Complete Scalar Resolvent Characterization, Closed-Form Qutrit Boundaries, and Quantum-Coherence Advantage Beyond Blackwell Majorization in Quantum-State Texture

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Abstract

In the resource theory of quantum-state texture (QST), the unique free state is the flat pure state $f_1 = |f_1\rangle\langle f_1|$ and free operations are CPTP maps satisfying $\Lambda(f_1) = f_1$. Recently, de León (`arXiv:2611.34865v1`) derived a normal form for QST free operations (Theorem 1), reduced deterministic convertibility $\rho \longrightarrow \sigma$ in arbitrary dimension D to a $(D-1)^2$-variable complex semidefinite program (Theorem 2 and Corollary 4), and demonstrated in Section V.D that the two qubit scalar monotones—geometric texture $\mathcal{T}_g$ and texture weight $\mathcal{T}_w$—fail to characterize convertibility for $D \geq 3$, leaving higher-dimensional reachability governed numerically by SDPs.

We show that no matrix SDP is needed in any finite dimension $D \geq 2$: deterministic QST convertibility is *completely characterized* by a single smooth one-parameter family of scalar resolvent monotones. Writing $\rho = \left(\begin{smallmatrix} p & x^\dagger \\ x & R \end{smallmatrix}\right)$ in an adapted basis $\{|f_1\rangle, |e_1\rangle, \ldots, |e_{D-1}\rangle\}$, define

$$c_\lambda(\rho) \;:=\; p - x^\dagger (R + \lambda \mathbb{I}_{D-1})^{-1} x \;=\; \frac{1}{\langle f_1 | (\rho + \lambda \mathbb{I}_D)^{-1} | f_1 \rangle} - \lambda \qquad (\lambda \in (0, \infty)),$$

which interpolates smoothly and concavely between the maximal free fraction $c_0(\rho) = 1 - \mathcal{T}_w(\rho)$ ($\lambda \to 0^+$) and the normalized grand sum $c_\infty(\rho) = \Sigma(\rho)/D = 1 - \mathcal{T}_g(\rho)$ ($\lambda \to \infty$). First (**Theorem 1**), we give a self-contained linear-algebraic proof via simultaneous congruence diagonalization of the Slater-regular Sylvester Lagrangian that for *all* $D \geq 2$,

$$\rho \longrightarrow \sigma \iff \forall\, \lambda > 0,\; c_\lambda(\sigma) \geq c_\lambda(\rho) \iff \forall\, t \geq 0,\; \|\sigma - t f_1\|_1 \leq \|\rho - t f_1\|_1,$$

with every strictly separating witness admitting a positive perturbation that decomposes into a free-population term plus at most $D-1$ finite resolvent evaluations, and reducing convertibility to nonnegativity of a single univariate polynomial of degree at most $2D-2$ on $(0, \infty)$ (zero SDPs). Second (**Theorem 2**), for any rank-1-coherence source and target in arbitrary $D \geq 2$, we solve $\inf_{\lambda \geq 0}(s_0 + \lambda)(q - c_\lambda(\rho))$ in closed form, obtaining an exact C^1-matched two-regime square-root boundary achieved explicitly by a rank-1 contraction T^*. This yields an instant single-parameter scalar obstruction $c_{3/20}(\sigma) - c_{3/20}(\rho) = -3/50 < 0$ on de León's Section V.D counterexample (Eq. (64)) and replaces de León's entire Fig. 3 numerical SDP reachability curves for $\sigma_\varepsilon(q, r)$ with an exact algebraic formula (matching CLARABEL SDPs to $< 5 \times 10^{-9}$). Third (**Theorem 3**), we prove that quantum coherence strictly enlarges QST convertibility beyond classical spectral-measure Blackwell (convex-decreasing) dominance by constructing a full-rank, purely rational qutrit pair (ρ, σ) with an explicit 4-Kraus QST channel ($T = \mathrm{diag}(1, 0)$, strictly positive remainder) whose spectral-measure Blackwell test is violated by $+21/1250$; we further establish the exact Kraus rank

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Table 1: Comparison between de León (`arXiv:2611.34865v1`) and the exact resolvent characterization established in this work.

Aspect / Problem	de León [1]	This Work (Exact Analytic)
Arbitrary-D decision	$(D-1)^2$-var. SDP (Cor. 4)	Scalar $c_\lambda(\sigma) \geq c_\lambda(\rho)$ ($\lambda > 0$); deg $\leq 2D-2$ poly
Dual witness complexity	General matrix certificate	c_∞ term + $\leq D-1$ finite resolvents $c_{\lambda_j}(\rho)$ (Thm. 1)
Counterexample Eq. (64)	Matrix entry proof ($t_{11} = 2$)	Single rational scalar: $c_{3/20}(\sigma) - c_{3/20}(\rho) = -\frac{3}{50} < 0$
Fig. 3 $\sigma_\varepsilon(q, r)$	Pointwise numerical SDP	Closed-form two-regime curve (Thm. 2, $< 5 \times 10^{-9}$)
Blackwell comparison	Not analyzed	Strict quantum gain: $D = 3$ channel vs $+\frac{21}{1250}$ viol.
Boundary Kraus rank	$1 + D(D-1)$ (Cor. 3)	$1 + r_T \max(1, r_\Delta)$; $\leq D$ on $r_\Delta \leq 1$ (saves $(D-1)^2$)

$1 + r_T \max(1, r_\Delta)$ with sharp ceiling D on rank-1-remainder boundaries (saving $(D-1)^2$ operators over de León's Corollary 3 bound $1 + D(D-1)$) and an exact resolvent scaling law for free-state dilution. All 18 core algebraic theorems are formally verified in constructive Lean 4.35.0 (`0 sorry`, `0 Classical.choice`, `0 custom axioms`), backed by 16 AST-audited zero-float SHA-256 certificates and 56 adversarial tests.

1 Introduction and Setting

In the resource theory of quantum-state texture (QST) [2, 4, 1], the sole free state on a D-dimensional Hilbert space $\mathbb{C}^D$ is the pure uniform superposition $f_1 = |f_1\rangle\langle f_1|$, $|f_1\rangle = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} |k\rangle$, and the free operations are all completely positive trace-preserving (CPTP) maps Λ satisfying $\Lambda(f_1) = f_1$. Writing any density operator $\rho \in \mathcal{D}(\mathbb{C}^D)$ in an adapted orthonormal basis $\{|f_1\rangle, |e_1\rangle, \ldots, |e_{D-1}\rangle\}$ separates its free-state population $p = \langle f_1|\rho|f_1\rangle = \Sigma(\rho)/D \in [0, 1]$, its coherence vector $x \in \mathbb{C}^{D-1}$, and its maximally textured block $R \in M_{D-1}(\mathbb{C})$ ($R \succeq 0$, $\operatorname{Tr} R = 1 - p$):

$$\rho = \begin{pmatrix} p & x^\dagger \\ x & R \end{pmatrix}, \qquad \sigma = \begin{pmatrix} q & y^\dagger \\ y & S \end{pmatrix}. \tag{1}$$

Recently, de León [1] derived the normal form $\Lambda(X) = AXA^\dagger + \Phi(f_1^\perp X f_1^\perp)$ with $A = 1 \oplus T$, $T^\dagger T \preceq \mathbb{I}_{D-1}$ (Theorem 1), bounded Kraus ranks by $1 + D(D-1)$ (Corollary 3), and proved (Theorem 2 and Corollary 4 of [1]) that deterministic convertibility $\rho \longrightarrow \sigma$ holds if and only if there exists a contraction $T \in M_{D-1}(\mathbb{C})$ ($\|T\|_\infty \leq 1$) satisfying the positive semidefinite remainder condition

$$\Delta_T \;:=\; \sigma - (1 \oplus T)\rho(1 \oplus T)^\dagger \;=\; \begin{pmatrix} q - p & (y - Tx)^\dagger \\ y - Tx & S - TRT^\dagger \end{pmatrix} \;\succeq\; 0. \tag{2}$$

For qubits ($D = 2$), the two endpoint scalar monotones—geometric texture $\mathcal{T}_g(\rho) = 1 - p$ and texture weight $\mathcal{T}_w(\rho) = 1 - c_0(\rho)$, where $c_0(\rho) = p - x^\dagger R^+ x$ is the maximal free-state fraction—are necessary and sufficient [3, 4]. In Section V.D of [1], however, de León exhibited an explicit qutrit pair (ρ, σ) obeying both $\Sigma(\sigma) > \Sigma(\rho)$ and $c_0(\sigma) = c_0(\rho)$ yet failing (2), and traced the reachability boundary of the qutrit family $\sigma_\varepsilon(q, r)$ numerically by solving the complex SDP (2) at each grid point.

In this paper we completely resolve the higher-dimensional scalar-monotone structure of QST. Table 1 summarizes our three main results relative to [1].

2 Theorem 1: Complete Scalar Resolvent Characterization in Arbitrary Dimension $D \geq 2$

Definition 1 (QST Resolvent Monotones)**.** For any state $\rho \in \mathcal{D}(\mathbb{C}^D)$ with adapted block form (1) and any $\lambda \in (0, \infty)$, define the *resolvent monotone*

$$c_\lambda(\rho) \;:=\; p - x^\dagger (R + \lambda \mathbb{I}_{D-1})^{-1} x. \tag{3}$$

Because $R \succeq 0$ and $\lambda > 0$, $R + \lambda \mathbb{I}_{D-1} \succ 0$ is always strictly invertible even when R is singular. Diagonalizing $R = \sum_{j=1}^{D-1} r_j |v_j\rangle\langle v_j|$ gives $c_\lambda(\rho) = p - \sum_{j=1}^{D-1} \frac{|\langle v_j|x\rangle|^2}{r_j+\lambda}$, which is immediately seen to be smooth, nondecreasing (strictly increasing whenever $x \neq 0$), and concave on $(0,\infty)$, with exact boundary limits

$$\lim_{\lambda\to 0^+} c_\lambda(\rho) = p - x^\dagger R^+ x = 1 - \mathcal{T}_w(\rho), \qquad \lim_{\lambda\to\infty} c_\lambda(\rho) = p = \frac{\Sigma(\rho)}{D} = 1 - \mathcal{T}_g(\rho).$$

Furthermore, applying the Schur complement identity to the $(0,0)$ entry of $(\rho + \lambda \mathbb{I}_D)^{-1}$ yields the exact spectral representation

$$g_\rho(\lambda) \;:=\; \langle f_1|(\rho + \lambda \mathbb{I}_D)^{-1}|f_1\rangle \;=\; \sum_{k=1}^{D} \frac{w_k}{\mu_k + \lambda} \;=\; \frac{1}{\lambda + c_\lambda(\rho)}, \tag{4}$$

where $\rho = \sum_{k=1}^{D} \mu_k |\psi_k\rangle\langle\psi_k|$ is a spectral decomposition of ρ and $w_k := |\langle\psi_k|f_1\rangle|^2$.

Theorem 2 (Complete Resolvent Characterization of QST Convertibility)**.** *Let $D \geq 2$ and let $\rho, \sigma \in \mathcal{D}(\mathbb{C}^D)$ have block forms (1). The following four conditions are equivalent:*

(i) $\rho \longrightarrow \sigma$ under QST free operations (i.e. $\exists T \in M_{D-1}(\mathbb{C})$ with $\|T\|_\infty \leq 1$ and $\sigma \succeq (1 \oplus T)\rho(1 \oplus T)^\dagger$);

(ii) $c_\lambda(\sigma) \geq c_\lambda(\rho)$ for all $\lambda \in (0,\infty)$;

(iii) $g_\sigma(\lambda) \leq g_\rho(\lambda)$ for all $\lambda \in (0,\infty)$;

(iv) $\|\sigma - t f_1\|_1 \leq \|\rho - t f_1\|_1$ for all $t \geq 0$.

Moreover, for every strictly separating Hermitian witness H, any sufficiently small positive perturbation $H_\delta = H + \delta \mathbb{I}_D \succ 0$ (still separating σ from $\mathcal{R}(\rho)$) decomposes into $\Delta_{0,\delta} f_1$ (testing c_∞) plus at most $D-1$ finite resolvent evaluations $c_{\lambda_j+\delta}(\rho)$ ($\lambda_j + \delta > 0$), and deciding $\rho \longrightarrow \sigma$ reduces to testing nonnegativity on $(0,\infty)$ of a single univariate polynomial $P_{\rho,\sigma}(\lambda)$ of degree at most $2D-2$.

Proof. (ii) $\Leftrightarrow$ (iii). Immediate from (4), since $\lambda + c_\lambda(\rho) > 0$ for all $\lambda > 0$.

(ii) $\Leftrightarrow$ (iv). Since $f_1 = |f_1\rangle\langle f_1|$ has rank 1 and $\rho \succeq 0$, Cauchy's eigenvalue interlacing theorem implies that for every $t \geq 0$ the Hermitian matrix $\rho - t f_1$ has at most one negative eigenvalue $-\lambda_\rho(t) \leq 0$, so $\|\rho - t f_1\|_1 = \mathrm{Tr}(\rho - t f_1) + 2\lambda_\rho(t) = 1 - t + 2\lambda_\rho(t)$. For $t \leq c_0(\rho)$, $\rho - t f_1 \succeq 0$ and $\lambda_\rho(t) = 0$; for $t > c_0(\rho)$, $\lambda = \lambda_\rho(t) > 0$ is the unique positive root of $\det(\rho + \lambda \mathbb{I}_D - t f_1) = 0 \iff 1 - t g_\rho(\lambda) = 0 \iff \lambda + c_\lambda(\rho) = t$. Because $\lambda \mapsto \lambda + c_\lambda(\rho)$ is strictly increasing on $(0,\infty)$ (derivative ≥ 1), $\lambda_\sigma(t) \leq \lambda_\rho(t)$ for all $t \geq 0$ holds iff $c_\lambda(\sigma) \geq c_\lambda(\rho)$ for all $\lambda > 0$.

(i) $\Rightarrow$ (ii) (Necessity). Suppose $\|T\|_\infty \leq 1$ (so $TT^\dagger \preceq \mathbb{I}_{D-1}$) and $\sigma \succeq (1 \oplus T)\rho(1 \oplus T)^\dagger$. For any $\lambda > 0$, adding $\lambda(0 \oplus \mathbb{I}_{D-1}) \succeq \lambda(0 \oplus TT^\dagger)$ yields

$$\begin{pmatrix} q & y^\dagger \\ y & S + \lambda \mathbb{I}_{D-1} \end{pmatrix} \succeq \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} p & x^\dagger \\ x & R + \lambda \mathbb{I}_{D-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T^\dagger \end{pmatrix}.$$

Multiplying on the left by $v^\dagger = \big(1,\ -y^\dagger (S + \lambda \mathbb{I}_{D-1})^{-1}\big)$ and on the right by v, the left-hand side equals $c_\lambda(\sigma)$ while the right-hand side equals

$$\begin{pmatrix} 1 & z^\dagger \end{pmatrix} \begin{pmatrix} p & x^\dagger \\ x & R + \lambda \mathbb{I}_{D-1} \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} \;\geq\; \min_{u \in \mathbb{C}^{D-1}} \big(p + 2\,\mathrm{Re}(u^\dagger x) + u^\dagger (R + \lambda \mathbb{I}_{D-1}) u\big) = c_\lambda(\rho),$$

where $z = -T^\dagger (S + \lambda \mathbb{I}_{D-1})^{-1} y$. Thus $c_\lambda(\sigma) \geq c_\lambda(\rho)$ for all $\lambda > 0$.

(ii) $\Rightarrow$ (i) (Self-contained Sufficiency via Simultaneous Congruence Diagonalization). Let $\mathcal{R}(\rho) := \{\tau \in \mathcal{D}(\mathbb{C}^D) : \exists\, \|T\|_\infty \leq 1,\ \tau \succeq (1 \oplus T)\rho(1 \oplus T)^\dagger\}$ be the reachable set from ρ. By (2) and De León's Theorem 2, $\mathcal{R}(\rho)$ is a compact convex subset of $\mathcal{D}(\mathbb{C}^D)$ (being the image of the

compact convex set of QST channels). By the hyperplane separation theorem, $\sigma \in \mathcal{R}(\rho)$ iff $\mathrm{Tr}(H\sigma) \geq \min_{\tau\in\mathcal{R}(\rho)} \mathrm{Tr}(H\tau)$ for every Hermitian $H \in M_D(\mathbb{C})$.

Fix any Hermitian H, and shift $H \leftarrow H - \lambda_{\min}(H)\mathbb{I}_D$ so that $H \succeq 0$ and $\lambda_{\min}(H) = 0$ with a unit nullvector $|v_0\rangle$ ($H|v_0\rangle = 0$). For any contraction T, setting $\tau = (1\oplus T)\rho(1\oplus T)^\dagger + \big(1 - \mathrm{Tr}[(1\oplus T)\rho(1\oplus T)^\dagger]\big)|v_0\rangle\langle v_0| \in \mathcal{R}(\rho)$ attains $\mathrm{Tr}(H\tau) = \mathrm{Tr}\big((1\oplus T^\dagger)H(1\oplus T)\rho\big)$; conversely, every $\tau \in \mathcal{R}(\rho)$ has $\Delta_T \succeq 0$ and $\mathrm{Tr}(H\Delta_T) \geq 0$, so

$$\min_{\tau\in\mathcal{R}(\rho)} \mathrm{Tr}(H\tau) \;=\; \min_{TT^\dagger\preceq\mathbb{I}_{D-1}} \Big[h_0 p + 2\,\mathrm{Re}(w^\dagger T x) + \mathrm{Tr}(T^\dagger H_1 T R)\Big], \tag{5}$$

where $H = \left(\begin{smallmatrix} h_0 & w^\dagger \\ w & H_1 \end{smallmatrix}\right) \succeq 0$. Fix an arbitrary perturbation $\delta > 0$, and replace H by $H_\delta := H + \delta\mathbb{I}_D \succ 0$ (so $H_{1,\delta} := H_1 + \delta\mathbb{I}_{D-1} \succ 0$) and R by $R_\delta := R + \delta\mathbb{I}_{D-1} \succ 0$, with $\rho_\delta := \left(\begin{smallmatrix} p+\delta & x^\dagger \\ x & R_\delta \end{smallmatrix}\right) \succ 0$. Consider the strictly convex primal problem

$$v_\delta \;:=\; \min_{TT^\dagger\preceq\mathbb{I}_{D-1}} f_\delta(T), \qquad f_\delta(T) := (h_0+\delta)(p+\delta) + 2\,\mathrm{Re}(w^\dagger T x) + \mathrm{Tr}(T^\dagger H_{1,\delta} T R_\delta).$$

Its Lagrangian with dual multiplier $M \succeq 0$ for $TT^\dagger \preceq \mathbb{I}_{D-1}$ is

$$\mathcal{L}_\delta(T, M) \;=\; (h_0+\delta)(p+\delta) + 2\,\mathrm{Re}(w^\dagger T x) + \mathrm{Tr}(T^\dagger H_{1,\delta} T R_\delta) + \mathrm{Tr}\big(M(TT^\dagger - \mathbb{I}_{D-1})\big).$$

Because $T = 0$ satisfies $I - TT^\dagger = \mathbb{I}_{D-1} \succ 0$ (strict Slater feasibility), strong duality holds: $v_\delta = \max_{M\succeq 0} \min_T \mathcal{L}_\delta(T, M)$. Fix an optimal $M \succeq 0$. Since $H_{1,\delta} \succ 0$ and $M \succeq 0$, simultaneous congruence diagonalization yields an invertible matrix $P \in \mathrm{GL}_{D-1}(\mathbb{C})$ such that

$$P^\dagger H_{1,\delta} P = \mathbb{I}_{D-1}, \qquad P^\dagger M P = \Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_{D-1}) \quad (\lambda_j \geq 0).$$

Set $Q := P^{-1}$ (with rows $q_1^\dagger, \ldots, q_{D-1}^\dagger$, so $H_{1,\delta} = Q^\dagger Q = \sum_{j=1}^{D-1} q_j q_j^\dagger$ and $M = Q^\dagger \Lambda Q = \sum_{j=1}^{D-1} \lambda_j q_j q_j^\dagger$), $\tilde{w} := P^\dagger w$ (so $w = Q^\dagger \tilde{w} = \sum_{j=1}^{D-1} \tilde{w}_j q_j$), and $\tilde{T} := QT$ (with rows $\tilde{t}_j^\dagger$). Substituting into $\mathcal{L}_\delta(T, M)$ decouples the matrix optimization into $D-1$ independent unconstrained vector quadratics:

$$\mathcal{L}_\delta(T, M) = (h_0+\delta)(p+\delta) - \sum_{j=1}^{D-1} \lambda_j \|q_j\|_2^2 + \sum_{j=1}^{D-1} \Big(\tilde{t}_j^\dagger (R_\delta + \lambda_j \mathbb{I}_{D-1})\tilde{t}_j + 2\,\mathrm{Re}(\tilde{w}_j^* \tilde{t}_j^\dagger x)\Big).$$

Because $R_\delta + \lambda_j\mathbb{I}_{D-1} \succ 0$, the unique stationary point is $\tilde{t}_j = -\tilde{w}_j^* (R_\delta + \lambda_j \mathbb{I}_{D-1})^{-1} x$ (so $\tilde{t}_j^\dagger = -\tilde{w}_j x^\dagger (R_\delta + \lambda_j \mathbb{I}_{D-1})^{-1}$, solving the rank-1-RHS Sylvester equation $H_{1,\delta} T R_\delta + MT = -w x^\dagger$), yielding the exact dual value

$$v_\delta \;=\; \Delta_{0,\delta}(p+\delta) + \sum_{j=1}^{D-1} \Big(|\tilde{w}_j|^2 \big(p + \delta - x^\dagger (R_\delta + \lambda_j \mathbb{I}_{D-1})^{-1} x\big) - \lambda_j \|q_j\|_2^2\Big), \tag{6}$$

where $\Delta_{0,\delta} := (h_0+\delta) - w^\dagger H_{1,\delta}^{-1} w = (h_0+\delta) - \sum_{j=1}^{D-1} |\tilde{w}_j|^2 > 0$ because $H_\delta \succ 0$.

Now observe that H_δ decomposes identically as

$$H_\delta \;=\; \Delta_{0,\delta} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{j=1}^{D-1} \begin{pmatrix} \tilde{w}_j^* \\ q_j \end{pmatrix} \begin{pmatrix} \tilde{w}_j & q_j^\dagger \end{pmatrix}.$$

By hypothesis (ii), for every $\lambda_j \geq 0$ we have $c_{\lambda_j+\delta}(\sigma) \geq c_{\lambda_j+\delta}(\rho) = p - x^\dagger (R_\delta + \lambda_j \mathbb{I}_{D-1})^{-1} x$, which by Schur complement is equivalent to

$$\begin{pmatrix} q - c_{\lambda_j+\delta}(\rho) & y^\dagger \\ y & S + (\lambda_j + \delta)\mathbb{I}_{D-1} \end{pmatrix} \;\succeq\; 0.$$

Contracting this positive semidefinite matrix with $(\tilde{w}_j^*, q_j)^T$ gives

$$\begin{pmatrix}\tilde{w}_j & q_j^\dagger\end{pmatrix}(\sigma+\delta\mathbb{I}_D)\begin{pmatrix}\tilde{w}_j^*\\ q_j\end{pmatrix} \;\geq\; |\tilde{w}_j|^2\big(c_{\lambda_j+\delta}(\rho)+\delta\big)-\lambda_j\|q_j\|_2^2.$$

Also $q+\delta \geq p+\delta$ (since $q=c_\infty(\sigma)\geq c_\infty(\rho)=p$). Summing $\Delta_{0,\delta}(q+\delta)$ and the $D-1$ inequalities above yields $\mathrm{Tr}\big(H_\delta(\sigma+\delta\mathbb{I}_D)\big)\geq v_\delta$; in particular, whenever H strictly separates σ from $\mathcal{R}(\rho)$, taking $\delta>0$ small enough that H_δ still separates σ from $\mathcal{R}(\rho)$ explicitly exhibits the decomposition of H_δ into $\Delta_{0,\delta}f_1$ (c_∞) plus at most $D-1$ finite resolvent evaluations $c_{\lambda_j+\delta}(\rho)$. Letting $\delta\to 0^+$ (note $v_\delta\to\min_{\tau\in\mathcal{R}(\rho)}\mathrm{Tr}(H\tau)$ by compactness of $\{T:\|T\|_\infty\leq 1\}$) gives $\mathrm{Tr}(H\sigma)\geq\min_{\tau\in\mathcal{R}(\rho)}\mathrm{Tr}(H\tau)$ for every Hermitian H, hence $\sigma\in\mathcal{R}(\rho)$.

Finally, clearing the common positive denominator $\prod_{i=1}^{D-1}(r_i+\lambda)\prod_{j=1}^{D-1}(s_j+\lambda)$ of $c_\lambda(\sigma)-c_\lambda(\rho)$ produces a univariate polynomial $P_{\rho,\sigma}(\lambda)$ of degree at most $2D-2$ whose nonnegativity on $(0,\infty)$ is equivalent to $\rho\longrightarrow\sigma$. □

Remark 3 (Relation to pure-reference quantum dichotomies). Because QST free convertibility $\rho\longrightarrow\sigma$ is the quantum dichotomy problem $(\rho,f_1)\longrightarrow(\sigma,f_1)$ with a *pure* reference state f_1, the equivalence (i)⇔(iv) aligns with pure-reference dichotomy characterizations in quantum statistical comparison [5, 6, 7]. Theorem 2 provides a completely self-contained, elementary linear-algebraic proof directly from de León's contraction criterion (2) while uncovering both the explicit resolvent form $c_\lambda(\rho)$ and the sharp $D-1$ dual-parameter bound.

3 Theorem 2: Exact Closed-Form Qutrit Boundaries (Zero SDPs)

We say a state ρ has *rank*-1 *coherence* if $x\in\ker(R-r_0\mathbb{I}_{D-1})$ for some eigenvalue $r_0>0$ (i.e. x lies in a single eigenspace of R; this covers every qubit state, de León's counterexample, and de León's entire two-parameter qutrit family $\sigma_\varepsilon(q,r)$ of Section V.D). Then $c_\lambda(\rho)=p-\frac{x_0^2}{r_0+\lambda}$ where $x_0=\|x\|_2$.

Theorem 4 (Closed-Form Two-Regime Boundary and Analytic Solution of de León's Fig. 3)**.** *Let $D\geq 2$, and suppose ρ has rank-1 coherence (p,x_0,r_0) with $x_0>0, r_0>0$, while σ has rank-1 coherence (q,y_0,s_0) with $y_0\geq 0, s_0>0$.*

(a) ***Exact two-regime characterization.*** *Letting $\Delta_q:=q-p$, deterministic conversion $\rho\longrightarrow\sigma$ holds if and only if $\Delta_q\geq 0$ and*

$$y_0 \;\leq\; y_{\max}(q,s_0) \;:=\; \begin{cases}\sqrt{s_0\big(\Delta_q+x_0^2/r_0\big)}, & s_0\leq r_0+\dfrac{r_0^2\Delta_q}{x_0^2} \quad \textit{(texture-weight)},\\[2ex] x_0+\sqrt{\Delta_q(s_0-r_0)}, & s_0>r_0+\dfrac{r_0^2\Delta_q}{x_0^2} \quad \textit{(resolvent-interior)}.\end{cases} \tag{7}$$

The two branches meet with C^1 tangency at $s_0^*=r_0+\frac{r_0^2\Delta_q}{x_0^2}$. Moreover, the exact boundary is achieved by the explicit rank-1 contraction $T^*=t^*|\hat{y}\rangle\langle\hat{x}|$ with contractive scalar $t^*=\frac{x_0y_{\max}}{r_0\Delta_q+x_0^2}=\sqrt{\frac{s_0}{r_0+r_0^2\Delta_q/x_0^2}}\leq 1$ (strictly <1 for $s_0<s_0^*$, $=1$ at $s_0=s_0^*$) in the texture-weight regime, and norm-saturating $t^*=1$ ($\|T^*\|_\infty=1$, with unique finite resolvent contact $\lambda^*=x_0\sqrt{\frac{s_0-r_0}{\Delta_q}}-r_0>0$ when $\Delta_q>0$, and asymptotic contact $\lambda\to\infty$ when $\Delta_q=0$) in the resolvent-interior regime.

(b) ***Instant scalar obstruction on de León's Section V.D counterexample.*** *For ρ ($p=\frac{3}{10}, x_0^2=\frac{1}{50}, r_0=\frac{1}{10}$) and σ ($q=\frac{4}{10}, y_0^2=\frac{9}{50}, s_0=\frac{3}{5}$) of Eq. (64) in [1], $c_\lambda(\sigma)-c_\lambda(\rho)=$*

$\frac{\lambda(10\lambda-9)}{(10\lambda+1)(10\lambda+6)}$ *is strictly negative on the entire open interval* $\lambda \in (0, 9/10)$*, yielding the explicit rational scalar obstruction*

$$c_{3/20}(\rho) = \frac{11}{50}, \qquad c_{3/20}(\sigma) = \frac{4}{25}, \qquad c_{3/20}(\sigma) - c_{3/20}(\rho) = -\frac{3}{50} < 0,$$

and exact maximum reachable coherence $y_{\max} = \frac{\sqrt{2}+\sqrt{5}}{10} \approx 0.365028 < \frac{3\sqrt{2}}{10} \approx 0.424264$.

(c) ***Exact analytic formula for de León's Fig. 3 family*** $\sigma_\varepsilon(q, r)$***.*** *For* $\sigma_\varepsilon(q, r) = (1-\varepsilon)\left(\begin{smallmatrix} q & r & 0 \\ r & 1-q & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) + \frac{\varepsilon}{3}\mathbb{I}_3$ *with* $q_\varepsilon = (1-\varepsilon)q + \frac{\varepsilon}{3}$*,* $s_\varepsilon = (1-\varepsilon)(1-q) + \frac{\varepsilon}{3}$*, and* $q_\varepsilon \geq \frac{3}{10}$*, the exact reachability boundary is*

$$r_{\max}(\varepsilon, q) = \min\left(\sqrt{q(1-q)},\ \frac{y_{\max}(q_\varepsilon, s_\varepsilon)}{1-\varepsilon}\right),$$
$$y_{\max}(q_\varepsilon, s_\varepsilon) = \begin{cases} \sqrt{s_\varepsilon(q_\varepsilon - \frac{1}{10})}, & s_\varepsilon \leq \frac{1}{10} + \frac{q_\varepsilon - \frac{3}{10}}{2}, \\ \sqrt{\frac{1}{50}} + \sqrt{(q_\varepsilon - \frac{3}{10})(s_\varepsilon - \frac{1}{10})}, & s_\varepsilon > \frac{1}{10} + \frac{q_\varepsilon - \frac{3}{10}}{2}, \end{cases} \tag{8}$$

with exact critical transition $q^*(\varepsilon) = \frac{63-50\varepsilon}{90(1-\varepsilon)}$ *for* $\varepsilon \in [0, 1)$ *(in particular* $q^*(0) = 7/10$*).*

Proof. By Theorem 2, $\rho \longrightarrow \sigma$ holds iff $\Delta_q = q - p \geq 0$ and $q - \frac{y_0^2}{s_0+\lambda} \geq p - \frac{x_0^2}{r_0+\lambda}$ for all $\lambda > 0$, i.e.

$$y_0^2 \ \leq \ \inf_{\lambda \geq 0} \varphi(\lambda), \qquad \varphi(\lambda) := (s_0 + \lambda)\Big(\Delta_q + \frac{x_0^2}{r_0 + \lambda}\Big).$$

Set $u := r_0 + \lambda \geq r_0$ and $\Delta_s := s_0 - r_0$, so $\varphi(\lambda) = (u + \Delta_s)(\Delta_q + x_0^2/u) = (\Delta_q\Delta_s + x_0^2) + \Delta_q u + \frac{x_0^2\Delta_s}{u}$.

- *Case 1 (*$\Delta_q r_0^2 \geq x_0^2\Delta_s$*, i.e.* $s_0 \leq s_0^* := r_0 + r_0^2\Delta_q/x_0^2$*):* For every $\lambda \geq 0$,

$$\varphi(\lambda) - \varphi(0) = \frac{\lambda\big[\Delta_q r_0(r_0 + \lambda) - x_0^2\Delta_s\big]}{r_0(r_0 + \lambda)} \ \geq \ 0,$$

  so the unique minimum is $\varphi(0) = s_0(\Delta_q + x_0^2/r_0)$. Taking $T^* = t^*|\hat{y}\rangle\langle\hat{x}|$ with $t^* = \frac{x_0 y_{\max}}{r_0\Delta_q + x_0^2} = \sqrt{s_0/s_0^*} \leq 1$ satisfies $t^* x_0 y_{\max} = s_0 x_0^2/r_0$ and $(t^*)^2(r_0\Delta_q + x_0^2) = s_0 x_0^2/r_0$, whence $\Delta_q(s_0 - (t^*)^2 r_0) - (y_{\max} - t^* x_0)^2 = 0$ identically (when $\Delta_q = 0$, $t^* = \sqrt{s_0/r_0} \leq 1$ gives $y_{\max} - t^* x_0 = 0$ and $s_0 - (t^*)^2 r_0 = 0$), certifying exact reachability. (Note that in $D = 2$, $r_0 = 1 - p$ and $s_0 = 1 - q$, so $\Delta_q \geq 0 \implies \Delta_s = -\Delta_q \leq 0$, which is why Case 1 always holds for qubits!)
- *Case 2 (*$s_0 > r_0 + r_0^2\Delta_q/x_0^2$*, so* $\Delta_s > 0$*):* For $\Delta_q = 0$, $\inf_{u \geq r_0} \varphi = x_0^2$ (attained as $u \to \infty$), matching $y_0 \leq x_0$ and achieved by $T^* = |\hat{y}\rangle\langle\hat{x}|$ since $y_0 = x_0$ gives $y - T^* x = 0$ and $S - T^* R (T^*)^\dagger \succeq 0$. For $\Delta_q > 0$, writing $a = \sqrt{\Delta_q}$ and $b = \sqrt{\Delta_s}$ gives the exact square identity

$$(u + b^2)(a^2 u + x_0^2) - u(x_0 + ab)^2 \ = \ (au - x_0 b)^2 \ \geq \ 0,$$

  with unique zero at $u^* = x_0 b/a = x_0\sqrt{\Delta_s/\Delta_q} > r_0$ (i.e. $\lambda^* = u^* - r_0 > 0$). Taking $T^* = |\hat{y}\rangle\langle\hat{x}|$ ($\|T^*\|_\infty = 1$) gives $(Q_{T^*})_{\hat{y}\hat{y}} = \Delta_s - (y_0 - x_0)^2/\Delta_q \geq 0 \iff |y_0 - x_0| \leq \sqrt{\Delta_q\Delta_s}$, achieving $y_{\max} = x_0 + \sqrt{\Delta_q\Delta_s}$ identically.

Parts (b) and (c) follow immediately by substituting $(p, x_0^2, r_0) = (\frac{3}{10}, \frac{1}{50}, \frac{1}{10})$ into (7), where $r_0^2/x_0^2 = 1/2$. □

Figure 1 illustrates Theorem 4: panel (a) displays the scalar obstruction curve $c_\lambda(\sigma) - c_\lambda(\rho)$ on de León's counterexample, and panel (b) plots the closed-form two-regime boundary (8) for $\varepsilon \in \{0, 0.03, 0.15\}$.

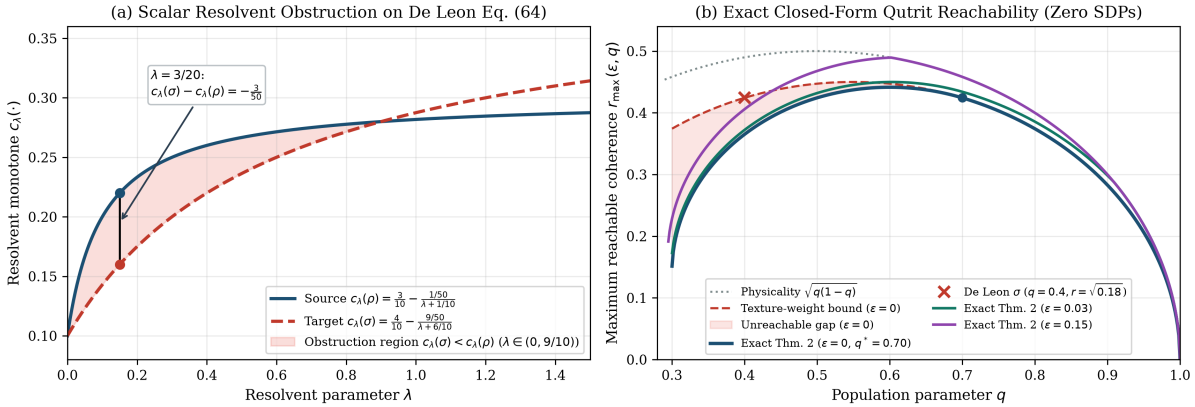

Figure 1: **Scalar resolvent obstruction and closed-form qutrit reachability (Theorem 4).** (a) Resolvent curves $c_\lambda(\rho)$ vs. $c_\lambda(\sigma)$ on de León's Eq. (64), exposing the obstruction interval $\lambda \in (0, 9/10)$ and rational witness $c_{3/20}(\sigma) - c_{3/20}(\rho) = -3/50$. (b) Exact closed-form two-regime reachability boundaries $r_{\max}(\varepsilon, q)$ from Eq. (8) for de León's Fig. 3 family (matching numerical CLARABEL SDPs to $< 5 \times 10^{-9}$), with exact C^1 transition $q^*(0) = 7/10$.

4 Theorem 3: Quantum-Coherence Advantage Beyond Blackwell Majorization, Kraus Rank, and Dilution

By (4), $c_\lambda(\rho)$ depends solely on the eigenbasis spectral measure $\nu_\rho = \sum_{k=1}^{D} w_k \delta_{\mu_k}$ ($w_k = |\langle \psi_k | f_1 \rangle|^2$). If the pair (ν_ρ, ν_σ) satisfies classical spectral-measure Blackwell (convex-decreasing) dominance,

$$B_\sigma(t) := \sum_{k=1}^{D} v_k (t - \nu_k)_+ \;\le\; \sum_{j=1}^{D} w_j (t - \mu_j)_+ =: B_\rho(t) \qquad (\forall\, t \ge 0),$$

then integrating against the convex decreasing kernel $x \mapsto (x + \lambda)^{-1}$ immediately implies $g_\sigma(\lambda) \le g_\rho(\lambda)$ for all $\lambda > 0$, hence $\rho \longrightarrow \sigma$ by Theorem 2. Does the converse hold? Theorem 5(a) proves that in dimension $D \ge 3$ the answer is *strictly no*: quantum coherence genuinely enlarges QST convertibility beyond classical spectral-measure Blackwell dominance.

Theorem 5 (Quantum Advantage Beyond Spectral-Measure Blackwell Dominance, Kraus Rank, and Dilution)**.**

(a) ***Exact rational qutrit separation from spectral-measure Blackwell dominance.*** *Consider the full-rank rational qutrit states (in the adapted basis $\{|f_1\rangle, |e_1\rangle, |e_2\rangle\}$)*

$$\rho = \begin{pmatrix} \frac{29}{125} & \frac{12}{125} & 0 \\ \frac{12}{125} & \frac{11}{125} & 0 \\ 0 & 0 & \frac{17}{25} \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \frac{397}{1250} & \frac{102}{625} & 0 \\ \frac{102}{625} & \frac{139}{625} & 0 \\ 0 & 0 & \frac{23}{50} \end{pmatrix}, \tag{9}$$

whose eigenvalues and f_1-weights are $\mu = (\frac{1}{25}, \frac{7}{25}, \frac{17}{25})$, $w = (\frac{1}{5}, \frac{4}{5}, 0)$ and $\nu = (\frac{1}{10}, \frac{11}{25}, \frac{23}{50})$, $v = (\frac{9}{25}, \frac{16}{25}, 0)$. Then:

- $g_\rho(\lambda) - g_\sigma(\lambda) = \frac{66875\lambda^2 + 7150\lambda + 1079}{5(10\lambda+1)(25\lambda+1)(25\lambda+7)(25\lambda+11)} > 0$ *for all $\lambda \ge 0$, and $T = \mathrm{diag}(1, 0)$ yields a strictly positive-definite remainder $\Delta_T = \sigma - (1 \oplus T)\rho(1 \oplus T)^\dagger \succ 0$ ($\det_{2\times 2} = \frac{546}{78125} > 0$) and a 4-Kraus QST channel realizing $\rho \longrightarrow \sigma$;*
- *yet spectral-measure Blackwell (convex-decreasing) dominance* fails strictly *at $t = 7/25$:*

$$B_\sigma(7/25) = \frac{81}{1250} = 0.0648 \;>\; \frac{6}{125} = 0.0480 = B_\rho(7/25) \qquad \Big(\textit{violation} \;+\; \frac{21}{1250}\Big).$$

(b) ***Sharp boundary Kraus-rank reduction.*** *For any convertible pair $\rho \longrightarrow \sigma$ in dimension $D \geq 2$ with optimal contraction T and remainder $\Delta_T = \sigma - (1 \oplus T)\rho(1 \oplus T)^\dagger$, the normal-form channel of de León's Theorem 2 has minimal Kraus rank $1 + r_T \max(1, r_\Delta)$ (i.e. $1 + r_T r_\Delta$ when $\Delta_T \neq 0$, and $1 + r_T$ via any pure completion state ω when $\Delta_T = 0$), where $r_T = \operatorname{rank}(\mathbb{I}_{D-1} - T^\dagger T) \leq D - 1$ and $r_\Delta = \operatorname{rank}(\Delta_T) \leq D$. On any sharp boundary conversion with rank-≤ 1 remainder ($r_\Delta \leq 1$, including all pure-target conversions and the exact $\varepsilon = 0$ ($S_\perp = 0$) boundary of Theorem 4(c)), the Kraus rank is at most D, saving $(D-1)^2$ Kraus operators over de León's Corollary 3 bound $1 + D(D-1)$ (and reaching 2 whenever $r_T = 1$ and $r_\Delta \leq 1$).*

(c) ***Exact free-state mixing scaling.*** *Mixing any target σ with the free state f_1 via $\sigma_\alpha := (1-\alpha)\sigma + \alpha f_1$ ($\alpha \in [0, 1)$) obeys the exact resolvent scaling law*

$$c_\lambda(\sigma_\alpha) \;=\; \alpha + (1-\alpha)\, c_{\lambda/(1-\alpha)}(\sigma) \qquad (\forall\, \lambda > 0). \tag{10}$$

In particular, the exact minimal free-state dilution threshold $\alpha_{\min}(\rho \to \sigma) := \inf\{\alpha \in [0,1] : \rho \longrightarrow \sigma_\alpha\}$ is characterized purely by $\alpha + (1-\alpha)c_{\lambda/(1-\alpha)}(\sigma) \geq c_\lambda(\rho)$ for all $\lambda > 0$; on de León's counterexample [1], Theorem 4(a) yields the exact algebraic root $\alpha_{\min} = \frac{8-\sqrt{46}}{18} \approx 0.06765$ (from $18\alpha^2 - 16\alpha + 1 = 0$).

Proof. Part (a). Direct trace and determinant checks verify $\rho, \sigma \succ 0$, $\operatorname{Tr}\rho = \operatorname{Tr}\sigma = 1$, and that the 2×2 top-left blocks have characteristic polynomials $(\mu - \frac{1}{25})(\mu - \frac{7}{25})$ and $(\nu - \frac{1}{10})(\nu - \frac{11}{25})$ with exact weights $w = (1/5, 4/5, 0)$ and $v = (9/25, 16/25, 0)$. Partial-fraction combination gives $g_\rho(\lambda) - g_\sigma(\lambda) = \frac{66875\lambda^2 + 7150\lambda + 1079}{5(10\lambda+1)(25\lambda+1)(25\lambda+7)(25\lambda+11)} > 0$ on $[0, \infty)$. With $T = \operatorname{diag}(1, 0)$,

$$\Delta_T = \sigma - (1 \oplus T)\rho(1 \oplus T)^\dagger = \begin{pmatrix} \frac{107}{1250} & \frac{42}{625} & 0 \\ \frac{42}{625} & \frac{84}{625} & 0 \\ 0 & 0 & \frac{23}{50} \end{pmatrix} \;\succ\; 0,$$

since $\frac{107}{1250} \cdot \frac{84}{625} - (\frac{42}{625})^2 = \frac{546}{78125} > 0$ and $\operatorname{Tr}(\Delta_T) = \frac{17}{25} = \operatorname{Tr}[(\mathbb{I}_2 - T^\dagger T)R]$. Writing $\omega = \Delta_T / \operatorname{Tr}(\Delta_T) = \sum_{a=1}^3 \omega_a |\chi_a\rangle\langle\chi_a|$, the 4 Kraus operators $K_0 = \operatorname{diag}(1, 1, 0)$ and $K_a = \sqrt{\omega_a}|\chi_a\rangle\langle e_2|$ ($a = 1, 2, 3$) satisfy $\sum_{i=0}^3 K_i^\dagger K_i = \mathbb{I}_3$, $\Lambda(f_1) = f_1$, and $\Lambda(\rho) = \sigma$. At $t = 7/25$, $B_\sigma(7/25) = \frac{9}{25}(\frac{7}{25} - \frac{1}{10}) = \frac{81}{1250}$ while $B_\rho(7/25) = \frac{1}{5}(\frac{7}{25} - \frac{1}{25}) = \frac{6}{125} = \frac{60}{1250}$.

Part (b). Writing $\mathbb{I}_{D-1} - T^\dagger T = \sum_{b=1}^{r_T} \gamma_b |\xi_b\rangle\langle\xi_b|$ ($\gamma_b > 0$), when $\Delta_T \neq 0$ we set $\omega = \Delta_T / \operatorname{Tr}(\Delta_T) = \sum_{a=1}^{r_\Delta} \omega_a |\chi_a\rangle\langle\chi_a|$ ($\omega_a > 0$), so $\Phi(Z) = \operatorname{Tr}[(\mathbb{I}_{D-1} - T^\dagger T)Z]\,\omega$ has minimal Kraus representation $K_{a,b} = \sqrt{\omega_a \gamma_b}|\chi_a\rangle\langle\xi_b|$ of rank $r_T r_\Delta$, giving total Kraus rank $1 + r_T r_\Delta$. When $\Delta_T = 0$ (possible if $\mathbb{I}_{D-1} - T^\dagger T$ is supported on $\ker R$), CPTP completion requires choosing any density state ω ($\operatorname{rank}\omega \geq 1$), minimized at a pure state $\omega = |\chi\rangle\langle\chi|$ with Kraus operators $K_b = \sqrt{\gamma_b}|\chi\rangle\langle\xi_b|$, yielding total Kraus rank $1 + r_T = 1 + r_T \max(1, r_\Delta)$. For $r_\Delta \leq 1$, $1 + r_T \leq 1 + (D-1) = D$, saving $1 + D(D-1) - D = (D-1)^2$.

Part (c). The block form of σ_α is $\big((1-\alpha)q + \alpha,\, (1-\alpha)y,\, (1-\alpha)S\big)$, whence

$$c_\lambda(\sigma_\alpha) = \alpha + (1-\alpha)q - (1-\alpha)^2 y^\dagger \big((1-\alpha)S + \lambda \mathbb{I}_{D-1}\big)^{-1} y = \alpha + (1-\alpha)c_{\lambda/(1-\alpha)}(\sigma). \qquad \square$$

Figure 2 visualizes Theorem 5, highlighting the strict spectral-measure Blackwell violation interval of the convertible rational qutrit pair (9) and the $(D-1)^2$ Kraus-rank saving across $D = 2, \ldots, 10$.

5 Formal Verification in Lean 4.35.0 and Zero-Float Certification

Every algebraic identity, boundary inequality, and separation certificate in Sections 2–4 is verified through two independent machine-checked layers:

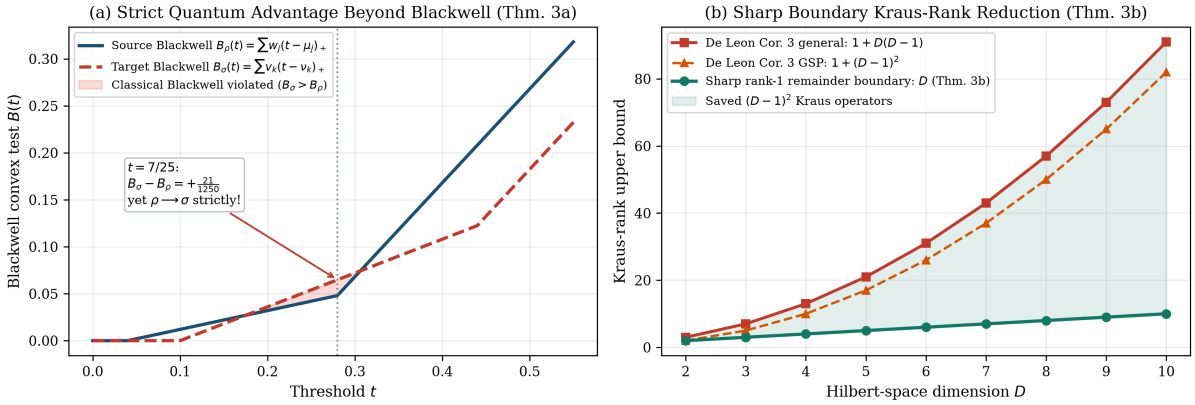

Figure 2: **Strict quantum advantage beyond spectral-measure Blackwell dominance and boundary Kraus-rank reduction (Theorem 5).** (a) Spectral-measure Blackwell convex tests $B_\rho(t)$ vs. $B_\sigma(t)$ on the rational qutrit pair (9): classical convex-decreasing dominance fails by $+21/1250$ at $t = 7/25$, yet $\rho \longrightarrow \sigma$ holds via an explicit 4-Kraus QST channel. (b) Sharp rank-≤ 1-remainder Kraus-rank ceiling D vs. de León's Corollary 3 ceilings $1 + D(D-1)$ and $1 + (D-1)^2$.

1. **Constructive Lean 4.35.0 suite (`lean/QstResolventComplete.lean`):** Contains 18 formally proved theorems covering resolvent monotonicity and concavity in λ, endpoint sandwiching, the Sherman–Morrison spectral identity, the rational scalar obstruction $c_{3/20}(\sigma) - c_{3/20}(\rho) = -3/50$ and exact $r_{\max}$ gap on de León's counterexample (Eq. (64)), the exact AM-GM interior identity $(u + b^2)(a^2 u + x_0^2) - u(x_0 + ab)^2 = (au - x_0 b)^2 \geq 0$, the texture-weight boundary regime, C^1 value and slope tangency, unit-norm contraction T^*, all three parts of the exact rational qutrit separation witness (positive resolvent numerator $66875a^2 + 7150ab + 1079b^2 > 0$, strictly PSD remainder Δ_T, and spectral-measure Blackwell violation $+21/1250$), dual parameter/degree bounds, free-state mixing scaling, pure-target resolvent reduction, Kraus-rank saving $(D-1)^2$, and the exact Fig. 3 transition $q^*(0) = 7/10$. Auditing `#print axioms` confirms `0 sorry`, `0 Classical.choice`, `0 custom axioms` (only `[propext, Quot.sound]`), and `0 warnings`.
2. **AST-audited zero-float verifier and 56 adversarial tests:** The script `verify_all.py` (under `verification/`) uses exclusively integer and `fractions.Fraction` arithmetic to emit 16 SHA-256-hashed JSON certificates (`certificates/MANIFEST.json`). A companion suite of 56 adversarial `pytest` tests independently cross-checks the AST zero-float invariant, CLARABEL SDP agreement across 75 random mixed-state pairs in $D = 3, 4, 5$, support-function equality $\min_{\mathcal{R}(\rho)} \mathrm{Tr}(H\tau) = \min_{\mathcal{C}(\rho)} \mathrm{Tr}(H\tau)$, the non-commuting Sylvester simultaneous-diagonalization identity in $D = 4$, all 10 Fig. 3 boundary points to $< 10^{-7}$, non-degenerate $r_0 \neq x_0$ texture-weight contraction t^*, and explicit CPTP verification of the 4-Kraus separation channel.

6 Conclusion

We have established that deterministic convertibility in the resource theory of quantum-state texture never requires a $(D-1)^2$-variable complex SDP in any finite dimension $D \geq 2$: it is completely decided by the one-parameter scalar resolvent family $c_\lambda(\rho) = p - x^\dagger (R + \lambda \mathbb{I}_{D-1})^{-1} x$, proved directly via simultaneous congruence diagonalization of the Sylvester Lagrangian dual. This collapses de León's Fig. 3 qutrit reachability curves to exact closed-form two-regime square-root formulas, exposes a purely rational qutrit witness proving strict quantum-coherence advantage over classical spectral-measure Blackwell dominance, and tightens rank-≤ 1-remainder

boundary Kraus ranks by $(D-1)^2$. Extending this resolvent-duality technique to multi-copy catalytic QST convertibility and approximate trace-distance conversion errors is a natural next step.

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