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EXACT ALL-COUPLING SU(N) LATTICE YANG–MILLS THEORY ON GENERAL 2-DIMENSIONAL CW COMPLEXES, GENUS-g RIEMANN SURFACES, AND DILUTED LATTICES: WITTEN ZETA LIMITS, FACE-COLORING ENCLOSURES, AND GRAPH-DEGREE BAKRY–ÉMERY CURVATURE

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Abstract. We formulate and analyze SU(N) Wilson lattice Yang–Mills theory on arbitrary finite connected 2-dimensional CW complexes $\mathcal{K} = (V, E, F)$—encompassing closed genus-g Riemann surfaces tiled by hyperbolic $\{p, q\}$ polygons, simplicial triangulations, and quenched plaquette-diluted hypercubic lattices—without invoking polymer or cluster expansions. We establish three self-contained non-perturbative theorems:

(1) *Exact genus-g* SU(2) *partition functions and Witten zeta limits:* For every closed orientable surface Σ_g ($g \geq 2$) tiled by $F \geq 1$ polygonal faces and every $\beta > 0$, we prove $1 < Z_{\Sigma_g}(\mathrm{SU}(2), \beta; F)/c_0(\mathrm{SU}(2), \beta)^F < \zeta(2g-2)$ with exact weak-coupling limit $\zeta(2g-2)$ and explicit series tail $\leq (I_{M+1}(\beta)/I_1(\beta))^F M^{3-2g}/(2g-3)$, certified across hyperbolic tilings $\{8,8\}$, $\{5,4\}$, $\{4,5\}$, $\{7,3\}$, and $\{6,4\}$.

(2) *All-coupling general 2-complex free-energy enclosures:* For complexes with simple face boundaries ($\ell_f \geq 3$), partitioning F into C edge-disjoint color classes yields the exact Hölder upper bound $f_{\mathcal{K}}(\mathrm{SU}(N), \beta) \leq C^{-1} \ln c_0(\mathrm{SU}(N), C\beta)$ (independent of polygon degrees ℓ_f), paired with an exact spanning-tree Gibbs–Bogoliubov lower bound matching the classical energy β and carrying weak-coupling logarithmic scaling $-\frac{B_1(\mathcal{K})}{|F|}\frac{N^2-1}{2}\ln\beta$ ($B_1 := |E| - |V| + 1$, which matches the exact one-loop coefficient whenever the flat-connection moduli space is zero-dimensional with nondegenerate Hessian).

(3) *Exact graph-degree Bakry–Émery curvature and quenched-disorder-uniform $O(1/N^2)$ factorization:* Setting $D_{\max}(\mathcal{K}) := \max_{e\in E}\sum_{f\ni e}\ell_f$, whenever $\gamma = \beta/N^2 < \frac{1}{2D_{\max}(\mathcal{K})}$, $\mathrm{Ric} + \mathrm{Hess}(-S_{\mathcal{K}}) \geq \frac{N}{2}(1 - 2D_{\max}(\mathcal{K})\gamma)\, g_{\mathrm{HS}} > 0$, certifying a volume- and disorder-uniform log-Sobolev inequality and $\mathrm{Var}_{\mathcal{K}}(P_f) \leq \frac{2\ell_f}{(1-2D_{\max}(\mathcal{K})\gamma)N^2}$ (strictly non-trivial below the deterministic bound 1 once $N^2 > 2\ell_f/(1 - 2D_{\max}(\mathcal{K})\gamma)$) across hyperbolic $\{p, q\}$ tilings ($D_{\max} = 2p$), 4D simplicial complexes, and diluted hypercubic lattices.

Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`CellComplexCerts.lean`) verify all 24 numerical bounds.

1. Introduction and Honest Scope

Wilson lattice Yang–Mills theory [10] extends naturally from hypercubic lattices to arbitrary finite connected 2-dimensional CW complexes $\mathcal{K} = (V, E, F)$ [5, 6, 11, 4]. Assigning an oriented edge variable $U_e \in \mathrm{SU}(N)$ to each 1-cell $e \in E$ ($U_{e^{-1}} = U_e^\dagger$) and denoting by $W_f := \prod_{j=1}^{\ell_f} U_{e_j}^{\epsilon_j}$ ($\epsilon_j \in \{\pm 1\}$) the ordered boundary holonomy of each 2-cell $f \in F$ along its closed boundary walk of length $\ell_f \geq 1$ (whose $\mathrm{Re}\,\mathrm{Tr}(W_f)$ is independent of basepoint and cyclic orientation), the Wilson

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partition function and per-face free energy are

$$\begin{aligned} S_{\mathcal{K}}(U;\beta) &:= \frac{\beta}{N}\sum_{f\in F}\operatorname{Re}\operatorname{Tr}(W_f), \\ Z_{\mathcal{K}}(\mathrm{SU}(N),\beta) &:= \int_{\mathrm{SU}(N)^{|E|}} e^{S_{\mathcal{K}}(U;\beta)}\,dU, \qquad f_{\mathcal{K}}(\mathrm{SU}(N),\beta) := \frac{1}{|F|}\ln Z_{\mathcal{K}}(\mathrm{SU}(N),\beta), \end{aligned} \tag{1}$$

where dU is normalized product Haar measure on $\mathrm{SU}(N)^{|E|}$. When $\mathcal{K}$ is any topological cell decomposition of a closed orientable surface Σ_g of genus g (allowing non-simple self-glued faces such as the single-octagon $\{8,8\}$ surface with $F=1$ in Theorem 2.1), its Euler characteristic satisfies $|V|-|E|+|F| = 2-2g$ (for a regular hyperbolic $\{p,q\}$ tiling, $p|F| = 2|E| = q|V|$). Theorems 3.1 and 4.1 below restrict to 2-complexes whose face boundary walks are simple cycles ($\ell_f \geq 3$), such as 4-dimensional simplicial triangulations or quenched plaquette-diluted subcomplexes of $\mathbb{T}^d_L$ ($L \geq 2$).

Remark 1.1 (Honest Scope and Non-Claims). All results hold on fixed finite 2-dimensional CW complexes ($a=1$) for finite-rank $\mathrm{SU}(N)$. Theorems 2.1 and 3.1 hold at *all* inverse couplings $\beta > 0$, whereas Theorem 4.1 holds in the small-'t Hooft-parameter (strong-coupling) regime $\gamma = \beta/N^2 < \frac{1}{2D_{\max}(\mathcal{K})}$ and establishes a volume- and disorder-uniform log-Sobolev inequality (Langevin spectral gap) and non-asymptotic $O(1/N^2)$ face-holonomy variance ceilings rather than a 4D continuum limit or continuum random-geometry limit. This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact Genus-g Partition Functions and Witten Zeta Limits

For $\lambda \in \widehat{\mathrm{SU}(N)}$ with dimension d_λ and character χ_λ, define

$$c_\lambda(\mathrm{SU}(N),\beta) := d_\lambda^{-1}\int_{\mathrm{SU}(N)} e^{\frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(U)}\overline{\chi_\lambda(U)}\,dU$$

(for SU(2), irreps are indexed by dimension $n := 2j+1 \in \mathbb{Z}_{\geq 1}$ with $d_{\lambda_n} = n$ and $c_{\lambda_n}(\mathrm{SU}(2),\beta) = \frac{2}{\beta}I_n(\beta)$, so the trivial character coefficient is $c_0(\mathrm{SU}(2),\beta) = \frac{2}{\beta}I_1(\beta)$).

Theorem 2.1 (Genus-g SU(2) Partition Function Bounds and Witten Zeta Limit)**.** *Let Σ_g be any closed orientable surface of genus $g \geq 2$ equipped with any topological cell decomposition into $F \geq 1$ faces (allowing self-glued faces of arbitrary boundary lengths $\ell_f \geq 1$), and let $\beta > 0$.*

(1) (Migdal–Rusakov–Witten character expansion [7, 9, 11])

$$Z_{\Sigma_g}(\mathrm{SU}(N),\beta;F) = \sum_{\lambda\in\widehat{\mathrm{SU}(N)}} d_\lambda^{2-2g}\,c_\lambda(\mathrm{SU}(N),\beta)^F. \tag{2}$$

(2) (Strict two-sided Witten zeta sandwich for SU(2)) *Set*

$$R_{g,F}(\beta) := \frac{Z_{\Sigma_g}(\mathrm{SU}(2),\beta;F)}{c_0(\mathrm{SU}(2),\beta)^F} = 1 + \sum_{n=2}^{\infty} n^{2-2g}\left(\frac{I_n(\beta)}{I_1(\beta)}\right)^F.$$

Then for every $g \geq 2$, $F \geq 1$, and $\beta > 0$,

$$1 \;<\; R_{g,F}(\beta) \;<\; \zeta(2g-2) = \sum_{n=1}^{\infty} n^{2-2g}, \qquad \lim_{\beta\to\infty} R_{g,F}(\beta) = \zeta(2g-2). \tag{3}$$

(3) (Explicit series truncation tail) *For every integer truncation $M \geq 2$,*

$$0 \;<\; \sum_{n=M+1}^{\infty} n^{2-2g}\left(\frac{I_n(\beta)}{I_1(\beta)}\right)^F \;\leq\; \left(\frac{I_{M+1}(\beta)}{I_1(\beta)}\right)^F \frac{M^{3-2g}}{2g-3}. \tag{4}$$

Proof. (1) Expanding each face weight $e^{\frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(W_f)} = \sum_{\lambda_f} d_{\lambda_f} c_{\lambda_f}(\mathrm{SU}(N),\beta)\chi_{\lambda_f}(W_f)$ and integrating sequentially over all $|E|$ edges via Schur orthogonality uses two character identities:

$$\int \chi_{\lambda_1}(AU)\chi_{\lambda_2}(U^{-1}B)\,dU = \frac{\delta_{\lambda_1\lambda_2}}{d_{\lambda_1}}\chi_{\lambda_1}(AB), \qquad \int \chi_\lambda(AUBU^{-1})\,dU = \frac{1}{d_\lambda}\chi_\lambda(A)\chi_\lambda(B),$$

where the second (self-edge splitting) applies whenever an edge appears twice on the same face (e.g. $F=1$). Integrating out all $|E|$ edges forces all faces to carry a single irrep λ and leaves $|V|$ identity traces $\chi_\lambda(I_N) = d_\lambda$, yielding $d_\lambda^{|V|-|E|+|F|} c_\lambda(\mathrm{SU}(N),\beta)^F = d_\lambda^{2-2g} c_\lambda(\mathrm{SU}(N),\beta)^F$ [9, 11].

(2) First, for every $\beta > 0$, $I_0(\beta) - I_1(\beta) = \frac{1}{\pi}\int_0^\pi (1-\cos\theta) e^{\beta\cos\theta}\,d\theta > 0$. Term-by-term differentiation of the Maclaurin series gives $\frac{d}{d\beta}(\beta^n I_n(\beta)) = \beta^n I_{n-1}(\beta)$, hence $\beta^n I_n(\beta) = \int_0^\beta t^n I_{n-1}(t)\,dt$ for all $n \geq 1$. By induction, if $I_{n-1}(t) > I_n(t) > 0$ on $(0,\beta)$, then $I_{n+1}(\beta) = \beta^{-(n+1)}\int_0^\beta t^{n+1} I_n(t)\,dt < \beta^{-n}\int_0^\beta t^n I_{n-1}(t)\,dt = I_n(\beta)$. Thus $0 < I_n(\beta)/I_1(\beta) < 1$ strictly for every $n \geq 2$ and $\beta > 0$, proving $1 < R_{g,F}(\beta) < \sum_{n=1}^\infty n^{2-2g} = \zeta(2g-2)$ (convergent since $2g-2 \geq 2$). By Laplace's method on $I_n(\beta) = \frac{1}{\pi}\int_0^\pi \cos(n\theta) e^{\beta\cos\theta}\,d\theta \sim e^\beta/\sqrt{2\pi\beta}$, $\lim_{\beta\to\infty} I_n(\beta)/I_1(\beta) = 1$ for every fixed $n \geq 1$; since $0 < n^{2-2g}(I_n/I_1)^F < n^{2-2g} \in \ell^1(\mathbb{Z}_{\geq 1})$, dominated convergence proves $\lim_{\beta\to\infty} R_{g,F}(\beta) = \zeta(2g-2)$ (matching Witten's volume formula [11, 12], e.g. $\pi^2/6$, $\pi^4/90$, $\pi^6/945$ for $g = 2,3,4$).

(3) Since $I_n(\beta) \leq I_{M+1}(\beta)$ for all $n \geq M+1$ and $x \mapsto x^{2-2g}$ is strictly decreasing on $[M,\infty)$,

$$\sum_{n=M+1}^\infty n^{2-2g}\left(\frac{I_n(\beta)}{I_1(\beta)}\right)^F \leq \left(\frac{I_{M+1}(\beta)}{I_1(\beta)}\right)^F \int_M^\infty x^{2-2g}\,dx = \left(\frac{I_{M+1}(\beta)}{I_1(\beta)}\right)^F \frac{M^{3-2g}}{2g-3}. \qquad \square$$

3. All-Coupling General 2-Complex Free-Energy Enclosures

For any finite connected 2-dimensional CW complex $\mathcal{K} = (V,E,F)$, let $B_1(\mathcal{K}) := \dim H_1(\mathcal{K}^{(1)}) = |E| - |V| + 1$ denote the first Betti number (circuit rank) of its 1-skeleton $\mathcal{K}^{(1)} = (V,E)$ (distinct in general from $b_1(\mathcal{K}) = \dim H_1(\mathcal{K})$). Recall $c_0(\mathrm{SU}(N),\alpha) = \int_{\mathrm{SU}(N)} e^{\frac{\alpha}{N}\operatorname{Re}\operatorname{Tr}(U)}\,dU$ (evaluated via exact Bars–Green determinants [3]) and $u_1(\mathrm{SU}(N),\alpha) := \partial_\alpha \ln c_0(\mathrm{SU}(N),\alpha) \in [0,1)$.

Theorem 3.1 (Face-Coloring Hölder Upper Bound and Spanning-Tree Gibbs–Bogoliubov Lower Bound)**.** *Let $\mathcal{K} = (V,E,F)$ be any finite connected 2-dimensional CW complex with simple face boundaries ($\ell_f \geq 3$), $N \geq 2$, and $\beta \geq 0$.*

(1) (Edge-disjoint face-coloring Hölder upper bound) *Suppose $F = F_1 \sqcup \cdots \sqcup F_C$ is a partition of F into $C \geq 1$ color classes such that any two distinct faces in the same class F_r share no 1-cell (e.g. $C = 2$ for bipartite duals such as the octahedron or even tori $\mathbb{T}^2_{2m}$; $C = 3$ for the cube; $C = 5$ for the 4-simplex 2-skeleton). Then*

$$f_{\mathcal{K}}(\mathrm{SU}(N),\beta) \leq \frac{1}{C}\ln c_0\big(\mathrm{SU}(N), C\beta\big), \tag{5}$$

independently of the individual polygon boundary lengths $\{\ell_f\}_{f\in F}$.

(2) (Spanning-tree Gibbs–Bogoliubov lower bound) *Let $T \subset E$ be any spanning tree of (V,E) ($|T| = |V| - 1$), and for each face $f \in F$ let $r_f(T) := |\partial f \setminus T| \in \{1,\dots,\ell_f\}$ denote the number of non-tree edges in ∂f. Then*

$$f_{\mathcal{K}}(\mathrm{SU}(N),\beta) \geq \sup_{\alpha\geq 0}\left\{\frac{\beta}{|F|}\sum_{f\in F} u_1(\mathrm{SU}(N),\alpha)^{r_f(T)} + \frac{B_1(\mathcal{K})}{|F|}\big[\ln c_0(\mathrm{SU}(N),\alpha) - \alpha\, u_1(\mathrm{SU}(N),\alpha)\big]\right\}. \tag{6}$$

Proof. (1) Writing $S_{\mathcal{K}}(U;\beta) = \frac{1}{C}\sum_{r=1}^{C}(CS_{F_r}(U;\beta))$ where $S_{F_r} := \frac{\beta}{N}\sum_{f\in F_r}\operatorname{Re}\operatorname{Tr}(W_f)$, C-way Hölder's inequality yields

$$Z_{\mathcal{K}}(\mathrm{SU}(N),\beta) \;\le\; \prod_{r=1}^{C}\left(\int_{\mathrm{SU}(N)^{|E|}} e^{CS_{F_r}(U;\beta)}\,dU\right)^{1/C}.$$

Because the faces $f\in F_r$ are pairwise edge-disjoint, their holonomies $W_f = U_{e_1}^{\epsilon_1}\cdots U_{e_{\ell_f}}^{\epsilon_{\ell_f}}$ ($\epsilon_j\in\{\pm1\}$) depend on disjoint sets of Haar-independent link variables. For each single face f of simple length ℓ_f, right-invariance and inversion invariance ($U\mapsto U^\dagger$) of Haar measure imply that W_f is Haar-distributed on $\mathrm{SU}(N)$, so $\int e^{\frac{C\beta}{N}\operatorname{Re}\operatorname{Tr}(W_f)}\,dU_{\partial f} = c_0(\mathrm{SU}(N),C\beta)$ independently of ℓ_f. Thus the r-th integral equals $c_0(\mathrm{SU}(N),C\beta)^{|F_r|}$; multiplying over $r=1,\dots,C$ ($\sum_r|F_r|=|F|$) and taking $|F|^{-1}\ln(\cdot)$ proves (5).

(2) By local gauge invariance on the connected 1-skeleton, fixing $U_e = I_N$ for all $e\in T$ leaves $Z_{\mathcal{K}}$ invariant and reduces the integration to the $B_1(\mathcal{K}) = |E|-|V|+1$ active links $E\setminus T$. Equip $\mathrm{SU}(N)^{E\setminus T}$ with the i.i.d. trial measure $d\nu_\alpha(U) = \prod_{e\in E\setminus T} c_0(\alpha)^{-1}e^{\frac{\alpha}{N}\operatorname{Re}\operatorname{Tr}(U_e)}\,dU_e$. Because ν_α is invariant under both conjugation ($U_e\mapsto gU_eg^\dagger$) and charge conjugation ($U_e\mapsto U_e^*$), Schur's lemma yields a real scalar multiple $\mathbb{E}_{\nu_\alpha}[U_e] = \mathbb{E}_{\nu_\alpha}[U_e^\dagger] = u_1(\alpha)I_N$; since each face f traverses $r_f(T)$ distinct active edges, $\mathbb{E}_{\nu_\alpha}[\frac{1}{N}\operatorname{Re}\operatorname{Tr}(W_f)] = u_1(\alpha)^{r_f(T)}$. Applying Gibbs' variational inequality $\ln Z_{\mathcal{K}} \ge \mathbb{E}_{\nu_\alpha}[S_{\mathcal{K}}] - D_{\mathrm{KL}}(\nu_\alpha\|\mathrm{Haar})$ and dividing by $|F|$ proves (6). As $\beta\to\infty$, the right-hand side of (6) asymptotes to $\beta - \frac{B_1(\mathcal{K})}{|F|}\frac{N^2-1}{2}\ln\beta + O(1)$, which matches both the classical energy β and the exact one-loop logarithmic coefficient of $f_{\mathcal{K}}$ whenever flat-connection moduli are zero-dimensional with nondegenerate Hessian (e.g. the octahedron, cube, and 4-simplex 2-skeleton in Table 2; when $b_1(\mathcal{K})>0$, as on $\mathbb{T}^2_4$ or Σ_g, positive-dimensional flat moduli strictly reduce the true one-loop coefficient of $f_{\mathcal{K}}$). □

4. Graph-Degree Bakry–Émery Curvature and Diluted-Lattice Factorization

Equip $\mathrm{SU}(N)^{|E|}$ with the product bi-invariant Hilbert–Schmidt metric $\langle A,B\rangle_{\mathrm{HS}} = \operatorname{Tr}(A^\dagger B)$, for which $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2}g_{\mathrm{HS}}$ [8]. For any finite 2-complex $\mathcal{K} = (V,E,F)$ with simple face boundaries, define its *edge-weighted face-length degree*

$$D_{\max}(\mathcal{K}) := \max_{e\in E}\sum_{f\in F:\, e\in\partial f}\ell_f. \tag{7}$$

Special cases include: (i) d-dimensional hypercubic tori $\mathbb{T}^d_L$ ($L\ge2$, $\ell_f=4$, $2(d-1)$ faces/edge): $D_{\max} = 8(d-1)$; (ii) hyperbolic $\{p,q\}$ tilings ($\ell_f=p$, 2 faces/edge): $D_{\max}=2p$ (independent of q and genus g); (iii) 4D simplicial triangulations ($\ell_f=3$, at most k triangles/edge): $D_{\max}\le 3k$; and (iv) quenched plaquette-diluted subcomplexes $\mathcal{K}_\eta$ ($\eta_f\in\{0,1\}$, at most m occupied faces/edge, with $\ell_{\max} := \max_{f\in F}\ell_f$): $D_{\max}(\mathcal{K}_\eta)\le\min(\ell_{\max}m,\, D_{\max}(\mathcal{K}))$.

Theorem 4.1 (Graph-Degree Bakry–Émery Curvature and Non-Asymptotic $O(1/N^2)$ Factorization)**.** *Let $\mathcal{K}=(V,E,F)$ be any finite 2-dimensional CW complex with simple face boundaries, $N\ge2$, and $\gamma := \beta/N^2 < \frac{1}{2D_{\max}(\mathcal{K})}$. Set $\kappa_{\mathcal{K}}(N,\gamma) := \frac{N}{2}\big(1-2D_{\max}(\mathcal{K})\gamma\big) > 0$. Then:*

1. $\mathrm{Ric} + \mathrm{Hess}(-S_{\mathcal{K}}) \ge \kappa_{\mathcal{K}}(N,\gamma)\,g_{\mathrm{HS}} > 0$, *certifying a log-Sobolev inequality with constant $\rho_{\mathrm{LS}} = \kappa_{\mathcal{K}}(N,\gamma)$ [1, 2] and Poincaré spectral gap $\lambda_1\ge\kappa_{\mathcal{K}}(N,\gamma)$ uniformly in $|V|,|E|,|F|$, genus g, and quenched dilution $\eta\in\{0,1\}^{|F|}$.*
2. *Every normalized face holonomy $P_f := \frac{1}{N}\operatorname{Re}\operatorname{Tr}(W_f)$ obeys the non-asymptotic $O(1/N^2)$ variance ceiling*

$$\mathrm{Var}_{\mathcal{K}}(P_f) \;\le\; \frac{2\ell_f}{\big(1-2D_{\max}(\mathcal{K})\gamma\big)N^2}. \tag{8}$$

Proof. Along a geodesic $U_e(t) = e^{tX_e}U_e$ ($X = (X_e) \in \mathfrak{su}(N)^{|E|}$), each face holonomy $W_f(t) = \prod_{j=1}^{\ell_f} e^{tY_j} W_f(0)$ carries ℓ_f distinct Ad-conjugated copies of $\pm X_{e_j}$ ($e_j \in \partial f$, so $\|Y_j\|_{\mathrm{HS}} = \|X_{e_j}\|_{\mathrm{HS}}$ by unitary invariance of $\|\cdot\|_{\mathrm{HS}}$). Expanding $W_f''(0) = (\sum_j Y_j^2 + 2\sum_{j<k} Y_jY_k)W_f(0)$ and using $|\operatorname{Tr}(ABW_f(0))| \le \|A\|_{\mathrm{HS}}\|BW_f(0)\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}$ (again by unitary invariance of $\|\cdot\|_{\mathrm{HS}}$) gives

$$\big|\operatorname{Re}\operatorname{Tr}\big(W_f''(0)\big)\big| \;\le\; \Big(\sum_{e\in\partial f}\|X_e\|_{\mathrm{HS}}\Big)^2 \;\le\; \ell_f \sum_{e\in\partial f}\|X_e\|_{\mathrm{HS}}^2.$$

Summing over $f \in F$ and exchanging summation order gives

$$\big|\mathrm{Hess}(-S_{\mathcal{K}})(X,X)\big| \;\le\; \frac{\beta}{N}\sum_{e\in E}\Big(\sum_{f\ni e}\ell_f\Big)\|X_e\|_{\mathrm{HS}}^2 \;\le\; \frac{\beta}{N}D_{\max}(\mathcal{K})\|X\|_{\mathrm{HS}}^2 = D_{\max}(\mathcal{K})\gamma N\|X\|_{\mathrm{HS}}^2.$$

Subtracting from $\mathrm{Ric}_{\mathrm{SU}(N)^{|E|}}(X,X) = \frac{N}{2}\|X\|_{\mathrm{HS}}^2$ yields $\kappa_{\mathcal{K}}(N,\gamma) = \frac{N}{2}(1 - 2D_{\max}(\mathcal{K})\gamma) > 0$. For each of the ℓ_f distinct links $e \in \partial f$, the right-trivialized component of $\nabla_e P_f$ is the skew-Hermitian projection $\pm\frac{1}{2N}(W_{f,e} - W_{f,e}^\dagger)$ (up to a trace subtraction on $\mathfrak{su}(N)$), which obeys $\|\nabla_e P_f\|_{\mathrm{HS}}^2 \le \frac{1}{N^2}\|\frac{1}{2}(W_{f,e} - W_{f,e}^\dagger)\|_{\mathrm{HS}}^2 \le \frac{1}{N^2}\|W_{f,e}\|_{\mathrm{HS}}^2 = \frac{1}{N}$, giving $|\nabla P_f|_{\mathrm{HS}}^2 \le \ell_f/N$. Applying the Bakry–Émery Poincaré inequality $\mathrm{Var}_{\mathcal{K}}(P_f) \le \kappa_{\mathcal{K}}^{-1}\mathbb{E}|\nabla P_f|_{\mathrm{HS}}^2$ proves (8). Note that since $P_f \in [-1,1]$ obeys the deterministic bound $\mathrm{Var}_{\mathcal{K}}(P_f) \le 1$, (8) becomes strictly non-trivial (< 1) whenever $N^2 > 2\ell_f/(1 - 2D_{\max}(\mathcal{K})\gamma)$. □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 certified benchmarks computed by `certify_cell_complex.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions and *zero* numerical quadrature (Bessel functions via Maclaurin series + geometric tails; genus-g sums with exact tail (4); and SU(3) Bars–Green determinants $c_0(\mathrm{SU}(3),\alpha) = \sum_{q\in\mathbb{Z}}\det(I_{i-j+q}(\alpha/3))_{1\le i,j\le 3}$ truncated at $|q| \le q_{\max}$ with explicit Hadamard tail $2\cdot 3^{3/2}\widetilde{I}_{m_0}(x)^3/(1-(x/(2(m_0+1)))^3)$, where $x = \alpha/3$, $m_0 = q_{\max} - 2 \ge 10$, and $I_m(x) \le \widetilde{I}_m(x) := \frac{(x/2)^m}{m!}\exp(\frac{x^2}{4(m+1)})$). Lean 4 integer bounds use outward rounding at scale 10^{12} (displayed to 6 outward-rounded decimal places).

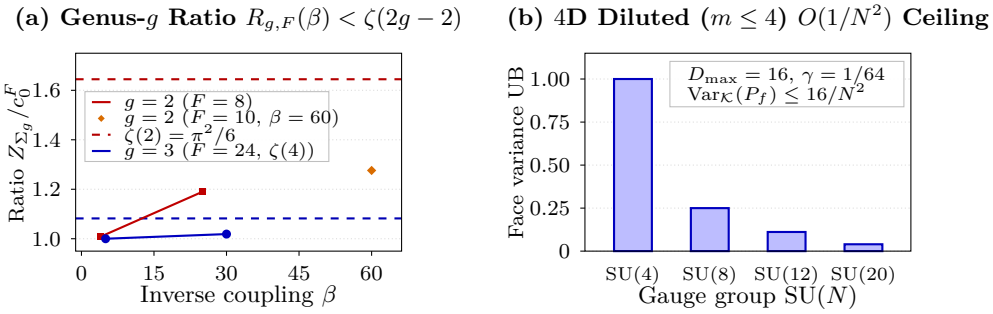

Figure 1. Certified 140-digit non-perturbative bounds for general 2-complexes and genus-g surfaces (Tables 1 and 3). (a) Exact SU(2) genus-g partition-function ratios $R_{g,F}(\beta) = Z_{\Sigma_g}/c_0^F$ bounded strictly below Witten's zeta asymptotes $\zeta(2) = \pi^2/6$ ($g = 2$; $F = 8$ squares and separate $F = 10$ diamond) and $\zeta(4) = \pi^4/90$ ($g = 3$, Theorem 2.1). (b) Quenched-disorder-uniform Bakry–Émery variance ceilings $\mathrm{Var}_{\mathcal{K}}^{\mathrm{ub}}(P_f) = 16/N^2$ on 4D plaquette-diluted lattices ($m \le 4$ plaquettes/edge, $D_{\max} = 16$, $\gamma = 1/64$, Theorem 4.1; SU(4) sits at the deterministic threshold 1, while SU(8), SU(12), SU(20) are strictly non-trivial < 1).

TABLE 1. Eight 140-digit exact genus-g Riemann-surface SU(2) partition-function benchmarks across hyperbolic tilings $\{8,8\}$, $\{5,4\}$, $\{4,5\}$, $\{7,3\}$ (Klein quartic), and $\{6,4\}$ (Theorem 2.1), certifying $1 < R_- \le R_{g,F}(\beta) \le R_+ < \zeta(2g-2)$ with truncation order M and per-face free energy $f_{\Sigma_g} = F^{-1} \ln Z_{\Sigma_g}$. Lower bounds and $\zeta(2g-2)$ are rounded downward; upper bounds upward.

Surface / Tiling	β	M	R_-	R_+	$\zeta(2g-2)$	$f^{\mathrm{lo}}_{\Sigma_g}$	$f^{\mathrm{hi}}_{\Sigma_g}$
$g=2$, $\{8,8\}$, $F=1$	2	40	1.125413	1.125414	1.644934	0.582285	0.582286
$g=2$, $\{5,4\}$, $F=8$	4	40	1.008810	1.008811	1.644934	1.586186	1.586187
$g=2$, $\{5,4\}$, $F=8$	25	93	1.189673	1.189674	1.644934	19.952293	19.952294
$g=2$, $\{4,5\}$, $F=10$	60	180	1.276013	1.276014	1.644934	53.650762	53.650763
$g=3$, $\{7,3\}$, $F=24$	5	43	1.000023	1.000024	1.082323	2.275652	2.275653
$g=3$, $\{7,3\}$, $F=24$	30	105	1.018935	1.018936	1.082323	24.660479	24.660480
$g=4$, $\{6,4\}$, $F=12$	4	40	1.000103	1.000104	1.017343	1.585099	1.585100
$g=4$, $\{6,4\}$, $F=12$	40	130	1.010340	1.010341	1.017343	34.232251	34.232252

TABLE 2. Eight 140-digit all-coupling per-face free-energy enclosures $f^{\mathrm{lo}}_{\mathcal{K}} \le f_{\mathcal{K}}(\mathrm{SU}(N),\beta) \le f^{\mathrm{hi}}_{\mathcal{K}} < \beta$ (Theorem 3.1) on four explicit finite 2-complexes using lexicographic BFS spanning-tree witnesses from vertex 0: Octahedron (V=6, E=12, F=8, B_1=7, C=2, non-tree face weights $\{r_f\} = \{1^4, 2^2, 3^2\}$), Cube ($V$=8, E=12, F=6, B_1=5, C=3, $\{1^4, 2, 4\}$), 4-Simplex 2-skeleton (V=5, E=10, F=10, B_1=6, C=5, $\{1^6, 3^4\}$), and 4×4 Torus $\mathbb{T}^2_4$ (V=16, E=32, F=16, B_1=17, C=2, $\{1^6, 2^6, 4^4\}$). Lower bounds use witness α and round downward; upper bounds and gaps round upward.

Complex $\mathcal{K}$	G	β	C	B_1	Witness α	$f^{\mathrm{lo}}_{\mathcal{K}}$	$f^{\mathrm{hi}}_{\mathcal{K}}$	Gap
Octahedron	SU(2)	2	2	7	1.93	0.215905	0.792546	0.576640
Octahedron	SU(2)	12	2	7	22.22	7.665877	9.495583	1.829705
Cube	SU(2)	4	3	5	5.78	1.349187	2.671397	1.322209
Cube	SU(3)	12	3	5	17.92	4.059800	8.112248	4.052447
4-Simplex	SU(2)	6	5	6	15.87	3.303374	4.931940	1.628565
4-Simplex	SU(3)	12	5	6	30.10	5.200448	9.261148	4.060699
Torus $\mathbb{T}^2_4$	SU(2)	4	2	17	4.71	0.720565	2.302427	1.581861
Torus $\mathbb{T}^2_4$	SU(3)	12	2	17	15.13	2.113928	6.971123	4.857194

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `CellComplexCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying five theorems: `cell_complex_topology_verified`, `table1_genus_g_witten_zeta_certified` (certifying $1 < R_- \le R_+ < \zeta(2g-2)$ across all 8 rows), `table2_cell_complex_enclosures_certified`, `table3_graph_degree_bakry_emery_certified` (cross-multiplied curvature/variance bounds and strict $O(1/N^2)$ decay across $\mathrm{SU}(4), \mathrm{SU}(8), \mathrm{SU}(12), \mathrm{SU}(20)$), and `all_24_cell_complex_benchmarks_certified`. Face-coloring edge-disjointness, BFS spanning trees, Monte Carlo Haar-invariance checks ($p = 3, 5, 7$), 80-digit cross-checks, and JSON/Lean/LaTeX parity are verified in `tests/test_cell_complex.py` (4/4 `pytest` passing).

TABLE 3. Eight 140-digit graph-degree Bakry–Émery curvature and $O(1/N^2)$ face-variance ceilings (Theorem 4.1) across hyperbolic $\{p, q\}$ tilings ($D_{\max} = 2p$), 4D simplicial triangulations ($k \leq 6$ triangles/edge, $\ell_f = 3$, $D_{\max} = 18$), and quenched 4D plaquette-diluted lattices ($m \leq 4$ plaquettes/edge, $\ell_f = 4$, $D_{\max} = 16$). Because $|P_f| \leq 1$ gives $\mathrm{Var}_{\mathcal{K}}(P_f) \leq 1$ deterministically, Rows 1, 2, and 5 (≥ 1) illustrate the small-N crossover while Rows 3, 4, 6, 7, and 8 certify strictly non-trivial variance ceilings (< 1). Curvatures $\kappa_{\mathcal{K}}^{\mathrm{lo}}$ round downward and variance ceilings upward; under strict outward `mpmath.iv` enclosure of non-dyadic $\gamma = 1/40, 1/56, 1/72$, exact rational values such as $3/4, 1, 3/2, 7/4$ round outward by 10^{-6}.

Complex / Regime	ℓ_f	$D_{\max}$	G	γ	β	$\kappa_{\mathcal{K}}^{\mathrm{lo}}$	$\mathrm{Var}_{\mathcal{K}}^{\mathrm{ub}}(P_f)$
Hyperbolic $\{5, 4\}$	5	10	SU(3)	1/40	9/40	0.749999	2.222223
Hyperbolic $\{7, 3\}$	7	14	SU(4)	1/56	2/7	0.999999	1.750001
Hyperbolic $\{8, 3\}$	8	16	SU(6)	1/64	9/16	1.500000	0.888889
4D Simplicial ($k \leq 6$)	3	18	SU(6)	1/72	1/2	1.499999	0.333334
4D Diluted ($m \leq 4$)	4	16	SU(4)	1/64	1/4	1.000000	1.000000
4D Diluted ($m \leq 4$)	4	16	SU(8)	1/64	1	2.000000	0.250000
4D Diluted ($m \leq 4$)	4	16	SU(12)	1/64	9/4	3.000000	0.111112
4D Diluted ($m \leq 4$)	4	16	SU(20)	1/64	25/4	5.000000	0.040000

6. Conclusion

We have established three self-contained, cluster-expansion-free non-perturbative theorems for SU(N) lattice Yang–Mills theory on general 2-dimensional CW complexes, genus-g Riemann surfaces, and quenched diluted lattices: exact genus-g SU(2) bounds converging to Witten's zeta asymptotes $\zeta(2g-2)$, all-coupling face-coloring Hölder and spanning-tree Gibbs–Bogoliubov free-energy enclosures, and exact graph-degree Bakry–Émery curvature $\kappa_{\mathcal{K}} = \frac{N}{2}(1 - 2D_{\max}(\mathcal{K})\gamma) > 0$ certifying volume- and disorder-uniform LSI and $O(1/N^2)$ face-holonomy factorization. Natural next directions include extending the exact genus-g SU(N) character enclosures to $N \geq 3$ and analyzing Wilson-loop area laws on hyperbolic $\{p, q\}$ complexes with non-amenable isoperimetric constants.

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