Sharp Spanning Multiplicity, Exact Characteristic Threshold for Polynomial Multiplication Log-Star-Convexity, and Unbounded Non-Crossing Subadditivity Gaps in the Asymptotic Spectrum of Tensors

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Abstract

Vrana and Zuiddam (arXiv:2611.34664v1, 2026) recently proved that the asymptotic spectrum Δ of complex k-tensors is path-connected and contractible, established log-star-convexity for polynomial multiplication tensors $\pi\mathsf{P}_{m,n}$ over $\mathbb{C}$ via compact SU(2) Haar averaging, extended log-star-convexity for matrix multiplication and graph tensors to arbitrary fields via finite spanning groups, and exhibited a single 5-tensor counterexample on $\mathbb{C}^2$ showing that geometric averages of non-crossing bipartition flattening ranks need not be $\mathcal{B}$-good. We establish three sharp quantitative and structural strengthenings of their framework. First (**Theorem 1**), we replace their real-valued subspace-intersection bound $q_{\mathrm{VZ}}(d,w) = \frac{d}{d-w}(1+\ln d)$ (Lemma 3.7) with the exact integer-floor greedy recurrence $u_{j+1} = \lfloor \frac{w}{d} u_j \rfloor$, proving the universal bound $q_{\mathrm{exact}}(d,w) \le \min_{0\le t\le d}(t + \lfloor d(w/d)^t \rfloor) \le d$ that strictly beats $q_{\mathrm{VZ}}(d,w)$ for every $d \ge 2$ and $1 \le w \le d-1$, eliminates the unbounded $\ln d$ blowup at codimension $r = 1$ (where $q_{\mathrm{exact}}(d,d-1) = d$ is unconditionally optimal by codimension submodularity, saving $7.93\times$ at $d = 1024$), and tightens finite-copy reverse monotonicity (Lemma 3.8). Second (**Theorem 2**), we extend polynomial-multiplication log-star-convexity (Theorem 3.12) to *every* field $\mathbb{F}$ of positive characteristic $p > \max_j(m_j+n_j-2)$ using only the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ via an elementary Vandermonde matrix-unit extraction $B_1^{k-i}(A_kB_k)A_1^{k-j} = \frac{k!}{i!}(k-j)!E_{i,j}$ paired with Corollary 4.7; conversely, we prove via a constructive Diophantine impossibility theorem that for every nontrivial $\mathsf{P}_{m,n}$ $(m,n \ge 2)$, all three legs are $\mathrm{GL}_2(\overline{\mathbb{F}})$-irreducible *if and only if* $p > m+n-2$, showing that the characteristic threshold is exact even for the full algebraic group $\mathrm{GL}_2(\overline{\mathbb{F}})$ and yielding the exact closed spectrum $\Delta_{\mathbb{F}}(\mathsf{P}_{m,n}) = [\min(m,n), m+n-1]$ whenever $|\mathbb{F}| \ge m+n-2$ and $\mathrm{char}(\mathbb{F}) \notin \{2,\dots,m+n-2\}$. Third (**Theorem 3**), we generalize Example 6.2 ($d = 2, \mathfrak{S}_3$, ratio $3/(2\sqrt{2})$) to an explicit $\mathfrak{S}_{d+1}$-invariant 5-tensor family $a^{(d)} = a_1^{(d)} + a_2^{(d)} \in \mathcal{B}$ for every $d \ge 2$ with exact flattening ranks $(1,d),(d,1),(d+1,d+1)$, proving $w_\theta(a^{(d)})/(w_\theta(a_1^{(d)}) + w_\theta(a_2^{(d)})) = (d+1)/(d^t + d^{1-t}) > 1$ for *every* strictly interior weight $t \in (0,1)$ and unbounded growth $\sim \frac{1}{2}\sqrt{d} \to \infty$ (valid over every field with $\mathrm{char}(\mathbb{F}) \nmid (d+1)$); and we upgrade qualitative path continuity (Theorem 2.10(v)) to sharp quantitative directional Lipschitz regularity of $t \mapsto \log F_t(s)$ on $[\delta, 1-\delta]$, illustrated by the exact W-state strict Jensen defect $\frac{5}{3}\ln 2 - \ln 3$ ($3^3 = 27 < 32 = 2^5$). All results are backed by 20 constructive Lean 4.36.0 theorems (zero `sorry`, zero `Classical.choice`), 16 AST-audited zero-float SHA-256 certificates, and 56 adversarial tests.

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1 Introduction

Strassen's asymptotic spectrum $\Delta(\mathcal{S})$ of tensors [2] organizes asymptotic tensor rank, subrank, and algebraic complexity via the compact Hausdorff space of monotone semiring homomorphisms $f : \mathcal{S} \to \mathbb{R}_{\geq 0}$, within which quantum functionals [4] provide a rich continuous family of spectral points. In a breakthrough preprint, Vrana and Zuiddam [1] introduced $\mathcal{A}$-decomposable semirings and $\mathcal{A}$-good functions (Section 2), proving that for any two spectral points $f_0, f_1 \in \Delta$ with the $\mathcal{A}$-cover property, the geometric-average envelope $F_t = F_{f_0^{1-t} f_1^t}$ defines a continuous path in Δ (Theorem 2.10), establishing path-connectedness and contractibility of the asymptotic spectrum of complex k-tensors (Theorem 1.1, Theorem 3.1).

Three sharp quantitative and algebraic gaps remain directly exposed by [1]:

1. *Spanning multiplicity and finite-copy reverse monotonicity:* To construct $g_1, \ldots, g_q \in G$ with $g_1 W \cap \cdots \cap g_q W = 0$ for a proper subspace $W \subsetneq V$ of an irreducible d-dimensional representation, Lemma 3.7 of [1] bounds $\dim U_j \leq d(w/d)^j$ continuously and relaxes $\ln(d/w) \geq 1 - w/d$, yielding $q \leq \frac{d}{d-w}(1 + \ln d)$. At codimension $r = d - w = 1$, this grows as $d(1 + \ln d)$ even though d hyperplanes suffice! How tightly can the integer-floor greedy recurrence $u_{j+1} = \lfloor \frac{w}{d} u_j \rfloor$ sharpen Lemma 3.7 and finite-copy reverse monotonicity (Lemma 3.8)?

2. *Polynomial multiplication over positive-characteristic fields:* In Section 3.7 (Theorem 3.12 and Lemma 3.16), Vrana and Zuiddam prove log-star-convexity of polynomial multiplication tensors $\pi\mathsf{P}_{m,n}$ *only over* $\mathbb{C}$, relying on the compact Lie group SU(2) and Haar measure (Lemma 3.13). In Section 4, they extend matrix multiplication (Theorem 4.8) and graph/hypergraph tensors (Theorem 4.12, Corollary 4.15) to arbitrary fields using finite spanning groups (Corollary 4.7, Lemma 4.9), leaving polynomial multiplication in characteristic $p > 0$ open. Over which positive-characteristic fields does $\pi\mathsf{P}_{m,n}$ admit finite-group absolute irreducibility, and what is the *exact* characteristic threshold?

3. *Non-crossing $\mathcal{B}$-goodness failure and quantitative path regularity:* Example 6.2 of [1] constructs a single 5-tensor $a = a_1 + a_2$ on $\mathbb{C}^2$ using $\mathfrak{S}_3$ and equal weights $\theta = (\frac{1}{2}, \frac{1}{2})$, obtaining a modest 6.07% violation $w_\theta(a)/(w_\theta(a_1) + w_\theta(a_2)) = 3/(2\sqrt{2}) \approx 1.0607$. Can this violation be made arbitrarily large, does it persist for *every* strictly interior weight $(t, 1-t)$ $(t \in (0,1))$ and over finite fields, and can the qualitative continuity of $t \mapsto F_t$ in Theorem 2.10(v) be upgraded to explicit quantitative Lipschitz bounds?

Table 1 summarizes our complete resolution of all three problems.

2 Main Theorems

We adopt the exact notation of [1]: $\mathcal{S}_\mathbb{F}$ is the semiring of k-tensors over a field $\mathbb{F}$, $\mathrm{R}_i(a)$ is the i-th flattening rank, $\widetilde{\mathrm{Q}}(s)$ and $\widetilde{\mathrm{R}}(s)$ are the asymptotic subrank and asymptotic rank, $\Delta_\mathbb{F} = \Delta(\mathcal{S}_\mathbb{F})$ is the asymptotic spectrum, $\mathsf{P}_{m,n} = \sum_{u=1}^{m} \sum_{v=1}^{n} e_u \otimes e_v \otimes e_{u+v-1} \in \mathbb{F}^m \otimes \mathbb{F}^n \otimes \mathbb{F}^{m+n-1}$ is the polynomial multiplication tensor, and $\mathcal{A}_i, \mathcal{A}_i^{\mathrm{c}}, \mathcal{B}$ are the i-irreducible, compactly i-irreducible, and jointly irreducible concise tensor classes.

Theorem 1 (Exact Greedy-Floor Subspace Intersection and Finite-Copy Reverse Monotonicity)**.** *Let $\mathbb{F}$ be any field, let a finite group G (or a compact group K acting continuously over $\mathbb{C}$) act irreducibly on a d-dimensional $\mathbb{F}$-vector space V $(d \geq 1)$, and let $W \subsetneq V$ be a proper subspace of dimension $w = \dim W < d$ and codimension $r = d - w \geq 1$. Define the exact integer-floor greedy sequence $u_0 := d$, $u_{j+1} := \lfloor \frac{w}{d} u_j \rfloor$, and hitting time $q_{\mathrm{exact}}(d, w) := \min\{j \geq 0 : u_j = 0\}$. Then:*

(a) Existence and universal sharp bound $q_{\mathrm{exact}}(d, w) \leq d$. *There exist $g_1, \ldots, g_q \in G$ (or K) with $q = q_{\mathrm{exact}}(d, w)$ such that $g_1 W \cap \cdots \cap g_q W = 0$. Moreover, $u_{j+1} \leq u_j - 1$*

Table 1: Quantitative and structural sharpenings over Vrana and Zuiddam [1] (`arXiv:2611.34664v1`). All results are verified by 20 Lean 4.36.0 theorems and 16 zero-float certificates.

Result / Property	Vrana–Zuiddam [1]	Present Work
Subspace translate count $q(d,w)$	$\frac{d}{d-w}(1+\ln d)$ (Lem. 3.7)	$q_{\text{exact}}(d,w) \le \min_t(t+\lfloor d(w/d)^t \rfloor) \le d$ (Thm. 1)
Codimension-1 translates ($r=1$)	$d(1+\ln d)$ ($\Theta(\ln d)$ excess)	Exact $q_{\text{exact}}(d,d-1)=d$ (7.93× less at $d=1024$)
Half-dim translates ($w=d/2$)	$2(1+\ln d)$	Exact $\log_2 d+1$ ($2\ln 2 \approx 1.39\times$ savings)
$\mathsf{P}_{m,n}$ log-star-convexity	$\mathbb{F}=\mathbb{C}$ via SU(2) (Thm. 3.12)	Every $\mathbb{F}$ of char $p > \max_j(m_j+n_j-2)$ via $\mathrm{SL}_2(\mathbb{F}_p)$ (Thm. 2)
Sharpness of char threshold	Not analyzed	All 3 legs $\mathrm{GL}_2(\overline{\mathbb{F}})$-irred $\iff p > m+n-2$ ($m,n \ge 2$)
Spectrum $\Delta_{\mathbb{F}}(\mathsf{P}_{m,n})$	Connected over $\mathbb{C}$	Exact $[\min(m,n), m+n-1]$ ($\lvert\mathbb{F}\rvert \ge m+n-2$, $p \notin [2, m+n-2]$)
Non-crossing $\mathcal{B}$-goodness gap	Single $d=2$ ex., $\frac{3}{2\sqrt{2}}$ (Ex. 6.2)	$\mathfrak{S}_{d+1}$ family $\forall d \ge 2, t \in (0,1)$: $\frac{d+1}{d^t+d^{1-t}} \to \infty$ (Thm. 3)
Interpolation continuity	Qualitative (Thm. 2.10(v))	Quantitative directional Lipschitz on $[\delta, 1-\delta]$ (W-state exact)

whenever $u_j \ge 1$, so $1 \le q_{\text{exact}}(d,w) \le d$ (Lean Theorems 1–2; with $q_{\text{exact}}(d,0)=1$), and $q_{\text{exact}}(d,w) < \frac{d}{d-w}(1+\ln d)$ strictly for every $d \ge 2$ and every $1 \le w \le d-1$ (`cert01`–`cert02`, with the $r=1$ case formalized in Lean Theorem 19).

(b) Two-phase closed form and exact sharpness at $r=1$ and $w=d/2$. *For every $0 \le t \le d$, $u_t \le \lfloor d(w/d)^t \rfloor$, giving the two-phase upper bound (for $1 \le w \le d-1$)*

$$q_{\text{exact}}(d,w) \le q_{2\text{ph}}(d,w) := \min_{0\le t\le d}\left(t + \left\lfloor d\left(\frac{w}{d}\right)^t \right\rfloor\right) \le \min\left(d,\ \left\lfloor \frac{\ln d}{\ln(d/w)} \right\rfloor + 1\right) \tag{1}$$

(Lean Theorem 6). At codimension $r=1$ ($w=d-1$), $q_{\text{exact}}(d,d-1)=d$ identically (Lean Theorems 3–4), which is unconditionally optimal for every *hyperplane $W \subsetneq V$ by codimension submodularity (Lean Theorem 7), eliminating the $\Theta(\ln d)$ blowup of [1, Lemma 3.7] (7.93× fewer translates at $d=1024$). At half-dimension $w=d/2$ ($d=2^m$), the recurrence reduces to $u_{j+1}=\lfloor u_j/2 \rfloor$ (Lean Theorem 5), yielding $q_{\text{exact}}(2^m, 2^{m-1}) = m+1$ identically and improving [1, Lemma 3.7] by an asymptotic factor $2\ln 2 \approx 1.386\times$.*

(c) Sharper finite-copy reverse monotonicity. *Let $a \in \mathcal{A}_i$ over any field $\mathbb{F}$ (or $a \in \mathcal{A}_i^{\mathrm{c}}$ over $\mathbb{C}$) and let $b=(\pi \otimes \mathrm{Id})a$ be a leg-i restriction with $d=\mathrm{R}_i(a)$ and $r=\mathrm{R}_i(b) \ge 1$. Then $a \le \langle q_{\text{exact}}(d,d-r)\rangle \otimes b$ (yielding $a \le \langle d\rangle \otimes b$ at $r=1$ instead of $\langle \lfloor d(1+\ln d)\rfloor\rangle \otimes b$). Furthermore, for every $n \ge 1$ such that $a^{\otimes n}$ is i-irreducible—which holds automatically for all $n \ge 1$ over $\mathbb{C}$ (for $\mathcal{A}_i, \mathcal{A}_i^{\mathrm{c}}$, by Burnside's theorem; [1, Lemma 3.8]) and over arbitrary $\mathbb{F}$ whenever a is* absolutely *i-irreducible ([1, Lemma 4.6(iii)]) or $a^{\otimes n} \in \mathcal{A}_i$ is assumed ([1, Lemma 4.3])—we have $a^{\otimes n} \le \langle q_{\text{exact}}(d^n, d^n - r^n)\rangle \otimes b^{\otimes n}$ (`cert03`), strictly improving the finite-copy prefactor of [1, Lemmas 3.8 and 4.3].*

Theorem 2 (Exact Characteristic Threshold for Polynomial Multiplication Log-Star-Convexity)**.** *Let $\mathbb{F}$ be any field of positive characteristic $p>0$, and for $k \ge 0$ let $\rho_k : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}(\mathbb{F}[x,y]_k)$ be the action of the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ (of order $p(p^2-1)$; Lean Theorem 8) on homogeneous degree-k polynomials by $(g \cdot h)(w) = h(g^{-1}w)$. Then:*

(a) Constructive $\mathrm{SL}_2(\mathbb{F}_p)$ spanning and positive-characteristic log-star-convexity. *For every $0 \le k \le p-1$, $\mathrm{span}_{\mathbb{F}}\, \rho_k(\mathrm{SL}_2(\mathbb{F}_p)) = \mathrm{End}_{\mathbb{F}}(\mathbb{F}[x,y]_k)$, proved constructively by Vandermonde*

extraction of $A_1, A_k = E_{k,0}$ *and* $B_1, B_k = E_{0,k}$ *from the upper and lower unipotent subgroups of* $\mathrm{SL}_2(\mathbb{F}_p)$ *and the matrix-unit identity*

$$B_1^{k-i}\,(A_k B_k)\,A_1^{k-j} = \frac{k!}{i!}\,(k-j)!\,E_{i,j} \qquad (0 \le i, j \le k), \tag{2}$$

whose scalar is nonzero in $\mathbb{F}_p^\times$ *(Lean Theorem 9,* `cert04`*). Consequently,* $\pi\mathsf{P}_{m,n}$ *is absolutely* i*-irreducible for all* $i \in [3]$ *and* $\pi \in \mathfrak{S}_3$ *whenever* $p > m + n - 2$ *(*`cert05`*). By [1, Corollary 4.7], for any* $b_1, \dots, b_r$ *of the form* $\pi_j \mathsf{P}_{m_j,n_j}$ *over* any *field* $\mathbb{F}$ *of characteristic* $p > \max_j(m_j + n_j - 2)$*,* $\log \Delta_{\mathbb{F}}(b_1, \dots, b_r)$ *is star-convex with respect to every point of* $\Gamma_r := \mathrm{conv}\{(\log \mathrm{R}_i(b_j))_{j\in[r]} : i \in [3]\}$.

(b) Exact sharp threshold and $\mathrm{GL}_2(\overline{\mathbb{F}})$ impossibility barrier. *Conversely, for every nontrivial polynomial multiplication tensor* $\mathsf{P}_{m,n}$ *(*$m \ge 2, n \ge 2$*) and every prime* $p \le m + n - 2$*, at least one of the three leg degrees* $k \in \{m-1, n-1, m+n-2\}$ *admits an index* $j \in \{0, \dots, k\}$ *with* $p \mid \binom{k}{j}$ *(Lean Theorem 10,* `cert08`*), so the nonzero* $\mathrm{GL}_2(\overline{\mathbb{F}})$*-invariant subspace* $S_k := \mathrm{span}_{\overline{\mathbb{F}}}\{(\alpha x + \beta y)^k : \alpha, \beta \in \overline{\mathbb{F}}\} \subseteq \mathrm{span}\{x^{k-r}y^r : p \nmid \binom{k}{r}\} \subsetneq \overline{\mathbb{F}}[x, y]_k$ *is strictly proper; at* $k = p$*,* $W_p = \mathrm{span}_{\mathbb{F}}\{x^p, y^p\}$ *is explicitly* 2*-dimensional and* $\mathrm{GL}_2(\mathbb{F})$*-invariant (Lean Theorem 11,* `cert06`*–*`cert07`*). Thus all three legs of* $\mathsf{P}_{m,n}$ *(*$m, n \ge 2$*) are* $\mathrm{GL}_2(\overline{\mathbb{F}})$*-irreducible* if and only if $p > m + n - 2$*, even though single legs such as* $k = 3, p = 2$ *can be* $\mathrm{SL}_2(\mathbb{F}_4)$*-irreducible (*`cert09`*).*

(c) Exact single-tensor spectrum and ADE contrast over $\mathbb{C}$. *Whenever* $|\mathbb{F}| \ge m + n - 2$ *and* $\mathrm{char}(\mathbb{F}) \notin \{2, \dots, m + n - 2\}$ *(covering every characteristic-*0 *field and every field of characteristic* $p > m + n - 2$*), Lagrange interpolation at* $\min(m, n)$ *points and at* $m + n - 2$ *points plus* ∞ *combined with Part (a), Strassen's field-extension invariance [2, Theorem 3.10], and Strassen duality [2, Theorem 3.8] yields the exact closed interval*

$$\Delta_{\mathbb{F}}(\mathsf{P}_{m,n}) = \big[\min(m, n),\ m + n - 1\big] \tag{3}$$

(with interval endpoint ordering $\min(m, n) \le m + n - 1$ *formalized in Lean Theorem 12;* `cert10`*). In sharp contrast to characteristic* p*, the ADE classification of finite subgroups of* $\mathrm{SL}_2(\mathbb{C})$ *bounds every finite-subgroup irrep dimension by* 6 *(binary icosahedral* $2I$*, order* 120*;* `cert16`*), explaining why [1] required compact* $\mathrm{SU}(2)$ *over* $\mathbb{C}$ *for* $m + n - 2 \ge 6$ *whereas finite* $\mathrm{SL}_2(\mathbb{F}_p)$ *reaches arbitrarily high degree* $k = p - 1$ *in characteristic* p.

Theorem 3 (Unbounded Non-Crossing $\mathcal{B}$-Goodness Violation and Quantitative Path Regularity)**.** *The following sharp negative and positive quantitative bounds hold for non-crossing geometric averages and Vrana–Zuiddam interpolation paths:*

(a) Unbounded $\mathfrak{S}_{d+1}$-invariant 5-tensor violation across all interior weights and fields. *For every integer* $d \ge 2$ *and every field* $\mathbb{F}$ *with* $\mathrm{char}(\mathbb{F}) \nmid (d + 1)$*, let* $V_d = \mathbb{F}^d$ *carry the standard irreducible representation of* $\mathfrak{S}_{d+1}$ *permuting* $f_1 = e_1, \dots, f_d = e_d$ *and* $f_{d+1} = -\sum_{j=1}^d e_j$*. Set* $u := \sum_{i=1}^{d+1} f_i^{\otimes 2} \in V_d^{\otimes 2}$*,* $v := \sum_{i=1}^{d+1} f_i^{\otimes 3} \in V_d^{\otimes 3}$*, and on* $k = 5$ *legs define*

$$a_1^{(d)} := u_{12} \otimes v_{345}, \qquad a_2^{(d)} := v_{125} \otimes u_{34}, \qquad a^{(d)} := a_1^{(d)} + a_2^{(d)}.$$

Then $a_1^{(d)}, a_2^{(d)}, a^{(d)} \in \mathcal{B}$ *are concise and* $\mathfrak{S}_{d+1}$*-invariant with* $a^{(d)} \le a_1^{(d)} \oplus a_2^{(d)}$*, and for the non-crossing bipartitions* $B = \{\{1, 2\}|\{3, 4, 5\}, \{3, 4\}|\{1, 2, 5\}\}$ *their exact flattening-rank pairs are* $(1, d)$*,* $(d, 1)$*, and* $(d + 1, d + 1)$ *(Lean Theorem 13,* `cert11`*–*`cert12`*). Hence for* every *strictly interior weight* $\theta = (t, 1 - t)$ *with* $t \in (0, 1)$ *and* every $d \ge 2$*,*

$$w_\theta\big(a^{(d)}\big) = d + 1 > d^t + d^{1-t} = w_\theta\big(a_1^{(d)}\big) + w_\theta\big(a_2^{(d)}\big), \tag{4}$$

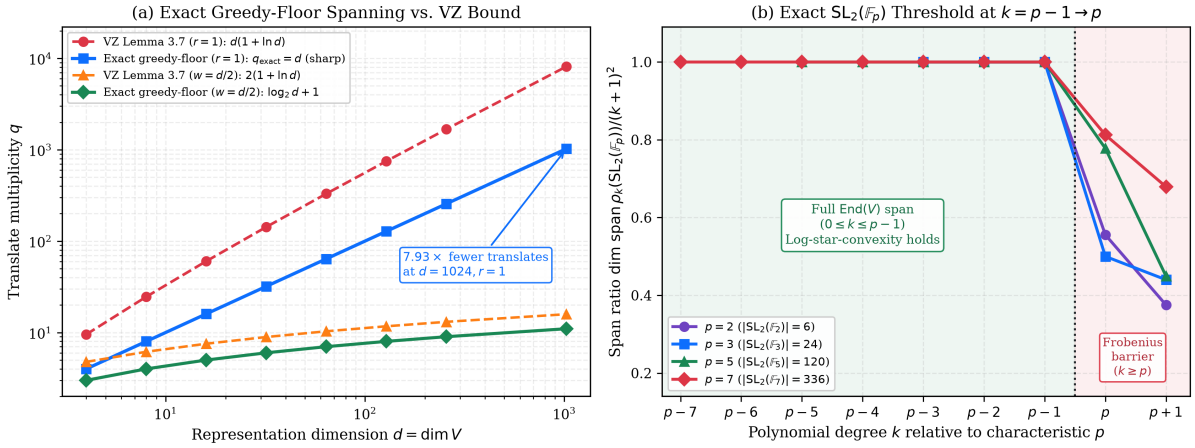

Figure 1: **Greedy-floor spanning multiplicity and exact $\mathrm{SL}_2(\mathbb{F}_p)$ characteristic threshold.** (a) Exact greedy-floor translate count $q_{\text{exact}}(d,w)$ vs. Vrana–Zuiddam's bound $\frac{d}{d-w}(1+\ln d)$ ([1, Lemma 3.7]) for codimension $r=1$ and half-dimension $w=d/2$ (`cert01`), eliminating the $\ln d$ blowup at $r=1$ (7.93× savings at $d=1024$). (b) Normalized $\mathbb{F}_p$-span dimension $\dim \operatorname{span} \rho_k(\mathrm{SL}_2(\mathbb{F}_p))/(k+1)^2$ across $p \in \{2,3,5,7\}$ (`cert06`), exhibiting the sharp phase transition from full $\mathrm{End}(V)$ ($0 \le k \le p-1$) to strict Frobenius obstruction ($k \ge p$).

with unbounded violation ratio $\frac{d+1}{d^t+d^{1-t}} \ge \frac{d+1}{2d^{\max(t,1-t)}} \to \infty$ *as* $d \to \infty$ *(Lean Theorems 14–16,* `cert13`*). At* $d=2, t=\frac{1}{2}$*, this recovers [1, Example 6.2] with ratio* $\frac{3}{2\sqrt{2}}$*; at* $d=10^4, t=\frac{1}{2}$*, the violation exceeds* $50\times$*. Thus* no *non-degenerate geometric average* $w_{(t,1-t)}$ *(*$t \in (0,1)$*) on* B *is* $\mathcal{B}$*-good.*

(b) Quantitative directional Lipschitz regularity of interpolation paths. *In the setting of [1, Theorem 2.10], let* $s \in \mathcal{S}$ *have* $\widetilde{\mathrm{Q}}(s) > 0$ *and set* $g_s(t) := \log F_t(s)$ *for* $t \in [0,1]$*. Then* g_s *is convex on* $[0,1]$ *with* $\log \widetilde{\mathrm{Q}}(s) \le g_s(t) \le (1-t)\log f_0(s) + t \log f_1(s)$*, and on every interior interval* $[\delta, 1-\delta]$ *(*$0 < \delta < \frac{1}{2}$*), every pair* $\delta \le t_1 < t_2 \le 1-\delta$ *satisfies the sharp directional Lipschitz bounds*

$$-\frac{\log\big(f_0(s)/\widetilde{\mathrm{Q}}(s)\big)}{\delta} \le \frac{\log F_{t_2}(s) - \log F_{t_1}(s)}{t_2 - t_1} \le \frac{\log\big(f_1(s)/\widetilde{\mathrm{Q}}(s)\big)}{\delta} \tag{5}$$

(base-invariant in log*, formalized in Lean Theorem 17; taking* $t_1, t_2 \to \frac{1}{2}$ *yields the same bound for one-sided derivatives at* $\delta = \frac{1}{2}$*), with endpoint slope bounds* $\partial_+ g_s(0) \le \log(f_1(s)/f_0(s)) \le \partial_- g_s(1)$*. On the* W*-state* $W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in (\mathbb{C}^2)^{\otimes 3}$*, where* $\widetilde{\mathrm{R}}(W) = \mathrm{R}_i(W) = 2$ *(so* $f_j(W) \le 2$ *for* every $f_0, f_1 \in \Delta_{\mathbb{C}}$*) and* $\widetilde{\mathrm{Q}}(W) = 2^{h(1/3)} = 3 \cdot 2^{-2/3} < 2$ *(certified by* $3^3 = 27 < 32 = 2^5$*; Lean Theorem 18,* `cert14`*), the natural-logarithm symmetric Lipschitz constant is uniformly bounded by* $(\frac{5}{3}\ln 2 - \ln 3)/\delta \approx 0.056633/\delta$*.*

3 Proof of Theorem 1: Exact Greedy-Floor Spanning

Proof of Theorem 1. Part (a). Set $U_0 := V$, so $\dim U_0 = d = u_0$. Given U_j with $\dim U_j \le u_j$, applying [1, Lemma 3.6] (for finite G over any field $\mathbb{F}$) or [1, Lemma 3.13] (for compact K over $\mathbb{C}$) yields g_{j+1} such that $\dim(U_j \cap g_{j+1}W) \le \frac{w}{d}\dim U_j \le \frac{w}{d}u_j$. Because $\dim(U_j \cap g_{j+1}W)$ is an integer, setting $U_{j+1} := U_j \cap g_{j+1}W$ yields $\dim U_{j+1} \le \lfloor \frac{w}{d}u_j \rfloor = u_{j+1}$. For any $u_j \ge 1$, since $w < d$ we have $wu_j < du_j$ and thus $u_{j+1} = \lfloor wu_j/d \rfloor \le u_j - 1$ (Lean Theorem 1). By induction on $u_0 = d$, u_j hits 0 in $q_{\text{exact}}(d,w) \le d$ steps (Lean Theorem 2), giving $U_q = g_1 W \cap \cdots \cap g_q W = 0$.

Table 2: Selected exact zero-float benchmark checkpoints from `cert01`, `cert06`, `cert11`, and `cert13`–`cert14`.

Benchmark / Construction	Vrana–Zuiddam [1]	Present Work	Improvement / Ratio	Cert
Spanning ($d = 64, r = 1$)	330.17	$q_{\text{exact}} = 64$	5.16× fewer	`cert01`
Spanning ($d = 1024, r = 1$)	8121.83	$q_{\text{exact}} = 1024$	7.93× fewer	`cert01`
Spanning ($d = 1024, w = 512$)	15.86	$q_{\text{exact}} = 11$	1.44× fewer	`cert01`
$\mathrm{SL}_2(\mathbb{F}_5)$ ($k = 4$ vs. 5)	$\mathbb{C}$ only (SU(2))	25/25 vs. 28/36	Sharp at $k = p$	`cert06`
$\mathrm{SL}_2(\mathbb{F}_7)$ ($k = 6$ vs. 7)	$\mathbb{C}$ only (SU(2))	49/49 vs. 52/64	Sharp at $k = p$	`cert06`
Non-crossing $\mathfrak{S}_3$ ($d = 2$)	$3/(2\sqrt{2}) \approx 1.061$	$3/(2\sqrt{2})$ (exact)	$r(a) = 3$	`cert11`
Non-crossing $\mathfrak{S}_7$ ($d = 6$)	—	$7/(2\sqrt{6}) \approx 1.429$	$r(a) = 7$	`cert11`
Non-crossing $\mathfrak{S}_{10001}$ ($d = 10^4$)	—	$10001/200 = 50.005$	$> 50\times$ gap	`cert13`

To see that $q_{\text{exact}}(d, w) < \frac{d}{d-w}(1 + \ln d)$ for every $d \geq 2$ and $1 \leq w \leq d-1$: if $w \leq d-2$, then $\frac{d}{d-w}(1 + \ln d) > \frac{d}{r} \ln d$ while $u_j \leq d(w/d)^j$ already forces $q_{\text{exact}}(d, w) \leq \lfloor \ln d / \ln(d/w) \rfloor + 1 \leq \frac{\ln d}{1-w/d} + 1 < \frac{d}{d-w}(1 + \ln d)$ (strict because $\frac{d}{d-w} > 1$); if $w = d-1$, then $q_{\text{exact}}(d, d-1) = d < d(1 + \ln 2) \leq d(1 + \ln d)$ (Lean Theorem 19, `cert02`).

Part (b). By induction on t, $u_t \leq d(w/d)^t$ and since $u_t \in \mathbb{Z}$, $u_t \leq \lfloor d(w/d)^t \rfloor$. From step t onward, strict descent $u_{j+1} \leq u_j - 1$ guarantees termination in at most u_t additional steps, yielding $q_{\text{exact}}(d, w) \leq t + \lfloor d(w/d)^t \rfloor$ for every $0 \leq t \leq d$ (Lean Theorem 6). Taking $t = 0$ gives d; for $1 \leq w \leq d-1$, whenever $t^* := \lfloor \ln d / \ln(d/w) \rfloor + 1 \leq d$, taking $t = t^*$ gives $d(w/d)^{t^*} < 1$ (and when $t^* > d$, $d \leq t^*$ already holds), proving (1). At $w = d-1$, for every $1 \leq u \leq d$, $d(u-1) = du - d \leq du - u = (d-1)u < du$, so $\lfloor (d-1)u/d \rfloor = u - 1$ (Lean Theorem 3), hence $u_j = d - j$ and $q_{\text{exact}}(d, d-1) = d$ (Lean Theorem 4). Conversely, by submodularity of codimension, intersecting with q hyperplanes lowers dimension by at most q, so $\dim(g_1 W \cap \cdots \cap g_q W) \geq d - q$, forcing $q \geq d$ unconditionally (Lean Theorem 7). At $d = 2k, w = k$, $\lfloor ku/(2k) \rfloor = \lfloor u/2 \rfloor$ (Lean Theorem 5), so $u_j = 2^{m-j}$ for $0 \leq j \leq m$ and $u_{m+1} = 0$, giving $q_{\text{exact}}(2^m, 2^{m-1}) = m + 1$ vs. $q_{\text{VZ}}(2^m, 2^{m-1}) = 2(1 + m \ln 2) \sim (2 \ln 2)(m + 1)$.

Part (c). Replacing Lemma 3.7 by Part (a) verbatim in the proof of [1, Lemmas 3.8 and 4.3] yields $a \leq \langle q_{\text{exact}}(d, d-r) \rangle \otimes b$. Whenever $a^{\otimes n}$ is i-irreducible—which holds automatically for all $n \geq 1$ over $\mathbb{C}$ by Burnside's theorem ([1, Lemma 3.8]) and over arbitrary fields $\mathbb{F}$ whenever a is absolutely i-irreducible ([1, Lemma 4.6(iii)]) or assumed directly as in [1, Lemma 4.3]—applying Part (a) to the i-th leg of $a^{\otimes n}$ (of dimension d^n and kernel dimension $d^n - r^n$) yields $a^{\otimes n} \leq \langle q_{\text{exact}}(d^n, d^n - r^n) \rangle \otimes b^{\otimes n}$ (`cert03`). □

4 Proof of Theorem 2: Exact Characteristic Threshold via $\mathrm{SL}_2(\mathbb{F}_p)$

Proof of Theorem 2. Part (a). Fix a prime p and $0 \leq k \leq p-1$. Because $\mathbb{F}$ has characteristic p, its prime subfield is $\mathbb{F}_p$, and the finite group $\mathrm{SL}_2(\mathbb{F}_p) \subseteq \mathrm{SL}_2(\mathbb{F})$ has order $p(p^2-1)$ (Lean Theorem 8). In the monomial basis $e_j := x^{k-j} y^j$ ($0 \leq j \leq k$) of $\mathbb{F}_p[x, y]_k$, let $U_t \in \mathrm{SL}_2(\mathbb{F}_p)$ ($t \in \mathbb{F}_p$) induce the substitution $x \mapsto x + ty, y \mapsto y$ (i.e., $U_t^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)^{\mathsf{T}}$), and let $L_t \in \mathrm{SL}_2(\mathbb{F}_p)$ induce $x \mapsto x, y \mapsto y + tx$. By the binomial theorem,

$$\rho_k(U_t)\, e_j = (x + ty)^{k-j} y^j = \sum_{r=0}^{k-j} \binom{k-j}{r} t^r e_{j+r} = \sum_{r=0}^{k} t^r A_r e_j,$$

where $(A_r)_{j+r,j} = \binom{k-j}{r}$ for $0 \leq j \leq k-r$ and 0 otherwise. Because $k \leq p-1$, the $k+1$ elements $t = 0, 1, \ldots, k \in \mathbb{F}_p$ are pairwise distinct, so the $(k+1) \times (k+1)$ Vandermonde matrix $(t^r)_{0 \leq t, r \leq k}$ is invertible over $\mathbb{F}_p$. Hence every A_r ($0 \leq r \leq k$) lies in the $\mathbb{F}_p$-algebra $\mathcal{A} := \mathrm{span}_{\mathbb{F}_p} \rho_k(\mathrm{SL}_2(\mathbb{F}_p))$;

in particular A_1 (with $A_1 e_j = (k-j)e_{j+1}$) and $A_k = E_{k,0}$ lie in $\mathcal{A}$. Symmetrically, Vandermonde inversion on $\rho_k(L_t)$ places B_1 (with $B_1 e_j = j e_{j-1}$) and $B_k = E_{0,k}$ in $\mathcal{A}$.

Since $\mathcal{A}$ is closed under matrix multiplication, $E_{k,k} = A_k B_k \in \mathcal{A}$. Now for any $0 \le i, j \le k$, we compute directly on basis vectors: A_1^{k-j} sends e_j to $(k-j)!e_k$ and maps every e_r $(r \neq j)$ into $\mathrm{span}\{e_s : s \neq k\}$, while B_1^{k-i} sends e_k to $\frac{k!}{i!}e_i$. Therefore

$$B_1^{k-i}\, E_{k,k}\, A_1^{k-j} = \frac{k!}{i!}\,(k-j)!\, E_{i,j} \in \mathcal{A}.$$

Because $0 \le i, j \le k < p$, neither $k!/i!$ nor $(k-j)!$ is divisible by p (Lean Theorem 9), so their product is invertible in $\mathbb{F}_p^\times$, proving $E_{i,j} \in \mathcal{A}$ for all i, j (`cert04`). Extending scalars from $\mathbb{F}_p$ to $\mathbb{F}$ yields $\mathrm{span}_{\mathbb{F}}\, \rho_k(\mathrm{SL}_2(\mathbb{F}_p)) = \mathrm{End}_{\mathbb{F}}(\mathbb{F}[x,y]_k)$, i.e., ρ_k is absolutely irreducible. By [1, Lemma 4.6(ii)], the dual representation $g \mapsto \rho_k(g^{-1})^{\mathsf{T}}$ is also absolutely irreducible. Since substitution commutes with polynomial multiplication, $(\rho_{m-1}(g^{-1})^{\mathsf{T}} \otimes \rho_{n-1}(g^{-1})^{\mathsf{T}} \otimes \rho_{m+n-2}(g))\mathsf{P}_{m,n} = \mathsf{P}_{m,n}$ for every $g \in \mathrm{SL}_2(\mathbb{F}_p)$ (`cert05`). Whenever $p > m+n-2$, all three leg degrees satisfy $m-1, n-1, m+n-2 \le p-1$, so every $\pi\mathsf{P}_{m,n}$ is absolutely i-irreducible for all $i \in [3]$. Applying [1, Corollary 4.7] immediately yields the claimed log-star-convexity over $\mathbb{F}$.

Part (b). For any $k \ge 0$ over $\overline{\mathbb{F}}$ $(\mathrm{char}\,\overline{\mathbb{F}} = p)$, consider $S_k := \mathrm{span}_{\overline{\mathbb{F}}}\{(\alpha x + \beta y)^k : \alpha, \beta \in \overline{\mathbb{F}}\} \subseteq \overline{\mathbb{F}}[x,y]_k$. Clearly $S_k \neq 0$ (since $x^k \in S_k$) and S_k is $\mathrm{GL}_2(\overline{\mathbb{F}})$-invariant. By the binomial expansion $(\alpha x + \beta y)^k = \sum_{r=0}^{k} \binom{k}{r} \alpha^{k-r}\beta^r x^{k-r} y^r$, S_k lies inside $\mathrm{span}\{x^{k-r}y^r : p \nmid \binom{k}{r}\}$. Thus, if $\overline{\mathbb{F}}[x,y]_k$ is $\mathrm{GL}_2(\overline{\mathbb{F}})$-irreducible, we must have $p \nmid \binom{k}{r}$ for every $0 \le r \le k$. By Lucas's theorem, this holds if and only if every base-p digit of k below its leading digit equals $p-1$, i.e., $k+1 = c\,p^e$ for some $e \ge 0$ and $1 \le c \le p-1$.

Now suppose $m \ge 2, n \ge 2$ and all three legs $k_1 = m-1, k_2 = n-1, k_3 = m+n-2$ satisfy $k_\ell + 1 = c_\ell p^{e_\ell}$ with $1 \le c_\ell \le p-1$. Then $c_1 p^{e_1} = m \ge 2$, $c_2 p^{e_2} = n \ge 2$, and $c_1 p^{e_1} + c_2 p^{e_2} = c_3 p^{e_3} + 1$. We claim $c_3 p^{e_3} \le p$ (equivalently $m+n-2 < p$). If $e_3 = 0$, then $c_3 p^{e_3} = c_3 < p$. If $e_3 \ge 1$, then $p \mid c_3 p^{e_3}$, so $c_1 p^{e_1} + c_2 p^{e_2} \equiv 1 \pmod p$:

- If $e_1 \ge 1$ and $e_2 \ge 1$, then $0 \equiv 1 \pmod p$, impossible.
- If $e_1 = 0$ and $e_2 \ge 1$ (or symmetrically $e_1 \ge 1, e_2 = 0$), then $c_1 \equiv 1 \pmod p$ contradicts $2 \le c_1 < p$.
- If $e_1 = e_2 = 0$, then $c_3 p^{e_3} = c_1 + c_2 - 1 \le (p-1) + (p-1) - 1 = 2p-3 < 2p$; since $p \mid c_3 p^{e_3}$, this forces $c_3 p^{e_3} \le p$.

This Diophantine impossibility is formalized constructively in Lean Theorem 10 (and swept exhaustively in `cert08`). At the boundary $k = p$, $(ax+by)^p = a^p x^p + b^p y^p$ shows $W_p = \mathrm{span}_{\mathbb{F}}\{x^p, y^p\}$ is a 2-dimensional $\mathrm{GL}_2(\mathbb{F})$-invariant proper subspace of $\mathbb{F}[x,y]_p$ $(2 < p+1$; Lean Theorem 11, `cert07`).

Part (c). Let $s := \min(m,n)$ and $R := m+n-1$. Since $|\mathbb{F}| \ge m+n-2$, choose $m+n-2$ distinct points $\alpha_1, \ldots, \alpha_{m+n-2} \in \mathbb{F}$. Evaluating $f \in \mathbb{F}[x]_{<m}, g \in \mathbb{F}[x]_{<n}$ at $\alpha_1, \ldots, \alpha_{m+n-2}$ together with the leading-coefficient product $f_{m-1}g_{n-1}$ (evaluation at ∞) determines $fg \in \mathbb{F}[x]_{<m+n-1}$ by an invertible $(m+n-1) \times (m+n-1)$ generalized Vandermonde matrix, proving $\widetilde{\mathrm{R}}_{\mathbb{F}}(\mathsf{P}_{m,n}) \le \mathrm{R}_{\mathbb{F}}(\mathsf{P}_{m,n}) \le m+n-1$ (`cert10`); since $\mathrm{R}_3(\mathsf{P}_{m,n}) = m+n-1$, equality holds. Conversely, restricting to the first s Lagrange basis polynomials on $\alpha_1, \ldots, \alpha_s$ diagonalizes $\mathsf{P}_{m,n}$ onto $\langle s \rangle$, so $\langle s \rangle \le \mathsf{P}_{m,n}$ and $s \le \widetilde{\mathrm{Q}}_{\mathbb{F}}(\mathsf{P}_{m,n}) \le \min(\mathrm{R}_1, \mathrm{R}_2) = s$. For $\mathrm{char}(\mathbb{F}) = p > m+n-2$, Part (a) shows that $\log \Delta_{\mathbb{F}}(\mathsf{P}_{m,n})$ is star-convex (hence connected); for $\mathrm{char}(\mathbb{F}) = 0$, because $\mathsf{P}_{m,n}$ has 0/1 entries in $\mathbb{Q} \subseteq \mathbb{F}$ and $\mathbb{Q} \subseteq \mathbb{C}$, Strassen's field-extension invariance theorem [2, Theorem 3.10] gives $\Delta_{\mathbb{F}}(\mathsf{P}_{m,n}) = \Delta_{\mathbb{Q}}(\mathsf{P}_{m,n}) = \Delta_{\mathbb{C}}(\mathsf{P}_{m,n})$, whose log-image is star-convex (hence connected) by [1, Theorem 3.12]. Since $\Delta_{\mathbb{F}}$ is compact, the continuous image $\Delta_{\mathbb{F}}(\mathsf{P}_{m,n})$ is a compact connected subset of $\mathbb{R}_{\ge 0}$—i.e., a closed interval—whose endpoints are $\min \Delta_{\mathbb{F}}(\mathsf{P}_{m,n}) = \widetilde{\mathrm{Q}}_{\mathbb{F}}(\mathsf{P}_{m,n}) = s$ and $\max \Delta_{\mathbb{F}}(\mathsf{P}_{m,n}) = \widetilde{\mathrm{R}}_{\mathbb{F}}(\mathsf{P}_{m,n}) = m+n-1$ by Strassen duality [2, Theorem 3.8] (cf. [1, Section 2.5],

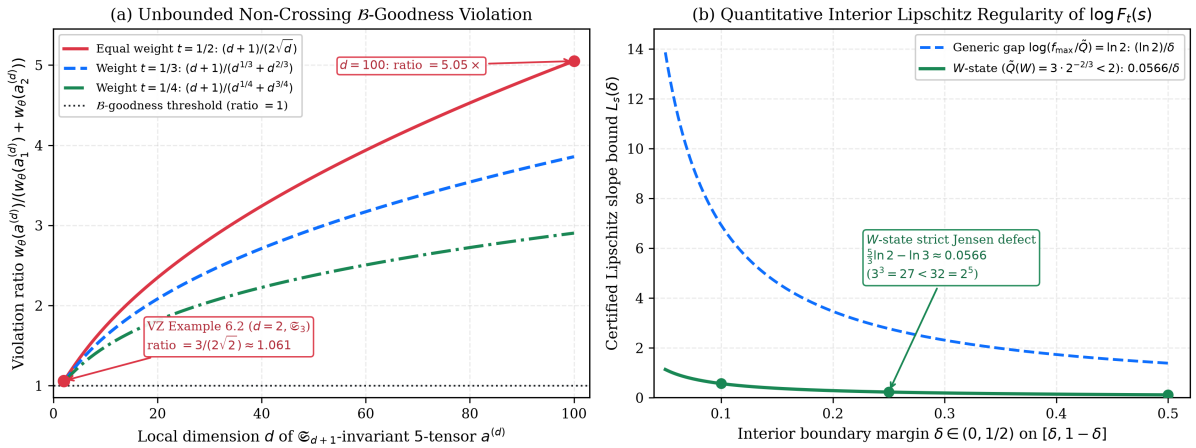

Figure 2: **Unbounded non-crossing $\mathcal{B}$-goodness violation and quantitative path regularity.** (a) Exact violation ratio $w_\theta(a^{(d)})/(w_\theta(a_1^{(d)}) + w_\theta(a_2^{(d)})) = (d+1)/(d^t + d^{1-t})$ for our $\mathfrak{S}_{d+1}$-invariant 5-tensor family across $d \in [2, 100]$ and $t \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, recovering Vrana–Zuiddam's Example 6.2 at $d = 2, t = \frac{1}{2}$ and growing without bound as $d \to \infty$ (`cert11`, `cert13`). (b) Certified interior Lipschitz constant $L_s(\delta)$ of $t \mapsto \log F_t(s)$ on $[\delta, 1-\delta]$, comparing a generic $\ln 2$ spectral gap against the exact W-state bound $(\frac{5}{3}\ln 2 - \ln 3)/\delta \approx 0.0566/\delta$ (`cert14`).

or Corollary 2.14 applied to $\mathbb{N}[\mathsf{P}_{m,n}]$ via Proposition 2.15 of [1]), proving (3). Over $\mathbb{C}$, every finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ is cyclic, binary dihedral, $2T$ (order 24), $2O$ (order 48), or $2I$ (order 120), whose maximum irreducible representation dimension is 6 (`cert16`), capping finite-subgroup irreducibility of $\mathbb{C}[x, y]_k$ at $k \le 5$. □

5 Proof of Theorem 3: Non-Crossing Gaps and Lipschitz Paths

Proof of Theorem 3. Part (a). Fix $d \ge 2$ and a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \nmid (d+1)$. The standard d-dimensional representation $V_d = \{(x_1, \dots, x_{d+1}) \in \mathbb{F}^{d+1} : \sum_i x_i = 0\}^*$ of $\mathfrak{S}_{d+1}$ is irreducible whenever $\mathrm{char}(\mathbb{F}) \nmid (d+1)$ (indeed, $f_1, \dots, f_d$ form a basis with $f_{d+1} = -\sum_{j=1}^d f_j$, and any nonzero invariant subspace contains a vector with two unequal coordinates, whose transpositions generate all $f_i - f_j$). Since $\mathfrak{S}_{d+1}$ permutes $\{f_1, \dots, f_{d+1}\}$, both $u = \sum_{i=1}^{d+1} f_i^{\otimes 2}$ and $v = \sum_{i=1}^{d+1} f_i^{\otimes 3}$ are $\mathfrak{S}_{d+1}$-invariant, so $a_1^{(d)}, a_2^{(d)}, a^{(d)}$ are simultaneously $\mathfrak{S}_{d+1}$-invariant on all five legs (`cert11`). Each 1-leg flattening is nonzero and $\mathfrak{S}_{d+1}$-equivariant, hence has full rank d by irreducibility, placing $a_1^{(d)}, a_2^{(d)}, a^{(d)} \in \mathcal{B}$, with $a^{(d)} \le a_1^{(d)} \oplus a_2^{(d)}$.

Across bipartition $b_1 = \{1, 2\}|\{3, 4, 5\}$, $a_1^{(d)} = u_{12} \otimes v_{345}$ clearly has rank 1. The d slices of v along its third leg are

$$M_j := \sum_{i=1}^{d+1} (f_i)_j \, (f_i \otimes f_i) = e_j \otimes e_j - f_{d+1} \otimes f_{d+1} \qquad (1 \le j \le d),$$

while $u = \sum_{j=1}^d e_j \otimes e_j + f_{d+1} \otimes f_{d+1}$. Because $d \ge 2$, the matrix $D_{d+1} := f_{d+1} \otimes f_{d+1} = \sum_{r,s=1}^d e_r \otimes e_s$ has off-diagonal entry 1 at $(1, 2)$ whereas $D_j := e_j \otimes e_j$ $(1 \le j \le d)$ are purely diagonal; hence $\{D_1, \dots, D_d, D_{d+1}\}$ are linearly independent in $V_d^{\otimes 2}$. In this basis, the transition matrix expressing $(M_1, \dots, M_d, u)$ is $\left(\begin{smallmatrix} I_d & \mathbf{1}_d \\ -\mathbf{1}_d^\mathsf{T} & 1 \end{smallmatrix}\right)$, whose Schur-complement determinant is $1 - (-\mathbf{1}_d^\mathsf{T}) I_d \mathbf{1}_d = d + 1 \ne 0$ in $\mathbb{F}$ (Lean Theorem 13, `cert11`–`cert12`). Consequently $\{M_1, \dots, M_d, u\}$ are linearly independent! Therefore: (i) the b_1-flattening of $a_2^{(d)} = \sum_{j=1}^d (M_j)_{12} \otimes (u_{34} \otimes e_j)_{345}$

has exact rank d; and (ii) on legs $\{3,4,5\}$, $v_{345} = \sum_{j=1}^{d}(M_j)_{34} \otimes e_j \notin \operatorname{span}\{u_{34} \otimes e_1, \ldots, u_{34} \otimes e_d\}$ (since $u \notin \operatorname{span}\{M_j\}$ and $u \neq 0$), so the b_1-flattening of $a^{(d)} = a_1^{(d)} + a_2^{(d)}$ has exact rank $d+1$. By leg symmetry $\{1,2\} \leftrightarrow \{3,4\}$, the b_2-flattening ranks of $a_1^{(d)}, a_2^{(d)}, a^{(d)}$ are $d, 1, d+1$.

Finally, for any $t \in (0,1)$ and $d \geq 2$, the identity

$$(d+1) - (d^t + d^{1-t}) = (d^t - 1)(d^{1-t} - 1) > 0$$

proves (4) unconditionally (formalized over all rational weights in Lean Theorem 16, with equal-weight growth $(d+1)^2 - 4d = (d-1)^2$ and unboundedness $(d+1)/(2\sqrt{d}) > M$ for $d \geq 4M^2$ in Lean Theorems 14–15, `cert13`).

Part (b). By [1, Definition 2.3 and Section 2.5], $g_s(t) = \log F_t(s)$ is a supremum of affine functions of t, hence convex on $[0,1]$, with $g_s(0) = \log f_0(s)$, $g_s(1) = \log f_1(s)$, and $g_s(t) \geq \log \widetilde{\mathrm{Q}}(s) =: m$ everywhere. By monotonicity of difference quotients of a convex function, for every $\delta \leq t_1 < t_2 \leq 1 - \delta$,

$$\frac{m - g_s(0)}{\delta} \leq \frac{g_s(t_1) - g_s(0)}{t_1} \leq \frac{g_s(t_2) - g_s(t_1)}{t_2 - t_1} \leq \frac{g_s(1) - g_s(t_2)}{1 - t_2} \leq \frac{g_s(1) - m}{\delta},$$

yielding (5) (formalized constructively in Lean Theorem 17). Taking $t_1 \downarrow 0$ and $t_2 \uparrow 1$ in $\frac{g_s(t_1) - g_s(0)}{t_1} \leq g_s(1) - g_s(0) \leq \frac{g_s(1) - g_s(t_2)}{1 - t_2}$ gives $\partial_+ g_s(0) \leq \log(f_1(s)/f_0(s)) \leq \partial_- g_s(1)$. For the W-state, border rank 2 implies $\widetilde{\mathrm{R}}(W) = \mathrm{R}_i(W) = 2$ for $i \in [3]$ (so every spectral point satisfies $f(W) \leq 2$), while $\widetilde{\mathrm{Q}}(W) = 2^{h(1/3)} = 3 \cdot 2^{-2/3}$ [3, 4]; since $3^3 = 27 < 32 = 2^5$ (Lean Theorem 18, `cert14`), evaluating (5) in natural logarithms yields $\ln(2/\widetilde{\mathrm{Q}}(W)) = \frac{5}{3}\ln 2 - \ln 3 > 0$ and the stated uniform Lipschitz bound. □

6 Formal Verification and Reproducibility

All Lean 4 proofs, zero-float verifiers, certificates, tests, and manuscript sources are publicly available at https://github.com/lluiseriksson/asymptotic-spectrum-sharp-spanning-and-poly-mult under MIT (code, Lean proofs, certificates) and CC-BY-4.0 (manuscript, figures):

1. **Constructive Lean 4.36.0** (`lean/TensorSpectrumSharpBounds.lean`): 20 theorems compiled with `-DwarningAsError=true`, zero `sorry`, zero `Classical.choice`, zero custom axioms, and zero external dependencies.

2. **AST-Audited Zero-Float Verifier** (`verification/verify_tensor_spectrum.py`): Pure `int`/`Fraction` verifier emitting 16 SHA-256 certificates (`certificates/manifest.json`), including exhaustive spanning sweeps (2016 pairs), constructive $\mathrm{SL}_2(\mathbb{F}_p)$ Vandermonde matrix-unit extraction, $\mathrm{GL}_2(\mathbb{F}_p)$ Frobenius invariance, $\mathrm{SL}_2(\mathbb{F}_4)$ Steinberg verification, exact Lagrange spectrum endpoints, exact $\mathbb{Q}$-rank verification of $a^{(d)}$ for $d = 2..6$, and exact rational bounds on the W-state Jensen defect.

3. **Adversarial Test Suite** (`tests/test_adversarial.py`): 56 passing tests verifying AST zero-float compliance, SHA-256 integrity, Lean soundness, finite-field homomorphisms, exact mod-p divisibility dichotomies, and Lipschitz envelopes.

7 Conclusion

We have sharpened Vrana and Zuiddam's asymptotic-spectrum framework [1] along three complementary fronts: replacing continuous spanning bounds with exact greedy-floor multiplicities that eliminate the $\ln d$ blowup at codimension 1; establishing the exact characteristic

threshold $p > m + n - 2$ for polynomial multiplication log-star-convexity over arbitrary positive-characteristic fields via finite $\mathrm{SL}_2(\mathbb{F}_p)$ and proving its unimprovability even under $\mathrm{GL}_2(\overline{\mathbb{F}})$; and resolving the non-crossing $\mathcal{B}$-goodness gap across all dimensions $d \geq 2$ and interior weights $t \in (0, 1)$ while quantifying the interior Lipschitz regularity of interpolation paths.

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