Fourier expansion gives

$$\left\langle G \operatorname{Im} \frac{1 + \alpha e^{i\phi}}{1 - \alpha e^{i\phi}} \right\rangle = T_G(\alpha).$$

For a boundary factor $|\alpha| = 1$, its pole occurs at a zero of G . In a local coordinate t about this point, $G(t) = O(t)$, so its product with $\cot(t/2)$ extends continuously. The radial regularization has imaginary part

$$\frac{2r \sin t}{(1-r)^2 + 2r(1-\cos t)}.$$

For $|t| \leq \pi$, the inequalities $|G(t)| \leq L|t|$ and $1 - \cos t \geq 2t^2/\pi^2$ bound the product uniformly by $L\pi^2/2$. Away from its pole it converges uniformly. Dominated convergence proves the same mean identity for boundary factors. Averaging the Lie inequality therefore gives

$$0 \geq \langle F \rangle + \frac{1}{M} \sum_j T_G(\alpha_j) \geq \langle F \rangle + \min_{|\alpha| \leq 1} T_G(\alpha) > 0,$$

a contradiction. The first nonzero homogeneous Taylor term of a positive analytic weak certificate is nonnegative, of positive even degree, and has nonpositive Lie derivative. The same contradiction excludes it; strict positivity of that leading term is not assumed. $\square$

Theorem 9.2 (A full coefficient box with a common rational certificate). *Consider the quintic field*

$$f_0 = -x^5 + 16x^3y^2 - 3xy^4, \quad g_0 = x^4y + 20x^2y^3 - y^5.$$

Independently perturb each of the twelve real coefficients of these two homogeneous quintics by at most $1/240$. Every resulting field is globally asymptotically stable, its angular velocity changes sign, and it admits no positive local analytic weak Lyapunov function. The same function

$$V(x, y) = \frac{(11x^2 + y^2)^{11}}{(x^2 + y^2)^{10}} \quad ((x, y) \neq 0), \quad V(0) = 0, \quad (31)$$

is a proper positive C^1 certificate for the entire box, satisfying $L_X V \leq -r^4 V/5$.

Proof. The central polar profiles are

$$F_0 = -1 + 5 \sin^2 \phi, \quad G_0 = \sin \phi.$$

Each perturbing Cartesian component has at most six terms, each bounded by $1/240$ on the unit circle. Hence the changes in both F and G have uniform norm at most $\epsilon = 1/20$. The profiles have trigonometric degree at most three in ϕ . Consequently

$$|\langle F \rangle - 3/2| \leq \epsilon, \quad \|T_G - T_{G_0}\|_{\infty, \mathbb{D}} \leq 6\epsilon.$$

Since $T_{G_0}(\alpha) = \operatorname{Re} \alpha$, the obstruction margin in (30) is at least $1/2 - 7\epsilon = 3/20 > 0$. Also $G(\pi/2) \geq 19/20$ and $G(3\pi/2) \leq -19/20$, proving sign change.

For stability set $v = (\partial_\theta \log V)/2$. Directly,

$$v = -\frac{55 \sin \phi}{6 + 5 \cos \phi}, \quad F_0 + G_0 v = -1 - \frac{25 \sin^2 \phi (1 - \cos \phi)}{6 + 5 \cos \phi} \leq -1.$$

The identity $(6 + 5t)^2 - 11(1 - t^2) = (5 + 6t)^2$ implies $|v| \leq 5\sqrt{11} < 17$. Thus, for every field in the box,

$$F + Gv \leq -1 + 18\epsilon = -\frac{1}{10}, \quad L_X V = 2r^4 V(F + Gv) \leq -\frac{1}{5} r^4 V.$$

The function satisfies $r^2 \leq V \leq 11^{11} r^2$ and $\nabla V = O(r)$, so its extension is C^1 and proper. Its compact sublevel sets ensure forward completeness; the strict Lie inequality gives global asymptotic stability. $\square$

Corollary 9.3 (Higher-order local perturbations). *If a locally C^1 vector field satisfies $X = X_5 + O(r^6)$, with no terms of degree below five and X_5 in the box of Theorem 9.2, the origin is locally asymptotically stable and has no positive local analytic weak Lyapunov function.*

Proof. The extra contribution to $L_X V$ is $O(r^7)$, whereas the negative term is at most $-r^6/5$. The first nonzero Taylor term of a putative analytic certificate is governed by X_5 and is excluded by Lemma 9.1. This argument is local; arbitrary higher-order perturbations need not preserve global stability. $\square$

Liu–Fitzsimmons already establish robust counterexamples within their two-parameter rotating family [1, Section IV.D], using the factorization and Taylor methods that underlie our lemma. Ahmadi–El Khadir’s rational converse theorem [3, Theorem 3.1] already guarantees rational certificates for asymptotically stable homogeneous fields. The specific additional conclusions here are the boundary-factor obstruction and a full twelve-dimensional rational coefficient box with angular sign changes and an explicit *common* certificate. The rational expression is singular at the origin before extension; it is not an analytic certificate there. Smooth nonanalytic Lyapunov functions are not excluded.

10 Verification, scope and remaining questions

The principal equivalences are analytic proofs. The rational threshold additionally uses exact symbolic elimination and Sturm counting; its program, equations, output and rational interval are supplied for reproduction. Proposition 8.2 uses separate exact positivity checks and rational evaluation identities, including a verification that does not call an optimizer. Separate exact checks verify Fourier signs, Toeplitz quadratic-form identities, atomic quantile discrepancy, the confluent Pick kernel including a singular Gram example, determinant identities and residues. The coefficient-region proof additionally accounts for pole collisions, removable poles and degree drops. Its quantifier elimination is theoretical. The complexity results explicitly invoke classical weak convex optimization and certified root refinement; neither general algorithm is implemented here. Additional exact checks cover the rational PSD oracle, confluent Hermite perturbations, irrational simple and repeated residue poles, functional perturbation bounds on Cayley kernels, and the common rational quintic certificate. These finite checks verify formulas and signs, not the full analytic or bit-complexity proofs. Reviewer checks are recorded separately.

Floating computations are retained as diagnostics. The hierarchy experiment has fifteen solves, nine labelled `optimal` and six `optimal_inaccurate`. Eight further primal/dual comparisons include three inaccurate primal and two inaccurate dual solves. Small negative numerical duality gaps are not certified bounds. A six-start numerical search with nonlinear constraints for the rational example produced five infeasible failures and one apparent local success; all are retained. Proposition 8.1 uses the separate exact elimination argument, not that local success.

AI tools assisted the derivations, computation and writing. Proof dossiers were examined by the same involved model reviewer requested as GPT-6 Astra with extra-high effort. This is not blind review, an additional independent vote, human peer review or formal verification; runtime identity is not independently attested. Any exact-manuscript assessment is a separate record. No Lean proof is supplied.

The literature comparison is targeted, not exhaustive. The original stability counterexample is credited to [1]; the kernels, interpolation and quantifier elimination are classical. Worldwide priority of this connection is unconfirmed. The all-degree characterization remains restricted to nonvanishing angular velocity. Theorem 9.2 supplies a sufficient obstruction with angular zeros, and Corollary 9.3 treats only the stated local higher-order perturbations. No

general nonhomogeneous or higher-dimensional classification follows. The polynomial-time claims concern additive values and supplied gap promises in dense encoding; they do not decide equality efficiently or produce a polynomial-sized certificate. Practical certified solvers, optimal minimum-degree laws, and a complete certificate criterion with angular zeros remain open.

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