Stationary concentration and local return in Boolean Doob chains

Lluis Eriksson

Supplement to *Rank defects and spectral stability of Boolean Doob chains*

Abstract

For half-correlated Boolean noise, we study how a reversible change of transition probabilities affects the ordered spectrum when the stationary coordinate marginals are exactly uniform through increasing order. At the stationary moment endpoint we prove recovery under an exponential cutoff on the critical density tail. For every stationary moment order strictly above three and a fixed positive density floor, we identify a set of vertices on which local one-step return characterizes full spectral recovery. Its killed eigenvalue also determines the second eigenvalue up to exponential accuracy. Interpolating stationary densities in an existing shell construction gives chains with any prescribed finite number of correct leading noise bands and a persistent error lower in the spectrum. Without coordinate matching, the endpoint relaxation time is at most $\exp(C\sqrt{L\log(2L)})$ in dimension L. Above the endpoint the spectral gap has a dimension-independent positive lower bound. For a two-valued stationary density, a bounded cubic moment gives a dimension-independent gap. The entire ordered spectrum varies from ordinary noise by at most $C_b\|\pi-1\|_3$ without matching, and a two-coordinate example attains linear order. We extend both estimates to finitely many stationary levels whose uniform masses have comparable logarithms, with an explicit sufficient ratio bound. The gap also holds with variable heights under separate cubic envelopes on the exceptional sets. With unrestricted ratios, three exact stationary values suffice for exponential trapping even as the density tends to one in a fixed norm of order greater than three. Under linear-order marginal matching the trap persists at exponentially small distance from a three-valued law, whereas vanishing log-uniform distance from a two-valued law rules out an exponential trapping rate under the same floor and cubic cap. For stationary laws on binary linear quotients, fractional packing of coordinate bases gives a separate sufficient condition for exponential recovery. Three full bases suffice at the second moment and two suffice at any stationary moment order above 15/7. The stationary minimum may tend to zero.

1 Concentration required for failure at the stationary endpoint

Let μ be uniform on the binary L-cube. Write $P = P_{1/2}$, $Q = P_{1/\sqrt{2}}$, $g = Ph$, $\pi = hg$, and $U = h/g$, with $h > 0$ and $E_\mu\pi = 1$. Let $K(x,y) = P(x,y)h(y)/g(x)$ and let Δ denote the maximum absolute difference of the ordered eigenvalues of K and P, including multiplicities. Assume $\pi \geq b > 0$, $\|\pi\|_p \leq B$, and exact uniformity of all stationary coordinate marginals through order $k \to \infty$. Throughout this section

$$p = (17 + 4\sqrt{13})/9, \qquad q = 6p/(p-3), \qquad 1/3 = 1/p + 2/q.$$

All norms use μ. All constants b, B are uniform along the sequence.

Endpoint concentration criterion. Fix rates, independently of dimension, satisfying

$$0 < \tau < \frac{6\log 2}{q-12}, \qquad 0 < \sigma < \frac{6\log 2 + 12\tau}{p}.$$

Define

$$A_k = \{g < e^{-\tau k},\ \pi > e^{\sigma k}\}, \qquad d_* = \limsup_{k\to\infty} \|\pi \mathbf{1}_{A_k}\|_p.$$

If $d_* < b$, then

$$\limsup_{k\to\infty} \Delta \ \le\ \frac{B^2 d_*}{(b-d_*)^2}. \tag{EC1}$$

In particular, vanishing of the critical stationary moment on A_k suffices for full spectral recovery at any diverging matching order. A condition involving the stationary density alone is

$$E_\mu[\pi^p \mathbf{1}_{\{\pi > e^{\sigma k}\}}] \longrightarrow 0. \tag{EC2}$$

Uniform integrability of π^p implies (EC2). The converse fails even among exactly matched chains with a fixed stationary floor.

Since τ may approach its upper bound, (EC2) suffices for any fixed $\sigma < 6\log 2/(6-p)$ with $\sigma > 0$. Indeed $6q/[p(q-12)] = 6/(6-p)$. In particular it is enough that $\limsup k^{-1}\log_2 \|\pi\|_\infty < 6/(6-p)$. For example, when π is constant on cosets of a linear map of rank s, each atom of the quotient has probability 2^{-s} and $\|\pi\|_\infty \le B 2^{s/p}$. Thus $\limsup s/k < 6p/(6-p)$ suffices, regardless of ambient dimension or the ranks of disjoint column families.

Critical exponents without iteration. Put $1/u = 1/p + 1/q$ and $v = 1 + 4(u-1)$. Substitution of $9p^2 - 34p + 9 = 0$ gives $q = 4(1+v) - 1$. Forward hypercontractivity for P sends L^u to L^v contractively. Reverse hypercontractivity with source exponent $-v$ and target exponent $-q$ gives $\|1/g\|_q \le \|1/h\|_v$: the ratio $(1+v)/(1+q)$ is $1/4$. The floor implies $1/h = g/\pi \le g/b$. Consequently,

$$H := \|1/g\|_q \le b^{-1}\|g\|_v \le b^{-1}\|h\|_u. \tag{EC3}$$

These inequalities hold in each finite dimension. Simply replacing $\|h\|_u$ by BH would give no uniform bound.

What matching controls. Let $a = Eh$, $F = Qh$ and $V = \|F - a\|_2^2 = 1 - a^2$. By Cauchy–Schwarz under $h\,d\mu$, $1 \le aE\pi g$. The high-degree part of $g = QF$ above order k has norm at most $2^{-(k+1)/2}\sqrt{V}$ in L^2. Matching annihilates the lower nonconstant part against $\pi - 1$. Since $p > 2$, it follows that $V \le a(B+1)2^{-(k+1)/2}\sqrt{V}$. This includes $V = 0$ and yields $\|g-1\|_2 \le C_B 2^{-k/2}$ using $1 - a \le V$. In fact $\|g-1\|_3 \le C_B 2^{-k/2}$ because $g - a = Q(F-a)$ and Q is contractive from L^2 to L^3.

A low denominator forces high denominators among its neighbors. The floor gives $g = P(\pi/g) \ge bP(1/g)$. For $T \ge T_0 = 4/b$, put $S_T = \{g > bT/2\}$. If $g(x)^{-1} > T$, then

$$P_x(S_T^c) \le (bT/2)P(1/g)(x) < 1/2.$$

The cubic estimate for $g - 1$ gives $\mu(S_T) \le C_{B,b} 2^{-3k/2} T^{-3}$. For any $1 < r < 4$, choose $m = 3/(4-r) > 1$ and $\ell = 4m - 3$. Forward hypercontractivity of P from L^m to L^ℓ and Markov's inequality give

$$\mu\{g^{-1} > T\} \le 2^\ell \|P\mathbf{1}_{S_T}\|_\ell^\ell \le 2^\ell \mu(S_T)^r \le C_{B,b,r}\, 2^{-3rk/2} T^{-3r}. \tag{EC4a}$$

This additional use of the floor controls the geometry of the exceptional set before applying (EC3).

Write $D = \{g < e^{-\tau k}\}$. Choose $r < 4$ close enough to four that both $(q-3r)\tau < (3r/2)\log 2$ and $p\sigma < (3r/2)\log 2 + 3r\tau$. This is possible by the two strict rate conditions and $q > 12$. Integrating (EC4a) up to $T = e^{\tau k}$ bounds the integral of g^{-q} on $\{T_0 < 1/g \le e^{\tau k}\}$ by $C2^{-3rk/2}(1+e^{(q-3r)\tau k}) = o(1)$. On $\{1/g \le T_0\}$ the reciprocal is bounded and converges to one in probability. Consequently

$$\|g^{-1}\mathbf{1}_{D^c}\|_q \to 1, \qquad \|(g^{-2}-1)\mathbf{1}_{D^c}\|_{q/2} \to 0. \tag{EC4}$$

On $D \setminus A_k$, $\|\pi\mathbf{1}_{D\setminus A_k}\|_p \le e^{\sigma k}\mu(D)^{1/p} = o(1)$ by (EC4a) at $T = e^{\tau k}$. Hence $d_* = \limsup \|\pi\mathbf{1}_D\|_p$.

Absorption on the exceptional set. Split $h = \pi/g$ over D^c and D in (EC3). Hölder gives

$$bH \le B\|g^{-1}\mathbf{1}_{D^c}\|_q + \|\pi\mathbf{1}_D\|_p H.$$

If $d_* < b$, then $\limsup H \le B/(b-d_*)$ by (EC4). A second application of Hölder gives $\|U\mathbf{1}_D\|_3 \le \|\pi\mathbf{1}_D\|_p H^2$. Also $\|\pi\mathbf{1}_D\|_3 \le B\mu(D)^{1/3-1/p} \to 0$ and, by (EC4), $\|(U-\pi)\mathbf{1}_{D^c}\|_3 \to 0$. This proves the bound on the right of (EC1) for $\limsup \|U-\pi\|_3$.

For completeness, $\|QM_{U-\pi}Q\| \le \|U-\pi\|_3$ by forward 2-to-3 hypercontractivity. For $M_{\pi-1}$ split each factor of Q at Walsh degree $j = \lfloor k/2 \rfloor$. Matching kills the low-low term. Writing $p' = p/(p-1)$, the remaining high factors have 2-to-$2p'$ norm at most θ^{j+1} where $\theta = \sqrt{(2p'-1)/2} < 1$: factor $Q = P_{1/\sqrt{2p'-1}}P_\theta$ and apply the Walsh tail bound to P_θ. The same factorization bounds the low factors by one. The three terms therefore contribute at most $3(B+1)\theta^{j+1}$. Finally QM_UQ and the symmetric conjugate of K are the two Gram products of $M_{\sqrt{U}}Q$. Min–max gives (EC1).

Necessary joint concentration. If $\Delta \ge \eta > 0$ along a subsequence, then for *every* fixed admissible pair τ, σ along that subsequence,

$$\liminf \|\pi\mathbf{1}_{A_k}\|_p \ge t_\eta, \qquad B^2 t_\eta = \eta(b-t_\eta)^2, \quad 0 < t_\eta < b.$$

Indeed the right-hand side of (EC1) is strictly increasing from zero to infinity on $[0, b)$. A further subsequence below t_η would contradict (EC1). This localizes both the large stationary density and the small denominator to the same vertices. The bound is on the integral of the pth power; their stationary probability can tend to zero.

A separating example. Take $n = \lfloor \sqrt{k} \rfloor$ disjoint blocks of $k+1$ bits. Let S be the event that every block parity is positive, $m = 2^{-n}$, and prescribe

$$\pi = 1 + m^{-1/p}(\mathbf{1}_S - m).$$

This density has mean one and a fixed positive floor for all sufficiently large k. Its nonconstant Walsh terms have degree at least $k+1$ by the block-parity expansion. It is realizable as hPh by positive symmetric diagonal scaling of the strictly positive matrix P with row sums $\pi(x)$. Here $\log\|\pi\|_\infty = O(\sqrt{k})$, so (EC2) holds. Yet $E[\pi^p\mathbf{1}_S] \to 1$ while $\mu(S) \to 0$, and $E[\pi^{p+\epsilon}\mathbf{1}_S] \to \infty$ for every fixed $\epsilon > 0$. The complement contributes a bounded critical moment. This example separates the hypotheses on stationary integrability; the theorem allows general stationary laws beyond this particular block geometry.

Relation to earlier work and review context. The forward and reverse inequalities are classical inputs (Bonami; Mossel, Oleszkiewicz and Sen). The realization in the example uses Sinkhorn's prescribed-row-sum theorem. The programme's stationary-moment paper and its subsequent matching-order estimate supply the setting and the matching-energy argument. The new step is absorption at the exact fixed exponent using the critical mass on the common exceptional set. It gives a conditional endpoint theorem. Whether the original floor, moment bound and matching alone force recovery is open. This source appendix has an author-side analytic check. Exact-PDF assessment, independent review and publication remain pending.

2 A local return test for the full ordered spectrum

Keep the half-noise model, the uniform matching through order k, the stationary floor b and the moment bound B of the preceding section. In this section any fixed finite $p > 3$ is allowed. Put $q = 6p/(p-3)$, choose $0 < \sigma < \tau < (\log 2)/q$, and set

$$D = \{g < e^{-\tau k}\}, \qquad A = D \cap \{\pi > e^{\sigma k}\}.$$

Write $\nu = \pi\mu$, $s = \|\mathbf{1}_A K \mathbf{1}_A\|_{2\to 2,\nu}$ and

$$R_A = \max_{\varnothing \neq E \subseteq A} \Pr_\nu(X_1 \in E \mid X_0 \in E).$$

Both quantities are zero for empty A. There exist $c, C > 0$, depending only on the displayed fixed parameters, for which

$$s/4 - Ce^{-ck} \le \Delta \le s + Ce^{-ck}, \qquad |\lambda_2(K) - \max(1/2, s)| \le Ce^{-ck}, \qquad R_A \le s \le \sqrt{2R_A - R_A^2}. \tag{LR1}$$

In particular the full ordered spectrum converges precisely when every subset of this specified exceptional set has vanishing one-step return, uniformly over those subsets. The second eigenvalue can converge to $1/2$ even when the criterion for the full spectrum fails; the formula for λ_2 by itself only detects s above $1/2$. The two-sided bound on Δ does not require knowing which noise eigenvalue is displaced.

There is a more detailed reconstruction with multiplicities. Form a list from the ordinary cube eigenvalues (each 2^{-j} occurs $\binom{L}{j}$ times) and all eigenvalues of the killed matrix K_{AA} on A. Sort in decreasing order and keep the first 2^L entries. Each entry differs from the corresponding eigenvalue of K by at most Ce^{-ck}. The additional matrix has order $|A| \le C_B 2^{L-k}$, with an empty matrix allowed.

Isolating the multiplier that survives matching. The matching-energy proof above gives $\|g - 1\|_2 \le C_B 2^{-k/2}$ also for this p. The floor and Jensen give $1/g \le Pg/b$, and $Pg = P_{(1/2)^{3/2}} Qh$ with $\|Qh\|_2 = 1$. Forward hypercontractivity therefore gives $\|1/g\|_9 \le 1/b$. On $D^c \cap \{g < 1/2\}$, $\mu\{g < 1/2\} \le C_B 2^{-k}$ and $g^{-2} \le e^{2\tau k}$. Since $q\tau < \log 2$, the $L^{q/2}$ norm of $(g^{-2} - 1)$ on that part decays exponentially. On $\{g \ge 1/2\}$ the function $g^{-2} - 1$ is bounded, and its absolute value is at most $16|g - 1|$; interpolation with the L^2 estimate gives exponential decay in $L^{q/2}$ there too. Thus $\|(U - \pi)\mathbf{1}_{D^c}\|_3 \le Ce^{-ck}$ by $1/3 = 1/p + 2/q$.

On the remaining low-denominator vertices outside A, use the ninth inverse moment directly:

$$\|U\mathbf{1}_{D\setminus A}\|_3^3 \le e^{3\sigma k} \int_D g^{-6}\, d\mu \le b^{-9} e^{-3(\tau-\sigma)k}.$$

Also $\|\pi \mathbf{1}_D\|_3 \le B\mu(D)^{1/3-1/p} \le Ce^{-ck}$. Together these bounds give $\|U\mathbf{1}_{A^c} - \pi\|_3 \le Ce^{-ck}$. The signed low/high Walsh estimate for $\pi - 1$ above uses only $p > 3$. By 2-to-3 hypercontractivity,

$$\|QM_UQ - (P+S)\| \le Ce^{-ck}, \qquad S = QM_{U\mathbf{1}_A}Q. \tag{LR2}$$

The two Gram products of $M_{\sqrt{U}\mathbf{1}_A}Q$ identify $\|S\| = s$. In particular S is positive semidefinite with $s \le 1$.

Separation from the ordinary low eigenmodes. Let Π project onto Walsh degrees at most $m = \lfloor \delta k \rfloor$, where $0 < \delta < 1$ is fixed with $\delta \log 7 < (\log 2)/12$. The floor bound above gives $\|U\|_{3/2} \le B/b^2$ since $1/p + 2/9 < 2/3$. For a unit vector z in this low space, $\|Qz\|_8 \le 7^{m/2}$: apply the ordinary 2-to-8 inequality to a polynomial with each degree j multiplied by $7^{j/2}$. Hölder with reciprocal exponents $2/3, 1/4, 1/12$ gives

$$\langle z, Sz\rangle \le (B/b^2)\, 7^m\, \mu(A)^{1/12} \le Ce^{-ck}.$$

Positivity bounds the norm of the off-diagonal block of S by $\|\Pi S \Pi\|^{1/2}\|S\|^{1/2}$. On the complementary space P has norm at most $2^{-(m+1)}$. Consequently QM_UQ differs in operator norm by at most Ce^{-ck} from the block diagonal operator

$$P|_{\operatorname{ran}\Pi} \;\oplus\; S_H, \qquad S_H = (I-\Pi)S(I-\Pi)|_{\operatorname{ran}(I-\Pi)}.$$

Its high-block norm s_H differs from s by at most Ce^{-ck} as well. For large k, the low block contains 1 once and $1/2$ with its ordinary multiplicity. The high block is positive with top eigenvalue $s_H \le 1$. This proves the second-eigenvalue estimate by min–max and Gram conjugation. The upper bound on Δ follows directly from (LR2).

To verify the reconstruction, work on two copies of the cube space and compare $P \oplus S$ to $(\Pi P \Pi) \oplus ((I-\Pi)S(I-\Pi))$. The norm error is exponentially small by the same two block estimates. After permuting subspaces the latter is the displayed block diagonal operator with 2^L extra zeros. Also S has the nonzero eigenvalues of K_{AA} with their multiplicities, by the Gram identity. Thus min–max on the doubled space identifies the first 2^L sorted entries with the claimed precision. The size bound follows from $\mu(A) \le C_B 2^{-k}$.

For the lower bound, if $s_H \le 4\,2^{-m}$ it is absorbed by the error. Otherwise choose j with $s_H/4 < 2^{-j} \le s_H/2$, so $1 \le j \le m$. In the ordered noise spectrum, let i be the first index at level 2^{-j}. The low block already supplies the $i-1$ eigenvalues strictly above this level, all at least 2^{1-j}. The high block supplies one more at least 2^{1-j}. Its addition raises the ith eigenvalue by at least 2^{-j}. This proves the first bound despite the growing noise multiplicities. Enlarging C covers the finitely many smaller matching orders.

A test phrased entirely in one-step probabilities. Testing with an indicator gives $R_A \le s$. For the reverse bound take a nonnegative unit top vector f for the killed chain, extended by zero outside A. Under its full stationary pair law J, integration over the level sets of f^2 gives

$$\tfrac{1}{2}E_J|f(X)^2 - f(Y)^2| \ge (1-R_A)\|f\|_{2,\nu}^2.$$

Indeed every nonempty positive level set lies in A, and its outward flow is at least $1-R_A$ times its stationary mass. Cauchy–Schwarz on $(f(X)-f(Y))(f(X)+f(Y))$ bounds the left side by $\sqrt{1-s^2}$ using stationarity and reversibility. This proves the last part of (LR1).

What nearly complete trapping requires in two ordinary noise steps. For $x \in A$, the floor bound $P(1/g)(x) \le g(x)/b$ implies $P_x(A) \le e^{-2\tau k}/b$. Yet the stationary conditional return under K may be much larger. For nonempty $E \subseteq A$, Cauchy–Schwarz under P_x and the floor give

$$K_x(E^c) \ge \frac{b}{g(x)} P_x(\mathbf{1}_{E^c}/g) \ge \frac{b\, P_x(E^c)^2}{g(x)\, Pg(x)}.$$

If $\lambda_2(K) \to 1$, (LR1) supplies maximizing sets E_k with conditional return tending to one. Averaging this last bound for $x \in E_k$ gives $E_{\nu(\cdot|E_k)}[(g\,Pg)^{-1}] \to 0$; hence $g\,Pg$ diverges in probability under those conditional laws. Conversely a uniform upper bound $g\,Pg \le M$ on A gives $s \le 1 - b(1 - o(1))/M$ by the maximum killed row sum, for nonempty A. This rules out convergence of the second eigenvalue to one in that regime.

A family with a correct spectral gap and incorrect lower bands. At half noise the programme's matched shell construction provides some fixed $p_0 > 3$, a fixed floor, $k = \lfloor \beta L \rfloor$, $\|\pi_{\mathrm{trap}} - 1\|_{p_0} \le e^{-c_0 L}$, and $\lambda_2 \to 1$. Fix any $0 < a_0 < 1$. Interpolate the stationary densities by $\pi_t = 1 + t(\pi_{\mathrm{trap}} - 1)$, $0 \le t \le 1$. They share all the matching, floor and moment estimates with uniform constants. At each fixed dimension positive symmetric diagonal scaling gives the unique $h_t > 0$ satisfying $h_t P h_t = \pi_t$ (here Ph_t is evaluated first). This scaling is continuous in t: each $h_t(x)$ is bounded above by $\sqrt{\sup_t \pi_t(x)/P(x,x)}$, and the floor divided by the largest such upper bound bounds every coordinate below. Any subsequential limit solves the same scaling equations, so uniqueness gives continuity.

Set $N(t) = \|Q M_{h_t/(Ph_t)} Q - P\|$. This continuous quantity is zero at $t = 0$. By (LR2), $|N(t) - s(t)| \le Ce^{-ck}$ uniformly in t, even though the exceptional vertex set can change discontinuously. By (LR1), $s(1) \to 1$. The intermediate value theorem gives t_L with $N(t_L) = a_0$ for every sufficiently large L. For these chains

$$|\lambda_2(K) - \max(1/2, a_0)| \le Ce^{-ck}, \qquad \Delta \ge a_0/4 - Ce^{-ck}.$$

Moreover for every fixed j with $2^{-j} > a_0$, all the ordered eigenvalues in degrees 0 through j, counted with the reference multiplicities, are within Ce^{-ck} of their reference values. Indeed the low block contains all those eigenvalues exactly, and the entire high block is below 2^{-j} for large L. In particular any prescribed finite number of leading Walsh bands can be asymptotically correct while the full ordered spectrum has a fixed error. The available p_0 comes from the matched shell result below the stationary endpoint. This interpolation constructs no counterexample at the endpoint itself.

Scope and provenance. The coarea estimate is the Dirichlet form argument underlying Goel, Montenegro and Tetali's spectral profile. The low-degree norm bound uses classical Boolean hypercontractivity. The contribution here is the exponential operator approximation on a specific joint tail and the eigenvalue count that converts a killed mode there into a defect of the full ordered spectrum. The unrestricted endpoint still requires controlling returns for all subsets of that tail. This is an analytic appendix with no exact-PDF assessment.

3 Global gap bounds without coordinate matching

Retain $P = P_{1/2}$, $g = Ph$, $\pi = hg$, $E_\mu \pi = 1$, $\pi \ge b > 0$ and $\|\pi\|_p \le B$. In this section there is no assumption on any nonconstant Walsh coefficient. Write $p_* = (17 + 4\sqrt{13})/9$ and $\gamma = 1 - \lambda_2(K)$, for $L \ge 1$.

Theorem. At $p = p_*$,

$$\gamma \;\geq\; \exp[-C_{b,B}\sqrt{L\log(2L)}]. \tag{G1}$$

In particular a gap of order $\exp(-cL)$ for fixed $c > 0$ is impossible at this exponent. For every fixed $p > p_*$ there is a dimension-independent lower bound $\gamma \geq c_{b,B,p} > 0$. There are also $\epsilon_0, C > 0$ depending only on b, B such that, uniformly for $p_* - \epsilon_0 \leq p \leq p_*$,

$$\log(1/\gamma) \leq C[(p_* - p)L + \sqrt{L\log(2L)}]. \tag{G2}$$

The estimate controls cuts containing any number of distinct levels of h. At equality it leaves open both a dimension-independent gap and recovery of the ordered spectrum under arbitrarily slow matching.

Macroscopic cuts. Let J be the stationary edge probability and $\nu = \pi\mu$. The two-function reverse hypercontractive inequality at correlation $1/2$ with both input exponents $1/2$ gives

$$J(E, E^c) = \langle h\mathbf{1}_E, P(h\mathbf{1}_{E^c})\rangle_\mu \geq \|h\mathbf{1}_E\|_{1/2}\|h\mathbf{1}_{E^c}\|_{1/2}.$$

Here $\|f\|_{1/2} = (E_\mu\sqrt{|f|})^2$. For each nonempty set F, $h \geq b/g$ and conditional Jensen imply

$$\|h\mathbf{1}_F\|_{1/2} \geq b\mu(F)^2(E_\mu[g^{-1/2} \mid F])^2 \geq b\mu(F)^2/E_\mu[g \mid F] \geq b\mu(F)^{7/3}.$$

The last step uses $\|g\|_3 = \|Q(Qh)\|_3 \leq \|Qh\|_2 = 1$. If $0 < \nu(E) \leq 1/2$, then $\mu(E^c) \geq (2B)^{-p/(p-1)}$ by Hölder, and $\nu(E) \leq B\mu(E)^{1-1/p}$. Consequently

$$\Phi(E) := J(E, E^c)/\nu(E) \geq c_{b,B,p}\,\mu(E)^{4/3+1/p}. \tag{G3}$$

The constant stays positive uniformly on compact exponent intervals above one. If $\|K_{FF}\| \leq 1/2$ for all $\mu(F) \leq e^{-s}$, all such small cuts have $\Phi(F) \geq 1/2$ by the indicator test. The remaining cuts satisfy (G3). Cheeger's inequality $\gamma \geq \frac{1}{2}\inf_{0<\nu(E)\leq 1/2}\Phi(E)^2$ now yields $\log(1/\gamma) \leq C_{b,B,p}(1+s)$. This step tests actual stationary returns, without postulating where the returning vertices or their ordinary-noise neighbors lie.

Moment budget for the small cuts. Writing $U = h/g$, the floor and Jensen give $1/g \leq Pg/b$ and $\|1/g\|_9 \leq 1/b$. Given a bound H_j on $\|1/g\|_{q_j}$, put $1/u_j = 1/p + 1/q_j$. Hölder bounds $\|h\|_{u_j} \leq BH_j$. Forward hypercontractivity bounds $\|g\|_{4u_j-3} \leq BH_j$; $1/h \leq g/b$ and reverse hypercontractivity then give

$$q_0 = 9, \qquad q_{j+1} = F_p(q_j) = 16pq_j/(p+q_j) - 9, \qquad H_{j+1} \leq (B/b)H_j.$$

All forward input exponents exceed one in a fixed neighborhood of p_*. The larger root $q_\infty(p)$ of $q^2 + (9-15p)q + 9p = 0$ is the limit of this increasing iteration there: $F_p(9) > 9$, F_p is increasing, and $F_p(q) > q$ between its two fixed points. Let $1/t_\infty(p) = 1/p + 2/q_\infty(p)$. Substitution gives $t_\infty(p_*) = 3$, and $F'_{p_*}(q_\infty(p_*)) = 16((p_*-3)/(p_*+3))^2 < 1$ (the prime denotes derivative in q). After a fixed number of iterations the distances to the limiting root contract by a common factor less than one throughout a sufficiently small closed neighborhood of p_*. Indeed the root is smooth and the finite iterates are continuous; first choose a contracting neighborhood and an iterate entering its interior at p_*, then shrink the parameter interval. Thus $O(1+\log(1/\delta))$ iterations and $\|U\|_{(1/p+2/q_j)^{-1}} \leq BH_j^2$ yield

$$\|U\|_{t_\infty(p)-\delta} \leq C\delta^{-A} \quad (0 < \delta \leq \delta_0). \tag{G4}$$

All constants here are uniform on that interval. The simple attracting root has positive derivative in p because $\partial_p F_p(q) > 0$. It follows that t_∞ is smooth and increasing, with $0 \le 3 - t_\infty(p) \le C_0(p_* - p)$ below p_*.

A cut of prescribed rarity. We recall the sparse-input estimate derived from Polyanskiy and Samorodnitsky, Theorem 7 and Remark 3. For $1 \le s \le L/32$, $m = s/L$, and $\mu(\operatorname{supp} f) \le e^{-s}$ it gives

$$\|Qf\|_2 \le \|f\|_a, \qquad a = 3/2 - \kappa m$$

for an absolute $\kappa > 0$. To see the uniform choice of exponents, use time $(\log 2)/2$ and entropy parameter ηm for sufficiently small fixed $\eta > 0$. The support bound gives $\log(\|f\|_a/\|f\|_1) \ge (1 - 1/a)s$. Their exponent ODE has rate at least $2 + c\eta m$ since the increasing rate function has expansion $2 + x/3 + O(x^2)$ at zero. The output exponent is at least $1 + 2(a-1)\exp(c\eta m(\log 2)/2)$, which reaches two when κ is sufficiently small. Absolute values cover signed inputs.

Let $t = t_\infty(p) - dm$ and $1/a_t = 1/2 + 1/(2t)$. There are absolute constants (or constants uniform on the chosen interval) $d, c_1, c_2 > 0$ such that if $p \le p_*$ and $p_* - p \le c_1 m$, then $1/a - 1/a_t \ge c_2 m$. Indeed $1/a - 2/3 \ge (4\kappa/9)m$, whereas $1/a_t - 2/3 = (3-t)/(6t) \le (C_0(p_* - p) + dm)/(6t)$; shrink d, c_1 and keep $t > 2$. Also take $dm \le \delta_0$. For z of unit $L^2(\mu)$ norm and $\mu(F) \le e^{-s}$, two applications of Hölder now give

$$\|Q(\mathbf{1}_F \sqrt{U} z)\|_2^2 \le \mu(F)^{2(1/a - 1/a_t)} \|U\|_t \le C m^{-A} \exp(-2c_2 s^2/L).$$

Squaring and the Gram identity are accounted for in the displayed squared norm. Taking the supremum bounds the stationary killed norm $\|K_{FF}\|$.

Choose $s = D[(p_* - p)L + \sqrt{L \log(2L)}]$ with D large enough that $p_* - p \le c_1 s/L$ and that the exponential dominates $C(L/s)^A$ with product at most $1/2$ for all sufficiently large L. Then choose ϵ_0 small enough that $s \le L/32$ for all such large L, uniformly for $0 \le p_* - p \le \epsilon_0$. Equation (G3) proves (G2) and its endpoint specialization (G1). For bounded dimensions (G3) and the single-atom lower bound $\mu(E) \ge 2^{-L}$ cover the remaining cases.

For fixed $p > p_*$ decrease p if necessary to an exponent strictly between p_* and the upper end of the same neighborhood. Then (G4) gives $\|U\|_t \le M$ for a fixed $t > 3$. Ordinary hypercontractivity and the Gram identity give $\|K_{FF}\| \le \|U\mathbf{1}_F\|_3 \le M\mu(F)^{1/3 - 1/t}$. A fixed value of s thus suffices in (G3), proving the dimension-independent claim.

The constants on the upper side can also be tracked. For $\varepsilon = p - p_* > 0$ sufficiently small the smooth root has strictly positive derivative, so $t_\infty(p) \ge 3 + c_3\varepsilon$. Choose $t = 3 + c_3\varepsilon/2$ in (G4), which gives $\|U\|_t \le C\varepsilon^{-A}$. Since $1/3 - 1/t \ge c_4\varepsilon$, the preceding small-cut test holds at $s = D\varepsilon^{-1}\log(2/\varepsilon)$ for a fixed sufficiently large D. Independently, monotonicity of probability norms makes (G1) applicable at this same p. Hence

$$\log(1/\gamma) \le C \min\{\sqrt{L \log(2L)},\ \varepsilon^{-1}\log(2/\varepsilon)\}.$$

Relation to the matched examples. The published shell and its supercubic source extension demonstrate exponential trapping at lower stationary exponents. Their existence does not identify p_* as the sharp transition. The present bound controls *all* cuts at p_* without exact matching. The small-cut estimate uses the programme's endpoint multiplier argument. The large-cut estimate uses the classical reverse Boolean inequality. Their combination yields a global relaxation bound rather than a comparison limited to exceptional sets selected by matching. This result has been checked analytically in the drafting context.

4 A single stationary spike at the cubic moment

Keep half noise and put $Y = 1/g$. Suppose $\pi = a$ off a set E of uniform mass $m > 0$, and that

$$b \le a \le 1, \qquad a \le \pi \le M, \qquad M m^{1/3} \le H. \tag{S1}$$

The constants b, H are fixed across dimensions. The shape of E and the variation of π inside this envelope are arbitrary. Every two-valued stationary density with a positive floor and bounded cubic norm satisfies (S1), taking its higher level for E. The constant density gives ordinary noise by uniqueness of positive scaling.

Theorem. Under (S1) the spectral gap is at least $c(b, H) > 0$, without coordinate matching. For a two-valued stationary density there are two stronger conclusions. The ordered spectral error satisfies $\Delta \le C_b \|\pi - 1\|_3$ without matching, and its linear order is sharp. If $p > 3$ is fixed, $\|\pi\|_p \le B$, and the stationary marginals match through order k, then the ordered spectral error is at most Ce^{-ck}, with constants depending only on b, B, p.

Two-step averaging on the spike. The floor $\pi \ge a$ and Jensen give $Y \le Pg/a$. Since $Pg = P^2(\pi Y)$, writing $T = P^2$ gives the pointwise inequality $Y \le TY + (M/a)T(Y\mathbf{1}_E)$. The unweighted term has coefficient one. Iteration and positivity give, for $j = 0, 1, 2$,

$$P^j Y \le P^j T^n Y + (M/a) \sum_{i=1}^{n} P^j T^i (Y \mathbf{1}_E). \tag{S2}$$

As before $\|Y\|_9 \le 1/b$. Choose $n = 1 + \lceil \log_{16}(1 + \log(1/m)) \rceil$ and $v = 1 + 8 \cdot 16^n$. Forward hypercontractivity gives $\|P^j T^n Y\|_v \le 1/b$, so its L^8 norm on E is at most $m^{1/8 - 1/v}/b \le e m^{1/8}/b$. For $i \ge 1$, $P^j T^i$ contracts $L^{23/16}$ into L^8: two ordinary noise steps have correlation $1/4$. An input supported on E gains a factor m^d relative to its L^8 norm, with $d = 16/23 - 1/8 > 1/3$. For m sufficiently small in terms of b, H, $(M/a) n m^d \le (H/b) n m^{d - 1/3} \le 1/2$. In (S2) with $j = 0$ absorb this term, then apply the result for $j = 1, 2$. With $\mu_E = \mu(\cdot \mid E)$ this gives $\|P^j Y\|_{8, \mu_E} \le C_b$.

Now $g = P(\pi Y)$ and $Pg = T(\pi Y)$. The spike contributes at most CMm^d to $\|Pg\|_{8, \mu_E}$ by the same two-step bound. For $\|g\|_{3, \mu_E}$, use the $L^{3/2}$-to-L^3 contraction of P on $Y\mathbf{1}_E$ and its conditional L^8 bound; this contribution is at most $CMm^{1/3}$. The baseline contributions are controlled by (S2), using $\pi Y \le aY + MY\mathbf{1}_E$. We obtain

$$\|Y\|_{8, \mu_E} + \|g\|_{3, \mu_E} + \|Pg\|_{8, \mu_E} \le C_{b, H}. \tag{S3}$$

All subsets of the spike. For nonempty $F \subseteq E$ put $\theta = \nu(F)/(Mm)$ and $r(F) = J(F, F)/\nu(F)$. Then $E_{\mu_E}[\mathbf{1}_F \pi / M] = \theta$. Forward hypercontractivity of Q from $L^{3/2}$ to L^2 and Hölder in μ_E give

$$J(F, F) = \|Q(h\mathbf{1}_F)\|_2^2 \le M^2 m^{4/3} \|Y\|_{8, \mu_E}^2 \|\mathbf{1}_F \pi / M\|_{24/13, \mu_E}^2 \le CM^2 m^{4/3} \theta^{13/12}.$$

Thus $r(F) \le CH\theta^{1/12}$. If $r(F) \ge 1/2$, then $\theta \ge c > 0$. The density of $\nu(\cdot \mid F)$ relative to μ_E is at most $1/c$, so (S3) bounds its mean of $g\,Pg$. Also $E_{\mu_E} P\mathbf{1}_E \le m^{1/3}$ by the same forward inequality on the indicator. Hence $E_{\nu(\cdot|F)} P\mathbf{1}_{F^c} \ge 1/2$, reducing the fixed threshold on m if necessary. Pointwise Cauchy–Schwarz under P_x and the floor give $K_x(F^c) \ge b[P_x(F^c)]^2/(g(x)Pg(x))$. A second Cauchy–Schwarz under $\nu(\cdot \mid F)$ yields

$$1 - r(F) \ge b \, \frac{(E_{\nu(\cdot|F)} P\mathbf{1}_{F^c})^2}{E_{\nu(\cdot|F)}[g\,Pg]} \ge c_{b, H} > 0.$$

In the other branch the exit is already at least $1/2$. The coarea estimate in (LR1), whose killed-chain part uses only reversibility and positivity, gives $\|K_{EE}\| \le 1 - c'_{b,H}$.

The global gap. For any cut D, the positive Gram multipliers for $D \cap E$ and $D \setminus E$ add. On E^c, $U = aY^2$ has a bounded $L^{9/2}$ norm. Thus

$$\|K_{DD}\| \le \|K_{EE}\| + \|U\mathbf{1}_{D\setminus E}\|_3 \le 1 - c'_{b,H} + C_b\mu(D)^{1/9}.$$

All sufficiently small cuts have return bounded away from one, at a threshold independent of m, L. Larger cuts of stationary mass at most one half obey (G3) at $p = 3$, as $E_\mu \pi^3 \le 1 + H^3$. If m exceeds the smallness threshold used above, M is bounded in terms of b, H, and the $L^{9/2}$ argument applies on the whole cube. Cheeger's inequality proves the global gap in both cases.

Full recovery for two stationary values. Now $\pi = M$ on E. In the small-m regime (S3) gives $\|U\mathbf{1}_E\|_3 \le CMm^{1/3}$. Write $\delta = \|\pi - 1\|_3$. We prove the quantitative bound $\Delta \le C_b\delta$. As $\Delta \le 1$, it suffices to take δ small in terms of b. With $u = Eh$ and $V = 1 - u^2 = \|Qh - u\|_2^2$, Cauchy–Schwarz gives $1 = (Ehg)^2 \le uE(hg^2) = uE(\pi g)$. Subtracting u^2 gives $V \le u\delta\sqrt{V}$ since $\|g - u\|_2 \le \sqrt{V}$. Thus $\sqrt{V} \le \delta$ and $1 - u \le \delta^2$. Applying forward hypercontractivity to $Qh - u$ yields $\|g - u\|_3 \le \delta$ and $\|Pg - u\|_9 \le \delta$. On $\{g \ge b/2\}$ the inverse square is Lipschitz in g. On $A = \{g < b/2\}$, Jensen gives $Pg \ge b/g > 2$, so $Y \le 2(Pg - u)/b$ there and $\mu(A) \le \delta^9$. Consequently

$$\|Y^2 - 1\|_3 \le C_b(\delta + \delta^2 + \delta^3).$$

If $M \le 2$, multiplication by π now gives $\|U - \pi\|_3 \le C_b\delta$. If $M > 2$, then $Mm^{1/3} \le 2(M-1)m^{1/3} \le 2\delta$. Hence $m \le \delta^3$, and for sufficiently small δ we may use (S3) with $H = 2$. On E both $\|U\mathbf{1}_E\|_3$ and $\|\pi\mathbf{1}_E\|_3$ are $O_b(\delta)$, and off E we have $\pi = a \le 1$. In both cases $\|U - 1\|_3 \le C_b\delta$. The Gram comparison $\Delta \le \|QM_{U-1}Q\| \le \|U - 1\|_3$ finishes the upper bound.

The linear order is sharp. In dimension two let $\chi = \chi_1\chi_2$ be the parity character. For $0 < t < 1$ take $h = (1 + t\chi)/\sqrt{1 + t^2/4}$. Then $\pi = 1 + \epsilon\chi$, with $\epsilon = (5t/4)/(1 + t^2/4)$, and $U = (1+t\chi)/(1+t\chi/4)$. On the span of χ_1, χ_2, Q multiplies by $1/\sqrt{2}$ and M_U has eigenvector $\chi_1+\chi_2$ with eigenvalue $(1+t)/(1+t/4)$. Thus QM_UQ has a nonleading eigenvalue $(1+t)/(2(1+t/4)) < 1$. As the ordinary second eigenvalue is $1/2$,

$$\liminf_{t\downarrow 0} \frac{\Delta}{\|\pi - 1\|_3} \;\ge\; \lim_{t\downarrow 0} \frac{3t}{8(1 + t/4)\epsilon} \;=\; \frac{3}{10}.$$

The example obeys every fixed floor $b < 1$ for sufficiently small t.

Visibility in stationary moment measurements. The same example shows a loss of one order when a spectral edge is read from a single coordinate. In the stationary inner product define $C_j(i, \ell) = \langle \chi_i, K^j\chi_\ell\rangle_\nu$, $i, \ell = 1, 2$. These are population lag moments. The coordinates are centered by symmetry. The functions $\chi_1 \pm \chi_2$ are orthogonal (their supports are disjoint), with squared norms $2(1 \pm \epsilon)$. Directly applying K on their supports gives eigenvalues $\lambda_\pm = (1 \pm t)/(2(1 \pm t/4))$. Consequently

$$C_j(1,1) = \tfrac{1}{2}[(1 + \epsilon)\lambda_+^j + (1 - \epsilon)\lambda_-^j], \qquad C_j(1,2) = \tfrac{1}{2}[(1 + \epsilon)\lambda_+^j - (1 - \epsilon)\lambda_-^j].$$

The diagonal differs from 2^{-j} by $O(t^2)$ uniformly over every integer $j \ge 0$. Indeed the expression is even, and its second derivative is uniformly bounded on $|t| \le 1/4$: each eigenvalue lies in

a compact subinterval of $(0,1)$, and the polynomial factors in j from two differentiations are dominated by geometric decay. The order is attained at lag one, where expansion gives $C_1(1,1) = 1/2 + 3t^2/8 + O(t^4)$. In contrast the two-channel population pencil (C_1, C_0) gives $\lambda_\pm$ exactly, since its two common directions have quotients $\lambda_\pm$. The ordinary second eigenvalue moves by at least a constant times t. Thus even the entire single-coordinate autocorrelation sequence in absolute sup norm cannot yield a locally Lipschitz estimate of the spectral edge on this family. A two-channel pencil resolves the splitting with bounded conditioning as $t \to 0$ since C_0 has eigenvalues $1 \pm \epsilon$. This compares population observables, with no sampling model assumed. It connects to the programme's two-channel spectral moment certificates, which use a different covariance and sampling model.

Matching above the cubic order. For the exponential estimate let $D = \{|g-1| > 2^{-k/8}\}$. The matching-energy argument gives $\mu(D) \le C_B 2^{-3k/4}$. On D^c, $\|(U-\pi)\mathbf{1}_{D^c}\|_3 \le C_B 2^{-k/8}$ for large k. If $m \le e^{-\alpha k}$, then $Mm^{1/3} \le Bm^{1/3-1/p}$ is exponentially small, and the $L^{9/2}$ bound on E^c handles $D \setminus E$. If $m > e^{-\alpha k}$ then $M \le Be^{\alpha k/p}$ and $\|(U-\pi)\mathbf{1}_D\|_3 \le C_b M \mu(D)^{1/9}$. Choose $0 < \alpha < p\log(2)/12$. Both bounds decay exponentially; bounded k is covered by contraction. Finally $QM_{\pi^{-1}}Q$ is exponentially small by the matched low/high split for $p > 3$.

Scope. The stationary-moment manuscript treats a two-level potential. Here the two values belong to hPh; the reconstructed h may take many values. The published shell shows why cubic convergence alone cannot yield this conclusion for an unrestricted stationary law. The argument for the gap permits multiple heights on the exceptional set under (S1) and controls each of its subsets. The unrestricted stationary endpoint remains open.

5 Several exceptional stationary levels

Let $E_1, \dots, E_N$ be disjoint nonempty subsets of the cube, $m_i = \mu(E_i)$ and $m = \max_i m_i$. Assume $\pi = a$ off their union, $b \le a \le 1$, and $a \le \pi \le M_i$ on E_i. Fix N, b, H, κ independently of dimension, with $b > 0$, and assume

$$M_i m_i^{1/3} \le H, \qquad m_i \ge m^\kappa, \qquad 1 \le \kappa < 21 - 8\sqrt{5}. \tag{F1}$$

The restriction compares logarithms of uniform masses. The mass and height ratios may diverge. The sets have arbitrary geometry.

Theorem. Under (F1) the spectral gap is bounded below by a positive constant depending only on N, b, H, κ. If π is constant on each E_i, then $\Delta \le C_{N,b,\kappa}\|\pi - 1\|_3$ without coordinate matching. For arbitrary allowed heights, cubic convergence gives full recovery, and uniform integrability of the cubed densities suffices under diverging matching order. With a fixed $p > 3$ and $\|\pi\|_p \le B$, matching through order k instead gives $\Delta \le Ce^{-ck}$ with constants depending only on N, b, H, κ, p, B; for exact levels one may take $H = B$.

For example, take disjoint sets of masses 2^{-n} and 2^{-2n} in dimension $2n$ for $n \ge 2$. Add $m_i^{-1/3}$ to a common background on each set, choosing $a = 1 - \sum_i m_i^{2/3}$ for normalization. For large n the floor exceeds $1/2$ and the cubic moment stays bounded. Both sets together violate any fixed single-spike envelope (S1): the maximum times their combined mass to the power $1/3$ grows at least as $2^{n/3}$. Here (F1) applies with $\kappa = 2$.

Control between different levels. Set $r = 12\sqrt{5} - 15 > 9$, $s = (r+15)/16$, $T = P^2$, and $\eta = 1/s - 1/3 - \kappa/r$. Then $\eta > 0$ because $r(16/(r+15) - 1/3) = 21 - 8\sqrt{5}$. Differentiation shows this choice maximizes the right side over $r > 6$; this is a limit of the estimate, not a necessary restriction for a gap. For every $\ell \geq 1$, $j \geq 0$, $P^j T^\ell$ contracts L^s into L^r. Let $Z = \max_i \|Y\|_{r,\mu_i}$, $Y = 1/g$, $\mu_i = \mu(\cdot \mid E_i)$. The contribution from an individual level to any other satisfies

$$M_i \|P^j T^\ell (Y \mathbf{1}_{E_i})\|_{r,\mu_v} \leq M_i m_i^{1/s} m_v^{-1/r} Z \leq H m^\eta Z. \tag{F2}$$

By Jensen $Y \leq Pg/a$, hence $Y \leq TY + a^{-1} \sum_i M_i T(Y \mathbf{1}_{E_i})$. Iterate $n = 1 + \lceil \log_{16}(1 + \log(1/\min_i m_i)) \rceil$ times as in (S2). The unrestricted term $P^j T^n Y$ has $L^{1+8\cdot 16^n}$ norm at most $1/b$, so its conditional L^r norm on each level is at most e/b. By (F2), $Z \leq e/b + (nNH/b) m^\eta Z$. For all sufficiently small m, $n = O_\kappa(1 + \log(1 + \log(1/m)))$ and the coefficient is at most $1/2$. Applying the iterated bound also with $j = 2$ controls TY on every level. Since $Pg = T(\pi Y) \leq aTY + \sum_i M_i T(Y \mathbf{1}_{E_i})$, we have

$$\max_i \left(\|Y\|_{r,\mu_i} + \|Pg\|_{r,\mu_i} \right) \leq C. \tag{F3}$$

The conditional estimate does not require pointwise control on g. When m is bounded below, all the M_i are bounded by (F1) and we use the ambient $\|Y\|_9 \leq 1/b$ instead.

Cut escape, including transitions between levels. We prove the uniform gap by contradiction along a sequence. If m is bounded below, (G3) and the global $L^{9/2}$ bound on $U = \pi Y^2$ give the gap as in the one-spike proof. Pass therefore to $m \to 0$. Extract a subsequence along which every ratio m_i/m_j has a limit in $[0, \infty]$. Group indices with finite nonzero limiting ratio. Each group C denotes the union of its levels, with mass m_C and height bound $M_C = \max_{i \in C} M_i$ (indices in this maximum refer to the group). Along this subsequence $M_C m_C^{1/3} \leq H_C < \infty$, and (F3) holds also under $\mu_C = \mu(\cdot \mid C)$.

For $\varnothing \neq F \subset C$ write $v_F = \nu(F)$, $\theta_F = v_F/(M_C m_C)$ and $R_F = J(F,F)/v_F$. As in (S3) and its cut estimate, using $\|Y\|_{8,\mu_C} \leq C$ gives

$$R_F \leq C M_C m_C^{1/3} \theta_F^{1/12}.$$

Thus if R_F stays bounded below then θ_F and $M_C m_C^{1/3}$ both stay bounded below, and v_F is comparable to $m_C^{2/3}$. Moreover $\nu(\cdot \mid F)$ has density at most $1/\theta_F$ relative to μ_C. Under this conditional stationary law, the expectations of Pg and $P(h\mathbf{1}_F)$ are uniformly bounded: for the latter,

$$\|P(h\mathbf{1}_F)\|_{3,\mu_C} \leq m_C^{-1/3} M_C \|Y \mathbf{1}_C\|_{3/2} \leq C M_C m_C^{1/3}.$$

Also $E_{\mu_C} P \mathbf{1}_C \leq m_C^{1/3}$ by ordinary hypercontractivity. Consequently, for a fixed sufficiently large A and all large dimensions, at least half the conditional stationary mass has $Pg \leq A$, $P(h\mathbf{1}_F) \leq A$ and $P\mathbf{1}_{F^c} \geq 1/2$, by Markov's inequality. On those vertices if $g > 2A$, then $K_x(F) \leq 1/2$. Otherwise the row escape bound from (S3) and its proof gives $K_x(F^c) \geq b/(8A^2)$. Hence R_F stays bounded away from one.

If the union admitted cuts F with $R_F \to 1$, split them into $F_C = F \cap C$. Pass to limits of their diagonal return fractions R_{F_C}, taking zero for empty pieces. The positive semidefinite edge form gives

$$\sqrt{R_F} \leq \sum_C \sqrt{R_{F_C}\, \nu(F_C)/\nu(F)}.$$

Groups with vanishing diagonal return contribute $o(1)$. Of those with positive limiting return, the one with the largest mass dominates all the others: their cut masses are comparable to their respective $m_C^{2/3}$, and distinct groups have mass ratios tending to zero or infinity. The dominant summand stays bounded away from one by the preceding row argument. This contradicts $R_F \to 1$ (if there are no such groups the sum tends to zero). The killed norm on the union is therefore bounded away from one along the extracted subsequence by (LR1). Off the union, $\|U\|_{9/2}$ restricted there is at most $1/b^2$. Adding Gram multipliers as in the one-spike proof controls all small cuts. Larger cuts obey (G3) since $E\pi^3 \le 1 + NH^3$. Cheeger's inequality now precludes a sequence with gap tending to zero.

Full spectral estimates. For exact levels take $M_i = \pi|_{E_i}$. Put $\delta = \|\pi - 1\|_3$. For the Lipschitz estimate it suffices to take $\delta \le 1$, when $H = 2$ works. Away from levels above two, the preceding section proved $\|\pi(Y^2 - 1)\mathbf{1}_{\{\pi \le 2\}}\|_3 \le C_b\delta$. On each higher level, $M_i m_i^{1/3} \le 2\delta$; (F3) gives $\|U\mathbf{1}_{E_i}\|_3 \le C M_i m_i^{1/3}$. If m is bounded below the same conditional L^6 control follows from $\|Y\|_9 \le 1/b$ and (F1). Sum the finitely many contributions and use the Gram inequality $\Delta \le \|U - 1\|_3$. The parity example in the previous section retains sharpness of the linear order for the full class.

Variable heights within levels. The gap statement allowed variable heights from the outset. The same hypotheses yield a quantitative transfer for the entire spectrum even when constant heights are unavailable. Put $\alpha = 1 - 6/r > 0$. For any set D in the small-m regime, conditional Hölder on E_i with $1/u = 1/3 - 2/r$ and (F3) give

$$\|U\mathbf{1}_{E_i \cap D}\|_3 \le C m_i^{1/3} \|\pi \mathbf{1}_D\|_{u,\mu_i} \le C(M_i m_i^{1/3})^{1-\alpha} \|\pi \mathbf{1}_{E_i \cap D}\|_3^{\alpha}.$$

Here $3/u = \alpha$. Off the exceptional union use the ambient $L^{9/2}$ bound. When m stays above the absorption threshold, the same ambient bound holds everywhere. For either regime,

$$\|U\mathbf{1}_D\|_3 \le C_{N,b,H,\kappa}(\|\pi\mathbf{1}_D\|_3^{\alpha} + \mu(D)^{1/9}). \tag{F4}$$

In particular full recovery without matching holds if $\delta = \|\pi - 1\|_3 \to 0$. For a rate on $\{\pi > 2\}$, in the small-m regime use the preceding levelwise inequality there (the background is at most one) and $\pi \le 2|\pi - 1|$. On $\{\pi \le 2\}$ use the inverse estimate from the one-spike section. In the remaining, bounded-height regime use that inverse estimate globally. This gives $\Delta \le C(\delta + \delta^{\alpha})$.

There is also a matched result at the cubic order: uniform integrability of π^3 under μ and $k \to \infty$ imply full spectral recovery for variable heights satisfying (F1). Indeed (F4) transfers that uniform integrability to U^3. The matched bad set $D = \{|g - 1| > 2^{-k/8}\}$ has exponentially small mass since the cubic cap implies a bounded second moment. The L^3 norm of $U - \pi$ on D therefore tends to zero; outside D it is exponentially small. To bound $QM_{\pi-1}Q$ at order three let A project onto Walsh degrees at most $\lfloor k/2 \rfloor$. The low-low block vanishes by matching of products. In each remaining block at least one high factor $Q(I - A)$ has L^2 operator norm at most $2^{-k/4}$. Truncate $\pi - 1$ at a fixed height $T > 1$, keeping its values on $\{\pi \le T\}$. The bounded part in each such block costs at most $(T + 1)2^{-k/4}$, and the tail costs at most $\|\pi\mathbf{1}_{\{\pi > T\}}\|_3$ by $Q : L^2 \to L^3$ and the L^2 contractions $A, I - A$ before Q. Take first the limit in k, then in T. Min–max proves the claim. For $p > 3$ and $\|\pi\|_p \le B$, the same proof is exponential: in (F4) use $\|\pi\mathbf{1}_D\|_3 \le B\mu(D)^{1/3-1/p}$, and take $T = 2^{k/8}$ in the spectral blocks, whose tail is at most $B^{p/3}T^{1-p/3}$. No bound on the within-level oscillation is needed.

The published trapping family has a fixed floor and converges in cubic norm. This theorem implies that it admits no representation satisfying (F1) with uniform parameters. The theorem uses ordinary Boolean hypercontractivity; Polyanskiy and Samorodnitsky give stronger estimates on small supports. Whether such an improvement removes the logarithmic mass comparison is open here.

6 Three stationary values suffice for exponential trapping

The rarity comparison in (F1) cannot be discarded, even for exact levels. For every $\kappa > 64/9$ and every fixed floor $0 < b < 1$ there are a fixed $p > 3$, $\sigma > 0$ and arbitrarily large dimensions with exactly three stationary density values such that

$$\pi \geq b, \qquad \|\pi - 1\|_p \leq e^{-\sigma L}, \qquad 1 - \lambda_2(K) \leq e^{-\sigma L}. \tag{T1}$$

All one-coordinate stationary marginals are uniform. The two exceptional levels have uniform masses $m_S < m_C$ with $m_S \geq m_C^{\kappa}$. Together with (S1), this identifies three as the smallest number of stationary values allowing loss of the uniform gap under a fixed floor and bounded cubic moment. The assertions about the number of levels do not impose higher-order marginal matching.

Ordinary shell probabilities. Write $z(x) = L^{-1}\sum x_i$ and $I(t) = ((1+t)\log(1+t) + (1-t)\log(1-t))/2$. For attainable $t > 0$, $\mu\{|z| = t\} = \exp[-LI(t) + O(\log L)]$ and $I(t) = t^2/2 + O(t^4)$. From $z(x) = s$, one half-noise step has magnetization mean $s/2$. For $|t|, |s| = O(d)$ write $w = t - s/2$. The probability of landing at the exact attainable t is between

$$\exp[-L(2w^2/3 + O(d^3)) - O(\log L)] \quad \text{and} \quad \exp[-L(2w^2/3 - O(d^3))]. \tag{T2}$$

For the lower bound the two groups of input signs have output-plus probabilities $3/4$ and $1/4$. Raise each by $w/2$ (a negative shift is allowed). Round the first group's output count to an integer and choose the second to give exactly the prescribed total. Each group has size comparable to L. Stirling's bounds for both binomial terms give the two relative entropies with $O(\log L)$ error. Each divergence is $2w^2/3 + O(d^3)$ by Taylor expansion. For the upper bound, the centered log moment generating function per output sign is $3v^2/8 + O(|v|^3)$ for a tilt v of either sign; use $v = 4w/3$. The constants are uniform for the shell locations used below.

Two prescribed exceptional densities. Choose rational $u < 3/8$ sufficiently close to $3/8$ that $u^{-2} < \kappa$. Choose k_0 strictly between

$$\tfrac{2}{3}(u - 1/4)^2 \quad \text{and} \quad \min\{\tfrac{2}{3}(1/2 - u)^2,\ u^2/12,\ (u - 1/8)^2/2\}. \tag{T3}$$

Such an interval exists: at $3/8$ the first two square-distance rates coincide, the other two upper bounds are strictly larger, and moving u to the left separates the first two rates. Choose a fixed $\omega > 0$ small enough that $\frac{2}{3}(u - 1/4 + \omega/2)^2 < k_0$ and $\omega < 1/2 - u$. Choose $e > 0$ so small that each of the gaps in (T3), the new gap after adding $\omega/2$, the number $\omega^2/2$, and $(7/8)^2/2 - 1/6$ exceeds $10e$. Take $0 < \gamma < \min(e, 1/32)$. Finally take rational $d > 0$ sufficiently small that all strict inequalities below survive the $O(d^3)$ errors in (T2). Take L along multiples making both positive shells attainable. Set

$$S = \{|z| = d\}, \qquad C = \{|z| = ud\}, \qquad B = \{\, ||z| - d/2| \leq \omega d \,\}.$$

For $x \in S$, $q = P_x(S)$ is independent of x. By (T2) and Hoeffding's bound,

$$e^{-(d^2/6+O(d^3))L-O(\log L)} \le q \le 2e^{-d^2L/8}.$$

Prescribe the density itself by

$$M_C = e^{k_0 d^2 L}, \qquad M_S = e^{-\gamma d^2 L}/q, \qquad \pi = M_C \mathbf{1}_C + M_S \mathbf{1}_S + a\mathbf{1}_{(S\cup C)^c}, \quad a = \frac{1 - m_C M_C - m_S M_S}{1 - m_C - m_S}.$$

Since $k_0 < u^2/12$, $M_C m_C^{1/3}$ decays exponentially. The corresponding exponent for $M_S m_S^{1/3}$ is at most $-\gamma d^2 + O(d^3)$, also strictly negative. Thus $a \to 1$ and the cubic distance decays exponentially. The same strict inequalities hold with $1/p$ in place of $1/3$ for some fixed $p > 3$ near three. The upper bound on q ensures $M_S > M_C > 1$ eventually. Sign reversal gives exact one-coordinate marginals. The limit of $\log m_S / \log m_C$ is $I(d)/I(ud)$, which tends to u^{-2} as $d \downarrow 0$, so (after fixing sufficiently small d) $m_S \ge m_C^{\kappa}$ eventually.

Recovering the potential from the prescribed law. The unique positive solution $hPh = \pi$ follows by strict convexity and coercivity of the finite symmetric scaling functional in log coordinates (as in the published shell proof). Permutations and global sign reversal preserve this solution. In particular $h, g = Ph$ are constant on each of S, C, denoted by subscripts. Put $\delta_2 = \|\pi - 1\|_2$. The variance proof in the one-spike section uses only the stationary second norm of the perturbation and yields $\|g - 1\|_3 \le \delta_2 + \delta_2^2$. Here $\delta_2 = O(M_C\sqrt{m_C} + M_S\sqrt{m_S})$. After division by $m_C^{1/3}$ the two exponential rates are bounded above by $k_0 d^2 - I(ud)/6$ and $d^2/6 + O(d^3) - I(d)/2 + I(ud)/3$, both strictly negative for small d. Therefore $g_C = 1 + o(1)$ and $h_C = (1 + o(1))M_C$. On S positivity and $g_S \ge q h_S$ imply $h_S \le \sqrt{M_S/q} \le 1/q$.

For $y \in B$, the lower bound (T2) aimed at the appropriately signed component of C gives $P_y(C) \ge \exp[-(\frac{2}{3}(u - 1/4 + \omega/2)^2 d^2 + O(d^3))L - O(\log L)]$. Since $g(y) \ge h_C P_y(C)$ and $\pi(y) = a$, for large L,

$$\sup_B h \le e^{-4ed^2L}. \tag{T4}$$

This depression in the potential is a consequence of the two prescribed high stationary values.

Escape from the rarer level. For $x \in S$ let $D = P_x(h\mathbf{1}_{S^c})$. The smallness of h on B alone would not control atypical transitions. To bound them set $T = P^2$, $Y = 1/g$. The floor a gives $Y \le Pg/a = Th/a$ by Jensen. Because $h = aY$ off the exceptional levels,

$$Y \le TY + a^{-1}(h_C T\mathbf{1}_C + h_S T\mathbf{1}_S).$$

Iterate $n = 1 + \lceil \log_{16}(1 + L) \rceil$ times. The ambient bound $\|Y\|_9 \le 1/a$ and hypercontractivity give $\|T^n Y\|_\infty \le 2^{L/(1+8\cdot 16^n)}/a \le 2/a$. Bounding separately the landing regions C, B, and ordinary background outside B, and using positivity in the iterated terms, gives the explicit estimate

$$D \le h_C P_x(C) + \sup_B h + 2P_x(B^c) + \sum_{i=1}^{n} \left[h_C P_x^{2i+1}(C) + h_S P_x^{2i+1}(S)\right]. \tag{T5}$$

The one-step term for C decays with exponent at least $(\frac{2}{3}(1/2 - u)^2 - k_0)d^2 - O(d^3)$ by (T2), including the more distant opposite-sign component. The third term costs at most $4e^{-\omega^2 d^2 L/2}$. For every odd step $2i + 1 \ge 3$ the signs are independent given x, with magnetization mean of magnitude at most $d/8$. Hoeffding bounds the two probabilities in the sum by $2e^{-(u-1/8)^2 d^2 L/2}$ and

$2e^{-(7/8)^2 d^2 L/2}$ respectively. Multiplication by $h_C \sim M_C$ and $h_S \le 1/q$ preserves strict exponential decay. With the chosen slack and $n = O(\log L)$, (T5) gives $D \le e^{-ed^2 L}$.

Finally $g_S = qh_S + D$ and $g_S h_S = M_S$ imply $g_S \ge \sqrt{qM_S} = e^{-\gamma d^2 L/2}$. Hence $K_x(S^c) = D/g_S \le e^{-(e-\gamma/2)d^2 L}$ on the whole shell. The centered shell indicator in the Rayleigh quotient gives $1 - \lambda_2(K) \le K_x(S^c)/(1 - M_S m_S)$, proving (T1).

Exact matching and the number of approximate levels. There is a corresponding obstruction with exact matching through a positive fraction of the coordinates. Let π_0 be the three-valued law just constructed. For sufficiently small fixed $\beta > 0$, subtract from π_0 its nonconstant Walsh coefficients of orders at most $\lfloor \beta L \rfloor$ to obtain π_1. Each coefficient is bounded by $\|\pi_0 - 1\|_1$, and their number is at most $e^{LH(\beta)}$, where H is binary entropy in natural units. By (T1) we can choose $H(\beta) < \sigma/2$. Thus π_1 is exactly matched, normalized, and $\|\pi_1 - \pi_0\|_\infty \le e^{-\sigma L/2}$. Its floor tends to one and its supercubic norm distance still tends exponentially to zero.

For completeness, let two normalized densities v_0, v_1 have floor $b_0 > 0$ and logarithmic distance $\epsilon = \|\log(v_1/v_0)\|_\infty \le 1$. Along their straight-line interpolation, $\|\dot v_t/v_t\|_\infty \le e^\epsilon - 1 \le 2\epsilon$. Logarithmic differentiation of $h_t P h_t = v_t$ gives $(I + K_t)\,\partial_t \log h_t = \dot v_t/v_t$. The operator $A_t = (I - K_t)/2$ has absolute row sums at most one. Positivity of the noise quadratic form gives stationary L^2 norm at most $1/2$. Since the smallest stationary mass is at least $b_0 2^{-L}$, $\|A_t^j\|_{\infty\to\infty} \le \min(1, 2^{L/2-j}/\sqrt{b_0})$. The inverse is $\frac{1}{2}\sum_{j\ge 0} A_t^j$, so $\|\log(h_1/h_0)\|_\infty \le C_{b_0}(L+1)\epsilon$. In particular every stationary edge weight changes by a factor between $e^{-2C_{b_0}(L+1)\epsilon}$ and $e^{2C_{b_0}(L+1)\epsilon}$, and every variance, written as an infimum over constant centerings, changes by a factor between $e^{-\epsilon}$ and e^{ϵ}. Taking Dirichlet quotients gives

$$\left|\log \frac{\operatorname{gap}(K_1)}{\operatorname{gap}(K_0)}\right| \le 2C_{b_0}(L+1)\epsilon + \epsilon. \tag{T6}$$

Applied to the projected law this preserves the exponential gap upper bound. The three exact levels refer to the reference law; the matched law lies exponentially close to those levels.

Conversely, suppose a density is within $o(1)$ in log sup norm of a normalized two-valued reference with uniform floor and bounded cubic norm. The one-spike theorem and (T6) imply $\log(1/\operatorname{gap}(K)) = o(L)$; distance $O(1/L)$ gives a uniform positive gap. Thus an exponential trap with a fixed positive rate is impossible even at vanishing logarithmic distance from two such levels, whereas exponentially accurate three-level approximation already allows it under linear-order matching. The unrestricted sufficient stationary exponent is unchanged by this construction.

Consequences and comparison. For the two-exceptional-level cubic gap problem, (F1) proves sufficiency for $\kappa < 21 - 8\sqrt{5}$. Equation (T1) excludes uniform sufficiency for every $\kappa > 64/9$. Neither endpoint is decided. The earlier shell started with several potential bands and then corrected the stationary marginals through linear order. Here the stationary law has precisely three values by construction; the reciprocal estimate (T5) controls its reconstructed transition rows. Ordinary Hamming sphere noise estimates are studied, for example, by Kirshner and Samorodnitsky. The present argument does not decide the unrestricted matched endpoint.

7 Fractional basis packing and a direct inverse moment

Use the notation of the rank paper, with $P = P_{1/2}$, $Q = P_{1/\sqrt{2}}$, $g = Ph$, $\pi = hg$, $E\pi = 1$ and $U = h/g$. Let A have full row rank and let π be a function of Ax. By uniqueness of positive

symmetric scaling, h and g are functions of Ax as well. A fractional packing of weight t assigns weights $w_J \geq 0$ to column bases J of A, with $\sum_J w_J = t$ and $\sum_{J \ni i} w_J \leq 1$ for each input column i. Repeated columns have separate capacities.

Suppose $\|\pi\|_p \leq B$ for fixed $2 \leq p < \infty$, and the stationary marginals match the uniform law through degree k. If t is fixed and

$$\frac{1}{p} + \frac{2}{4^t - 1} < \frac{2^t - 1}{2^t + 1}, \tag{P1}$$

then the entire ordered spectrum differs from that of P by at most $C_{B,p,t} e^{-c_{p,t} k}$, with $c_{p,t} > 0$. No stationary minimum uniform in dimension is assumed. In particular, at $p = 2$ it suffices that $t > \log_2(2 + \sqrt{5})$, so three disjoint full bases suffice. At $t = 2$ it suffices that $p > 15/7$, which includes the programme's stationary moment endpoint. The strict boundary in (P1) is not decided.

Allocating noise time. On the quotient, the noise contributed by column i acts by retaining or translating by that column. Its multipliers are 1 and λ. Thus two factors in the same column multiply their correlations. Allocate a factor of correlation λ^{w_J} in each column of J to basis J, omitting zero weights. The unused capacity is another commuting noise factor. Each basis acts as ordinary product noise on the entire quotient in its own coordinates. Successive applications of forward hypercontractivity give contraction from $a > 1$ to $1 + (a-1)\lambda^{-2t}$. For a positive input and $0 < a < 1$, reverse hypercontractivity gives the lower bound from a to $1 - (1-a)\lambda^{-2t}$, including negative target orders. This follows by using the successive equality exponents of the classical inequalities. If an intermediate order vanishes, use the geometric mean by continuity; all functions are strictly positive on the finite quotient. Leftover noise contracts forward norms and increases the generalized reverse norm by Jensen. These steps use the usual time convention for Boolean noise (Bonami and Mossel, Oleszkiewicz and Sen, Remark 1.13). Fractional base packing itself is classical, for example Dynamic Matroids, Base Packing and Covering.

Start with the potential. The normalization $\|Qh\|_2 = 1$ implies $\|g\|_2 \leq 1$. Since $E\pi = 1$ and $\|\pi\|_2 \leq B$, Cauchy–Schwarz gives $\mu\{\pi \geq 1/2\} \geq 1/(4B^2)$. Also $\mu\{g > 4B\} \leq 1/(16B^2)$. On an intersection of measure at least $3/(16B^2)$, we thus have $h = \pi/g \geq 1/(8B)$. For every fixed $a > 0$, this bounds $\|h\|_a$ from below solely in terms of a, B. In particular, for $0 < q < 4^t - 1$, use $a = 1 - (1+q)4^{-t}$ and the reverse estimate on Ph, directly, to obtain $\|g^{-1}\|_q \leq C_{B,t,q}$. This spends the full noise time on the potential and requires neither a floor on π nor matching in this step.

Ambient operator comparison. Choose $q < 4^t - 1$ and $s > 1$ so that

$$\frac{1}{p} + \frac{2}{q} < \frac{1}{s} < \frac{2^t - 1}{2^t + 1}.$$

Then $U = \pi/g^2$ is uniformly bounded in L^v for $1/v = 1/p + 2/q < 1/s$. For any nonnegative nonzero quotient weight W, the unique positive unit leading eigenvector of QM_WQ is invariant under all translations in $\ker A$. Forward hypercontractivity on this vector, with $2s/(s-1) < 1 + 2^t$, bounds $\|QM_WQ\| \leq \|W\|_s$ on the full cube. Zero weights satisfy the same bound.

The matching-energy estimate of the rank paper gives $\|g - 1\|_2 \leq C_B 2^{-k/2}$. On $E_k = \{|g-1| > 2^{-k/8}\}$ the measure is at most $C_B 2^{-3k/4}$. Hölder bounds $\|(U - \pi)\mathbf{1}_{E_k}\|_s$ by a constant times $\mu(E_k)^{1/s - 1/v}$. Outside E_k, $|U - \pi| \leq C 2^{-k/8}\pi$ for large k. The multiplier bound applied to $|U - \pi|$ therefore gives exponential decay of $\|QM_{U-\pi}Q\|$.

By the strict inequality for $1/p$, take $\lambda > 1/\sqrt{2}$ sufficiently close to $1/\sqrt{2}$ that $2p/(p-1) \leq 1 + \lambda^{-2t}$; take $\lambda < 1$. Set $\theta = 1/(\sqrt{2}\lambda) < 1$. The same positive multiplier argument gives $\|P_\lambda M_{|\pi-1|} P_\lambda\| \leq B+1$, hence the same bound for its signed counterpart. In $Q = P_\lambda P_\theta$ split the P_θ factors at degree $j = \lfloor k/2 \rfloor$. The low-low term against $\pi - 1$ is zero by matching; the other three terms each have norm at most $(B+1)\theta^{j+1}$. Thus $\|QM_UQ - P\|$ decays exponentially. The two Gram products of $M_{\sqrt{U}}Q$ identify the spectrum of the first operator with that of the stationary Doob chain. The claimed comparison follows by min–max. For $p = 2$, writing $u = 2^t$, (P1) reduces to $u^2 - 4u - 1 > 0$. For $t = 2$ it reduces to $1/p < 7/15$.

Separation from the four-family rank estimate. In each of $n \geq 2$ independent rank-two blocks use five columns of types $(1,0), (0,1), (1,1)$ with multiplicities $2, 2, 1$. Assign total weights $3/2, 1/2, 1/2$ to the three distinct pairs of types in the order $(1,2), (1,3), (2,3)$, spreading each uniformly over the available copies. Each column uses exactly its unit capacity. Normalize the block weights to probabilities, take their product across blocks, and multiply by $5/2$. This is a fractional packing of global bases of weight $5/2$. The coordinate count $5n$ and the basis size $2n$ also give the matching upper bound. The full code forces the four-family rank defect to be at least $\lceil 2n - 5n/4 \rceil$ by the support formula of the rank paper.

For an explicit matched law, choose z in the code nonzero in every block and a basis $e_1, \ldots, e_{2n}$ consisting of words supported on single blocks. Take $0 < \epsilon_n < 1$ tending to zero and set

$$\pi = 1 - \frac{1 - \epsilon_n}{2n+1} \left(\chi_z + \sum_{i=1}^{2n} \chi_{z+e_i} \right).$$

Every nonzero word in one block weighs at least three. All displayed characters therefore have degree at least $3(n-1)$. They are distinct, their differences span the code, and π has mean one and takes values between ϵ_n and two, attaining the lower bound at the origin. Finite positive symmetric scaling realizes this law. Taking $k = 3n - 4$ makes the old four-family bound vacuous, whereas (P1) gives exponential recovery at $p = 2$. The example separates the two sufficient estimates on the intrinsic code. It makes no exclusivity claim about all other stability methods. The unrestricted endpoint without quotient information remains open.

Model assessments and review context

Codex assisted with the mathematical derivation, comparison with primary sources, and source checks in the programme's research chat. This supplement uses the previously constructed shell as an explicit input to the interpolation argument. Its typeset version awaits a first exact-PDF assessment. No numerical score or independent referee report has been obtained for this supplement.