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EXACT ALL-MASS, ALL-COUPLING FREE-ENERGY, CHIRAL-CONDENSATE, AND DUAL-BRANCH BAKRY–ÉMERY VENEZIANO LARGE-N BOUNDS FOR d-DIMENSIONAL SU(N) LATTICE QCD WITH DYNAMICAL STAGGERED (KOGUT–SUSSKIND) FERMIONS

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Abstract. Whereas hopping-expansion analyses of dynamical Wilson fermions are restricted to heavy bare masses ($2d\kappa_f < 1$, i.e. bare diagonal parameter $1/(2\kappa_f) > d$), the anti-Hermitian staggered (Kogut–Susskind) Dirac operator $D_{\mathrm{KS}}(U;m_0) = m_0 I + D_{\mathrm{hop}}(U)$ on a d-dimensional periodic hypercubic torus $\mathbb{T}^d_L$ ($d \geq 2$, L even) satisfies $\det D_{\mathrm{KS}}(U;m_0) > 0$ strictly for *every* bare quark mass $m_0 \in (0,\infty)$ and every gauge configuration $U \in \mathrm{SU}(N)^{|E|}$. Without invoking polymer or cluster expansions, we establish three self-contained non-perturbative theorems valid across *all* bare masses $m_0 \in (0,\infty)$ and all flavor weights $N_{\mathrm{stag}} > 0$:

(1) *All-mass, all-coupling free-energy enclosures via Hermite quadratic spectral brackets:* Using exact operator norm $\|D_{\mathrm{hop}}(U)\|_{\mathrm{op}} \leq d$, exact gauge-independent second trace $\mathrm{Tr}(A(U)) = \frac{d}{2}VN$ ($A := -D^2_{\mathrm{hop}}$), and exact fourth-power plaquette trace $\mathrm{Tr}(A(U)^2) = \frac{d(4d-1)}{8}VN - \frac{1}{2}\sum_p \mathrm{Re}\,\mathrm{Tr}(U_p)$ on even $L \geq 6$, Hermite quadratic interpolation of $\frac{1}{2}\ln(m_0^2+t)$ at nodes $(0,\frac{d}{2},\frac{d}{2})$ and $(\frac{d}{2},\frac{d}{2},d^2)$ brackets $f_{\mathrm{KS}}(d,N,\beta,N_{\mathrm{stag}},m_0)$ between shifted pure-gauge enclosures at $\beta^{\mathrm{lo}}_{\mathrm{eff}}(m_0) > \beta^{\mathrm{hi}}_{\mathrm{eff}}(m_0) > \beta$ across all $m_0 > 0$ and $\beta \geq 0$, including light quarks $m_0 \ll d$.

(2) *All-mass dynamical plaquette and chiral-condensate enclosures:* A two-line exact rational spectral identity for $\frac{m_0}{m_0^2+t}$ combined with finite-L convexity certifies strictly positive staggered chiral-condensate brackets $\langle\bar\chi\chi\rangle_{\mathrm{KS}} \in [\Sigma_-,\Sigma_+]$ strictly above the Jensen floor $\Sigma_J = \frac{m_0}{m_0^2+d/2}$ across light ($m_0 < d$) and heavy ($m_0 > d$) quarks in 2D and 4D SU(2)–SU(4) QCD.

(3) *All-mass dual-branch Bakry–Émery curvature and light-quark 't Hooft/Veneziano factorization:* Combining an unsubtracted $O(1/m_0^2)$ geodesic Hessian bound with an exact $O(1/m_0^4)$ bound obtained by subtracting $\frac{1}{2m_0^2}\frac{d^2}{dt^2}\mathrm{Tr}(D_{\mathrm{hop}}(U(t))^2) \equiv 0$ certifies a volume-uniform log-Sobolev inequality and $O(1/N^2)$ plaquette factorization $\mathrm{Var}_{\mathrm{KS}}(P_p) \leq \frac{4}{\kappa_{\mathrm{KS}}N}$ on all even $L \geq 4$, reaching light quarks $m_0 = 1 < d = 4$ in 't Hooft scaling and $m_0 = 3 < d = 4$ in Veneziano scaling ($N_{\mathrm{stag}} = N/2$).

Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`StaggeredQcdCerts.lean`) verify all 24 numerical bounds.

1. Introduction and Honest Scope

Coupling d-dimensional ($d \geq 2$) SU(N) Wilson lattice gauge theory [12] to $N_{\mathrm{stag}} > 0$ flavors (one unrooted staggered flavor corresponds to $2^{d/2}$ continuum tastes in even d, while fractional $N_{\mathrm{stag}} > 0$ covers rooted staggered formulations) of dynamical Kogut–Susskind (staggered) fermions [6, 9, 5, 8, 4] on a periodic hypercubic torus $\mathbb{T}^d_L$ (L even, $V = L^d$, $|E| = dV$, $|P| = \binom{d}{2}V$) yields the Euclidean partition function

$$\begin{aligned} Z_{\mathrm{KS}}(\mathbb{T}^d_L;N,\beta,N_{\mathrm{stag}},m_0) &:= \int_{\mathrm{SU}(N)^{|E|}} e^{S_W(\beta;U)}\big(\det D_{\mathrm{KS}}(U;m_0)\big)^{N_{\mathrm{stag}}}\,dU, \\ S_W(\beta;U) &:= \frac{\beta}{N}\sum_{p\in P}\mathrm{Re}\,\mathrm{Tr}(U_p), \qquad D_{\mathrm{KS}}(U;m_0) := m_0 I + D_{\mathrm{hop}}(U), \end{aligned} \tag{1}$$

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where dU is normalized product Haar measure on $\mathrm{SU}(N)^{|E|}$, $m_0 > 0$ is the bare quark mass, and the anti-Hermitian staggered hopping operator $D_{\mathrm{hop}}(U)$ acts on $\ell^2(\mathbb{T}_L^d;\mathbb{C}^N)$ by

$$\big(D_{\mathrm{hop}}(U)\psi\big)(x) := \frac{1}{2}\sum_{\mu=1}^{d}\eta_\mu(x)\Big(U_{x,\mu}\,\psi(x+e_\mu) - U^\dagger_{x-e_\mu,\mu}\,\psi(x-e_\mu)\Big), \tag{2}$$

with standard Kogut–Susskind sign factors $\eta_1(x) := 1$ and $\eta_\mu(x) := (-1)^{x_1+\cdots+x_{\mu-1}}$ $(2 \le \mu \le d)$, obeying $\eta_\mu(x)\eta_\nu(x+e_\mu)\eta_\mu(x+e_\nu)\eta_\nu(x) = -1$ around every elementary plaquette $p = (x, \mu < \nu)$.

Write $f_{\mathrm{KS},L}(\beta, m_0) := V^{-1}\ln Z_{\mathrm{KS}}(\mathbb{T}_L^d; N, \beta, N_{\mathrm{stag}}, m_0)$ for the finite-volume per-site free energy and $f_{\mathrm{KS}}(d, N, \beta, N_{\mathrm{stag}}, m_0)$ for any subsequential thermodynamic limit as even $L \to \infty$ (all inequalities in Theorems 2.2(2) and 3.1(2) hold for both $\liminf_{L\to\infty}$ and $\limsup_{L\to\infty}$). Note that Lemma 2.1(1)–(2) and Theorem 4.1 hold identically on every finite even $L \ge 4$, while Lemma 2.1(3) and the pointwise identities Theorems 2.2(1) and 3.1(1) hold identically on every finite even $L \ge 6$; only the pure-gauge axial-gauge enclosures in Theorems 2.2(2) and 3.1(2) carry vanishing boundary terms ($O(1/L^2)$ in $d = 2$, $O(1/L)$ in $d \ge 3$).

Remark 1.1 (Honest Scope, Quantitative Limitations, and Non-Claims). All results hold at fixed lattice spacing ($a = 1$) for arbitrary bare quark mass $m_0 \in (0, \infty)$ (breaking completely past the $m_0 > d$ hopping-expansion convergence radius). Theorems 2.2 and 3.1 hold at *all* gauge couplings $\beta \ge 0$, whereas Theorem 4.1 requires $\kappa_{\mathrm{KS}} > 0$—in particular, the deep strong-coupling threshold $\beta < N^2/(16(d-1))$ ($\beta < N^2/48$ in 4D)—and certifies a volume-uniform log-Sobolev inequality and $O(1/N^2)$ factorization rather than spontaneous chiral symmetry breaking ($m_0 \to 0^+$) or a 4D continuum limit. Quantitatively, in 4D the pure-gauge axial-gauge bounds (8) have wide free-energy gaps (4.0455–23.5522 in Table 1), capping $P_+ = 1.000000$ in Table 2 Rows 5–8, though both $P_- > 0$ and the excess condensate $\Sigma_- - \Sigma_J > 0$ remain strictly nontrivial across all rows. This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. All-Mass Free-Energy Enclosures via Hermite Quadratic Spectral Brackets

Define the lattice-averaged plaquette $\overline{P}(U) := \frac{1}{|P|N}\sum_{p\in P}\mathrm{Re}\,\mathrm{Tr}(U_p) \in [-1,1]$ and the positive operator $A(U) := -D_{\mathrm{hop}}(U)^2 = D_{\mathrm{hop}}(U)^\dagger D_{\mathrm{hop}}(U)$.

Lemma 2.1 (Staggered Spectrum, Gauge-Independent Second Trace, and Fourth-Power Plaquette Trace)**.** *Fix $d \ge 2$, $N \ge 2$, and $U \in \mathrm{SU}(N)^{|E(\mathbb{T}_L^d)|}$.*

1. *For every even $L \ge 2$, $D_{\mathrm{hop}}(U)^\dagger = -D_{\mathrm{hop}}(U)$, $\|D_{\mathrm{hop}}(U)\|_{\mathrm{op}} \le d$, $0 \le A(U) \le d^2 I$, and for every $m_0 > 0$, $\det D_{\mathrm{KS}}(U; m_0) = (\det(m_0^2 I + A(U)))^{1/2} > 0$ strictly.*
2. *For every even $L \ge 4$, $\mathrm{Tr}(A(U)) = -\mathrm{Tr}(D_{\mathrm{hop}}(U)^2) = \frac{d}{2}VN$ identically (independent of U).*
3. *For every even $L \ge 6$,*

$$\mathrm{Tr}\big(A(U)^2\big) = \mathrm{Tr}\big(D_{\mathrm{hop}}(U)^4\big) = \frac{d(4d-1)}{8}\,VN - \frac{1}{2}\sum_{p\in P}\mathrm{Re}\,\mathrm{Tr}(U_p), \tag{3}$$

so the normalized spectral variance $\sigma^2(U) := \frac{1}{VN}\mathrm{Tr}\big((A(U) - \frac{d}{2}I)^2\big)$ satisfies

$$\sigma^2(U) = \frac{d(2d-1)}{8} - \frac{1}{2}\binom{d}{2}\overline{P}(U) \;\ge\; \frac{d}{8} \;>\; 0 \qquad (\forall U \in \mathrm{SU}(N)^{|E|}). \tag{4}$$

Proof. (1) Anti-Hermiticity $D_{\mathrm{hop}}(U)^\dagger = -D_{\mathrm{hop}}(U)$ is immediate from (2) and $\eta_\mu(x) \in \{\pm 1\}$. Writing $(T_\mu(U)\psi)(x) := \eta_\mu(x)U_{x,\mu}\psi(x+e_\mu)$, each $T_\mu(U)$ is unitary, so $\|D_{\mathrm{hop}}(U)\|_{\mathrm{op}} = \|\frac{1}{2}\sum_{\mu=1}^d(T_\mu - T_\mu^\dagger)\|_{\mathrm{op}} \le d$, giving $0 \le A(U) \le d^2 I$. The site-parity operator $(\epsilon\psi)(x) := (-1)^{x_1+\cdots+x_d}\psi(x)$ satisfies $\epsilon^2 = I$ and $\epsilon D_{\mathrm{hop}}(U)\epsilon = -D_{\mathrm{hop}}(U)$, so nonzero eigenvalues of $D_{\mathrm{hop}}(U)$ occur in conjugate pairs

$\pm i\lambda_j$ ($\lambda_j \in [0,d]$), whence $\det(m_0 I + D_{\rm hop}) = \prod_{j=1}^{VN}(m_0^2+\lambda_j^2)^{1/2} = (\det(m_0^2 I + A(U)))^{1/2} > 0$ for all $m_0 > 0$.

(2) On even $L \geq 4$, every length-2 closed walk backtracks along one of the $2d$ directed links incident to x, carrying $(+\frac{1}{2}\eta_\mu U_e)(-\frac{1}{2}\eta_\mu U_e^\dagger) = -\frac{1}{4}I_N$; summing over all $2d$ directions and V sites gives $\mathrm{Tr}(D_{\rm hop}^2) = -\frac{d}{2}VN$.

(3) On even $L \geq 6$, no length-4 closed walk wraps around $\mathbb{T}_L^d$. Every length-4 closed walk starting at x falls into one of three disjoint classes: (i) single-direction walks ($\binom{4}{2} = 6$ balanced sequences per direction μ, each with product $+\frac{1}{16}I_N$), contributing $\frac{6d}{16} = \frac{3d}{8}$ per site and color; (ii) two-direction backtracking tree walks for each unordered pair $\{\mu,\nu\}$ (8 sequential pairs $(a,-a,b,-b)$ plus 8 nested pairs $(a,b,-b,-a)$, each with product $+\frac{1}{16}I_N$ since $\eta_\mu^2 = \eta_\nu^2 = 1$), contributing $\frac{16}{16}\binom{d}{2} = \binom{d}{2}$; and (iii) elementary plaquette walks (4 corners $\times 2$ orientations per $p \in P$, each carrying two minus signs from backward steps and the staggered plaquette product $\eta_\mu(x)\eta_\nu(x+e_\mu)\eta_\mu(x+e_\nu)\eta_\nu(x) = -1$), contributing $4(-\frac{1}{16})(\mathrm{Tr}\,U_p + \mathrm{Tr}\,U_p^\dagger) = -\frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}(U_p)$. Summing gives $\frac{3d}{8} + \binom{d}{2} = \frac{d(4d-1)}{8}$, proving (3). Subtracting $(\frac{d}{2})^2 = \frac{d^2}{4}$ yields (4), with minimum $\frac{d(2d-1)}{8} - \frac{d(d-1)}{4} = \frac{d}{8} > 0$ at $\overline{P}(U) = 1$. □

Let

$$c_0(\mathrm{SU}(N),\alpha) := \int_{\mathrm{SU}(N)} e^{\frac{\alpha}{N}\,\mathrm{Re}\,\mathrm{Tr}(U)}\,dU = \sum_{q=-\infty}^{\infty} \det\big[I_{j-k+q}(\alpha/N)\big]_{j,k=1}^{N}$$

be the exact Bars–Green single-link integral [3] ($2I_1(\alpha)/\alpha$ for SU(2)). Set

$$u_1(\mathrm{SU}(N),\alpha) := \partial_\alpha \ln c_0(\mathrm{SU}(N),\alpha) \in [0,1), \qquad f_{\mathrm{YM},L}^{\rm site}(d,N,\beta') := V^{-1}\ln Z_{\rm YM}(\mathbb{T}_L^d;N,\beta').$$

Theorem 2.2 (All-Mass, All-Coupling Staggered QCD Free-Energy Enclosures)**.** *Fix $d \geq 2$, $N \geq 2$, $N_{\rm stag} > 0$, $\beta \geq 0$, and $m_0 \in (0,\infty)$. Define the strictly positive Hermite curvature coefficients*

$$\begin{aligned} |c_-(d,m_0)| &:= \frac{\frac{1}{2}\ln\!\Big(1+\frac{d}{2m_0^2}\Big) - \frac{d}{4(m_0^2+d/2)}}{(d/2)^2} \;>\; 0, \\ |c_+(d,m_0)| &:= \frac{\frac{d^2-d/2}{2(m_0^2+d/2)} - \frac{1}{2}\ln\!\Big(\frac{m_0^2+d^2}{m_0^2+d/2}\Big)}{(d^2-d/2)^2} \;>\; 0, \end{aligned} \tag{5}$$

which satisfy $0 < |c_+(d,m_0)| < |c_-(d,m_0)|$ for all $m_0 > 0$, together with the shifted couplings $\beta_{\rm eff}^{\rm lo} := \beta + \Delta\beta_{\rm lo}$ and $\beta_{\rm eff}^{\rm hi} := \beta + \Delta\beta_{\rm hi}$, where $\Delta\beta_{\rm lo} := \frac{N_{\rm stag}N}{2}|c_-(d,m_0)| > \Delta\beta_{\rm hi} := \frac{N_{\rm stag}N}{2}|c_+(d,m_0)| > 0$.

(1) *For every even $L \geq 6$ and every $U \in \mathrm{SU}(N)^{|E(\mathbb{T}_L^d)|}$,*

$$-|c_-(d,m_0)|\,\sigma^2(U) \;\leq\; \frac{1}{VN}\ln\det D_{\rm KS}(U;m_0) - \frac{1}{2}\ln\!\Big(m_0^2+\frac{d}{2}\Big) \;\leq\; -|c_+(d,m_0)|\,\sigma^2(U). \tag{6}$$

(2) *Consequently, writing $C_{\rm lo/hi} := N_{\rm stag}N\big[\frac{1}{2}\ln(m_0^2+\frac{d}{2}) - |c_\mp(d,m_0)|\frac{d(2d-1)}{8}\big]$, every subsequential thermodynamic-limit per-site free energy obeys*

$$\begin{aligned} f_{\rm KS}^{\rm lo} &:= C_{\rm lo} + f_{\rm YM}^{\rm lo}(d,N,\beta_{\rm eff}^{\rm lo}) \\ &\leq\; f_{\rm KS}(d,N,\beta,N_{\rm stag},m_0) \;\leq\; C_{\rm hi} + f_{\rm YM}^{\rm hi}(d,N,\beta_{\rm eff}^{\rm hi}) =: f_{\rm KS}^{\rm hi}, \end{aligned} \tag{7}$$

where in $d=2$, $f_{\rm YM}^{\rm lo}(2,N,\beta') = f_{\rm YM}^{\rm hi}(2,N,\beta') = \ln c_0(\mathrm{SU}(N),\beta')$ is exact, and for $d \geq 3$,

$$\begin{aligned} f_{\rm YM}^{\rm hi}(d,N,\beta') &:= (d-1)\ln c_0\Big(\mathrm{SU}(N),\tfrac{d}{2}\beta'\Big), \\ f_{\rm YM}^{\rm lo}(d,N,\beta') &:= \sup_{\alpha\geq 0}\left\{(d-1)\beta'\,u_1(\alpha)^2 + \binom{d-1}{2}\beta'\,u_1(\alpha)^4 + (d-1)\big[\ln c_0(\alpha) - \alpha u_1(\alpha)\big]\right\}. \end{aligned} \tag{8}$$

Proof. (1) Set $g(t) := \frac{1}{2}\ln(m_0^2+t)$ on $[0,d^2]$. Since $g'''(t) = (m_0^2+t)^{-3} > 0$ strictly on $[0,d^2]$, Hermite quadratic interpolation at nodes $(a,b,c) \in [0,d^2]^3$ satisfies $g(t) - Q(t) = \frac{g'''(\xi)}{6}(t-a)(t-b)(t-c)$ for some $\xi \in (0,d^2)$. Choosing $(a,b,c) = (0, \frac{d}{2}, \frac{d}{2})$ gives $t(t-\frac{d}{2})^2 \geq 0$ and $Q_-(t) = g(\frac{d}{2}) + g'(\frac{d}{2})(t-\frac{d}{2}) - |c_-|(t-\frac{d}{2})^2 \leq g(t)$; choosing $(a,b,c) = (\frac{d}{2}, \frac{d}{2}, d^2)$ gives $(t-\frac{d}{2})^2(t-d^2) \leq 0$ and $Q_+(t) = g(\frac{d}{2}) + g'(\frac{d}{2})(t-\frac{d}{2}) - |c_+|(t-\frac{d}{2})^2 \geq g(t)$. Both $|c_\pm| > 0$ and $|c_+| < |c_-|$ follow directly from strict concavity of g and $g''' > 0$ (since $Q_-(t) \leq g(t) \leq Q_+(t)$ for all $t \in [0,d^2]$). Evaluating at the eigenvalues $t_j = \lambda_j(U)^2 \in [0,d^2]$ of $A(U)$ and averaging over $1 \leq j \leq VN$ eliminates the linear term $g'(\frac{d}{2})(t_j - \frac{d}{2})$ identically by Lemma 2.1(2), yielding (6).

(2) Substituting (4) into (6), multiplying by $N_{\mathrm{stag}}VN$, exponentiating, and integrating against $e^{S_W(\beta;U)}dU$ yields the exact finite-volume inequality

$$C_{\mathrm{lo}} + f^{\mathrm{site}}_{\mathrm{YM},L}(d,N,\beta^{\mathrm{lo}}_{\mathrm{eff}}) \;\leq\; f_{\mathrm{KS},L}(\beta,m_0) \;\leq\; C_{\mathrm{hi}} + f^{\mathrm{site}}_{\mathrm{YM},L}(d,N,\beta^{\mathrm{hi}}_{\mathrm{eff}}).$$

In $d=2$, fixing a maximal spanning tree on $\mathbb{T}^2_L$ leaves $V-1$ independent plaquettes plus 2 Polyakov-loop variables determining the V-th plaquette U_{p_0} ($|\frac{\beta'}{N}\operatorname{Re}\operatorname{Tr} U_{p_0}| \leq \beta'$), giving $|f^{\mathrm{site}}_{\mathrm{YM},L}(2,N,\beta') - \ln c_0(\mathrm{SU}(N),\beta')| \leq (\beta' + \ln c_0(\beta'))/L^2 = O(1/L^2)$. For $d \geq 3$, d-way Hölder's inequality and Gibbs' variational principle in axial gauge $U_{x,1} = I$ on $L-1$ slices yield $f^{\mathrm{lo}}_{\mathrm{YM}}(\beta') - O(1/L) \leq f^{\mathrm{site}}_{\mathrm{YM},L}(d,N,\beta') \leq f^{\mathrm{hi}}_{\mathrm{YM}}(\beta') + O(1/L)$, and passing to $\liminf$ / $\limsup_{L\to\infty}$ proves (7). □

3. All-Mass Dynamical Plaquette and Staggered Chiral-Condensate Enclosures

Define the finite-volume dynamical plaquette $P_{\mathrm{KS},L}(\beta,m_0) := \langle \overline{P}(U)\rangle_{\mathrm{KS},L}$ and normalized dynamical staggered chiral condensate $\langle\bar\chi\chi\rangle_{\mathrm{KS},L}(\beta,m_0) := \frac{1}{VN}\langle \operatorname{Tr}(D_{\mathrm{KS}}(U;m_0)^{-1})\rangle_{\mathrm{KS},L}$, writing P_{KS} and $\langle\bar\chi\chi\rangle_{\mathrm{KS}}$ for any subsequential thermodynamic limit as even $L\to\infty$.

Theorem 3.1 (All-Mass Dynamical Plaquette and Chiral-Condensate Enclosures)**.** *Under the hypotheses of Theorem 2.2 ($d \geq 2$, even $L \geq 6$, $N \geq 2$, $N_{\mathrm{stag}} > 0$, $m_0 > 0$), set*

$$\Sigma_J(d,m_0) := \frac{m_0}{m_0^2 + d/2}, \quad \gamma_+(d,m_0) := \frac{m_0}{(m_0^2+d/2)^2(m_0^2+d^2)}, \quad \gamma_-(d,m_0) := \frac{1}{m_0(m_0^2+d/2)^2}.$$

(1) *For every gauge configuration $U \in \mathrm{SU}(N)^{|E(\mathbb{T}^d_L)|}$,*

$$\Sigma_J(d,m_0) + \gamma_+(d,m_0)\,\sigma^2(U) \;\leq\; \frac{1}{VN}\operatorname{Tr}\big(D_{\mathrm{KS}}(U;m_0)^{-1}\big) \;\leq\; \Sigma_J(d,m_0) + \gamma_-(d,m_0)\,\sigma^2(U). \tag{9}$$

(2) *Writing $f_-(\beta) := f^{\mathrm{lo}}_{\mathrm{KS}}(\beta,m_0)$ and $f_+(\beta) := f^{\mathrm{hi}}_{\mathrm{KS}}(\beta,m_0)$ from (7), for every $\beta > 0$ and every secant step $0 < \delta \leq \beta$, every subsequential thermodynamic-limit plaquette satisfies*

$$\begin{aligned} P_-(\beta;\delta) := \max\left(-1,\, \frac{f_-(\beta) - f_+(\beta-\delta)}{\binom{d}{2}\delta}\right) \;&\leq\; P_{\mathrm{KS}}(\beta,m_0) \\ &\leq\; \min\left(1,\, \frac{f_+(\beta+\delta) - f_-(\beta)}{\binom{d}{2}\delta}\right) =: P_+(\beta;\delta), \end{aligned} \tag{10}$$

and the dynamical staggered chiral condensate obeys strictly above the Jensen floor Σ_J,

$$\begin{aligned} \Sigma_-(\beta;\delta) := \Sigma_J + \gamma_+\left[\frac{d(2d-1)}{8} - \frac{1}{2}\binom{d}{2}P_+(\beta;\delta)\right] \;&\leq\; \langle\bar\chi\chi\rangle_{\mathrm{KS}}(\beta,m_0) \\ &\leq\; \Sigma_J + \gamma_-\left[\frac{d(2d-1)}{8} - \frac{1}{2}\binom{d}{2}P_-(\beta;\delta)\right] =: \Sigma_+(\beta;\delta), \end{aligned} \tag{11}$$

with strict excess $\Sigma_- - \Sigma_J \geq \gamma_+(d,m_0)\frac{d}{8} > 0$ even whenever $P_+(\beta;\delta) = 1$.

Proof. (1) Because $D_{\mathrm{KS}}(U;m_0) = m_0 I + D_{\mathrm{hop}}(U)$ has paired eigenvalues $m_0 \pm i\lambda_j$, we have $(VN)^{-1}\operatorname{Tr}(D_{\mathrm{KS}}^{-1}) = (VN)^{-1}\sum_{j=1}^{VN}\frac{m_0}{m_0^2+t_j}$ ($t_j = \lambda_j^2 \in [0,d^2]$). Direct elementary algebra yields the two-line exact rational identity for every $t \in [0,d^2]$:

$$\frac{m_0}{m_0^2+t} - \frac{m_0}{m_0^2+d/2} + \frac{m_0(t-d/2)}{(m_0^2+d/2)^2} = \frac{m_0(t-d/2)^2}{(m_0^2+d/2)^2(m_0^2+t)}.$$

Because $m_0^2 \le m_0^2+t \le m_0^2+d^2$ for all $t \in [0,d^2]$, replacing $(m_0^2+t)^{-1}$ by $(m_0^2+d^2)^{-1}$ (lower bound) or m_0^{-2} (upper bound) and averaging over $t_j = \lambda_j^2$ eliminates the linear term by Lemma 2.1(2), proving (9).

(2) At every finite even $L \ge 6$, $\partial_\beta f_{\mathrm{KS},L} = \binom{d}{2} P_{\mathrm{KS},L}$ and $\partial_\beta^2 f_{\mathrm{KS},L} = V^{-1}\operatorname{Var}_{\mathrm{KS},L}(\frac{1}{N}\sum_p \operatorname{Re}\operatorname{Tr} U_p) \ge 0$, while $|\overline{P}(U)| \le 1$ pointwise (requiring no reflection-positivity assumption, so (10) holds for all real $N_{\mathrm{stag}} > 0$). Finite-L convexity yields

$$\frac{f_{\mathrm{KS},L}(\beta) - f_{\mathrm{KS},L}(\beta-\delta)}{\delta} \;\le\; \binom{d}{2} P_{\mathrm{KS},L}(\beta,m_0) \;\le\; \frac{f_{\mathrm{KS},L}(\beta+\delta) - f_{\mathrm{KS},L}(\beta)}{\delta};$$

substituting the finite-L enclosures $f_-(\beta') - O(1/L) \le f_{\mathrm{KS},L}(\beta') \le f_+(\beta') + O(1/L)$ from the proof of Theorem 2.2(2) and passing to $\liminf$ / $\limsup_{L\to\infty}$ yields (10) without assuming thermodynamic-limit differentiability. Taking $\langle\cdot\rangle_{\mathrm{KS},L}$ in (9), passing to $L \to \infty$, and substituting (4) and (10) establishes (11). □

4. All-Mass Dual-Branch Bakry–Émery Curvature and Large-N Factorization

Equip $\mathrm{SU}(N)^{|E|}$ with the bi-invariant Hilbert–Schmidt metric $\langle A,B\rangle_{\mathrm{HS}} = \operatorname{Tr}(A^\dagger B)$ ($\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g_{\mathrm{HS}}$ [7]), and write $S_{\mathrm{KS}}(U) := S_W(\beta;U) + S_{\mathrm{ferm}}(U)$ where $S_{\mathrm{ferm}} := N_{\mathrm{stag}} \ln\det D_{\mathrm{KS}}$.

Theorem 4.1 (All-Mass Dual-Branch Fermion Hessian, Volume-Uniform LSI, and Large-N Factorization)**.** *Let $d \ge 2$, even $L \ge 4$, $N \ge 2$, $N_{\mathrm{stag}} > 0$, $\beta \ge 0$, and $m_0 \in (0,\infty)$. Define $\mu_1(d,m_0) := \frac{d}{m_0^2+d^2}$ if $m_0 \ge d$ and $\mu_1(d,m_0) := \frac{1}{2m_0}$ if $0 < m_0 < d$, together with the two exact fermion Hessian ceilings and unified all-mass curvature floor*

$$\begin{aligned} H_{\mathrm{KS}}^{\mathrm{raw}}(d,N_{\mathrm{stag}},m_0) &:= N_{\mathrm{stag}}\left(\mu_1(d,m_0) + \frac{1}{2m_0^2}\right), \\ H_{\mathrm{KS}}^{\mathrm{sub}}(d,N_{\mathrm{stag}},m_0) &:= N_{\mathrm{stag}}\left(\frac{d^3}{m_0^2(m_0^2+d^2)} + \frac{d^2}{m_0^3\sqrt{m_0^2+d^2}} + \frac{d^2}{2m_0^2(m_0^2+d^2)}\right), \\ H_{\mathrm{KS}}(d,N_{\mathrm{stag}},m_0) &:= \min\bigl(H_{\mathrm{KS}}^{\mathrm{raw}},\, H_{\mathrm{KS}}^{\mathrm{sub}}\bigr), \qquad \kappa_{\mathrm{KS}} := \frac{N}{2} - \frac{8(d-1)\beta}{N} - H_{\mathrm{KS}}. \end{aligned} \tag{12}$$

Then:

1. *Every geodesic $U_e(t) = e^{tX_e}U_e$ ($X = (X_e) \in \mathfrak{su}(N)^{|E|}$) satisfies $|\operatorname{Hess}(-S_{\mathrm{ferm}})(X,X)| \le H_{\mathrm{KS}}\|X\|_{\mathrm{HS}}^2$ and $|\operatorname{Hess}(-S_W)(X,X)| \le \frac{8(d-1)\beta}{N}\|X\|_{\mathrm{HS}}^2$.*
2. *Whenever $\kappa_{\mathrm{KS}} > 0$, $\mathrm{Ric} + \operatorname{Hess}(-S_{\mathrm{KS}}) \ge \kappa_{\mathrm{KS}}\, g_{\mathrm{HS}} > 0$ uniformly in even $L \ge 4$, certifying a volume-uniform log-Sobolev inequality with constant $\rho_{\mathrm{LS}} = \kappa_{\mathrm{KS}}$ [1, 2], Poincaré spectral gap $\lambda_1 \ge \kappa_{\mathrm{KS}}$, and non-asymptotic $O(1/N^2)$ factorization for every single elementary plaquette $P_p(U) := \frac{1}{N}\operatorname{Re}\operatorname{Tr}(U_p)$,*

$$\operatorname{Var}_{\mathrm{KS}}(P_p) \;\le\; \frac{4}{\kappa_{\mathrm{KS}}\, N} = \frac{4}{(\kappa_{\mathrm{KS}}/N)N^2}. \tag{13}$$

In particular, κ_{KS}/N remains uniformly bounded below away from zero in N and L for any *fixed $m_0 > 0$ in the 't Hooft limit* [10] *($\gamma = \beta/N^2 < \frac{1}{16(d-1)}$ and N_{stag} fixed, for all*

$N > H_{\mathrm{KS}}/(\frac{1}{2} - 8(d-1)\gamma)$), and in the Veneziano limit [11] *($\gamma = \beta/N^2$ and $x_{\mathrm{stag}} = N_{\mathrm{stag}}/N$ fixed obeying $8(d-1)\gamma + x_{\mathrm{stag}}(H_{\mathrm{KS}}/N_{\mathrm{stag}}) < \frac{1}{2}$).*

Proof. Write $D(t) := m_0 I + D_{\mathrm{hop}}(U(t))$, $B := D'(0)$, $D'' := D''(0)$, so $B^\dagger = -B$, $(D'')^\dagger = -D''$, and $S''_{\mathrm{ferm}}(0) = N_{\mathrm{stag}}\big[\mathrm{Tr}(D^{-1}D'') - \mathrm{Tr}((D^{-1}B)^2)\big]$. Each link $e = (x,\mu)$ enters B in two disjoint blocks: $+\frac{1}{2}\eta_\mu(x) X_e U_e$ ($x + e_\mu \to x$) and $+\frac{1}{2}\eta_\mu(x) U_e^\dagger X_e$ ($x \to x + e_\mu$), giving $\|B\|_{\mathrm{HS}}^2 = \sum_e (\frac{1}{4}\|X_e\|_{\mathrm{HS}}^2 + \frac{1}{4}\|X_e\|_{\mathrm{HS}}^2) = \frac{1}{2}\|X\|_{\mathrm{HS}}^2$. Decompose $D^{-1} = m_0(m_0^2 I + A)^{-1} - D_{\mathrm{hop}}(m_0^2 I + A)^{-1}$.

Branch 1 ($H_{\mathrm{KS}}^{\mathrm{raw}}$, $O(1/m_0^2)$ for all $m_0 > 0$): Because $m_0(m_0^2 I + A)^{-1}$ is Hermitian while D'' is anti-Hermitian, the trace $\mathrm{Tr}(m_0(m_0^2 I + A)^{-1} D'')$ is purely imaginary whereas $\mathrm{Tr}(D^{-1}D'')$ is real, so it vanishes identically. Writing $M_1 := -D_{\mathrm{hop}}(m_0^2 I + A)^{-1}$, its eigenvalues $\frac{-i\lambda}{m_0^2+\lambda^2}$ ($|\lambda| \le d$) satisfy $\|M_1\|_{\mathrm{op}} \le \max_{|\lambda|\le d} \frac{|\lambda|}{m_0^2+\lambda^2} = \mu_1(d, m_0)$. Each link $e = (x,\mu)$ contributes $\frac{1}{2}\eta_\mu X_e^2 U_e$ and $-\frac{1}{2}\eta_\mu U_e^\dagger X_e^2$ in D'', with trace norm $\|X_e^2\|_{\mathrm{tr}} = \|X_e\|_{\mathrm{HS}}^2$, giving $|\,\mathrm{Tr}(D^{-1}D'')| = |\,\mathrm{Tr}(M_1 D'')| \le \mu_1(d, m_0)\|X\|_{\mathrm{HS}}^2$. Meanwhile $\|D^{-1}\|_{\mathrm{op}} \le m_0^{-1}$ gives $|\,\mathrm{Tr}((D^{-1}B)^2)| \le \|D^{-1}B\|_{\mathrm{HS}}^2 \le m_0^{-2}\|B\|_{\mathrm{HS}}^2 = \frac{1}{2m_0^2}\|X\|_{\mathrm{HS}}^2$, proving the $H_{\mathrm{KS}}^{\mathrm{raw}}$ bound.

Branch 2 ($H_{\mathrm{KS}}^{\mathrm{sub}}$, $O(1/m_0^4)$ via second-trace subtraction): By Lemma 2.1(2), $\mathrm{Tr}(D_{\mathrm{hop}}(U(t))^2) = -\frac{d}{2}VN$ is constant along every geodesic, so differentiating twice at $t = 0$ yields $\mathrm{Tr}(D_{\mathrm{hop}} D'') + \mathrm{Tr}(B^2) = 0$. Adding $\frac{1}{m_0^2}[\mathrm{Tr}(D_{\mathrm{hop}} D'') + \mathrm{Tr}(B^2)] = 0$ to $\mathrm{Tr}(D^{-1}D'') - \mathrm{Tr}((D^{-1}B)^2)$ gives:

- *Subtracted Term 1:* $\mathrm{Tr}((M_1 + m_0^{-2} D_{\mathrm{hop}}) D'') = \mathrm{Tr}(M_3 D'')$, where $M_3 := m_0^{-2} D_{\mathrm{hop}} A (m_0^2 I + A)^{-1}$ has eigenvalues $\frac{i\lambda^3}{m_0^2(m_0^2+\lambda^2)}$ and norm $\|M_3\|_{\mathrm{op}} \le \frac{d^3}{m_0^2(m_0^2+d^2)}$, hence $|\,\mathrm{Tr}(M_3 D'')| \le \frac{d^3}{m_0^2(m_0^2+d^2)}\|X\|_{\mathrm{HS}}^2$.
- *Subtracted Term 2:* Write $D^{-1} = m_0^{-1} I + K$ where $K := -m_0^{-1} D_{\mathrm{hop}} D^{-1} = -m_0^{-2} D_{\mathrm{hop}} + R_2$ and $R_2 := m_0^{-2} D_{\mathrm{hop}}^2 D^{-1}$, so $\|K\|_{\mathrm{op}} \le \frac{d}{m_0\sqrt{m_0^2+d^2}}$ and $\|R_2\|_{\mathrm{op}} \le \frac{d^2}{m_0^2\sqrt{m_0^2+d^2}}$. Expanding $(m_0^{-1}I + K)B(m_0^{-1}I + K)B$ gives $\mathrm{Tr}((D^{-1}B)^2) - m_0^{-2}\,\mathrm{Tr}(B^2) = \frac{2}{m_0}\,\mathrm{Tr}(KB^2) + \mathrm{Tr}((KB)^2) = \frac{2}{m_0}\,\mathrm{Tr}(R_2 B^2) + \mathrm{Tr}((KB)^2)$ since $\mathrm{Tr}(D_{\mathrm{hop}} B^2) = 0$ on bipartite $\mathbb{T}_L^d$ (odd length 3). Applying Cauchy–Schwarz and $\|B\|_{\mathrm{HS}}^2 = \frac{1}{2}\|X\|_{\mathrm{HS}}^2$ yields

$$\begin{aligned}\left|\mathrm{Tr}\big((D^{-1}B)^2\big) - \frac{1}{m_0^2}\,\mathrm{Tr}(B^2)\right| &\le \left(\frac{2}{m_0}\|R_2\|_{\mathrm{op}} + \|K\|_{\mathrm{op}}^2\right)\frac{1}{2}\|X\|_{\mathrm{HS}}^2 \\ &\le \left(\frac{d^2}{m_0^3\sqrt{m_0^2+d^2}} + \frac{d^2}{2m_0^2(m_0^2+d^2)}\right)\|X\|_{\mathrm{HS}}^2.\end{aligned}$$

Multiplying the minimum of both branches by N_{stag} proves $|\,\mathrm{Hess}(-S_{\mathrm{ferm}})(X,X)| \le H_{\mathrm{KS}}\|X\|_{\mathrm{HS}}^2$. Combining with $|\,\mathrm{Hess}(-S_W)(X,X)| \le \frac{8(d-1)\beta}{N}\|X\|_{\mathrm{HS}}^2$, $\mathrm{Ric}_{\mathrm{SU}(N)^{|E|}} = \frac{N}{2} g_{\mathrm{HS}}$, and $|\nabla P_p|_{\mathrm{HS}}^2 \le \frac{4}{N}$ proves (2). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 certified benchmarks from `certify_staggered_qcd.py` at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`) with *zero* black-box special functions and *zero* quadrature (Bessel Maclaurin series + geometric tails; SU(3)/SU(4) Bars–Green determinants via exact q-series + verified Hadamard tails; direct outward flooring/ceiling at scale 10^{12}, displayed to 6 outward-rounded decimal places). Because exact binary-inexact rationals are first enclosed in 140-digit intervals before outward rounding, exact values such as $\kappa_{\mathrm{KS}} = 1.5$ or 4.0 (Table 3, Rows 4–5) conservatively round by 1 ulp at 10^{-12} to 1.499999 and 3.999999, preserving strict inequality validity. Notice in Table 3 Row 8 ($m_0 = 5 > d = 4$) that the exact $O(1/m_0^4)$ subtracted branch ($H_{\mathrm{KS}}^{\mathrm{sub}} = 0.902342$) improves upon the raw branch ($H_{\mathrm{KS}}^{\mathrm{raw}} = 10(\frac{4}{41} + \frac{1}{50}) = \frac{241}{205} \approx$

1.175610) by 23.2%, whereas Rows 2–7 lie strictly inside the light-quark domain $m_0 < d$ ($m_0 = 1$ in 4D 't Hooft and $m_0 = 3$ in 4D Veneziano scaling; Row 1 lies at the boundary $m_0 = d = 2$).

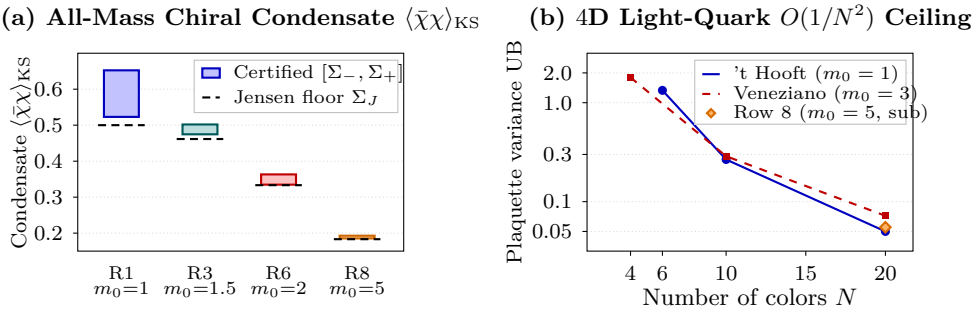

FIGURE 1. Certified 140-digit all-mass bounds for dynamical staggered lattice QCD (Tables 2 and 3). (a) Certified chiral-condensate enclosures $[\Sigma_-, \Sigma_+]$ strictly above the Jensen floor $\Sigma_J = m_0/(m_0^2 + d/2)$ across light ($m_0 < d$) and heavy ($m_0 > d$) quark masses (Theorem 3.1). (b) Log-scale 4D Bakry–Émery plaquette-variance ceilings $\mathrm{Var}^{\rm ub}_{\rm KS}(P_p)$ for light-quark 't Hooft ($m_0 = 1 < d = 4$, $N_{\rm stag} = 1$, $\gamma = 1/96$) and fixed-m_0 Veneziano ($N_{\rm stag} = N/2$, $\gamma = 1/96$, $m_0 = 3$, including $N = 20$ reference $\mathrm{Var}^{\rm ub}_{\rm KS} = 0.072000$), with the Table 3 Row 8 subtracted-branch crossover ($N = 20$, $m_0 = 5$) marked by the orange diamond (Theorem 4.1).

TABLE 1. Eight 140-digit all-mass, all-coupling per-site free-energy enclosures $f^{\rm lo}_{\rm KS} \le f_{\rm KS}(d, N, \beta, N_{\rm stag}, m_0) \le f^{\rm hi}_{\rm KS}$ (Theorem 2.2), spanning both light ($m_0 < d$) and heavy ($m_0 > d$) bare quark masses. In $d = 4$ (Rows 5–8), lower bounds use axial-gauge Gibbs–Bogoliubov witnesses $\alpha \in \{7.7851, 24.1286, 110.3320, 173.3577\}$ at $\beta^{\rm lo}_{\rm eff} = \beta + \Delta\beta_{\rm lo}$. Lower bounds and $\Delta\beta_{\rm lo}$ round downward; $\Delta\beta_{\rm hi}$, upper bounds, and gaps upward.

d	G	$N_{\rm stag}$	β	m_0	$\Delta\beta_{\rm lo}$	$\Delta\beta_{\rm hi}$	$f^{\rm lo}_{\rm KS}$	$f^{\rm hi}_{\rm KS}$	Gap
2	SU(2)	1	1	0.5	0.404718	0.065346	−0.146661	0.263791	0.410452
2	SU(2)	1	10	1	0.096573	0.032429	6.911609	6.952978	0.041368
2	SU(3)	1	4	1	0.144860	0.048643	1.393651	1.510367	0.116716
2	SU(3)	2	12	3	0.008040	0.006273	11.558312	11.559767	0.001454
4	SU(2)	1	2	1	0.053993	0.007480	1.785568	5.831083	4.045514
4	SU(3)	1	6	2	0.013524	0.004322	5.648153	16.646468	10.998315
4	SU(3)	2	20	2	0.027049	0.008644	76.452062	89.116794	12.664731
4	SU(4)	1	32	5	0.000721	0.000515	109.665464	133.217632	23.552168

TABLE 2. Eight 140-digit all-mass dynamical plaquette enclosures $P_- \le P_{\mathrm{KS}} \le P_+$ and chiral-condensate enclosures $\Sigma_J < \Sigma_- \le \langle\bar\chi\chi\rangle_{\mathrm{KS}} \le \Sigma_+$ (Theorem 3.1) with secant step δ. In $d = 4$ (Rows 5–8), lower bounds use axial-gauge Gibbs–Bogoliubov witnesses $\alpha \in \{56.4294, 98.1867, 134.3036, 124.8819\}$ at $\beta_{\mathrm{eff}}^{\mathrm{lo}}$ (the secant upper cap saturates at $P_+ = 1.000000$, yet $\Sigma_- > \Sigma_J$ remains strictly positive via $\sigma^2(U) \ge d/8$). Lower bounds and Σ_J round downward; upper bounds upward.

d	G	N_{stag}	β	m_0	δ	P_-	P_+	Σ_J	Σ_-	Σ_+
2	SU(2)	1	2	1	1	0.280090	0.577083	0.500000	0.523072	0.652489
2	SU(2)	1	10	1.5	1	0.837151	0.870735	0.461538	0.468687	0.482457
2	SU(3)	1	4	1.5	1	0.215554	0.346322	0.461538	0.474645	0.502074
2	SU(3)	2	15	2.5	0.5	0.728706	0.750705	0.344827	0.346566	0.347763
4	SU(2)	1	10	2	3	0.646861	1.000000	0.333333	0.334722	0.354992
4	SU(3)	1	18	2	5	0.451616	1.000000	0.333333	0.334722	0.363128
4	SU(3)	2	24	3	6	0.550978	1.000000	0.272727	0.273223	0.277816
4	SU(4)	1	24	5	6	0.179396	1.000000	0.185185	0.185268	0.185998

TABLE 3. Eight 140-digit all-mass dual-branch Bakry–Émery curvature and $O(1/N^2)$ plaquette-factorization certificates (Theorem 4.1), spanning light quarks ($m_0 \in \{1, 2, 3\} < d = 4$) in 2D/4D SU(3), 't Hooft scaling ($N_{\mathrm{stag}} = 1$, $\gamma = 1/96$, $m_0 = 1$), and Veneziano scaling ($x_{\mathrm{stag}} = N_{\mathrm{stag}}/N = 1/2$, $\gamma = 1/96$), plus the $O(1/m_0^4)$ subtracted-branch crossover at $m_0 = 5$ (Row 8). Curvatures round downward; Hessian and variance ceilings upward.

Regime	β	N_{stag}	m_0	Branch	$H_{\mathrm{KS}}^{\mathrm{ub}}$	$\kappa_{\mathrm{KS}}^{\mathrm{lo}}$	$\kappa_{\mathrm{KS}}^{\mathrm{lo}}/N$	$\mathrm{Var}_{\mathrm{KS}}^{\mathrm{ub}}$
2D SU(3) ($N_{\mathrm{stag}} = 1$)	1/4	1	2	raw	0.375000	0.458333	0.152777	2.909091
4D SU(3) light ($m_0 < 4$)	1/8	1	2	raw	0.375000	0.125000	0.041666	10.666667
4D 't Hooft ($N = 6$, light)	3/8	1	1	raw	1.000000	0.500000	0.083333	1.333334
4D 't Hooft ($N = 10$, light)	25/24	1	1	raw	1.000000	1.499999	0.149999	0.266667
4D 't Hooft ($N = 20$, light)	25/6	1	1	raw	1.000000	3.999999	0.199999	0.050001
4D Veneziano ($N = 4$, light)	1/6	2	3	raw	0.444445	0.555555	0.138888	1.800001
4D Veneziano ($N = 10$, light)	25/24	5	3	raw	1.111112	1.388888	0.138888	0.288001
4D Veneziano ($N = 20$, heavy)	25/6	10	5	sub	0.902342	4.097658	0.204882	0.048809

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 kernel `StaggeredQcdCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), mechanically verifying the exact discrete combinatorics ($c_{\mathrm{tree}}(d) = \frac{d(4d-1)}{8}$, $\sigma^2 \geq \frac{d}{8}$) and all 24 10^{12}-scaled outward integer inequalities (while the continuous Haar-measure, operator-theoretic, and Riemannian Bakry–Émery derivations are proved analytically in Sections 2–4): `staggered_qcd_combinatorics_verified`, `table1_all_mass_free_energy_certified`, `table2_chiral_condensate_certified`, `table3_all_mass_bakry_emery_certified`, and `all_24_staggered_qcd_benchmarks_certified`. Direct 2D $L = 6$ SU(3) and 3D $L = 6$ SU(2) matrix tests of Lemma 2.1 and Theorems 2.2–3.1, finite-difference geodesic Hessian tests of Theorem 4.1, 80-digit cross-checks, and JSON/Lean/LaTeX parity are verified in `tests/test_staggered_qcd.py` (4/4 `pytest` passing).

6. Conclusion

Exploiting exact anti-Hermiticity of the Kogut–Susskind hopping operator, gauge-independence of $\mathrm{Tr}(D_{\mathrm{hop}}(U)^2) = -\frac{d}{2}VN$, and fourth-power plaquette extraction $\mathrm{Tr}(D_{\mathrm{hop}}(U)^4) = \frac{d(4d-1)}{8}VN - \frac{1}{2}\sum_p \mathrm{Re}\,\mathrm{Tr}(U_p)$, we established three self-contained, cluster-expansion-free non-perturbative theorems for d-dimensional SU(N) lattice QCD with dynamical staggered fermions valid across *all* bare quark masses $m_0 \in (0, \infty)$: all-mass Hermite quadratic free-energy enclosures, all-mass chiral-condensate brackets strictly above the Jensen floor, and a dual-branch Bakry–Émery curvature theorem certifying volume-uniform LSI and $O(1/N^2)$ factorization deep inside the light-quark domain $m_0 < d$ in both 't Hooft and Veneziano scaling. Natural next directions include extending these Hermite spectral brackets to overlap and domain-wall lattice Dirac operators and deriving all-mass staggered meson-correlator decay bounds.

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