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CONSTRUCTIVE $\mathbb{Z}_2$-CENTER CONFINEMENT VS. ALL-COUPLING ADJOINT SCREENING AND TWO-PARAMETER VARIATIONAL FREE-ENERGY ENCLOSURES IN d-DIMENSIONAL SU(2) FUNDAMENTAL–ADJOINT (BHANOT–CREUTZ) LATTICE YANG–MILLS THEORY

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Abstract. We establish three completely self-contained, cluster-expansion-free, non-perturbative theorems for the d-dimensional ($d \geq 3$) SU(2) fundamental–adjoint (Bhanot–Creutz) lattice Yang–Mills theory with action $S_\Lambda(\beta_F, \beta_A) = \sum_{p\in\mathcal{P}(\Lambda)}(\beta_F \frac{1}{2}\operatorname{Tr} U_p + \beta_A \frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U_p)$:

(1) *All-$\beta_A \in (-\infty, +\infty)$ fundamental confinement strip:* Decomposing every link as $U_b = \sigma_b \tilde{U}_b$ ($\sigma_b \in \{\pm 1\}$), applying a four-line Ginibre inequality for signed $\mathbb{Z}_2$ gauge couplings, exact positive-weight high-temperature surface factorization, homological projection onto the $(1,2)$-plane, and Fuss–Catalan plaquette-BFS counting ($K_d := 8d-13$, $B_d := K_d^{K_d}/(K_d-1)^{K_d-1}$), we prove that on every finite free-BC box $\Lambda_L = \{0,\dots,L\}^d$, the fundamental string tension satisfies $\sigma_F(\beta_F, \beta_A) \geq -\ln(B_d \tanh|\beta_F|) > 0$ uniformly for *every* $\beta_A \in (-\infty, +\infty)$ whenever $|\beta_F| < \operatorname{artanh}(1/B_d)$ (including $|\beta_F| \leq 0.034$ in 3D and $|\beta_F| \leq 0.019$ in 4D). In particular, no bulk transition at arbitrarily large $\beta_A > 0$ (nor negative $\beta_A < 0$) can deconfine fundamental charges inside this strip.

(2) *All-coupling adjoint screening (string breaking) in $d \geq 3$:* Using Osterwalder–Seiler transfer-matrix positivity $\mathcal{T}_\mu \geq 0$ for $(\beta_F, \beta_A) \in [0,\infty)^2 \setminus \{(0,0)\}$ and a local, SO(3)-valued, gauge-covariant two-gluelump Rayleigh–Ritz trial state built from $\widetilde{F}(U_p) := \frac{1}{4i}(U_p^2 - (U_p^\dagger)^2)$, we prove an explicit, volume-uniform and distance-uniform upper bound on the static-adjoint potential $V_{\mathrm{adj},L_s}(R) \leq 8\ln(e^{\beta_F+\beta_A}/c_1(\beta_F,\beta_A)) + 2(8d-19)(2\beta_F + \frac{4}{3}\beta_A) < \infty$ for all $L_s \geq 4$ and all $R \in \{1,\dots,L_s-1\}$, establishing vanishing asymptotic adjoint string tension $\sigma_{\mathrm{adj}}(\beta_F,\beta_A) = 0$ across the entire non-negative quadrant (including pure SO(3) at $\beta_F = 0, \beta_A > 0$), alongside an exact reflection-positivity gluelump correlation inequality.

(3) *Two-parameter variational free-energy enclosures and zero-quadrature certification:* We derive an exact all-$\beta_A \in \mathbb{R}$ Jacobi–Anger Bessel convolution for $Z_{1\mathrm{p}}(\beta_F,\beta_A)$ and (u_F, u_A), prove $(d-1)$-ladder Hölder upper bounds and Schur-lemma two-parameter axial-gauge Gibbs–Bogoliubov lower bounds matching the exact one-loop asymptotics $(\beta_F+\beta_A) - \frac{3}{d}\ln(\beta_F + \frac{8}{3}\beta_A) + O(1)$, and certify 24 explicit points (8 all-β_A confinement strips and 16 two-parameter 3D/4D variational enclosures) via 160-digit zero-quadrature `mpmath.iv` interval arithmetic and standalone zero-import, zero-axiom Lean 4.36.0 (`BhanotCreutzCerts.lean`).

1. Introduction and Honest Scope

The two-parameter SU(2) fundamental–adjoint lattice gauge theory introduced by Bhanot and Creutz [1] (and studied extensively in [7, 10, 2, 6]) generalizes Wilson's lattice Yang–Mills action [11] by adding an adjoint plaquette coupling β_A:

$$S_\Lambda(\beta_F,\beta_A) := \sum_{p\in\mathcal{P}(\Lambda)} \left(\beta_F \frac{1}{2}\operatorname{Tr} U_p + \beta_A \frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U_p\right), \qquad (\beta_F,\beta_A) \in \mathbb{R}^2, \tag{1}$$

where for $U \in \mathrm{SU}(2)$ with eigenvalues $e^{\pm i\theta}$ ($\theta \in [0,\pi]$), $\frac{1}{2}\operatorname{Tr} U = \cos\theta = \frac{1}{2}\chi_{1/2}(U)$ and $\frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U = \frac{4}{3}\cos^2\theta - \frac{1}{3} = \frac{1+2\cos(2\theta)}{3} = \frac{1}{3}\chi_1(U)$. The model interpolates continuously between pure SU(2)

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Wilson gauge theory ($\beta_A = 0$), pure $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$ lattice gauge theory ($\beta_F = 0$), the center-frozen $\mathbb{Z}_2$ lattice gauge theory ($\beta_A \to +\infty$ at fixed β_F), and the monopole-suppressed negative-β_A regime ($\beta_A < 0$).

In four dimensions, numerical studies [1] reveal a line of first-order bulk transitions emerging from the pure-SO(3) axis ($\beta_F = 0, \beta_A \approx 2.5$) and terminating at a critical endpoint near $(\beta_F, \beta_A) \approx (1.5, 0.9)$. A central conceptual question in non-Abelian confinement physics [6] is how the $\mathbb{Z}_2$ center symmetry controls the sharp contrast between fundamental ($j = \frac{1}{2}$, non-trivial center charge) and adjoint ($j = 1$, trivial center charge) static sources across this entire (β_F, β_A) phase diagram.

In this paper (building on our constructive lattice Yang–Mills series [4, 3]), we prove three rigorous, non-perturbative, cluster-expansion-free theorems valid directly in $d \geq 3$ dimensions (with concrete 3D and 4D certifications):

- *Uniform-in-β_A fundamental confinement strip* (Section 2): For $|\beta_F| < \operatorname{artanh}(1/B_d)$ ($B_d = K_d^{K_d}/(K_d - 1)^{K_d - 1}$, $K_d = 8d - 13$), fundamental Wilson loops obey an exact area law $\sigma_F(\beta_F, \beta_A) \geq -\ln(B_d \tanh|\beta_F|) > 0$ for *every* $\beta_A \in (-\infty, +\infty)$, ruling out fundamental deconfinement at arbitrarily large positive or negative adjoint coupling.
- *All-coupling adjoint screening (string breaking)* (Section 3): On every even spatial torus $\mathbb{T}_{L_s}^{d-1}$ ($d \geq 3$, $L_s \geq 4$) and every $(\beta_F, \beta_A) \in [0, \infty)^2 \setminus \{(0,0)\}$, we prove an explicit L_s- and R-uniform upper bound $V_{\mathrm{adj}, L_s}(R) \leq C_d(\beta_F, \beta_A) < \infty$ on the static-adjoint potential, proving $\sigma_{\mathrm{adj}}(\beta_F, \beta_A) = 0$ unconditionally—including simultaneously at the exact same points where $\sigma_F(\beta_F, \beta_A) > 0$.
- *Two-parameter all-coupling variational free-energy enclosures* (Section 4): Extending the $(d-1)$-ladder axial-gauge decomposition and Schur-lemma axial-gauge Gibbs–Bogoliubov principle [4] to $(\beta_F, \beta_A) \in \mathbb{R}^2$ via an exact zero-quadrature Jacobi–Anger Bessel convolution, we certify 16 two-parameter 3D/4D free-energy brackets spanning pure SO(3), the confining strip, the near-endpoint crossover, negative β_A, and deep weak coupling (Section 5).

Remark 1.1 (Honest Scope and Non-Claims). All theorems apply to pure d-dimensional ($d \geq 3$) SU(2) fundamental–adjoint lattice Yang–Mills theory at fixed lattice spacing $a = 1$. Theorem 2.3 certifies fundamental confinement in the finite-width strip $|\beta_F| < \operatorname{artanh}(1/B_d)$ uniformly in $\beta_A \in \mathbb{R}$ (not along the weak-coupling continuum ray $\beta_F \to \infty$); Theorem 3.1 proves adjoint screening unconditionally for all $(\beta_F, \beta_A) \in [0, \infty)^2 \setminus \{(0,0)\}$ at fixed $a = 1$; and Theorems 4.2–4.3 provide exact fixed-lattice free-energy enclosures without assuming thermodynamic-limit existence. This paper does *not* prove existence of the 4D continuum limit, does *not* establish a 4D continuum mass gap, and does *not* claim any resolution of the Clay Millennium Yang–Mills Problem.

2. All-$\beta_A \in \mathbb{R}$ Fundamental Confinement Strip via $\mathbb{Z}_2$ Center Domination

Fix $d \geq 3$ and a finite box $\Lambda_L := \{0, \ldots, L\}^d \subset \mathbb{Z}^d$ with free boundary conditions. Let $E(\Lambda_L)$ and $\mathcal{P}(\Lambda_L)$ denote its undirected links and elementary plaquettes, and let $\langle \cdot \rangle_{\Lambda_L, \beta_F, \beta_A}$ denote expectation under the probability measure proportional to $\exp(S_{\Lambda_L}(\beta_F, \beta_A)) \prod_{b \in E(\Lambda_L)} dU_b$. For a closed oriented loop $C \subset E(\Lambda_L)$, write $W_C := \prod_{b \in C} U_b \in \mathrm{SU}(2)$ for its ordered holonomy and $\sigma_C := \prod_{b \in C} \sigma_b \in \{\pm 1\}$ for the corresponding $\mathbb{Z}_2$ Wilson loop.

Lemma 2.1 (Four-Line Ginibre Domination for Signed $\mathbb{Z}_2$ Gauge Couplings)**.** *Let $\{J_p : p \in \mathcal{P}(\Lambda_L)\}$ be any real plaquette couplings satisfying $|J_p| \leq \beta$ for a constant $\beta \geq 0$, and let $\langle \cdot \rangle_{\mathbb{Z}_2, J}$ and $\langle \cdot \rangle_{\mathbb{Z}_2, \beta}$ denote expectations on $\{\pm 1\}^{E(\Lambda_L)}$ with weights $\exp(\sum_p J_p \sigma_p)$ and $\exp(\beta \sum_p \tau_p)$ respectively. Then for every closed loop $C \subset E(\Lambda_L)$,*

$$\left| \langle \sigma_C \rangle_{\mathbb{Z}_2, J} \right| \;\leq\; \langle \sigma_C \rangle_{\mathbb{Z}_2, \beta}. \tag{2}$$

Proof. Let Z_J and Z_β be the respective partition functions. On the doubled configuration space $(\sigma, \tau) \in \{\pm 1\}^{E(\Lambda_L)} \times \{\pm 1\}^{E(\Lambda_L)}$, perform the bijective change of variables $t_b := \sigma_b \tau_b \in \{\pm 1\}$ (so

$\sigma_b = t_b\tau_b$, $\sigma_p = t_p\tau_p$, and $\sigma_C = t_C\tau_C$). For every plaquette p, the combined coupling $w_p(t) := \beta + J_p t_p$ satisfies $w_p(t) \geq \beta - |J_p| \geq 0$ for every $t \in \{\pm 1\}^{E(\Lambda_L)}$. Hence, for both choices of sign $\pm$,

$$Z_J Z_\beta\big(\langle\tau_C\rangle_{\mathbb{Z}_2,\beta} \pm \langle\sigma_C\rangle_{\mathbb{Z}_2,J}\big) = \sum_{t\in\{\pm1\}^{E(\Lambda_L)}} (1 \pm t_C) \left[\sum_{\tau\in\{\pm1\}^{E(\Lambda_L)}} \tau_C \exp\left(\sum_{p\in\mathcal{P}(\Lambda_L)} w_p(t)\,\tau_p\right)\right].$$

For every t, $1 \pm t_C \in \{0, 2\} \geq 0$, and expanding $\exp(\sum_p w_p(t)\tau_p) = \sum_{\{k_p\geq 0\}} \prod_p \frac{w_p(t)^{k_p}}{k_p!}\tau_p^{k_p}$ shows that summing over $\tau \in \{\pm1\}^{E(\Lambda_L)}$ yields a sum of non-negative terms (since $w_p(t) \geq 0$ and $\sum_\tau \tau_C \prod_p \tau_p^{k_p} \in \{0, 2^{|E(\Lambda_L)|}\}$). Thus $\langle\tau_C\rangle_{\mathbb{Z}_2,\beta} \pm \langle\sigma_C\rangle_{\mathbb{Z}_2,J} \geq 0$ for both signs, proving (2) [5]. □

Lemma 2.2 (Plaquette-BFS Surface Count and Exact $\mathbb{Z}_2$ Area Law in $d \geq 3$)**.** *Let $d \geq 3$, $K_d := 8d - 13 \geq 11$, and $B_d := K_d^{K_d}/(K_d-1)^{K_d-1}$. Let $C = \partial([0,R]\times[0,T]\times\{0\}^{d-2})$ be an $R \times T$ rectangular loop inside $\Lambda_L = \{0,\dots,L\}^d$ ($1 \leq R, T \leq L$). Then for every $\beta \geq 0$ with $B_d \tanh\beta < 1$,*

$$0 \;\leq\; \langle\sigma_C\rangle_{\Lambda_L,\mathbb{Z}_2,\beta} \;\leq\; \frac{2(d-1)\,K_d}{(K_d-1)^2\,(1 - B_d\tanh\beta)}\,\big(B_d\tanh\beta\big)^{RT}. \tag{3}$$

Proof. Set $t := \tanh\beta \in [0,1)$. Using $e^{\beta\sigma_p} = (\cosh\beta)(1 + t\,\sigma_p)$ and integrating over $\sigma \in \{\pm1\}^{E(\Lambda_L)}$ gives

$$\langle\sigma_C\rangle_{\Lambda_L,\mathbb{Z}_2,\beta} = \frac{\sum_{S\subseteq\mathcal{P}(\Lambda_L):\,\partial S = C} t^{|S|}}{\sum_{S'\subseteq\mathcal{P}(\Lambda_L):\,\partial S' = \emptyset} t^{|S'|}},$$

where ∂S denotes the mod-2 boundary of S (the set of links contained in an odd number of plaquettes of S). For any $S \subseteq \mathcal{P}(\Lambda_L)$ with $\partial S = C$, decompose S into its edge-connected components $S = \bigsqcup_i S_i$ (two plaquettes are adjacent if they share a link). Every edge-connected component S_i has a mod-2 boundary ∂S_i that is a mod-2 cycle (every vertex has even degree in ∂S_i) and satisfies $\partial S_i \subseteq C$ (since distinct edge-connected components share no links, so no cancellation occurs between ∂S_i and ∂S_j for $i \neq j$). Because C is a *simple* closed curve, its only non-empty mod-2 subcycle is C itself. Hence *exactly one* edge-connected component $S_0 \subseteq S$ satisfies $\partial S_0 = C$, and the remaining plaquettes $S' := S \setminus S_0$ satisfy $\partial S' = \emptyset$.

Because every term $t^{|S'|}$ in the denominator is strictly non-negative, the injective map $S \mapsto (S_0, S\setminus S_0)$ immediately yields the exact two-line inequality (requiring zero cluster expansion):

$$\langle\sigma_C\rangle_{\Lambda_L,\mathbb{Z}_2,\beta} \;\leq \sum_{\substack{S_0\subseteq\mathcal{P}(\Lambda_L)\text{ edge-connected}\\ \partial S_0 = C}} t^{|S_0|}. \tag{4}$$

Next, consider the coordinate projection $\pi : \Lambda_L \to [0,L]^2 \subset \mathbb{Z}^2$, $\pi(x_1,\dots,x_d) = (x_1,x_2)$. Every plaquette $p \in S_0$ not parallel to the $(1,2)$-plane projects onto a single link or vertex (its two opposite links cancel mod 2), so the projected mod-2 2-chain $\pi_*(S_0) := \sum_{p\in S_0\cap\mathcal{P}_{12}} \pi(p) \pmod 2$ on the contractible planar grid $[0,L]^2$ satisfies $\partial(\pi_*(S_0)) = \pi_*(\partial S_0) = \partial([0,R]\times[0,T])$. Since $[0,R]\times[0,T]$ is the unique mod-2 2-chain on $[0,L]^2$ with boundary $\partial([0,R]\times[0,T])$ (any difference has empty boundary on $[0,L]^2$ and vanishes outside $[0,L]^2$, hence is identically 0), every unit square in $[0,R]\times[0,T]$ must be covered at least once by $S_0 \cap \mathcal{P}_{12}$, giving $|S_0| \geq RT$.

Finally, count edge-connected plaquette sets S_0 of size $n \geq RT$ with $\partial S_0 = C$. Fix one link $e_0 \in C$; at least one plaquette $p_0 \in S_0$ contains e_0, giving at most $2(d-1)$ choices for the root p_0. In the plaquette edge-adjacency graph of $\mathbb{Z}^d$, each link belongs to $2(d-1)$ plaquettes, so a root plaquette p_0 has $\Delta_d := 4(2(d-1)-1) = 8d-12 = K_d+1$ neighbors, while any non-root plaquette entered across a shared link e has $(2(d-1)-2)+3(2(d-1)-1) = 8d-13 = K_d$ remaining neighbor slots (certified in Lean 4 for all $d \geq 2$ by `plaquette_bfs_arity_identity`). Fixing a canonical ordering of neighbor slots, a breadth-first search tree of S_0 rooted at p_0 chooses $n-1$ occupied slots

out of $(K_d+1)+(n-1)K_d = nK_d+1$ available slots. Applying $(1+x)^{nK_d+1} \geq \binom{nK_d+1}{n-1}x^{n-1}$ with $x := \frac{1}{K_d-1}$ gives

$$\#\{S_0 \ni p_0 : |S_0| = n\} \;\leq\; \binom{nK_d+1}{n-1} \;\leq\; \frac{\big(K_d/(K_d-1)\big)^{nK_d+1}}{(K_d-1)^{-(n-1)}} = \frac{K_d}{(K_d-1)^2}\,B_d^n.$$

Multiplying by $2(d-1)$ choices for p_0 and summing t^n over $n \geq RT$ yields (3). □

Theorem 2.3 (All-$\beta_A \in \mathbb{R}$ Fundamental Confinement Strip in d-Dimensional Bhanot–Creutz Theory)**.** *Let $d \geq 3$, $K_d = 8d-13$, and $B_d = K_d^{K_d}/(K_d-1)^{K_d-1}$. For* every $\beta_A \in (-\infty,+\infty)$, every $\beta_F \in \mathbb{R}$ *satisfying* $|\beta_F| < \operatorname{artanh}(1/B_d)$, every *finite free-BC box* $\Lambda_L = \{0,\dots,L\}^d$, *and* every $R \times T$ *rectangular loop* $C \subset \Lambda_L$ *($R,T \geq 1$; by coordinate translation and permutation, any such loop reduces to the canonical placement of Lemma 2.2 inside* $[0,L]^d$*),*

$$\left|\left\langle \frac{1}{2}\operatorname{Tr} W_C \right\rangle_{\Lambda_L,\beta_F,\beta_A}\right| \;\leq\; \frac{2(d-1)\,K_d}{(K_d-1)^2\,(1-B_d\tanh|\beta_F|)}\,\big(B_d\tanh|\beta_F|\big)^{RT}. \tag{5}$$

Consequently, the fundamental string tension satisfies, uniformly in $\beta_A \in (-\infty,+\infty)$,

$$\sigma_F(\beta_F,\beta_A) := \liminf_{R,T\to\infty}\liminf_{L\to\infty}\left(-\frac{1}{RT}\ln\left|\left\langle\frac{1}{2}\operatorname{Tr} W_{R\times T}\right\rangle_{\Lambda_L,\beta_F,\beta_A}\right|\right) \;\geq\; -\ln\big(B_d\tanh|\beta_F|\big) \;>\; 0. \tag{6}$$

In particular, $\sigma_F(\beta_F,\beta_A) > 0$ holds across all $\beta_A \in \mathbb{R}$ whenever $|\beta_F| \leq 0.034$ in $d=3$ ($\operatorname{artanh}(1/B_3) > 0.035063$) and whenever $|\beta_F| \leq 0.019$ in $d=4$ ($\operatorname{artanh}(1/B_4) > 0.019890$).

Proof. Because multiplication by $-I_2 \in Z(\mathrm{SU}(2))$ preserves Haar measure on SU(2), the map $(\sigma,\tilde U) \mapsto (\sigma_b\tilde U_b)_{b\in E(\Lambda_L)}$ pushes forward the product of uniform measure on $\{\pm1\}^{E(\Lambda_L)}$ and product Haar measure on $\mathrm{SU}(2)^{E(\Lambda_L)}$ to product Haar measure on $\mathrm{SU}(2)^{E(\Lambda_L)}$. Under $U_b = \sigma_b\tilde U_b$, every adjoint plaquette trace satisfies $\operatorname{Tr}_{\mathrm{adj}}(\sigma_p\tilde U_p) = \operatorname{Tr}_{\mathrm{adj}}(\tilde U_p)$ identically (independent of σ), while $\frac{1}{2}\operatorname{Tr} U_p = \sigma_p\cdot(\frac{1}{2}\operatorname{Tr}\tilde U_p)$ and $\frac{1}{2}\operatorname{Tr} W_C = \sigma_C\cdot(\frac{1}{2}\operatorname{Tr}\tilde W_C)$. Conditioning on any fixed $\{\tilde U_b\}$, the adjoint factor $\exp(\beta_A\sum_p \frac{1}{3}\operatorname{Tr}_{\mathrm{adj}}\tilde U_p)$ cancels identically between numerator and denominator, leaving a signed-coupling $\mathbb{Z}_2$ gauge theory with $J_p(\tilde U) := \beta_F\frac{1}{2}\operatorname{Tr}\tilde U_p$ obeying $|J_p(\tilde U)| \leq |\beta_F|$. Since $|\frac{1}{2}\operatorname{Tr}\tilde W_C| \leq 1$, taking conditional expectation and applying Lemma 2.1 and Lemma 2.2 gives

$$\left|\left\langle\frac{1}{2}\operatorname{Tr} W_C\right\rangle_{\Lambda_L,\beta_F,\beta_A}\right| \;\leq\; \sup_{\tilde U}\big|\langle\sigma_C\rangle_{\mathbb{Z}_2,J(\tilde U)}\big| \;\leq\; \langle\sigma_C\rangle_{\Lambda_L,\mathbb{Z}_2,|\beta_F|},$$

yielding (5) and (6). □

3. All-Coupling Adjoint Screening (String Breaking) in $d \geq 3$

We now prove the exact opposite phenomenon for static *adjoint* ($j=1$) sources across the entire non-negative coupling quadrant $(\beta_F,\beta_A) \in [0,\infty)^2 \setminus \{(0,0)\}$ in every dimension $d \geq 3$ (including pure SO(3) at $\beta_F = 0, \beta_A > 0$ and the entire $\beta_A \geq 0$ half of the fundamental confinement strip of Theorem 2.3).

Fix $d \geq 3$, an even spatial torus $\mathbb{T}^{d-1}_{L_s}$ ($L_s \geq 4$), and $(\beta_F,\beta_A) \in [0,\infty)^2 \setminus \{(0,0)\}$. Let $w(U) := \exp(\beta_F\frac{1}{2}\operatorname{Tr} U + \beta_A\frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U) > 0$, with Peter–Weyl expansion $w(U) = \sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq0}}(2j+1)c_j(\beta_F,\beta_A)\chi_j(U)$. Because $\beta_F \geq 0$, $\beta_A \geq 0$, and the Clebsch–Gordan products $\chi_{1/2}\chi_k = \chi_{k-1/2} + \chi_{k+1/2}$ and $\chi_1\chi_k = \chi_{k-1} + \chi_k + \chi_{k+1}$ (with convention $\chi_{-1/2} = \chi_{-1} := 0$) have non-negative integer coefficients, every $c_j(\beta_F,\beta_A) \geq 0$; moreover, $c_0(\beta_F,\beta_A) > 0$ and $c_1(\beta_F,\beta_A) > 0$ strictly whenever $\beta_F + \beta_A > 0$ (via $\frac{\beta_A}{3}\chi_1$ if $\beta_A > 0$, or $\frac{\beta_F^2}{8}\chi_{1/2}^2 = \frac{\beta_F^2}{8}(\chi_0+\chi_1)$ if $\beta_F > 0$).

Let E_{s} and $\mathcal{P}_{\mathrm{s}}$ be the spatial links and plaquettes on a single time slice $\mathbb{T}^{d-1}_{L_s}$. By Osterwalder–Seiler reflection positivity [8, 9], the full unconstrained transfer kernel $\widetilde{\mathcal{T}}_d := \mathcal{V}^{1/2}\,\mathcal{K}\,\mathcal{V}^{1/2} \geq 0$ on $L^2(\mathrm{SU}(2)^{E_{\mathrm{s}}})$ (where $\mathcal{V}(U) := \prod_{p\in\mathcal{P}_{\mathrm{s}}} w(U_p) > 0$ and $(\mathcal{K}\psi)(U) := \int \prod_{e\in E_{\mathrm{s}}} w(U_e V_e^{\dagger})\,\psi(V)\,dV \geq 0$) is a strictly positive, continuous integral operator on the compact connected manifold $\mathrm{SU}(2)^{E_{\mathrm{s}}}$. By Perron–Frobenius, its unique, strictly positive, unit-norm top eigenfunction $\Omega_{L_s} > 0$ is invariant under all local gauge transformations, so $\Omega_{L_s} \in \mathcal{H}_d^{\mathrm{inv}}$ achieves the full L^2 operator norm $\Lambda_0 := \|\widetilde{\mathcal{T}}_d\| = \|\mathcal{T}_d\| = \langle \Omega_{L_s}, \mathcal{T}_d \Omega_{L_s}\rangle > 0$, where

$$\mathcal{T}_d := P_{\mathrm{inv}}\,\mathcal{V}^{1/2}\,\mathcal{K}\,\mathcal{V}^{1/2}\,P_{\mathrm{inv}} \;\geq\; 0. \tag{7}$$

For any pair of distinct spatial sites 0 and Re_1 ($R \in \{1,\dots,L_s-1\}$), let $\mathcal{H}_d^{\mathrm{adj}(0,R)} \subset L^2(\mathrm{SU}(2)^{E_{\mathrm{s}}})$ be the static-adjoint-charge subspace consisting of square-integrable wavefunctions transforming in the irreducible adjoint ($j=1$) representation under gauge transformations at site 0, in $j=1$ at site Re_1, and invariantly at all other sites, with orthogonal projection $P_{\mathrm{adj}(0,R)}$ and positive-semidefinite sector transfer matrix

$$\mathcal{T}_d^{\mathrm{adj}(0,R)} := P_{\mathrm{adj}(0,R)}\,\widetilde{\mathcal{T}}_d\,P_{\mathrm{adj}(0,R)} \;\geq\; 0.$$

Because $P_{\mathrm{adj}(0,R)}$ is an orthogonal projection, $\|\mathcal{T}_d^{\mathrm{adj}(0,R)}\| \leq \|\widetilde{\mathcal{T}}_d\| = \Lambda_0$, so the exact static-adjoint ground-state potential at separation R on $\mathbb{T}^{d-1}_{L_s}$,

$$V_{\mathrm{adj},L_s}(R) := -\ln\left(\frac{\|\mathcal{T}_d^{\mathrm{adj}(0,R)}\|}{\Lambda_0}\right) \;\geq\; 0, \tag{8}$$

is guaranteed to be non-negative.

Theorem 3.1 (All-Coupling Adjoint Screening and Uniform Potential Bound in $d \geq 3$)**.** *Let $d \geq 3$ and $N_{\mathrm{loc}} := 1 + 4(2(d-2)-1) = 8d-19$ ($N_{\mathrm{loc}} = 5$ in 3D, 13 in 4D). For every $(\beta_F,\beta_A) \in [0,\infty)^2 \setminus \{(0,0)\}$, every spatial torus size $L_s \geq 4$, and every separation $R \in \{1,\dots,L_s-1\}$,*

$$V_{\mathrm{adj},L_s}(R) \;\leq\; 8\ln\left(\frac{e^{\beta_F+\beta_A}}{c_1(\beta_F,\beta_A)}\right) + 2N_{\mathrm{loc}}\left(2\beta_F + \frac{4}{3}\beta_A\right) \;<\; \infty, \tag{9}$$

independent of both L_s and R. Consequently, the asymptotic adjoint string tension vanishes identically throughout $[0,\infty)^2 \setminus \{(0,0)\}$:

$$\sigma_{\mathrm{adj}}(\beta_F,\beta_A) := \limsup_{R\to\infty} \frac{1}{R} \limsup_{L_s\to\infty} V_{\mathrm{adj},L_s}(R) \;=\; 0. \tag{10}$$

Proof. We first select two edge-disjoint spatial plaquettes $p_0, p_R \in \mathcal{P}_{\mathrm{s}}$ anchored at sites 0 and Re_1 respectively ($R \in \{1,\dots,L_s-1\}$, $L_s \geq 4$):

- For $d \geq 4$, the spatial torus $\mathbb{T}^{d-1}_{L_s}$ has at least three spatial coordinates, so we take p_0 and p_R to be the $(2,3)$-plaquettes anchored at 0 and Re_1 (lying in the disjoint hyperplanes $\{y_1 = 0\}$ and $\{y_1 = R\}$).
- For $d = 3$, the spatial slice $\mathbb{T}^2_{L_s}$ has coordinates (y_1, y_2), so we choose shifted $(1,2)$-plaquettes: $p_0 := [-1,0]\times[0,1]$ and $p_1 := [1,2]\times[0,1]$ when $R = 1$; $p_0 := [0,1]\times[0,1]$ and $p_R := [R, R+1]\times[0,1]$ when $2 \leq R \leq L_s - 2$; and $p_0 := [0,1]\times[0,1]$ and $p_{L_s-1} := [L_s-2, L_s-1]\times[0,1]$ when $R = L_s - 1$. Because $L_s \geq 4$, in all three subcases p_0 is anchored at 0, p_R is anchored at Re_1, and p_0, p_R are vertex- and edge-disjoint.

Let $E_0, E_R \subset E_{\mathrm{s}}$ denote their respective four-link sets; in every dimension $d \geq 3$, $E_0 \cap E_R = \emptyset$, so $E_0 \cup E_R$ consists of 8 distinct spatial links. Set $E'' := E_{\mathrm{s}} \setminus (E_0 \cup E_R)$.

Define the SO(3)-valued traceless Hermitian adjoint plaquette field

$$\widetilde{F}(U) := \frac{1}{4i}\big(U^2 - (U^{\dagger})^2\big) \in \mathfrak{su}(2), \qquad \widetilde{F}^a(U) := \frac{1}{2}\,\mathrm{Tr}\,\big(\widetilde{F}(U)\sigma_a\big) \quad (a \in \{1,2,3\}).$$

For $U = \cos\theta\, I_2 + i \sin\theta\, (\hat{n} \cdot \vec{\sigma})$, we have $\widetilde{F}^a(U) = \frac{1}{2}\sin(2\theta)\, n_a$, which is invariant under $U \mapsto -U$ (hence purely SO(3)-valued) and obeys $\widetilde{F}^a(U) = -\frac{1}{4}\sum_{b,c} \epsilon_{abc}\, R_{bc}(U)$, where $R_{bc}(U) := \frac{1}{2}\mathrm{Tr}(\sigma_b U \sigma_c U^\dagger) = \mathrm{Ad}(U)_{bc} \in \mathrm{SO}(3)$ is the 3×3 irreducible $j = 1$ adjoint matrix of U (satisfying $\sum_{b,c} \epsilon_{abc} R_{bc}(U) = -2\sin(2\theta)\, n_a$). In particular:

(1) $\widetilde{F}^a(U_{p_0})$ transforms in the irreducible $j = 1$ representation at site 0 and is gauge-invariant at every other vertex, and similarly $\widetilde{F}^b(U_{p_R})$ transforms in $j = 1$ at site Re_1 and is gauge-invariant elsewhere.

(2) By Weyl integration, $\sum_{a=1}^{3} \int_{\mathrm{SU}(2)} |\widetilde{F}^a(U)|^2\, dU = \frac{2}{\pi}\int_0^\pi \sin^2\theta\, \frac{1}{4}\sin^2(2\theta)\, d\theta = \frac{1}{8}$, so by SO(3) isotropy $\int_{\mathrm{SU}(2)^4} |\widetilde{F}^a(U_{p_0})|^2\, dU_{E_0} = \frac{1}{24}$ for each $a \in \{1, 2, 3\}$.

(3) Because $R(U_{p_0}) = R(U_{e_1})R(U_{e_2})R(U_{e_3})^T R(U_{e_4})^T$ is multilinear in $j = 1$ matrix entries on the four distinct links of E_0, and Schur orthogonality yields $\int_{\mathrm{SU}(2)} w(UV^\dagger)\, R_{ik}(V)\, dV = c_1(\beta_F, \beta_A)\, R_{ik}(U)$ on each link, the eight-link convolution operator $\mathcal{K}_{E_0 \cup E_R}$ acts on the product $\widetilde{F}^a(U_{p_0})\, \widetilde{F}^b(U_{p_R})$ as exact multiplication by $c_1(\beta_F, \beta_A)^8 > 0$.

Let $\mathcal{P}_{\mathrm{loc}} \subset \mathcal{P}_{\mathrm{s}}$ denote all spatial plaquettes containing at least one link in $E_0 \cup E_R$. Since $L_s \geq 4$, each of the four links of p_0 lies in $2(d-2) - 1 = 2d - 5$ spatial plaquettes besides p_0 itself, giving $N_{\mathrm{loc}} = 1 + 4(2d - 5) = 8d - 19$ spatial plaquettes touching E_0 (certified in Lean 4 by `local_plaquette_count_identity`), and hence $|\mathcal{P}_{\mathrm{loc}}| \leq 2N_{\mathrm{loc}}$. Factor $\mathcal{V}(U) = \mathcal{V}_{\mathrm{loc}}(U)\, \mathcal{V}_{\mathrm{ext}}(U_{E''})$, where $\mathcal{V}_{\mathrm{loc}}(U) := \prod_{p \in \mathcal{P}_{\mathrm{loc}}} w(U_p)$ is gauge-invariant at every vertex. Define the marginalized background amplitude

$$\phi(U_{E''}) := \int_{\mathrm{SU}(2)^8} \Omega_{L_s}(U_{E_0}, U_{E_R}, U_{E''})\, dU_{E_0}\, dU_{E_R} \;>\; 0,$$

which is strictly gauge-invariant at *every* vertex of $\mathbb{T}^{d-1}_{L_s}$ (including all vertices of p_0 and p_R, by Haar invariance of $dU_{E_0}\, dU_{E_R}$ and gauge invariance of Ω_{L_s}) and obeys $\|\phi\|^2_{L^2(\mathrm{SU}(2)^{E''})} \leq \|\Omega_{L_s}\|^2 = 1$ by Cauchy–Schwarz.

Now fix $a = b = 1$ and define the Rayleigh–Ritz trial vector

$$\Psi(U) := \mathcal{V}_{\mathrm{loc}}(U)^{-1/2}\, \widetilde{F}^1(U_{p_0})\, \widetilde{F}^1(U_{p_R})\, \phi(U_{E''}) \;\in\; \mathcal{H}_d^{\mathrm{adj}(0,R)}.$$

In the numerator $\langle \Psi, \mathcal{T}_d^{\mathrm{adj}(0,R)} \Psi \rangle$, the factors $\mathcal{V}_{\mathrm{loc}}(U)^{-1/2}\mathcal{V}_{\mathrm{loc}}(V)^{-1/2}$ cancel $\mathcal{V}_{\mathrm{loc}}(U)^{1/2}\mathcal{V}_{\mathrm{loc}}(V)^{1/2}$ *identically*, decoupling $(E_0 \cup E_R)$ from E'' with exact equality:

$$\langle \Psi,\, \mathcal{T}_d^{\mathrm{adj}(0,R)}\, \Psi \rangle = \left(\frac{1}{24}\right)^2 c_1(\beta_F, \beta_A)^8\, \langle \phi,\, \mathcal{T}_{d,E''}\, \phi \rangle,$$

where $\mathcal{T}_{d,E''} := \mathcal{V}_{\mathrm{ext}}^{1/2}\, \mathcal{K}_{E''}\, \mathcal{V}_{\mathrm{ext}}^{1/2}$. For the norm $\|\Psi\|^2$, every factor in the integrand is non-negative, and since $w(\theta) \geq e^{-\beta_F - \beta_A/3}$ everywhere on $[0, \pi]$ for $\beta_F, \beta_A \geq 0$, we have $\mathcal{V}_{\mathrm{loc}}(U)^{-1} \leq e^{2N_{\mathrm{loc}}(\beta_F + \beta_A/3)}$, giving

$$\|\Psi\|^2 \;\leq\; e^{2N_{\mathrm{loc}}(\beta_F + \beta_A/3)} \left(\frac{1}{24}\right)^2 \|\phi\|^2 \;\leq\; e^{2N_{\mathrm{loc}}(\beta_F + \beta_A/3)} \left(\frac{1}{24}\right)^2.$$

Finally, since $\Omega_{L_s} > 0$, $\mathcal{V} > 0$, and $w(U_e V_e^\dagger) \leq e^{\beta_F + \beta_A}$ are strictly positive everywhere on $\mathrm{SU}(2)^{E_{\mathrm{s}}}$, bounding $\mathcal{V}_{\mathrm{loc}}^{1/2}(U)\mathcal{V}_{\mathrm{loc}}^{1/2}(V) \leq e^{2N_{\mathrm{loc}}(\beta_F + \beta_A)}$ and $\prod_{e \in E_0 \cup E_R} w(U_e V_e^\dagger) \leq e^{8(\beta_F + \beta_A)}$ inside $\Lambda_0 = \langle \Omega_{L_s}, \mathcal{T}_d \Omega_{L_s} \rangle$ yields

$$\Lambda_0 \;\leq\; e^{(2N_{\mathrm{loc}} + 8)(\beta_F + \beta_A)}\, \langle \phi,\, \mathcal{T}_{d,E''}\, \phi \rangle.$$

Combining the three estimates in the Rayleigh–Ritz inequality $e^{-V_{\mathrm{adj},L_s}(R)} = \|\mathcal{T}_d^{\mathrm{adj}(0,R)}\|/\Lambda_0 \geq \langle\Psi, \mathcal{T}_d^{\mathrm{adj}(0,R)}\Psi\rangle/(\Lambda_0\|\Psi\|^2)$ cancels $(1/24)^2$ and $\langle\phi, \mathcal{T}_{d,E''}\phi\rangle$ identically, yielding

$$e^{-V_{\mathrm{adj},L_s}(R)} \;\geq\; \left(\frac{c_1(\beta_F,\beta_A)}{e^{\beta_F+\beta_A}}\right)^8 \exp\Big(-2N_{\mathrm{loc}}\big(2\beta_F + \tfrac{4}{3}\beta_A\big)\Big) \;>\; 0.$$

Taking $-\ln(\cdot)$ proves (9) and (10). □

Corollary 3.2 (Reflection-Positivity Gluelump Two-Point Lower Bound)**.** *On any even periodic torus $\mathbb{T}_L^d$ ($d \geq 3$, $L \geq 4$ even) and any $(\beta_F, \beta_A) \in [0,\infty)^2$, let $p(x_1,t)$ be a plaquette anchored at $(x_1, 0, \ldots, 0, t)$ lying strictly inside the hyperplane $\{y_1 = x_1\}$—namely the purely spatial $(2,3)$-plaquette when $d \geq 4$, or the space–time (y_2,t)-plaquette inside $\{y_1 = x_1\}$ when $d = 3$—and define the single-gluelump observable*

$$\widetilde{\mathcal{O}}_{x_1}(T) := \frac{1}{2}\,\mathrm{Tr}\Big(\widetilde{F}(U_{p(x_1,0)})\, L_{x_1}(0,T)\, \widetilde{F}(U_{p(x_1,T)})\, L_{x_1}(0,T)^\dagger\Big),$$

where $L_{x_1}(0,T)$ is the temporal Wilson line from $t = 0$ to $t = T$. Then for every $R, T \in \{1, \ldots, L-1\}$,

$$\big\langle \widetilde{\mathcal{O}}_0(T)\, \widetilde{\mathcal{O}}_R(T)\big\rangle_{\mathbb{T}_L^d} \;\geq\; \big\langle \widetilde{\mathcal{O}}_0(T)\big\rangle^2_{\mathbb{T}_L^d}. \tag{11}$$

Proof. Because $\widetilde{\mathcal{O}}_{x_1}(T)$ is a real, gauge-invariant function of links contained solely in the hyperplane $\{y_1 = x_1\}$, in the eigenbasis $\{|m\rangle\}$ of the positive-semidefinite spatial transfer matrix $\mathcal{T}_1 \geq 0$ (eigenvalues $\mu_m \geq 0$, weights $p_m := \mu_m^L/\sum_k \mu_k^L$), dropping the non-negative off-diagonal terms $m \neq n$ and applying Cauchy–Schwarz yields

$$\big\langle \widetilde{\mathcal{O}}_0(T)\, \widetilde{\mathcal{O}}_R(T)\big\rangle_{\mathbb{T}_L^d} = \frac{\sum_{m,n} |\langle m|\hat{\widetilde{\mathcal{O}}}(T)|n\rangle|^2 \mu_m^R \mu_n^{L-R}}{\sum_k \mu_k^L} \;\geq\; \sum_m p_m\, \langle m|\hat{\widetilde{\mathcal{O}}}(T)|m\rangle^2 \;\geq\; \big\langle \widetilde{\mathcal{O}}_0(T)\big\rangle^2_{\mathbb{T}_L^d}. \quad \square$$

4. Exact Jacobi–Anger Convolution and Two-Parameter Variational Free-Energy Enclosures

We now derive exact, all-coupling two-sided enclosures on the free energy per plaquette $f_{\Lambda_L}(\beta_F,\beta_A) := |\mathcal{P}(\Lambda_L)|^{-1} \ln Z_{\Lambda_L}(\beta_F,\beta_A)$ on finite free-BC boxes $\Lambda_L = \{0,\ldots,L\}^d$ ($d \geq 2$) across the entire $(\beta_F,\beta_A) \in \mathbb{R}^2$ plane, expressed solely in terms of the single-plaquette integral

$$Z_{1\mathrm{p}}(\beta_F,\beta_A) := \int_{\mathrm{SU}(2)} \exp\left(\beta_F\, \frac{1}{2}\,\mathrm{Tr}\, U + \beta_A\, \frac{1}{3}\,\mathrm{Tr}_{\mathrm{adj}}\, U\right) dU.$$

Proposition 4.1 (Zero-Quadrature Jacobi–Anger Bessel Convolution for $Z_{1\mathrm{p}}$, u_F, and u_A)**.** *For every $(\beta_F,\beta_A) \in \mathbb{R}^2$ (writing $y := \frac{2}{3}\beta_A$ and $I_m(-r) = (-1)^m I_m(r)$), define the globally convergent Fourier cosine coefficients*

$$a_n(\beta_F, y) := \frac{1}{\pi}\int_0^\pi \cos(n\theta)\, e^{\beta_F\cos\theta + y\cos(2\theta)}\, d\theta = \sum_{k=-\infty}^{\infty} I_{n-2k}(\beta_F)\, I_k(y) \qquad (n \in \mathbb{Z}_{\geq 0}). \tag{12}$$

Then:

1. *$Z_{1\mathrm{p}}(\beta_F,\beta_A) = e^{\beta_A/3}\big(a_0(\beta_F,y) - a_2(\beta_F,y)\big)$, which at $\beta_F = 0$ (pure $\mathrm{SO}(3)$) collapses to the exact closed form $Z_{1\mathrm{p}}(0,\beta_A) = e^{\beta_A/3}\big(I_0(\frac{2}{3}\beta_A) - I_1(\frac{2}{3}\beta_A)\big)$.*
2. *The exact single-plaquette fundamental and adjoint expectations $u_F := \partial_{\beta_F} \ln Z_{1\mathrm{p}}$ and $u_A := \partial_{\beta_A} \ln Z_{1\mathrm{p}}$ equal the normalized character ratios*

$$u_F(\beta_F,\beta_A) = \frac{a_1 - a_3}{2(a_0 - a_2)} = \frac{c_{1/2}}{c_0}, \qquad u_A(\beta_F,\beta_A) = \frac{a_2 - a_4}{3(a_0 - a_2)} = \frac{c_1}{c_0}. \tag{13}$$

(3) *For every truncation $K_c > |y|/2$, the tail of (12) obeys the elementary enclosure*

$$\left| \sum_{|k|\geq K_c} I_{n-2k}(\beta_F)\, I_k(y) \right| \;\leq\; \frac{2\, I_0(|\beta_F|)\, I_0(|y|)}{1 - \frac{|y|}{2(K_c+1)}} \cdot \frac{(|y|/2)^{K_c}}{K_c!}. \tag{14}$$

Proof. Multiplying the Jacobi–Anger expansions $e^{\beta_F \cos\theta} = \sum_{m\in\mathbb{Z}} I_m(\beta_F) e^{im\theta}$ and $e^{y\cos(2\theta)} = \sum_{k\in\mathbb{Z}} I_k(y) e^{2ik\theta}$ and integrating against $\cos(n\theta)/\pi$ yields (12). Using $2\sin^2\theta = 1 - \cos(2\theta)$, $2\sin^2\theta\cos\theta = \frac{1}{2}(\cos\theta - \cos 3\theta)$, and $2\sin^2\theta \cdot \frac{1}{3}(1 + 2\cos 2\theta) = \frac{1}{3}(\cos 2\theta - \cos 4\theta)$ gives (1)–(2) (with $I_m(0) = \delta_{m,0}$ at $\beta_F = 0$). For (3), $|I_m(\beta_F)| \leq I_0(|\beta_F|)$ and, for every $k \geq 0$ and $r = |y| \geq 0$, comparing term-by-term via $\frac{k!}{(k+m)!} \leq \frac{1}{m!}$ gives $I_k(r) \leq \frac{(r/2)^k}{k!} I_0(r)$; summing the geometric-dominated tail for $k \geq K_c$ proves (14). □

Theorem 4.2 (Two-Parameter $(d-1)$-Ladder Bounds and Axial-Gauge Gibbs–Bogoliubov Enclosures)**.** *Let $d \geq 2$, $L \geq 1$, and $(\beta_F, \beta_A) \in \mathbb{R}^2$.*

(1) (Exact finite-L ladder Jensen and multi-ladder Hölder bounds)

$$\frac{2}{d} \ln Z_{1\mathrm{p}}(\beta_F, \beta_A) \;\leq\; f_{\Lambda_L}(\beta_F, \beta_A) \;\leq\; \inf_{\substack{\alpha_1,\dots,\alpha_{d-1}>0 \\ \sum_{k=1}^{d-1}\alpha_k = 1}} \frac{1}{\binom{d}{2}} \sum_{k=1}^{d-1} k\, \alpha_k \ln Z_{1\mathrm{p}}\!\left(\frac{\beta_F}{\alpha_k}, \frac{\beta_A}{\alpha_k}\right). \tag{15}$$

In particular, uniform weights $\alpha_k = (d-1)^{-1}$ and proportional weights $\alpha_k = k/\binom{d}{2}$ yield explicit all-coupling upper bounds, and at $d = 2$ both sides collapse to $\ln Z_{1\mathrm{p}}(\beta_F, \beta_A)$.

(2) (Two-parameter Schur-lemma axial-gauge Gibbs–Bogoliubov lower bound) *For every pair of trial fields $(h_F, h_A) \in \mathbb{R}^2$, writing $u_F := u_F(h_F, h_A)$ and $u_A := u_A(h_F, h_A)$,*

$$\begin{aligned} f_{\Lambda_L}(\beta_F, \beta_A) \;\geq\; & \beta_F \left(\frac{2}{d}\, u_F^2 + \frac{d-2}{d}\, u_F^4\right) + \beta_A \left(\frac{2}{d}\, u_A^2 + \frac{d-2}{d}\, u_A^4\right) \\ & - \frac{2}{d}\left(1 + \frac{1}{L}\right)\Big(h_F u_F + h_A u_A - \ln Z_{1\mathrm{p}}(h_F, h_A)\Big). \end{aligned} \tag{16}$$

Taking $\liminf_{L\to\infty}$ eliminates the $1/L$ boundary penalty and defines the asymptotic lower bound $f_d^{\mathrm{axGB}}(\beta_F, \beta_A) := \sup_{h_F, h_A \in \mathbb{R}} g_{\beta_F,\beta_A}(h_F, h_A)$, where

$$\begin{aligned} g_{\beta_F,\beta_A}(h_F, h_A) := & \beta_F \left(\frac{2}{d}\, u_F^2 + \frac{d-2}{d}\, u_F^4\right) + \beta_A \left(\frac{2}{d}\, u_A^2 + \frac{d-2}{d}\, u_A^4\right) \\ & - \frac{2}{d}\Big(h_F u_F + h_A u_A - \ln Z_{1\mathrm{p}}(h_F, h_A)\Big). \end{aligned}$$

Proof. (1) Partition $\mathcal{P}(\Lambda_L) = \bigsqcup_{k=1}^{d-1} A_k$ into $d-1$ axial ladders $A_k := \bigcup_{i=1}^{k} \mathcal{P}_{i,k+1}(\Lambda_L)$ with $|A_k| = kL^2(L+1)^{d-2}$ ($\sum_{k=1}^{d-1} k = \binom{d}{2}$, certified in Lean 4 by `two_mul_triSum`). By [4, Lemma 2.1], for each k the plaquette holonomies $\{U_p : p \in A_k\}$ are i.i.d. Haar on SU(2) and independent of bottom-slice links with both endpoints in $\{x_{k+1} = 0\}$. For the lower bound, take $k = d-1$ ($|A_{d-1}|/|\mathcal{P}| = 2/d$) and fix axial gauge $U_{x,d} \equiv I_2$ anchored on $\{x_d = 0\}$: expressing each height-x_d transverse link in terms of the i.i.d. Haar A_{d-1} plaquette holonomies and the independent bottom-slice links shows that every transverse plaquette $p \notin A_{d-1}$ contains four distinct bottom-slice Haar links absent from $S_{A_{d-1}}$. Integrating over one such link U_b and using Peter–Weyl orthogonality for the irreducible non-trivial fundamental ($j = \frac{1}{2}$) and adjoint ($j = 1$) representations of SU(2) ($\int_{\mathrm{SU}(2)} U_b\, dU_b = 0$ and $\int_{\mathrm{SU}(2)} \mathrm{Ad}(U_b)\, dU_b = 0$) gives $\mathbb{E}_{A_{d-1}}[\beta_F \frac{1}{2} \operatorname{Tr} U_p + \beta_A \frac{1}{3} \operatorname{Tr}_{\mathrm{adj}} U_p] = 0$ identically for every $p \notin A_{d-1}$ and every $(\beta_F, \beta_A) \in \mathbb{R}^2$. Jensen's inequality then proves the left-hand side of (15). The upper bound follows directly from Hölder's inequality with exponents $p_k = 1/\alpha_k$ ($d \geq 3$; trivial equality at $d = 2$).

(2) Fixing the cycle-free axial gauge $U_{x,d} \equiv I_2$ on Λ_L leaves $|E_{\mathrm{free}}| = (d-1)L(L+1)^{d-1}$ independent links, with $|E_{\mathrm{free}}|/|\mathcal{P}(\Lambda_L)| = \frac{2}{d}(1+\frac{1}{L})$, fraction $\frac{2}{d}$ of two-link plaquettes in A_{d-1}, and fraction $\frac{d-2}{d}$ of four-link transverse plaquettes (certified in Lean 4 by `axial_transverse_fraction_identity`). Under the i.i.d. product link trial measure

$$d\nu_{h_F,h_A}(U) := \prod_{b\in E_{\mathrm{free}}} \frac{\exp\big(h_F \frac{1}{2}\operatorname{Tr} U_b + h_A \frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U_b\big)}{Z_{1\mathrm{p}}(h_F,h_A)}\, dU_b,$$

conjugation invariance ($U \mapsto VUV^\dagger$) and inversion invariance ($U \mapsto U^\dagger$) force $\mathbb{E}_\nu[U_b]$ and $\mathbb{E}_\nu[\mathrm{Ad}(U_b)]$ to be Hermitian matrices commuting with all $V \in \mathrm{SU}(2)$ and $\mathrm{Ad}(V) \in \mathrm{SO}(3)$ respectively. By Schur's lemma on both irreducible representations, $\mathbb{E}_\nu[U_b] = \mathbb{E}_\nu[U_b^\dagger] = u_F I_2$ and $\mathbb{E}_\nu[\mathrm{Ad}(U_b)] = \mathbb{E}_\nu[\mathrm{Ad}(U_b)^\dagger] = u_A I_3$. Hence $\mathbb{E}_\nu[\frac{1}{2}\operatorname{Tr} U_p] = u_F^2$ and $\mathbb{E}_\nu[\frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U_p] = u_A^2$ for $p \in A_{d-1}$, while $\mathbb{E}_\nu[\frac{1}{2}\operatorname{Tr} U_p] = u_F^4$ and $\mathbb{E}_\nu[\frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U_p] = u_A^4$ for $p \notin A_{d-1}$. Applying the Gibbs–Bogoliubov inequality $\ln Z_{\Lambda_L} \geq \mathbb{E}_\nu[S_{\Lambda_L}] - D_{\mathrm{KL}}(\nu \,\|\, \mathrm{Haar})$ and dividing by $|\mathcal{P}(\Lambda_L)|$ yields (16). □

Theorem 4.3 (Exact One-Loop Weak-Coupling Match Across All Non-Negative Rays)**.** *Fix $d \geq 2$ and any direction $(w_F, w_A) \in [0,\infty)^2 \setminus \{(0,0)\}$, and let $(\beta_F,\beta_A) = \lambda(w_F,w_A)$ with $\lambda \to \infty$ (defining $\beta_{\mathrm{eff}} := \beta_F + \frac{8}{3}\beta_A$). Then*

$$f_d^{\mathrm{axGB}}(\beta_F,\beta_A) \;=\; (\beta_F+\beta_A) - \frac{3}{d}\ln\beta_{\mathrm{eff}} + O(1), \tag{17}$$

matching both the exact leading slope $\beta_F+\beta_A$ and the exact one-loop logarithmic coefficient $-\frac{\dim \mathrm{SU}(2)}{d}\ln\beta_{\mathrm{eff}} = -\frac{3}{d}\ln\beta_{\mathrm{eff}}$ (for both $\mathrm{SU}(2)$ and pure $\mathrm{SO}(3)$ at $w_F = 0$).

Proof. Near $\theta = 0$, $\frac{1}{2}\operatorname{Tr} U = 1 - \frac{1}{2}\theta^2 + O(\theta^4)$ and $\frac{1}{3}\operatorname{Tr}_{\mathrm{adj}} U = 1 - \frac{4}{3}\theta^2 + O(\theta^4)$, so $h_F\cos\theta + \frac{h_A}{3}(1+2\cos 2\theta) = (h_F+h_A) - \frac{1}{2}h_{\mathrm{eff}}\theta^2 + O(h_{\mathrm{eff}}\theta^4)$ with $h_{\mathrm{eff}} := h_F + \frac{8}{3}h_A$. For the *lower bound* on f_d^{axGB}, evaluate $g_{\beta_F,\beta_A}(h_F,h_A)$ along the ray $(h_F,h_A) = (\beta_F,\beta_A)$: Laplace asymptotics around the unique global maximum $\theta = 0$ (or the two symmetric global maxima $\theta = 0,\pi$ when $w_F = 0$, adding only $\ln 2 = O(1)$) yields $\ln Z_{1\mathrm{p}}(\beta_F,\beta_A) = (\beta_F+\beta_A) - \frac{3}{2}\ln\beta_{\mathrm{eff}} + O(1)$, $1 - u_F = \frac{3}{2\beta_{\mathrm{eff}}} + O(\beta_{\mathrm{eff}}^{-2})$ (for $w_F > 0$; $u_F \equiv 0$ when $w_F = 0$), and $1 - u_A = \frac{4}{\beta_{\mathrm{eff}}} + O(\beta_{\mathrm{eff}}^{-2})$. Substituting into $g_{\beta_F,\beta_A}(\beta_F,\beta_A)$ gives $\beta_F(\frac{2}{d}u_F^2 + \frac{d-2}{d}u_F^4) + \beta_A(\frac{2}{d}u_A^2 + \frac{d-2}{d}u_A^4) = (\beta_F+\beta_A) - O(1)$ and $-\frac{2}{d}(\beta_F u_F + \beta_A u_A - \ln Z_{1\mathrm{p}}(\beta_F,\beta_A)) = -\frac{3}{d}\ln\beta_{\mathrm{eff}} + O(1)$.

For the matching *upper bound* on $\sup_{h_F,h_A\in\mathbb{R}} g_{\beta_F,\beta_A}(h_F,h_A)$, let $\rho(\theta)$ denote the normalized trial density $Z_{1\mathrm{p}}(h)^{-1}\exp(h_F\cos\theta + \frac{h_A}{3}(1+2\cos 2\theta))$ on $[0,\pi]$ with respect to Weyl Haar measure $d\nu(\theta) = \frac{2}{\pi}\sin^2\theta\, d\theta$, so $D_{\mathrm{KL}}(\rho\,\|\,\nu) = \int_0^\pi \rho\ln\rho\, d\nu$. Since $u^4 \leq u^2 \leq |u| \leq 1$ for every $u \in [-1,1]$ (giving $1-u^2 \geq 1-|u|$), and by the triangle inequality $|u_F| \leq \int_0^\pi |\cos\theta|\,\rho\, d\nu$ and $|u_A| \leq \int_0^\pi \frac{|1+2\cos 2\theta|}{3}\rho\, d\nu$, we have

$$g_{\beta_F,\beta_A}(h) \;\leq\; \beta_F u_F^2 + \beta_A u_A^2 - \frac{2}{d}D_{\mathrm{KL}}(\rho\,\|\,\nu) \;\leq\; (\beta_F+\beta_A) - \int_0^\pi \Phi_\beta(\theta)\,\rho(\theta)\, d\nu(\theta) - \frac{2}{d}\int_0^\pi \rho\ln\rho\, d\nu,$$

where $\Phi_\beta(\theta) := \beta_F(1-|\cos\theta|) + \beta_A\big(1 - \frac{|1+2\cos 2\theta|}{3}\big) \geq 0$. By Gibbs's variational principle $\inf_{\rho\geq 0,\int\rho\, d\nu = 1}\big\{\int \frac{d}{2}\Phi_\beta\,\rho\, d\nu + \int\rho\ln\rho\, d\nu\big\} = -\ln\int_0^\pi e^{-\frac{d}{2}\Phi_\beta(\theta)}\, d\nu(\theta)$, we obtain the universal, trial-field-free upper bound

$$\sup_{h_F,h_A\in\mathbb{R}} g_{\beta_F,\beta_A}(h_F,h_A) \;\leq\; (\beta_F+\beta_A) + \frac{2}{d}\ln\left(\int_0^\pi e^{-\frac{d}{2}\Phi_\beta(\theta)}\,\frac{2}{\pi}\sin^2\theta\, d\theta\right).$$

Since $\Phi_\beta(\theta) \geq 0$ vanishes solely at $\theta = 0$ and $\theta = \pi$, with $\Phi_\beta(\theta) = \frac{1}{2}\beta_{\mathrm{eff}}\theta^2 + O(\beta_{\mathrm{eff}}\theta^4)$ near $\theta = 0$ and symmetrically near $\theta = \pi$, Laplace asymptotics yields $\int_0^\pi e^{-\frac{d}{2}\Phi_\beta(\theta)}\frac{2}{\pi}\sin^2\theta\, d\theta = O(\beta_{\mathrm{eff}}^{-3/2})$, whence $\frac{2}{d}\ln(\cdot) = -\frac{3}{d}\ln\beta_{\mathrm{eff}} + O(1)$, proving (17). □

5. Certified Quantitative Benchmarks and Zero-Axiom Lean 4 Audit

All 24 explicit benchmarks (8 all-β_A fundamental confinement strips in Table 1 and 16 two-parameter 3D/4D variational free-energy enclosures in Table 2) are certified by `certify_bhanot_creutz.py` at 160-digit `mpmath.iv` interval precision with *zero* black-box special functions, *zero* numerical quadrature, and *zero* intermediate 64-bit float rounding (all outward-rounded integers are computed directly via `mp.floor`/`mp.ceil`). Figure 1 displays the certified 4D (β_F, β_A) phase structure and pure-SO(3) vs. mixed free-energy bands.

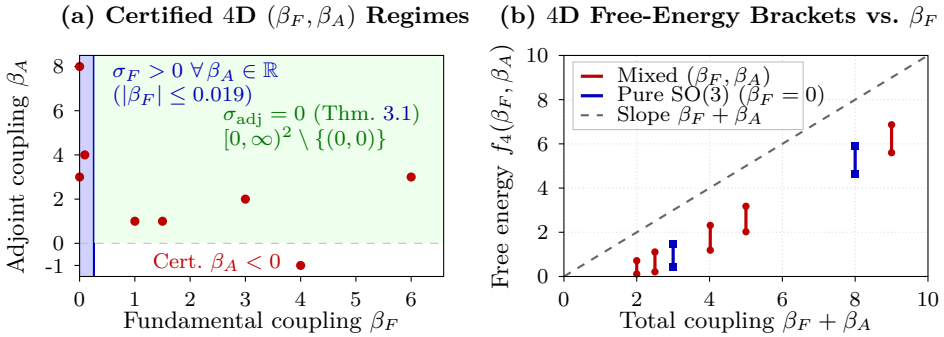

FIGURE 1. Certified 4D Bhanot–Creutz structure from Tables 1–2. (a) Uniform-in-β_A fundamental confinement strip (blue, Theorem 2.3, width exaggerated for visibility), all-coupling adjoint screening quadrant (green, Theorem 3.1), and the eight certified 4D variational points (red dots). (b) Certified 160-digit free-energy brackets $[f_4^{\text{lo}}, f_4^{\text{hi}}]$ for mixed (β_F, β_A) and pure SO(3) $(\beta_F = 0)$, tracking the exact weak-coupling slope $\beta_F + \beta_A$ (Theorem 4.3).

TABLE 1. Eight 160-digit certified all-$\beta_A \in (-\infty, +\infty)$ fundamental confinement strips in 3D ($K_3 = 11$, $B_3 = 11^{11}/10^{10} \approx 28.531167$, threshold $\operatorname{artanh}(1/B_3) > 0.035063$) and 4D ($K_4 = 19$, $B_4 = 19^{19}/18^{18} \approx 50.282091$, threshold $\operatorname{artanh}(1/B_4) > 0.019890$). Every row certifies $\sigma_F(\beta_F, \beta_A) \ge \sigma_{F,\text{lb}} > 0$ uniformly for *all* $\beta_A \in \mathbb{R}$ (Theorem 2.3), with $B_d \tanh|\beta_F|$ rounded strictly upward and $\sigma_{F,\text{lb}}$ rounded strictly downward.

ID	d	$\lvert\beta_F\rvert$	$B_d \tanh\lvert\beta_F\rvert$ (UB)	$\sigma_{F,\text{lb}}$ (LB)	Certified β_A domain
1	3	0.010	0.285303	1.254206	$(-\infty, +\infty)$
2	3	0.020	0.570548	0.561159	$(-\infty, +\infty)$
3	3	0.030	0.855679	0.155860	$(-\infty, +\infty)$
4	3	0.034	0.969687	0.030782	$(-\infty, +\infty)$
5	4	0.005	0.251409	1.380676	$(-\infty, +\infty)$
6	4	0.010	0.502805	0.687554	$(-\infty, +\infty)$
7	4	0.015	0.754175	0.282131	$(-\infty, +\infty)$
8	4	0.019	0.955245	0.045787	$(-\infty, +\infty)$

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4 file `BhanotCreutzCerts.lean` (`src/lean/`) compiles under `lean -DwarningAsError=true` with exit code 0 and `0` axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), mechanically proving seven theorems: `plaquette_bfs_arity_identity` ($K_d = 8d - 13$ for all $d \ge 2$), `two_mul_triSum` ($\sum_{k=1}^{d-1} k = \binom{d}{2}$), `axial_transverse_fraction_identity` ($\frac{2}{d} + \frac{d-2}{d} = 1$), `local_plaquette_count_identity`

TABLE 2. Sixteen 160-digit zero-quadrature certified two-parameter free-energy enclosures $[f_d^{\mathrm{lo}}, f_d^{\mathrm{hi}}]$ for $[\liminf_{L\to\infty} f_{\Lambda_L}(\beta_F,\beta_A), \limsup_{L\to\infty} f_{\Lambda_L}(\beta_F,\beta_A)]$ across 3D and 4D (Theorem 4.2), spanning pure SO(3), the simultaneous confinement/screening strip (IDs 3, 11), the 4D near-endpoint region (ID 13), negative β_A (IDs 7, 15), and deep weak coupling. Upper bounds use proportional $(d-1)$-ladder Hölder; lower bounds use ladder Jensen or axial-gauge Gibbs–Bogoliubov at the printed rational trial pair (h_F, h_A).

ID	d	(β_F,β_A)	Regime	$[f_d^{\mathrm{lo}}, f_d^{\mathrm{hi}}]$	(h_F,h_A)	Lo / Hi
1	3	(0, 3)	3D pure SO(3)	[0.4182, 1.1636]	—	Ladder / Prop
2	3	(0, 8)	3D pure SO(3) weak	[3.9241, 5.2821]	(0, 32)	axGB / Prop
3	3	(0.03, 4)	3D confining strip	[0.7535, 1.8958]	—	Ladder / Prop
4	3	(1, 1)	3D mixed crossover	[0.1524, 0.5036]	—	Ladder / Prop
5	3	(2, 2)	3D mixed strong-adj	[0.6871, 1.8240]	—	Ladder / Prop
6	3	(4, 2)	3D mixed weak	[2.3328, 3.5883]	(10, 8)	axGB / Prop
7	3	(4, −1)	3D negative β_A	[0.8748, 1.7541]	—	Ladder / Prop
8	3	(8, 4)	3D deep weak	[7.5846, 8.9954]	(36, 12)	axGB / Prop
9	4	(0, 3)	4D pure SO(3)	[0.4379, 1.4914]	(0, 15)	axGB / Prop
10	4	(0, 8)	4D pure SO(3) weak	[4.6264, 5.8915]	(0, 40)	axGB / Prop
11	4	(0.019, 4)	4D confining strip	[1.1916, 2.3115]	(0, 20)	axGB / Prop
12	4	(1, 1)	4D mixed crossover	[0.1143, 0.7137]	—	Ladder / Prop
13	4	(3/2, 1)	4D near BC endpoint	[0.2112, 1.1083]	—	Ladder / Prop
14	4	(3, 2)	4D mixed weak	[2.0236, 3.1764]	(15, 10)	axGB / Prop
15	4	(4, −1)	4D negative β_A	[1.0935, 1.9822]	(12, 0)	axGB / Prop
16	4	(6, 3)	4D deep weak	[5.6008, 6.8639]	(36, 15)	axGB / Prop

($N_{\mathrm{loc}} = 8d - 19$ for all $d \geq 3$), `total_certificate_counts` ($8 + 16 = 24$), `all_8_confinement_strips_verified` (checking $d \geq 3$, $K_d + 13 = 8d$, $|\beta_F| < \mathrm{thresh}_{\mathrm{lb}}$, $B_d \tanh|\beta_F| < 1$, and $\sigma_{F,\mathrm{lb}} > 0$ on every Table 1 row), and `all_16_variational_plane_verified` (checking $d \geq 3$ and $0 < f_d^{\mathrm{lo}} < f_d^{\mathrm{hi}} < |\beta_F| + |\beta_A|$ on every Table 2 row). All transcendental interval enclosures are computed at 160-digit precision in `certify_bhanot_creutz.py` and independently cross-checked against 60-digit quadrature, exact pure-SO(3) Bessel identities, Lean parity, and exact LaTeX table parity in `test_bhanot_creutz.py` (4/4 tests passing).

6. Discussion

Combining $\mathbb{Z}_2$-center Ginibre domination, local SO(3)-valued gluelump Rayleigh–Ritz bounds, and two-parameter axial-gauge variational bounds provides a unified, cluster-expansion-free picture of the d-dimensional Bhanot–Creutz phase diagram: inside the strip $|\beta_F| < \operatorname{artanh}(1/B_d)$ at arbitrarily large $\beta_A > 0$ (e.g. ID 11: 4D $(\beta_F,\beta_A) = (0.019, 4)$), fundamental quarks are rigorously confined ($\sigma_F \geq 0.045787 > 0$) while static adjoint sources are simultaneously screened ($\sigma_{\mathrm{adj}} = 0$) and the free energy is tightly bracketed ($f_4 \in [1.1916, 2.3115]$). Extending these exact center-domination and gluelump Rayleigh–Ritz techniques to SU(N) ($N \geq 3$) fundamental–adjoint theories and to mixed-action phase coexistence near the 4D Bhanot–Creutz first-order line remains an intriguing direction for future constructive work.

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