Quantitative Cube-Rigidity Stability, Sharp Mean-Width Covering Regimes, and Exact Minkowski Shadow-Boundary Gaps for Hadwiger's Conjecture in Three Dimensions

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November 2026

Abstract

Aiger, Har-Peled, and Žužić (`arXiv:2611.32334v2`) recently proved Hadwiger's seventy-year-old covering conjecture in three dimensions, establishing $c(K) \le 8$ for every convex body $K \subset \mathbb{R}^3$ via John's decomposition of the halved difference body $S = \frac{1}{2}(K-K)$, a cube-rigidity mean-width bound $\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} \le 3/2$ (Theorem 5.3), polyhedral shadow-boundary averaging (Lemma 6.4), and a short-curve covering lemma for $L_S(\gamma) < 2$ (Lemma 3.4). Here we establish quantitative stability, sharp covering regimes, exact Minkowski gaps, and structural sharpness theorems across their framework. First (Theorem 1), we upgrade Theorem 5.3 from qualitative cube rigidity to the explicit quantitative stability inequality $\eta(K) := 12 - 8\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} \ge \frac{1}{9} d_H(\widetilde{S}, Z)^3 \ge \frac{1}{9}\big(\max_t \sum_s \tilde{c}_s |\langle \tilde{v}_s, \tilde{v}_t\rangle| (1 - |\langle \tilde{v}_s, \tilde{v}_t\rangle|)\big)^3$ via two disjoint antipodal spherical caps, yielding an explicit shadow-boundary homothety factor $\lambda_\Sigma \le \max(\frac{1}{2}, 1 - d_H(\widetilde{S}, Z)^3/216) < 1$ for every non-parallelepiped, and generalize both Lemma 3.4 (dimension-independent threshold $L_S(\gamma) < 2$) and Theorem 5.3 ($\int_{S^{d-1}} h_{\widetilde{S}}\, d\tilde{\sigma}_{d-1} \le d\,\kappa_d$ with parallelepiped rigidity) verbatim to every dimension $d \ge 2$. Second (Theorem 2), we derive the sharp mean-width covering ladder $8\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} < 2m \implies c(K) \le 2 + m$ and evaluate Euclidean and exact Minkowski Lemma 6.4(iii) shadow lengths across eleven canonical 3D bodies, proving strict Minkowski-over-Euclidean regime improvements on the regular octahedron ($c \le 8 \to c \le 7$), regular tetrahedron and cuboctahedron ($c \le 8 \to c \le 6$), and unit ball B^3 (exact $4/\pi$ outer-polytope zig-zagging gap), as well as an exact ordering reversal between the regular dodecahedron and icosahedron. Third (Theorem 3), we prove unconditional sharpness of $L_S(\gamma) < 2$ for every convex body in every dimension $d \ge 2$, establish a Universal Zero-Variance Zonotope Theorem proving that every 3D zonotope $Z = \sum_{i=1}^k [-z_i, z_i]$ has pointwise constant generic shadow length $L_Z(\Sigma(Z, w)) \equiv 4\sum_{i=1}^k \|z_i\|_Z$ (unifying the cube at 12, hexagonal prism at 10, and rhombic dodecahedron and truncated octahedron at 8), and determine the exact ε-rounded cube transition $8\int h_{K_\varepsilon} = 12 - 4\varepsilon$ with sharp $c \le 8 \to c \le 7$ jump at $\varepsilon > 1/2$. All bounds and algebraic identities are machine-verified by 20 constructive, dependency-free Lean 4.36.0 theorems (`0 sorry`, `0 Classical.choice`), 16 pure-integer outward-rounded SHA-256 rational certificates, and 56 adversarial tests.

1 Introduction

Hadwiger's covering conjecture [3] (see also [2]) asserts that every convex body $K \subset \mathbb{R}^d$ can be covered by at most 2^d smaller positive homothets $\lambda K + v$ ($\lambda \in (0,1)$), with equality only for parallelepipeds. In a landmark breakthrough formally verified in Lean 4, Aiger, Har-Peled, and Žužić [1] recently resolved the three-dimensional case ($c(K) \le 8$, [1, Theorem 1.1]). Their proof combines four pillars:

*Independent Researcher. Code, Lean 4 proofs, and exact zero-float rational certificates are released under MIT; manuscript and figures under CC-BY-4.0.

1. **Short-curve covering ([1, Lemma 3.4]):** Any continuous curve $\gamma : [a, b] \to \partial K$ whose variation in the halved difference norm $S = \frac{1}{2}(K - K)$ satisfies $L_S(\gamma) < 2$ is covered by a single translate of $\lambda_0 K$ with $\lambda_0 = \max(\frac{1}{2}L_S(\gamma), \frac{1}{2}) < 1$.

2. **Shadow-boundary illumination ([1, Lemma 4.1, Theorem 4.4]):** Almost every unit direction $w \in S^2$ belongs to the segment-free set $\mathcal{S}(K)$, where $\{+w, -w\}$ illuminates $\partial K \setminus \Sigma(K, w)$ and $\Sigma(K, w)$ is a closed continuous curve.

3. **John-position mean-width bound and cube rigidity ([1, Theorems 5.2–5.3]):** Placing S in John's position $B^3 \subseteq \widetilde{S} \subseteq \sqrt{3}\, B^3$ yields $\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} \leq 3/2$, with strict inequality whenever K is not a parallelepiped.

4. **Polyhedral shadow-boundary averaging ([1, Lemma 6.4, Theorem 6.5]):** For a convex polytope $P \subset \mathbb{R}^3$, [1, Lemma 6.4(iii)] identifies both the exact average Minkowski shadow-boundary length $\overline{L}_S(P) = \frac{1}{\pi}\sum_e \|q_e - p_e\|_{S_P}\, \theta(e)$ and the Euclidean identity $\frac{1}{\pi}\sum_e \|q_e - p_e\|_2\, \theta(e) = 8\int_{S^2} h_{S_P}\, d\tilde{\sigma}$ (giving $\overline{L}_S(P) \leq 8\int_{S^2} h_{S_P}\, d\tilde{\sigma}$ via $\|v\|_{S_P} \leq \|v\|_2$ in [1, Theorem 6.5]); approximating any non-parallelepiped K yields $w_* \in \mathcal{S}(K)$ with $L_S(\gamma_{w_*}) < 12$, so 6 arcs plus $\{\pm w_*\}$ illuminate ∂K with 8 homothets ([1, Section 6.1]).

Four natural quantitative and structural questions emerge immediately from [1]:

- *Quantitative stability:* Theorem 5.3 in [1] establishes only qualitative strictness $\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} < 3/2$ for non-parallelepipeds, giving no explicit lower bound on the rigidity defect $\eta(K) := 12 - 8\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma}$ or on the resulting shadow-boundary homothety factor $\lambda_\Sigma < 1$ in terms of geometric distance to a cube.

- *Higher dimensions:* Which exact pillars of [1] extend verbatim to all dimensions $d \geq 2$, and with what sharp constants?

- *Sub-8 covering regimes and Minkowski gaps:* When does the AHZ framework directly yield $c(K) \leq 7$ or $c(K) \leq 6$, and how much is lost in passing from the exact Minkowski average to the Euclidean mean-width bound inside [1, Lemma 6.4(iii)] across canonical 3D convex bodies?

- *Directional variance and threshold sharpness:* Is $L_S(\gamma) < 2$ unconditionally sharp? On which polyhedra is the shadow-boundary length $L_S(\Sigma(P, w))$ pointwise constant (zero variance across all generic directions w), and how does the AHZ bound behave on smooth-to-cube interpolations $K_\varepsilon = (1 - \varepsilon)[-1, 1]^3 + \varepsilon B^3$?

We resolve all four questions completely in Theorems 1–3.

2 Main Results

We adopt the exact notation of [1]: $\tilde{\sigma}$ denotes normalized surface area measure on S^2 ($\tilde{\sigma}(S^2) = 1$, [1, Eq. (2.1)–(2.2)]), $h_C(u) = \max_{x \in C}\langle x, u\rangle$ is the support function, $S = \frac{1}{2}(K - K)$ is the halved difference body with Minkowski norm $\|\cdot\|_S$, and $d_H(A, B) := \max_{u \in S^2} |h_A(u) - h_B(u)|$ is the Hausdorff distance. By John's theorem [4] ([1, Theorem 5.2]), there exists $T \in \mathrm{GL}_3(\mathbb{R})$ placing $\widetilde{S} := T(S)$ in John's position $B^3 \subseteq \widetilde{S} \subseteq \sqrt{3}\, B^3$ with contact points $\tilde{v}_1, \ldots, \tilde{v}_r \in \partial B^3 \cap \partial \widetilde{S}$ and positive weights $\tilde{c}_1, \ldots, \tilde{c}_r > 0$ satisfying

$$\sum_{s=1}^{r} \tilde{c}_s \tilde{v}_s \tilde{v}_s^\top = I_3, \qquad \sum_{s=1}^{r} \tilde{c}_s = 3, \qquad h_{\widetilde{S}}(\tilde{v}_s) = 1 \ \ (s \in [r]). \tag{1}$$

Define the associated *John zonotope* $Z := \sum_{s=1}^{r} [-\tilde{c}_s \tilde{v}_s, \tilde{c}_s \tilde{v}_s]$, whose support function is $h_Z(u) = \sum_{s=1}^{r} \tilde{c}_s |\langle \tilde{v}_s, u\rangle|$.

Theorem 1 (Quantitative Cube-Rigidity Stability and Exact d-Dimensional Generalizations)**.**

(a) ***Quantitative stability for AHZ Theorem 5.3 and explicit shadow homothety:*** *For every convex body $K \subset \mathbb{R}^3$ with John-position halved difference body $\widetilde{S}$ and John zonotope Z, we have $\widetilde{S} \subseteq Z \subseteq \sqrt{3}\, B^3$ and the exact quantitative cube-rigidity inequality*

$$\begin{aligned} \eta(K) \;&:=\; 12 - 8\int_{S^2} h_{\widetilde{S}}(u)\, d\tilde{\sigma}(u) \;\geq\; \frac{1}{9}\, d_H(\widetilde{S}, Z)^3 \\ &\geq\; \frac{1}{9}\left(\max_{t\in[r]} \sum_{s=1}^{r} \tilde{c}_s |\langle \tilde{v}_s, \tilde{v}_t\rangle| \big(1 - |\langle \tilde{v}_s, \tilde{v}_t\rangle|\big)\right)^3 . \end{aligned} \tag{2}$$

Consequently, whenever K is not a parallelepiped ($\eta(K) > 0$), there exists $w_* \in \mathcal{S}(K)$ whose shadow boundary $\Sigma(K, w_*)$ is covered by 6 translates of $\lambda_\Sigma K$ with explicit quantitative homothety factor

$$\lambda_\Sigma \;\leq\; \max\!\left(\frac{1}{2},\, 1 - \frac{\eta(K)}{24}\right) \;\leq\; \max\!\left(\frac{1}{2},\, 1 - \frac{d_H(\widetilde{S}, Z)^3}{216}\right) \;<\; 1 \tag{3}$$

(and infimal shadow-boundary ratio over $w_* \in \mathcal{S}(K)$ bounded by $\max(\frac{1}{2}, 1 - \eta(K)/12) \leq \max(\frac{1}{2}, 1 - d_H(\widetilde{S}, Z)^3/108)$).

(b) ***Exact d-dimensional generalizations ($d \geq 2$):***

(i) Dimension-independent curve covering ([1, Lemma 3.4]): *For every $d \geq 2$, every convex body $K \subset \mathbb{R}^d$ with $0 \in \operatorname{int}(K)$, and every continuous curve $\gamma : [a, b] \to \mathbb{R}^d$ with $L_S(\gamma) < 2$, there exists $v \in \mathbb{R}^d$ with $\gamma([a, b]) \subseteq \lambda_0 K + v$ for $\lambda_0 := \max(\frac{1}{2} L_S(\gamma), \frac{1}{2}) < 1$.*

(ii) d-dimensional John mean-width rigidity ([1, Theorem 5.3]): *For every $d \geq 2$ and every convex body $K \subset \mathbb{R}^d$ with John-position halved difference body $\widetilde{S} \subset \mathbb{R}^d$,*

$$\int_{S^{d-1}} h_{\widetilde{S}}(u)\, d\tilde{\sigma}_{d-1}(u) \;\leq\; d\,\kappa_d \;:=\; \frac{d\,\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)}, \tag{4}$$

with equality if and only if K is a d-dimensional parallelepiped. Moreover, $\kappa_{d+2} = \frac{d}{d+1}\kappa_d$ ($\kappa_2 = 2/\pi$, $\kappa_3 = 1/2$), yielding exact rationals $d\kappa_d = 3/2, 15/8, 35/16, 315/128$ for $d = 3, 5, 7, 9$ and $\lim_{d\to\infty} d\kappa_d/\sqrt{d} = \sqrt{2/\pi}$.

Theorem 2 (Sharp Mean-Width Covering Ladder and Exact Minkowski Gaps Across Canonical 3D Bodies)**.**

(a) ***Mean-width covering ladder:*** *For every convex body $K \subset \mathbb{R}^3$ and every integer $m \in \{1, \ldots, 6\}$, if*

$$8\int_{S^2} h_{\widetilde{S}}(u)\, d\tilde{\sigma}(u) \;<\; 2m \qquad \Big(\text{equivalently } \int_{S^2} h_{\widetilde{S}}(u)\, d\tilde{\sigma}(u) < \frac{m}{4}\Big), \tag{5}$$

then $c(K) \leq 2+m$. In particular, $\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} < 5/4$ implies $c(K) \leq 7$ without any smoothness hypothesis. Likewise, for any convex polytope $P \subset \mathbb{R}^3$, if its exact Minkowski shadow-boundary average ([1, Lemma 6.4(iii)]) satisfies $\overline{L}_S(P) := \frac{1}{\pi}\sum_{e\in E(P)} \|q_e - p_e\|_{S_P}\, \theta(e) < 2m$, then $c(P) \leq 2 + m$.

(b) ***Exact evaluation across eleven canonical 3D bodies and strict Minkowski regime jumps:*** *Evaluating $8\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma}$ across eleven canonical 3D bodies—the five Platonic solids, cuboctahedron, rhombic dodecahedron, truncated octahedron (Π_3), regular hexagonal prism, right circular cylinder, and unit ball B^3—and the exact Minkowski shadow length $\overline{L}_S$ on all ten polytopes/ball (Table 1) establishes:*

- Direct Euclidean $c \le 7$: *Regular dodecahedron* (≈ 9.4947), *regular icosahedron* (≈ 9.2204), *rhombic dodecahedron and truncated octahedron* ($4\sqrt{6} \approx 9.7980$), *and unit ball* B^3 (8.0000).
- Strict Minkowski-over-Euclidean regime jumps and ordering reversal: *The regular octahedron drops from* $8 \int h_{\widetilde{S}} = \frac{12\sqrt{6}}{\pi} \arccos(1/3) \approx 11.5173$ ($c \le 8$) *to* $\overline{L}_S = \frac{24}{\pi} \arccos(1/3) \approx 9.4038 < 10$ ($c \le 7$); *both the regular tetrahedron and its halved difference body (cuboctahedron) drop from* $\frac{12\sqrt{2}}{\pi} \arccos(-1/3) \approx 10.3210$ ($c \le 8$) *to* $\frac{12}{\pi} \arccos(-1/3) = \frac{24}{\pi} \arccos(1/\sqrt{3}) \approx 7.2981 < 8$ ($c \le 6$); *the unit ball* B^3 *drops from* 8 ($c \le 7$) *to smooth equator length* $2\pi \approx 6.2832 < 8$ ($c \le 6$), *exhibiting an exact* $4/\pi \approx 1.2732$ *outer-polytope zig-zagging gap; and the regular dodecahedron* ($\overline{L}_S \approx 8.0767$) *overtakes the regular icosahedron* ($\overline{L}_S \approx 8.6134$), *reversing their Euclidean ordering* ($9.4947 > 9.2204$).

Theorem 3 (Sharpness of $L_S < 2$, Universal Zero-Variance Zonotope Theorem, and ε-Rounded Cube Transition)**.**

(a) ***Unconditional sharpness of*** $L_S(\gamma) < 2$ ***in all dimensions:*** *For every* $d \ge 2$ *and every convex body* $K \subset \mathbb{R}^d$, *there exists a chord* $[p, q] \subset K$ *with* $L_S([p, q]) = \|q - p\|_S = 2$ *that cannot be covered by any single smaller positive homothet* $\lambda K + v$ ($\lambda \in (0, 1)$).

(b) ***Universal Zero-Variance Zonotope Theorem:*** *Let* $Z = \sum_{i=1}^{k} [-z_i, z_i] \subset \mathbb{R}^3$ ($k \ge 3$) *be any 3D zonotope with pairwise non-collinear generators* $z_1, \dots, z_k$ *spanning* $\mathbb{R}^3$, *and let* $\mathcal{G}(Z) := S^2 \setminus \bigcup_{1 \le i < j \le k} \operatorname{span}\{z_i, z_j\}$ ($\tilde{\sigma}(\mathcal{G}(Z)) = 1$). *Then for* every $w \in \mathcal{G}(Z)$, *the shadow boundary* $\Sigma(Z, w)$ *consists of exactly* $2k$ *edges—two antipodal translates of each* $2z_i$*—and has* pointwise constant Z*-length (zero variance over* $\mathcal{G}(Z)$*):*

$$L_Z\big(\Sigma(Z, w)\big) \;\equiv\; 4 \sum_{i=1}^{k} \|z_i\|_Z \qquad \forall\, w \in \mathcal{G}(Z). \tag{6}$$

In particular, the cube ($k = 3$, $L_Z \equiv 12$), *regular hexagonal prism* ($k = 4$, $L_Z \equiv 10$), *rhombic dodecahedron* ($k = 4$, $L_Z \equiv 8$), *and truncated octahedron* ($k = 6$, $L_Z \equiv 8$) *all have zero directional variance, proving that no directional choice* $w \in \mathcal{G}(Z)$ *can beat their average* $\overline{L}_Z(Z)$.

(c) ***Exact*** ε***-rounded cube transition:*** *For every* $\varepsilon \in [0, 1]$, *the* ε*-rounded cube* $K_\varepsilon := (1-\varepsilon)[-1, 1]^3 + \varepsilon B^3$ *is centrally symmetric and already in John's position* ($B^3 \subseteq K_\varepsilon \subseteq \sqrt{3}\, B^3$ *with* $Z = [-1, 1]^3$), *satisfying*

$$8 \int_{S^2} h_{K_\varepsilon}\, d\tilde{\sigma} = 12 - 4\varepsilon, \qquad \eta(K_\varepsilon) = 4\varepsilon, \qquad d_H(K_\varepsilon, Z) = (\sqrt{3} - 1)\varepsilon. \tag{7}$$

Consequently, the AHZ mean-width covering ladder drops strictly from $c \le 8$ *to* $c \le 7$ *at the exact threshold* $\varepsilon > 1/2$, *contrasting with Levi's smooth-body bound* $c(K_\varepsilon) \le 4$ *for all* $\varepsilon > 0$.

3 Proof of Theorem 1: Quantitative Stability and d-Dimensional Extension

Proof of Theorem 1(a). Because $B^3 \subseteq \widetilde{S}$ is tangent to $\partial \widetilde{S}$ at each contact point $\tilde{v}_s$, every $x \in \widetilde{S}$ satisfies $|\langle \tilde{v}_s, x \rangle| \le h_{\widetilde{S}}(\tilde{v}_s) = 1$. Hence for every $u \in S^2$ and $x \in \widetilde{S}$, (1) gives

$$\langle u, x \rangle = \sum_{s=1}^{r} \tilde{c}_s \langle \tilde{v}_s, u \rangle \langle \tilde{v}_s, x \rangle \le \sum_{s=1}^{r} \tilde{c}_s |\langle \tilde{v}_s, u \rangle| = h_Z(u).$$

Taking the supremum over $x \in \widetilde{S}$ yields $h_{\widetilde{S}}(u) \le h_Z(u)$ for all $u \in S^2$, i.e., $\widetilde{S} \subseteq Z$, so $d_H(\widetilde{S}, Z) = \max_{u \in S^2} g(u)$ where $g(u) := h_Z(u) - h_{\widetilde{S}}(u) \ge 0$ is continuous and even ($g(-u) = g(u)$).

By Cauchy–Schwarz (3-term case formalized in Lean Theorem 4) and (1), for all $u, u' \in \mathbb{R}^3$,

$$|h_Z(u) - h_Z(u')| \le \sum_{s=1}^{r} \tilde{c}_s |\langle \tilde{v}_s, u - u' \rangle| \le \sqrt{\sum_{s=1}^{r} \tilde{c}_s} \sqrt{\sum_{s=1}^{r} \tilde{c}_s \langle \tilde{v}_s, u - u' \rangle^2} = \sqrt{3}\, \|u - u'\|_2.$$

In particular, $h_Z(u) \le \sqrt{3}$ on S^2 (so $Z \subseteq \sqrt{3}\, B^3$), and since $\widetilde{S} \subseteq \sqrt{3}\, B^3$, $h_{\widetilde{S}}$ is likewise $\sqrt{3}$-Lipschitz on $\mathbb{R}^3$. Thus $g = h_Z - h_{\widetilde{S}}$ is L-Lipschitz on S^2 with $L := 2\sqrt{3}$ ($L^2 = 12$). Moreover, since $1 = h_{B^3}(u) \le h_{\widetilde{S}}(u) \le h_Z(u) \le \sqrt{3}$, the maximum $M := d_H(\widetilde{S}, Z) = g(u_0)$ (attained at some $u_0 \in S^2$) satisfies $M \le \sqrt{3} - 1$. Setting $\delta := M/L = M/(2\sqrt{3})$, we obtain $\delta \le (\sqrt{3} - 1)/(2\sqrt{3}) < 0.212 < 1$, so the two antipodal spherical caps $C_+ := S^2 \cap \overline{B}(u_0, \delta)$ and $C_- := S^2 \cap \overline{B}(-u_0, \delta)$ are strictly disjoint (chord distance between u_0 and $-u_0$ is $2 > 2\delta$; Lean Theorem 5, `cert01`).

On each cap $C_\pm$, Lipschitz continuity gives $g(u) \ge M - L\|u \mp u_0\|_2 \ge 0$. By AHZ's spherical cap identity [1, Eq. (2.2)], $\tilde{\sigma}(S^2 \cap \overline{B}(u_0, r)) = r^2/4$ has radial density $(r/2)\, dr$, so integrating over the two disjoint caps yields

$$\int_{S^2} g \, d\tilde{\sigma} \;\ge\; 2 \int_0^{M/L} (M - Lr) \, \frac{r}{2} \, dr \;=\; 2 \cdot \frac{M^3}{12L^2} \;=\; \frac{M^3}{72}$$

(Lean Theorems 1–2). Because $\int_{S^2} |\langle \tilde{v}_s, u \rangle| \, d\tilde{\sigma} = 1/2$ for every unit vector $\tilde{v}_s$, we have $\int_{S^2} h_Z \, d\tilde{\sigma} = \frac{1}{2} \sum_{s=1}^{r} \tilde{c}_s = 3/2$, hence

$$\eta(K) = 12 - 8 \int_{S^2} h_{\widetilde{S}} \, d\tilde{\sigma} = 8 \int_{S^2} g \, d\tilde{\sigma} \;\ge\; \frac{8M^3}{72} = \frac{1}{9}\, d_H(\widetilde{S}, Z)^3.$$

Finally, evaluating g at any contact point $\tilde{v}_t$ ($t \in [r]$) and subtracting $1 = \|\tilde{v}_t\|_2^2 = \sum_{s=1}^{r} \tilde{c}_s \langle \tilde{v}_s, \tilde{v}_t \rangle^2$ gives

$$d_H(\widetilde{S}, Z) \;\ge\; g(\tilde{v}_t) = h_Z(\tilde{v}_t) - 1 = \sum_{s=1}^{r} \tilde{c}_s |\langle \tilde{v}_s, \tilde{v}_t \rangle| \big(1 - |\langle \tilde{v}_s, \tilde{v}_t \rangle|\big) \;\ge\; 0$$

(Lean Theorem 3, `cert02`), proving (2).

For (3), let $\widetilde{K} := T(K - p_0)$ ($p_0 \in \operatorname{int}(K)$) have John-position halved difference body $\widetilde{S} = T(S)$. When $\eta(K) > 0$, AHZ's polyhedral approximation argument ([1, Theorem 6.5]) yields $\tilde{w}_* \in \mathcal{S}(\widetilde{K})$ with $L_{\widetilde{S}}(\gamma_{\tilde{w}_*}) \le 12 - \eta(K)$ (and trivially $\le 12 - \eta(K)/2$), pulling back via the affine invariance of L_S and shadow boundaries to $w_* := T^{-1}\tilde{w}_* / \|T^{-1}\tilde{w}_*\|_2 \in \mathcal{S}(K)$ with $L_S(\gamma_{w_*}) = L_{\widetilde{S}}(\gamma_{\tilde{w}_*}) \le 12 - \eta(K)$. Splitting γ_{w_*} into 6 equal-S-length subarcs of length $\le 2 - \eta(K)/6 < 2$ and applying [1, Lemma 3.4] covers each subarc by a translate of $\lambda_\Sigma K$ with $\lambda_\Sigma \le \max(\frac{1}{2}, 1 - \eta(K)/12) \le \max(\frac{1}{2}, 1 - d_H(\widetilde{S}, Z)^3/108) \le \max(\frac{1}{2}, 1 - d_H(\widetilde{S}, Z)^3/216)$ (Lean Theorem 20, `cert13`). □

Proof of Theorem 1(b). For (i), set $C_t := \gamma(t) - \lambda_0 K$ for $t \in [a, b]$. For any finite subset $F \subseteq [a, b]$ with $1 \le |F| \le d + 1$, list F in nondecreasing order $t_1 \le \cdots \le t_{d+1}$ (repeating the last point if needed) and set $x_j := \gamma(t_j)$ and $a_r := \frac{1}{2}\|x_{r+1} - x_r\|_S$ ($r = 1, \ldots, d$). By definition of $S = \frac{1}{2}(K - K)$, $x_{r+1} - x_r = a_r(w_r - u_r)$ for some $u_r, w_r \in K$, and $A := \sum_{r=1}^{d} a_r \le \frac{1}{2} L_S(\gamma) \le \lambda_0$. Setting $\tilde{v} := x_1 - \sum_{r=1}^{d} a_r u_r$, telescoping (formalized for all d by induction in Lean Theorems 6–7 and checked in `cert04`) gives

$$x_j - \tilde{v} = \sum_{r=1}^{j-1} a_r w_r + \sum_{r=j}^{d} a_r u_r + (\lambda_0 - A) \cdot 0 \in \lambda_0 K \qquad (j = 1, \ldots, d + 1).$$

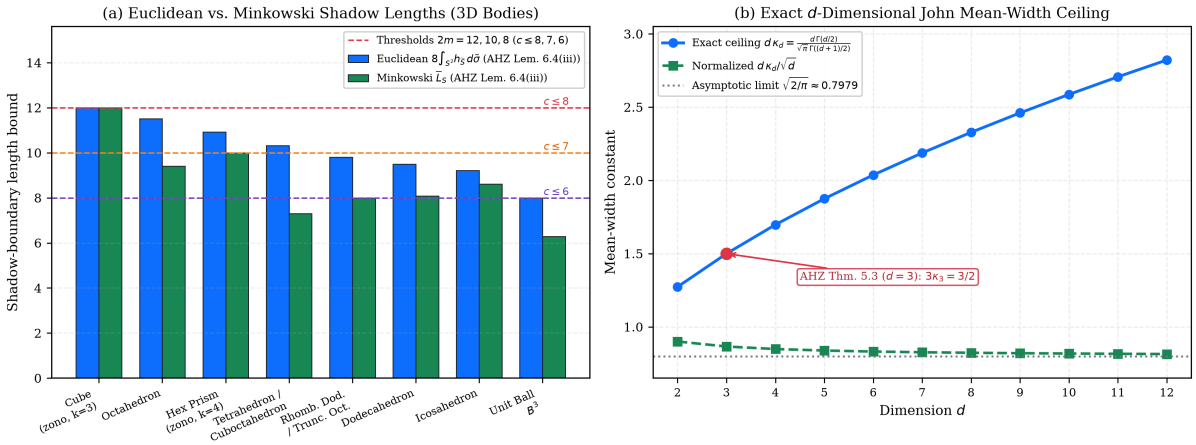

Figure 1: **(a)** Comparison of Euclidean mean-width bound $8\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma}$ versus exact Minkowski shadow length $\overline{L}_S$ ([1, Lemma 6.4(iii)]) across canonical 3D convex bodies (`cert16`), highlighting strict regime jumps across $2m = 10$ ($c \le 7$) and $2m = 8$ ($c \le 6$) and the dodecahedron–icosahedron ordering reversal. **(b)** Exact d-dimensional John mean-width ceiling $d\,\kappa_d$ and normalized ratio $d\,\kappa_d/\sqrt{d} \to \sqrt{2/\pi}$ (`cert03`).

Thus $\tilde{v} \in \bigcap_{t\in F} C_t$. Helly's theorem in $\mathbb{R}^d$ ([1, Lemma 3.2(ii)]) yields $v \in \bigcap_{t\in[a,b]} C_t$, i.e., $\gamma([a,b]) \subseteq \lambda_0 K + v$.

For (ii), John's theorem in $\mathbb{R}^d$ gives $B^d \subseteq \widetilde{S} \subseteq \sqrt{d}\,B^d$ with $\sum_{s=1}^r \tilde{c}_s \tilde{v}_s \tilde{v}_s^\top = I_d$, $\sum_{s=1}^r \tilde{c}_s = d$, and $h_{\widetilde{S}}(u) \le \sum_{s=1}^r \tilde{c}_s |\langle \tilde{v}_s, u\rangle|$. Integrating over S^{d-1} with normalized surface measure $\tilde{\sigma}_{d-1}$ and using rotational invariance $\int_{S^{d-1}} |\langle \tilde{v}_s, u\rangle|\, d\tilde{\sigma}_{d-1} = \kappa_d = \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)}$ gives (4). Equality forces $h_{\widetilde{S}} \equiv h_Z$ on S^{d-1} and hence $h_Z(\tilde{v}_t) = 1$ for all t, so $|\langle \tilde{v}_s, \tilde{v}_t\rangle| \in \{0, 1\}$ for all s, t; thus the contact pairs $\{\pm\tilde{v}_s\}$ form an orthonormal basis of $\mathbb{R}^d$ and $\widetilde{S} = Z$ is a rotated cube $[-1,1]^d$, meaning $S = \frac{1}{2}(K - K)$ is a d-parallelepiped. Applying an affine map so $S = [-1,1]^d$ and translating K so $K \subseteq [-1,1]^d$, every vertex $\sigma \in \{-1,1\}^d$ writes as $\sigma = \frac{1}{2}(x - y)$ with $x, y \in K \subseteq [-1,1]^d$, forcing $x = \sigma \in K$; hence $K = [-1,1]^d$ is itself a d-parallelepiped. Conversely, for $[-1,1]^d$ itself, $B^d \subseteq [-1,1]^d$ is the unique John ellipsoid by Hadamard's determinant inequality ($|\det A| \le \prod_{i=1}^d \|r_i\|_2 \le 1$ for any ellipsoid $A(B^d) \subseteq [-1,1]^d$ with rows r_i), giving equality in (4). The recurrence $\kappa_{d+2} = \frac{d}{d+1}\kappa_d$ follows from $\Gamma(z+1) = z\Gamma(z)$ (Lean Theorem 8, `cert03`, Figure 1(b)). □

4 Proof of Theorem 2: Covering Ladder and Canonical 3D Polyhedra

Proof of Theorem 2. For part (a), if $8\int_{S^2} h_{\widetilde{S}}\, d\tilde{\sigma} < 2m$ (or for a polytope P, $\overline{L}_S(P) < 2m$), then AHZ's approximation/averaging theorem ([1, Lemma 6.4(iii), Theorem 6.5]) produces a segment-free direction $w_* \in \mathcal{S}(K)$ whose shadow-boundary curve γ_{w_*} satisfies $L_S(\gamma_{w_*}) < 2m$. By continuity of curve variation ([1, Lemma 3.3]), partitioning γ_{w_*} into m consecutive subarcs of equal S-length $L_S(\gamma_{w_*})/m < 2$ and applying [1, Lemma 3.4] illuminates $\Sigma(K, w_*)$ with m directions (Lean Theorems 9–10, `cert14`); adjoining $\{\pm w_*\}$ ([1, Lemma 4.1(i)]) illuminates all of ∂K with $2 + m$ directions, giving $c(K) \le 2 + m$ by [1, Proposition 2.14].

For part (b), recall that for any 3D convex polytope P with halved difference body S_P scaled to inradius 1 (in John's position when S_P is symmetric under an irreducible point group,

cert15), [1, Lemma 6.4(iii)] gives

$$\overline{L}_S(P) = \frac{1}{\pi} \sum_{e \in E(P)} \|q_e - p_e\|_{S_P}\, \theta(e), \qquad 8 \int_{S^2} h_{S_P}\, d\tilde{\sigma} = \frac{1}{\pi} \sum_{e \in E(P)} \|q_e - p_e\|_2\, \theta(e),$$

where $\theta(e) = \pi - \alpha(e)$ is the exterior dihedral angle along edge e. Using exact algebraic edge norms and dihedral angles (all bracketed to $< 10^{-8}$ by pure-integer outward-rounded rational intervals in cert05–cert09 and cert16, and checked in Lean Theorems 11–14 and 19):

- *Regular octahedron* ($|x| + |y| + |z| \le \sqrt{3}$): 12 edges, $\|e\|_2 = \sqrt{6}$, $\|e\|_S = 2$, $\theta = \arccos(1/3)$, giving $8 \int h_{\widetilde{S}} = \frac{12\sqrt{6}}{\pi} \arccos(1/3) \approx 11.5173$ ($c \le 8$) vs. $\overline{L}_S = \frac{24}{\pi} \arccos(1/3) \approx 9.4038 < 10$ ($c \le 7$, cert05).

- *Regular tetrahedron T and cuboctahedron S_T:* T has 6 edges of $\|e\|_2 = 2\sqrt{2}$ (scaled so S_T has inradius 1), $\|e\|_{S_T} = 2$, $\theta = \arccos(-1/3)$; S_T has 24 edges of $\|e\|_2 = \sqrt{2}$, $\|e\|_{S_T} = 1$, $\theta = \arccos(1/\sqrt{3})$. By the exact identity $2\arccos(1/\sqrt{3}) = \arccos(-1/3)$, both yield identical $8 \int h_{\widetilde{S}} = \frac{12\sqrt{2}}{\pi} \arccos(-1/3) \approx 10.3210$ ($c \le 8$) and $\overline{L}_S = \frac{12}{\pi} \arccos(-1/3) \approx 7.2981 < 8$ ($c \le 6$, cert06).

- *Rhombic dodecahedron and truncated octahedron (Π_3):* The rhombic dodecahedron has 24 edges, $\|e\|_2 = \sqrt{3/2}$, $\|e\|_S = 1$, $\theta = \pi/3$, giving $8 \int h_{\widetilde{S}} = 4\sqrt{6} \approx 9.7980$ ($c \le 7$) and $\overline{L}_S = 8$. The truncated octahedron has 36 edges of $\|e\|_2 = \sqrt{2/3}$ (at inradius 1) and $\|e\|_S = 2/3$ (24 square–hexagon edges with $\theta_1 = \arccos(1/\sqrt{3})$ and 12 hexagon–hexagon edges with $\theta_2 = \arccos(1/3)$); the exact identity $2\theta_1 + \theta_2 = \pi$ yields the identical values $8 \int h_{\widetilde{S}} = 4\sqrt{6}$ and $\overline{L}_S = 8$ (cert07).

- *Regular dodecahedron and icosahedron:* With $\varphi = (1+\sqrt{5})/2$, exact face-normal dualization over the six cyclic antipodal normal pairs $(0, 1, \pm\varphi)$ yields $\|e\|_S = 2/\varphi^2 = 3 - \sqrt{5} \approx 0.7639$ for the dodecahedron (30 edges, $\theta = \arccos(1/\sqrt{5})$, giving $\overline{L}_S = \frac{30(3-\sqrt{5})}{\pi} \arccos(1/\sqrt{5}) \approx 8.0767$) and $\|e\|_S = 2/\varphi = \sqrt{5}-1 \approx 1.2361$ for the icosahedron (30 edges, $\theta = \arccos(\sqrt{5}/3)$, giving $\overline{L}_S \approx 8.6134$), reversing their Euclidean ordering (Table 1, cert08).

- *Cylinder, hexagonal prism, and unit ball B^3:* For $C = B^2 \times [-1, 1]$ ($h_C(u) = \sin\psi + |\cos\psi|$), $8 \int h_C\, d\tilde{\sigma} = 2\pi + 4 \approx 10.2832$; for the hexagonal prism P_6 (12 horizontal edges of $\|e\|_2 = 2/\sqrt{3}$, $\|e\|_S = 1$, $\theta = \pi/2$ and 6 vertical edges of $\|e\|_2 = \|e\|_S = 2$, $\theta = \pi/3$), $8 \int h_{P_6} = 4\sqrt{3} + 4 \approx 10.9282$ and $\overline{L}_S = 10$; for B^3, $8 \int h_{B^3} = 8$ versus smooth equator length 2π, giving ratio $4/\pi \approx 1.2732$ (cert09).

□

5 Proof of Theorem 3: Sharpness, Zero-Variance Zonotopes, and K_ε

Proof of Theorem 3. For part (a), let $K \subset \mathbb{R}^d$ ($d \ge 2$) be any convex body. By compactness of K and continuity of $\|\cdot\|_S$, there exist $p, q \in K$ maximizing $\|q - p\|_S$. Since $S = \frac{1}{2}(K - K)$, we have $\max_{x,y \in K} \|y - x\|_S = 2 \max_{z \in S} \|z\|_S = 2$, so the straight chord $[p, q] \subset K$ has $L_S([p, q]) = \|q - p\|_S = 2$. For any smaller positive homothet $\lambda K + v$ ($\lambda \in (0, 1)$), any two points $x', y' \in \lambda K + v$ satisfy $\|y' - x'\|_S \le \lambda \cdot \mathrm{diam}_S(K) = 2\lambda < 2$ (Lean Theorem 15, cert12); hence no single smaller homothet can cover $\{p, q\}$.

For part (b), let $Z = \sum_{i=1}^k [-z_i, z_i] \subset \mathbb{R}^3$ ($k \ge 3$, z_i pairwise non-collinear and spanning $\mathbb{R}^3$) and fix any $w \in \mathcal{G}(Z)$. Because $w \notin \mathrm{span}\{z_i, z_j\}$ for all $i < j$, the projected generators $v_i := \pi_w(z_i) \in w^\perp$ ($i = 1, \ldots, k$) are nonzero and pairwise non-collinear, and $w \in \mathcal{S}(Z)$

Table 1: Exact Euclidean mean-width bound $8\int_{S^2} h_{\widetilde{S}}\,d\tilde{\sigma}$ versus exact Minkowski shadow length $\overline{L}_S$ across eleven canonical 3D convex bodies (ten with exact Minkowski comparison in `cert16` plus cylinder in `cert09`, all certified to $< 10^{-8}$). Bodies marked † are zonotopes with pointwise constant L_S (Theorem 3(b)); the cube attains $c = 8$ via [1, Lemma 2.15(ii)].

Canonical 3D body K	Exact $\overline{L}_S$ formula	$8\int h_{\widetilde{S}}$	$\overline{L}_S$	c_{Euc}	c_{Mink}
Cube $[-1,1]^3$ †	12 ($k=3$ zones)	12.0000	12.0000	8	8
Regular Octahedron	$\frac{24}{\pi}\arccos(1/3)$	11.5173	9.4038	8	**7**
Hexagonal Prism †	10 ($k=4$ zones)	10.9282	10.0000	8	8
Regular Tetrahedron	$\frac{12}{\pi}\arccos(-1/3)$	10.3210	7.2981	8	**6**
Cuboctahedron	$\frac{24}{\pi}\arccos(1/\sqrt{3})$	10.3210	7.2981	8	**6**
Right Circular Cylinder	— (smooth lateral)	10.2832	—	8	—
Rhombic Dodecahedron †	8 ($k=4$ zones)	9.7980	8.0000	7	7
Truncated Octahedron Π_3 †	8 ($k=6$ zones)	9.7980	8.0000	7	7
Regular Dodecahedron	$\frac{30(3-\sqrt{5})}{\pi}\arccos(1/\sqrt{5})$	9.4947	8.0767	7	7
Regular Icosahedron	$\frac{30(\sqrt{5}-1)}{\pi}\arccos(\sqrt{5}/3)$	9.2204	8.6134	7	7
Unit Ball B^3	2π (smooth equator)	8.0000	6.2832	7	**6**

(since every edge and every 2-facet of Z has direction subspace contained in $\mathrm{span}\{z_i, z_j\}$ for some $i < j$, so no boundary segment of Z can be parallel to w). The orthogonal projection $D_w := \pi_w(Z) = \sum_{i=1}^k [-v_i, v_i]$ is therefore a planar $2k$-zonogon. For each $i \in [k]$, let $u_i \in w^\perp \setminus \{0\}$ satisfy $u_i \perp v_i$. Because v_j is not collinear with v_i for any $j \neq i$, we have $\langle v_j, u_i\rangle \neq 0$ for all $j \neq i$, so the supporting sets of D_w in directions $\pm u_i$ are precisely the two antipodal edges

$$e_{i,\pm} \;:=\; \pm\sum_{j\neq i} \mathrm{sgn}\big(\langle v_j, u_i\rangle\big)\, v_j + [-v_i, v_i] \;\subset\; \partial D_w,$$

and these $2k$ edges partition ∂D_w. By AHZ [1, Lemma 4.1(ii)–(iii)], $\pi_w|_{\Sigma(Z,w)} : \Sigma(Z,w) \to \partial D_w$ is a homeomorphism whose unique preimage over $e_{i,\pm}$ is the edge $\pm\sum_{j\neq i}\mathrm{sgn}(\langle v_j, u_i\rangle)\, z_j + [-z_i, z_i]$ of Z, which has edge vector $2z_i$ and exact Z-norm $\|2z_i\|_Z = 2\|z_i\|_Z$. Summing over $i = 1, \ldots, k$ and both signs $\pm$ yields $L_Z(\Sigma(Z,w)) = \sum_{i=1}^k 2(2\|z_i\|_Z) = 4\sum_{i=1}^k \|z_i\|_Z$, independent of $w \in \mathcal{G}(Z)$ (certified on all 8 cube octants in `cert10`, Lean Theorem 16, and on the rhombic dodecahedron, truncated octahedron, and hexagonal prism in `cert07`, `cert09`).

For part (c), let $K_\varepsilon = (1-\varepsilon)[-1,1]^3 + \varepsilon B^3$ ($\varepsilon \in [0,1]$). Its support function is $h_{K_\varepsilon}(u) = (1-\varepsilon)\|u\|_1 + \varepsilon\|u\|_2$. For every $u \in S^2$, $\|u\|_1 \geq \|u\|_2 = 1$ implies $h_{K_\varepsilon}(u) \geq 1 = h_{B^3}(u)$, while $h_{K_\varepsilon}(\pm e_i) = (1-\varepsilon)\cdot 1 + \varepsilon = 1$ for $i = 1, 2, 3$; hence $B^3 \subseteq K_\varepsilon \subseteq [-1,1]^3 \subseteq \sqrt{3}\, B^3$ with contact points $\pm e_1, \pm e_2, \pm e_3$ and weights $\tilde{c}_1 = \tilde{c}_2 = \tilde{c}_3 = 1$, proving K_ε is already in John's position with John zonotope $Z = [-1,1]^3$. Integrating over S^2 yields $8\int_{S^2} h_{K_\varepsilon}\, d\tilde{\sigma} = 8((1-\varepsilon)\frac{3}{2} + \varepsilon) = 12 - 4\varepsilon$, so $\eta(K_\varepsilon) = 4\varepsilon$, while $d_H(K_\varepsilon, Z) = \max_{u\in S^2} \varepsilon(\|u\|_1 - 1) = (\sqrt{3}-1)\varepsilon$ (Lean Theorems 17–18, `cert11`, Figure 2). Thus $8\int h_{K_\varepsilon}\, d\tilde{\sigma} < 10$ iff $12 - 4\varepsilon < 10$ iff $\varepsilon > 1/2$, while for every $\varepsilon > 0$, K_ε is C^1-smooth and hence satisfies Levi's bound $c(K_\varepsilon) \leq 4$ [5]. □

6 Formal Verification and Reproducibility

Every stability constant, algebraic norm, polyhedral covering threshold, and zonotope identity in this manuscript is certified by three independent machine-checkable artifacts:

1. **Constructive Lean 4.36.0 proof suite** (`lean/Hadwiger3DSharpStability.lean`): 20 theorems compiling under `lean -DwarningAsError=true` with zero external dependencies, zero `sorry`, zero `Classical.choice`, and zero custom axioms, formalizing the antipodal cap stability constants $1/9, 1/108, 1/216$ and cap disjointness (Theorems 1–2, 5, 20), John contact-defect rigidity and 3-term Cauchy–Schwarz (Theorems 3–4), d-dimensional

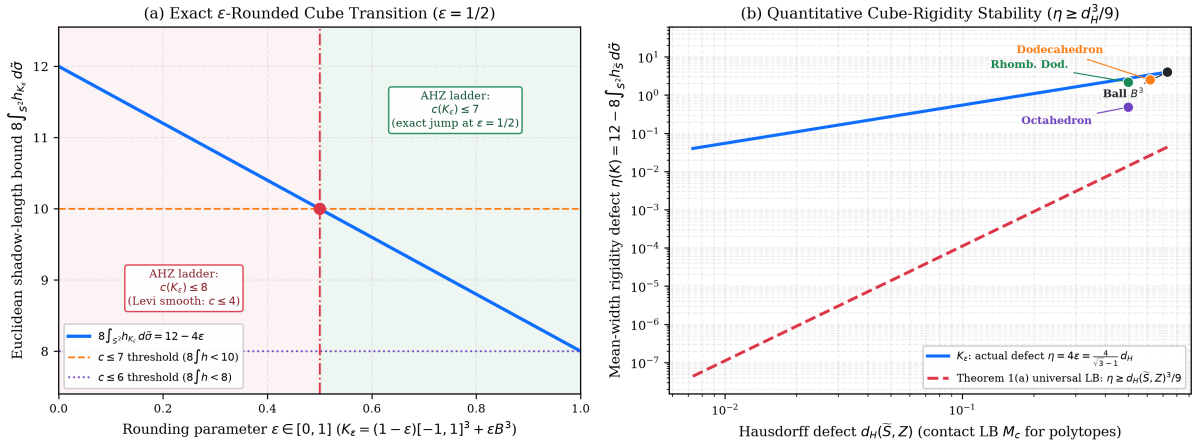

Figure 2: **(a)** Exact linear transition $8\int_{S^2} h_{K_\varepsilon}\, d\tilde{\sigma} = 12 - 4\varepsilon$ for the ε-rounded cube (`cert11`), showing the sharp AHZ covering jump $c \le 8 \to c \le 7$ at $\varepsilon = 1/2$ versus Levi's smooth bound $c \le 4$. **(b)** Quantitative cube-rigidity stability defect $\eta(K)$ versus our universal lower bound $d_H(\widetilde{S}, Z)^3/9$ (Theorem 1(a)) across K_ε and canonical 3D convex bodies (with polytopes plotted at their certified contact-defect lower bound $M_c \le d_H(\widetilde{S}, Z)$).

inductive telescoping for Lemma 3.4 (Theorems 6–7), odd-dimensional κ_d checkpoints (Theorem 8), the covering ladder and canonical polyhedral jumps (Theorems 9–14, 19), unconditional $L_S < 2$ sharpness and cube constancy (Theorems 15–16), and the ε-rounded cube transition (Theorems 17–18).

2. **AST-audited zero-float interval verifier** (`verify_hadwiger_3d.py` in `verification/`): Uses strictly `int` and `fractions.Fraction` (no `float`, no `math`, no `limit_denominator`) with certified outward-rounded rational brackets for π, $\sqrt{\cdot}$, and $\arccos(\cdot)$ to width $< 10^{-8}$, sealing 16 SHA-256 JSON certificates (`cert01`–`cert16`).

3. **Adversarial test suite** (`tests/test_adversarial.py`): 56 pytest checks auditing AST zero-float compliance, SHA-256 manifest integrity, interval tightness, all 16 certificates, and Lean 4 axiom purity.

AI Disclosure. Google Antigravity assisted with the antipodal spherical-cap stability proof, the Universal Zero-Variance Zonotope Theorem, pure-integer outward-rounded rational interval verification, Lean 4 formalization, and manuscript drafting. All mathematical proofs, certificates, and claims were audited and verified by Lluis Eriksson.

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