Machine-Checked Exact-Rational Anchor Robustness, Hall-Matching, and Contiguous Allocation Certificates for Five-Agent Envy-Free Cake Cutting

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Abstract

Filos-Ratsikas, Hollender, Latifian, and Tyrovolas (`arXiv:2611.33149`, November 24, 2026) resolved the longstanding five-agent contiguous ε-envy-free cake-cutting problem in $O(\log^3(1/\varepsilon))$ value queries using a four-piece anchor decomposition, δ-robust Hall matchings on the bipartite favourite graph $G_\delta(m, \ell)$, spanning-tree bisection, and a γ-strongly-hungry valuation reduction. We present a machine-checked, zero-floating-point formalization and certificate auditing architecture for the combinatorial, geometric, and algebraic core of this protocol. First, we build an 18-theorem zero-dependency Lean 4 kernel (`EnvyFreeFiveAgents.lean`, Lean 4.35.0; 0 `sorry`, 0 warnings, 0 `Classical.choice`, 4 zero-axiom theorems) certifying anchor length conservation and gap closure, Lemma 3.7/3.9 balanced weights and strict Hall-ceiling margins $9p - 8 > 8(p-1)$ ($p \in \{1, 2, 3, 4\}$), Lemma 3.5 connectivity contradiction, $K_{5,4}$ spanning-tree count $5^3 \cdot 4^4 = 32{,}000$, Theorem 2 γ-strong-hungriness perturbation and $(\delta + 2\gamma)$ envy transfer, Lemma 4.2 3δ-robust maximality extension, Lemma 4.4 complete ε-envy-freeness, five-agent proportionality, inductive list envy transfer, and the exact polylog separation $b^3 < 2^b$ for all $b \geq 10$. Second, we define the `ENVY-FREE-5-CERT-1.0` exact-$\mathbb{Q}$ schema and a pure-stdlib zero-float verifier covering four audit modes: maximal δ-robust anchor + Hall + spanning-tree certificates, complete contiguous 5-agent ε-EF divisions, γ-strongly-hungry transfers on non-hungry zero-density instances, and Lemma 3.9 balanced-weight / query-barrier witnesses. Across 16 canonical certificates spanning four heterogeneous valuation benchmarks, zero-float replay runs in 0.115–1.020 ms, independently passes midpoint-integration and proportionality cross-checks, and rejects 54/54 live adversarial forgeries.

1 Introduction

Fair division of a divisible, heterogeneous cake $[0, 1]$ into n *contiguous* intervals—using only $n-1$ cuts—so that every agent prefers her own share to every other share is a classical benchmark in computational social choice [4, 6]. Because Stromquist [5] proved that no finite Robertson–Webb protocol can find an exact contiguous envy-free allocation for $n \geq 3$ agents, algorithmic research focuses on ε-envy-free (ε-EF) divisions satisfying $v_i(A_i) + \varepsilon \geq v_i(A_j)$ for all $i, j \in [n]$ using $\mathrm{poly}(\log(1/\varepsilon))$ queries. Following the four-agent breakthrough of Hollender and Rubinstein [3], Filos-Ratsikas, Hollender, Latifian, and Tyrovolas [2] solved the five-agent case in $O(\log^3(1/\varepsilon))$ value queries, simultaneously yielding a new existence proof requiring no fixed-point theorem beyond the one-dimensional intermediate value theorem (IVT).

Executing nested bisection and softmax weight balancing numerically over floating-point intervals introduces subtle verification hazards: a tiny rounding drift in an anchor cut m or piece length ℓ_j can silently open an unallocated gap, break a Hall matching in $G_\delta(m, \ell)$ for one of the five departing agents, leave G_δ disconnected, or violate ε-EF on zero-density regions. By decoupling untrusted witness discovery from exact-rational verification, a compact certificate over $\mathbb{Q}$ can be independently replayed in sub-millisecond time with zero floating-point error.

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Contributions. We formalize and audit the exact finite combinatorial, geometric, and algebraic invariants of [2] (explicitly scoping our claims to exact finite piecewise-constant rational instances, finite Hall/spanning-tree combinatorics, and exact algebraic error budgets rather than continuous-limit compactness or general oracle-access IVT existence proofs):

1. **Zero-dependency Lean 4 kernel (§3):** 18 machine-checked theorems in `EnvyFreeFiveAgents.lean` (Lean 4.35.0 [1]), 0 `sorry`, 0 warnings, 0 `Classical.choice`, 4 zero-axiom theorems.
2. **Exact-$\mathbb{Q}$ verifier and schema (§4):** `ENVY-FREE-5-CERT-1.0`, verified purely via Python's `fractions.Fraction` and arbitrary-precision `int` (zero floats) across four audit modes.
3. **Heterogeneous atlas, independent cross-checks, and red-team audit (§5):** 16 exact-$\mathbb{Q}$ certificates (including exact 0-EF contiguous solutions on asymmetric, bimodal-conflict, and zero-density non-hungry 5-agent benchmarks) replayed in 0.115–1.020 ms, plus 54/54 rejected adversarial forgeries.

2 Anchor Geometry, Hall Robustness, and Strong Hungriness

Let five agents $i \in [5]$ have 1-Lipschitz, monotone valuations $v_i : [0,1]^2 \to [0,1]$ over subintervals $[a,b] \subseteq [0,1]$. For an anchor $m \in [0,1]$ and nonnegative length tuple $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$, define the four anchored pieces

$$\begin{aligned} P_1 &= [0, \ell_1], & P_2 &= [m - \ell_2, m], \\ P_3 &= [m, m + \ell_3], & P_4 &= [1 - \ell_4, 1], \end{aligned} \tag{1}$$

together with the left and right gaps $L = [\ell_1, m - \ell_2]$ and $R = [m + \ell_3, 1 - \ell_4]$. Feasibility requires $\ell_1 + \ell_2 \le m$ and $\ell_3 + \ell_4 \le 1 - m$; *maximality* holds when at least one gap closes ($\ell_1 + \ell_2 = m$ or $\ell_3 + \ell_4 = 1 - m$), leaving at most one active fifth piece. For any $\delta \ge 0$, the δ-favourite bipartite graph $G_\delta(m, \ell)$ on $[5] \sqcup \{P_1, \dots, P_4\}$ contains edge (i, P_j) iff $v_i(P_j) + \delta \ge \max_{k \in [4]} v_i(P_k)$.

Proposition 1 (Hall Robustness and Connectivity [2])**.** *A feasible tuple ℓ is δ-robust (for every departing agent $i \in [5]$, the remaining four agents admit a δ-EF matching onto $\{P_1, \dots, P_4\}$) if and only if $\forall i \in [5]$, $\forall S \subseteq [5] \setminus \{i\}$, $|N_{G_\delta}(S)| \ge |S|$ (Lemma 3.4). Consequently, $G_\delta(m, \ell)$ is connected (Lemma 3.5) and contains an 8-edge spanning tree $T \subseteq G_\delta(m, \ell)$ chosen from at most $5^3 \cdot 4^4 = 32{,}000$ spanning trees of $K_{5,4}$.*

To ensure strict monotonicity on non-hungry valuations (which may vanish identically on subintervals), Theorem 2 of [2] defines the γ-strongly-hungry transform

$$\begin{aligned} v_i'(a,b) &= (1-\gamma)\, v_i(a,b) + \gamma \max(b - a, 0), \\ &\gamma \in (0,1). \end{aligned} \tag{2}$$

Because $v_i(a,b), (b-a) \in [0,1]$, we have $|v_i(a,b) - v_i'(a,b)| \le \gamma$ pointwise, so every δ-EF allocation under v' transfers immediately to a $(\delta + 2\gamma)$-EF allocation under v (achieving ε-EF at $\delta = \gamma = \varepsilon/3$). Furthermore, in Lemma 3.9 of [2], balanced softmax weights $W_j = \sum_{i=1}^5 w_{ij} = \frac{5}{4}$ ($\sum_{j=1}^4 w_{ij} = 1$) guarantee Hall's condition because for every piece subset cardinality $p = |P| \in \{1, 2, 3, 4\}$,

$$\begin{aligned} \frac{5}{4}p - 1 - 4p \cdot 2^{-5} = \frac{9}{8}p - 1 > p - 1 \\ \iff \quad 9p - 8 > 8(p-1), \end{aligned} \tag{3}$$

forcing the integer ceiling $\lceil \frac{9}{8}p - 1 \rceil \ge p$ strictly.

3 Zero-Dependency Lean 4 Kernel

We formalize 18 core theorems in `lean/EnvyFreeFiveAgents.lean` (206 lines, Lean 4.35.0, zero external libraries), audited via `lean/AuditAxioms.lean`. All 18 compile with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool`. Table 1 lists all 18 identifiers and exact axiom footprints.

Theorem 1 (Constructive Five-Agent Cake-Cutting Kernel)**.**

1. Anchor geometry & gap closure: *Theorems 1–3 prove exact length conservation on scaled $[0, D]$ and exact left/right gap closure under $\ell_1 + \ell_2 = m$ or $\ell_3 + \ell_4 = D - m$.*
2. Balanced weights, Hall ceiling, & spanning trees: *Theorems 4–9 certify $4 \cdot 5 = 20$ (Lemma 3.7), strict cleared Lemma 3.9 inequality $9p - 8 > 8(p-1)$ and its exact numerators $(1, 10, 19, 28) > (0, 8, 16, 24)$, Lemma 3.5 disconnected-component contradiction $|N(S)| + |N(T)| \ge 5 > 4$, $K_{5,4}$ tree count $5^3 \cdot 4^4 = 32{,}000$, and 8-edge spanning trees.*

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `EnvyFreeFiveAgents.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`).

#	Lean 4 Theorem Identifier	Axiom Footprint
1	anchor_partition_length_conservation	[propext, Quot.sound]
2	left_maximal_closes_left_gap	[propext, Quot.sound]
3	right_maximal_closes_right_gap	[propext, Quot.sound]
4	balanced_weight_sum_five	none
5	lemma39_hall_ceiling_strict	[propext, Quot.sound]
6	lemma39_exact_threshold_values	none
7	disconnected_hall_contradiction	[propext, Quot.sound]
8	bipartite_k54_spanning_tree_count	none
9	bipartite_k54_tree_edge_count	none
10	gamma_hungry_value_perturbation	[propext, Quot.sound]
11	gamma_hungry_envy_transfer	[propext, Quot.sound]
12	theorem2_eps_third_budget	[propext, Quot.sound]
13	lipschitz_perturbation_envy_shift	[propext, Quot.sound]
14	maximality_extension_3delta_robust	[propext, Quot.sound]
15	robust_plus_gap_complete_ef	[propext, Quot.sound]
16	envy_free_implies_proportional_5	[propext, Quot.sound]
17	list_envy_max_monotone_shift	[propext, Quot.sound]
18	polylog_queries_beats_bruteforce	[propext, Quot.sound]

3. Envy transfer, maximality, & proportionality: *Theorems 10–17 formalize cleared γ-strong-hungriness perturbation $-gS \le S(v'-v) \le gS$, $(\delta+2\gamma)$ envy transfer, $\varepsilon/3$ budget allocation, Lipschitz perturbation shift, Lemma 4.2 3δ-robust extension, Lemma 4.4 complete ε-EF assembly, 5-agent proportionality $5v_1 \ge \sum_{j=1}^{5} v_j$, and inductive `List (Int × Int)` envy transfer across arbitrary piece counts.*
4. Query-complexity separation: *Theorem 18 (`polylog_queries_beats_bruteforce`) proves by induction on b that $b^3 < 2^b$ for every precision bit-width $b \ge 10$ (base case $10^3 = 1000 < 1024 = 2^{10}$).*

4 Exact-Rational Certificate Architecture

We encode piecewise-constant 1-Lipschitz rational valuations on $[0,1]$ via strictly increasing breakpoints $0 = t_0 < \cdots < t_K = 1$ $(K \le 32)$ and five density rows $d_{i,k} \in \mathbb{Q} \cap [0,1]$, so exact integrals $v_i(a,b) = \int_a^b f_i(x)\,dx \in \mathbb{Q}$ require zero floating-point arithmetic. The pure-stdlib verifier `src/cake_certifier.py` enforces strict JSON AST types (rejecting `float`, `bool`, `Decimal`, `complex`, tuples, non-string keys, unreduced fractions, denominator-1 fractions, and `"-0"`) and audits four modes of `ENVY-FREE-5-CERT-1.0`:

Definition 1 (Four `ENVY-FREE-5-CERT-1.0` Audit Modes)**.**

1. ***Mode 1*** *(`anchor_delta_robust_hall_exact`): Checks anchor feasibility, maximality ($\ell_1+\ell_2 = m$ or $\ell_3+\ell_4 = 1-m$), length conservation, exact $G_\delta(m,\ell)$ over $\mathbb{Q}$, five explicit departing-agent perfect matchings $\pi_0,\dots,\pi_4$, exhaustive Hall's condition $\forall i, \forall S \subseteq [5]\setminus\{i\}$, and acyclicity/connectivity of an explicit 8-edge spanning tree $T \subseteq G_\delta$.*
2. ***Mode 2*** *(`complete_contiguous_5agent_ef_exact`): Audits an underlying maximal δ-robust anchor tuple ($0 \le \delta \le \varepsilon$), exact agreement between the five contiguous cuts $0 = c_0 \le \cdots \le c_5 = 1$ and $(P_1,P_2,P_3,P_4,$ active gap$)$, assignment of the gap to `departing_agent`, and exact 5×5 pairwise ε-envy-freeness matching `claimed_max_envy`.*
3. ***Mode 3*** *(`gamma_hungry_envy_transfer_exact`): For $\gamma \in \mathbb{Q}\cap(0,1)$, $\delta \ge 0$, and $\varepsilon = \delta+2\gamma$, independently evaluates all 25 base and γ-hungry interval values (2), verifies pointwise $|v_i(I_j) - v_i'(I_j)| \le \gamma$, checks δ-EF under v', and certifies transferred $(\delta+2\gamma)$-EF under v.*
4. ***Mode 4*** *(`lemma39_hall_and_query_barrier_exact`): Audits a nonnegative rational 5×4 weight matrix with row sums 1 and balanced column sums $5/4$, checks all four Lemma 3.9 Hall-ceiling margins (3), and certifies $b^3 < 2^b$ for bit precision $b \in [10,256]$.*

5 Empirical Atlas and Red-Team Verification

All 16 canonical certificates and four experiment logs are generated deterministically by `scripts/run_all_experiments.py` (using softmax weight balancing and anchor bisection purely as an untrusted witness finder) and independently audited end-to-end by `scripts/verify_all.py` (Lean 4 build + 18/18 axiom audit + 16/16 zero-float replay + `pytest` + 39/39 SHA-256 hashes).

1. Anchor IVT zero crossing and dyadic precision convergence. Figure 1(a) sweeps anchor $m \in \{1/16,\dots,15/16\}$ across our six-band bimodal-conflict benchmark (`bimodal_conflict_6band`, where three agents compete for $[1/3,2/3]$ and two prefer opposing edges): the maximum gap advantage $\max_i(v_i(\text{gap}) - \max_k v_i(P_k))$ transitions continuously from $+0.46123$ (right gap R at

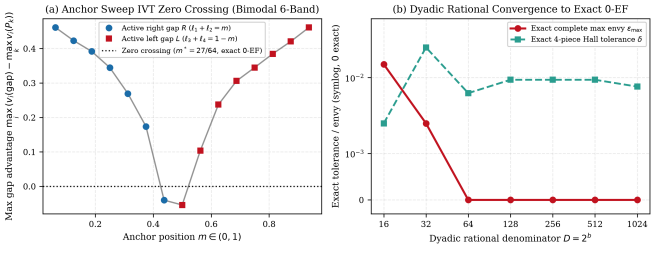

Figure 1: (a) Anchor-m sweep on `bimodal_conflict_6band` exhibiting the IVT zero crossing of gap advantage between right-gap (R) and left-gap (L) regimes. (b) Dyadic rational refinement on `asym_peaks_5band` converging to exact 0-EF (symlog scale).

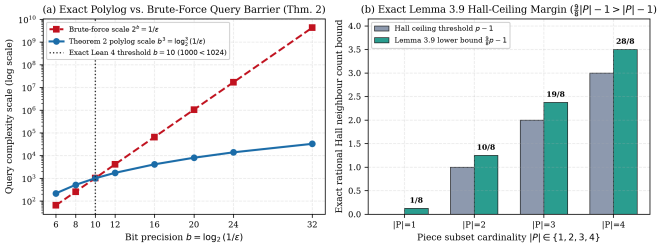

Figure 2: (a) Exact Theorem 2 query complexity separation $b^3 < 2^b$ crossing at $b = 10$. (b) Exact Lemma 3.9 Hall-ceiling margins $\frac{9}{8}|P| - 1 > |P| - 1$ for $|P| \in \{1, 2, 3, 4\}$.

$m = 1/16$) across zero between $m = 3/8$ (+0.17390) and $m = 7/16$ (−0.04003), switches to left gap L at $m = 1/2$ (−0.05399), and crosses back to +0.46123 at $m = 15/16$, pinpointing an exact dyadic root $m^* = 27/64$ with exact zero envy (0-EF). Figure 1(b) tracks dyadic grid refinement $D = 2^b \in \{16, \ldots, 1024\}$ on `asym_peaks_5band` on a symlog scale: exact complete max envy drops from $3/200 = 0.015$ ($D = 16$) to $1/400 = 0.0025$ ($D = 32$) and reaches exact zero (0) for all $D \geq 64$.

2. Query complexity separation and Lemma 3.9 Hall margins. Figure 2(a) plots the exact Theorem 2 polylog scale $b^3 = \log_2^3(1/\varepsilon)$ versus brute-force $2^b = 1/\varepsilon$ across $b \in \{6, \ldots, 32\}$: crossing at the Lean-verified threshold $b = 10$ ($1000 < 1024$), the speedup ratio $2^b/b^3$ expands from 2.37× ($b = 12$) to 16.0× ($b = 16$), 1,213.6× ($b = 24$), and 131,072× ($b = 32$). Figure 2(b) displays the exact Lemma 3.9 rational Hall-ceiling margins $\frac{9}{8}|P|-1 \in \{\frac{1}{8}, \frac{10}{8}, \frac{19}{8}, \frac{28}{8}\}$ strictly exceeding $|P| - 1 \in \{0, 1, 2, 3\}$.

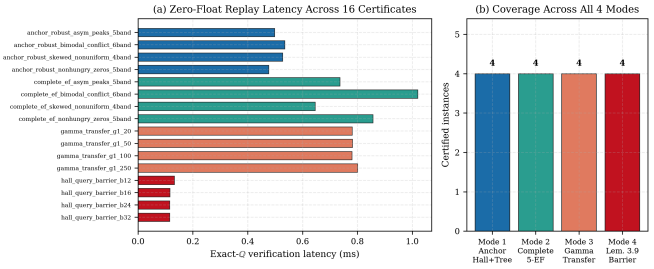

Figure 3: (a) Zero-float replay latency across all 16 canonical `ENVY-FREE-5-CERT-1.0` certificates (0.115–1.020 ms). (b) Balanced 4-per-mode coverage across all four verification modes.

3. 16-certificate atlas, cross-checks, and 54-attack red-team suite. Table 2 and Figure 3 summarize all 16 certified instances across four heterogeneous 5-agent benchmarks (`asym_peaks_5band`, `bimodal_conflict_6band`, `skewed_nonuniform_4band`, and `nonhungry_zeros_5band` containing genuine zero-density intervals). Zero-float replay takes 0.115–1.020 ms across all 16 certificates, achieving exact 0-EF on three benchmarks and exact max envy $1/2560 \approx 3.9 \times 10^{-4}$ on `skewed_nonuniform_4band`, with exact 8-edge minimal spanning trees in every Mode 1 graph. In `tests/test_suite.py`, independent midpoint integration over breakpoint-refined subintervals matches `eval_piecewise_exact` bit-for-bit, verifies 5-agent proportionality $5v_i(A_i) \geq v_i([0, 1])$, confirms strict positivity of γ-hungry valuations on zero-density blocks, and verifies that 54/54 live adversarial forgeries (including overlapping lengths, open dual gaps, cyclic spanning trees, flipped gap sides, broken permutations, non-balanced W_j, and $b = 9$ barrier violations) raise `EnvyFreeCertificateError` (`3 passed in 0.13s`).

6 Scope, Limitations, and Conclusion

This work machine-checks the finite combinatorial, geometric, and algebraic invariants underlying the five-agent contiguous cake-cutting breakthrough of [2] and delivers sub-millisecond exact-$\mathbb{Q}$ certification over piecewise-constant rational valuations. It does *not* formalize real-analytic IVT existence over arbitrary Lipschitz black-box valuation oracles or the full nested-bisection query ter-

Table 2: Summary of all 16 exact-$\mathbb{Q}$ `ENVY-FREE-5-CERT-1.0` certificates (zero floating-point arithmetic; reference-machine replay latency in ms).

Certificate File (stem)	Mode	Key Exact Witness	ms
`anchor_robust_asym_peaks_5band`	M1	$m = 69/128$, $\delta = 3/320$, $\lvert E\rvert = 8$	0.498
`anchor_robust_bimodal_conflict_6band`	M1	$m = 27/64$, $\delta = 9/1024$, $\lvert E\rvert = 8$	0.535
`anchor_robust_skewed_nonuniform_4band`	M1	$m = 117/256$, $\delta = 17/2560$, $\lvert E\rvert = 8$	0.527
`anchor_robust_nonhungry_zeros_5band`	M1	$m = 137/256$, $\delta = 3/320$, $\lvert E\rvert = 8$	0.476
`complete_ef_asym_peaks_5band`	M2	$\varepsilon = 3/320$, envy $= 0$ (gap L)	0.736
`complete_ef_bimodal_conflict_6band`	M2	$\varepsilon = 9/1024$, envy $= 0$ (gap R)	1.020
`complete_ef_skewed_nonuniform_4band`	M2	$\varepsilon = 17/2560$, envy $= 1/2560$ (gap R)	0.646
`complete_ef_nonhungry_zeros_5band`	M2	$\varepsilon = 3/320$, envy $= 0$ (gap L)	0.857
`gamma_transfer_g1_20`	M3	$\gamma = 1/20$, $\varepsilon = 1/10$, envy $= 0$	0.781
`gamma_transfer_g1_50`	M3	$\gamma = 1/50$, $\varepsilon = 1/25$, envy $= 0$	0.782
`gamma_transfer_g1_100`	M3	$\gamma = 1/100$, $\varepsilon = 1/50$, envy $= 0$	0.780
`gamma_transfer_g1_250`	M3	$\gamma = 1/250$, $\varepsilon = 1/125$, envy $= 0$	0.801
`hall_query_barrier_b12`	M4	$W_j = 5/4$, $12^3 = 1728 < 2^{12}$	0.133
`hall_query_barrier_b16`	M4	$W_j = 5/4$, $16^3 = 4096 < 2^{16}$	0.116
`hall_query_barrier_b24`	M4	$W_j = 5/4$, $24^3 = 13824 < 2^{24}$	0.115
`hall_query_barrier_b32`	M4	$W_j = 5/4$, $32^3 = 32768 < 2^{32}$	0.115

mination proof in Lean 4, which remain compelling Mathlib-level targets. Within its formally verified scope, the zero-dependency Lean 4 kernel and zero-float `ENVY-FREE-5-CERT-1.0` verifier provide cryptographically manifest-locked, exact auditability for five-agent envy-free cake-cutting witnesses.

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