

Fat Gauss Fibres and Tjurina–Milnor Defects Forced by Osculating Absorption

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Abstract

Let $X^d \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ be a smooth non-linear hypersurface, let $\gamma_X : X \rightarrow X^\vee$ be its Gauss normalization, and let Γ_η be the scheme-theoretic fibre over a tangent hyperplane $\eta = [W]$. If h_p is the isolated germ of the tangent section $X \cap W$ at a point p of the fibre, the first-jet description of the conormal incidence gives

$$\hat{\mathcal{O}}_{\Gamma_\eta, p} \cong \mathbb{C}[[z_1, \dots, z_d]]/(h_p, \partial h_p).$$

Thus the fibre length is $\sum_p \tau(h_p)$, whereas the classical formula of Dimca gives $\text{mult}_\eta(X^\vee) = \sum_p \mu(h_p)$. We record the exact defect identity

$$\text{mult}_\eta(X^\vee) - \text{length}(\Gamma_\eta) = \sum_p (\mu_p - \tau_p).$$

We also compute $\text{Fitt}_0 \Omega_{X/X^\vee, p} = (\det \text{Hess } h_p)$. These local identities are classical or immediate consequences of classical first-jet and singularity theory; they are not claimed as new. The equality $\text{mult}_\eta(X^\vee) = \text{length}(\Gamma_\eta)$ occurs precisely when all section germs are analytically quasihomogeneous.

The application proved here is to point-span osculating absorption. If the complete $\mathcal{O}_X(m)$ point span of the reduced fibre absorbs order s , $1 \leq s \leq m$, then $h_p \in \mathfrak{m}_p^{s+1}$ and the Gauss fibre contains $\text{Spec}(\mathcal{O}_{X, p}/\mathfrak{m}_p^s)$ at every support. Consequently

$$\text{length}(\Gamma_\eta) \geq \binom{d+s-1}{d} |Z| \geq \binom{d+s-1}{d} \binom{d+m}{d},$$

and the ramification equation vanishes to order at least $d(s-1)$ at each support. Finally, when $d(s-1) > s+1$, we construct smooth hypersurfaces for which the reduced fibre has the extremal size $N = \binom{d+m}{d}$ and, for every $0 \leq q \leq N$,

$$\text{mult}_\eta(X^\vee) = s^d N, \quad \text{length}(\Gamma_\eta) = s^d N - q.$$

The construction uses finite contact determinacy and prescribed jets. Its degree threshold is sufficient rather than minimal, and no exhaustive priority claim is made.

1 Statement and scope

Let $X^d \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ be a smooth hypersurface of degree $D \geq 2$. Write

$$\gamma_X : X \longrightarrow X^\vee \subset (\mathbb{P}^{d+1})^\vee$$

for the Gauss morphism. It is finite and birational, hence is the normalization of the dual hypersurface; see, for example, Tevelev [13, Chapter 1] and the proof recalled in [8, Lemma 2.1]. For $\eta = [W] \in X^\vee$, put

$$\Gamma_\eta = X \times_{X^\vee} \text{Spec } \kappa(\eta), \quad Z = (\Gamma_\eta)_{\text{red}}.$$

All fibre schemes in this paper are fibres over the reduced complex point η ; no claim about a scheme structure on a varying contact locus is implicit.

Fix $m \geq 1$, put $H = \mathcal{O}_X(1)$, and let

$$S_Z = \operatorname{Im}\left(H^0(Z, H^m|_Z)^* \longrightarrow H^0(X, H^m)^*\right).$$

Principal-parts evaluation defines the affine order- s osculating block

$$\widehat{\operatorname{Osc}}_p^s(H^m) = \operatorname{Im}\left(H^0((s+1)p, H^m|_{(s+1)p})^* \longrightarrow H^0(X, H^m)^*\right).$$

Definition 1.1. For $1 \leq s \leq m$, the reduced fibre Z is *point-span s -osculating-absorbing* if

$$\widehat{\operatorname{Osc}}_p^s(H^m) \subseteq S_Z \quad (p \in Z).$$

The first theorem separates three invariants that should not be conflated: the cardinality of the reduced fibre, the length of the fibre scheme, and the multiplicity of the dual. For an isolated analytic hypersurface germ h , our conventions are

$$\mu(h) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{z\}}{(\partial h)}, \quad \tau(h) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{z\}}{(h, \partial h)}.$$

Theorem 1.2 (Gauss fibre, Tjurina length, and dual defect). *Let $\eta = [W] \in X^\vee$ and let $h_p \in \mathbb{C}\{z_1, \dots, z_d\}$ be a local equation of the tangent section $X \cap W$ at $p \in Z$. Then*

$$\begin{aligned} \widehat{\mathcal{O}}_{\Gamma_\eta, p} &\cong \mathbb{C}[[z_1, \dots, z_d]] / (h_p, \partial_1 h_p, \dots, \partial_d h_p), \\ \operatorname{length}(\Gamma_\eta) &= \sum_{p \in Z} \tau(h_p), \\ \operatorname{mult}_\eta(X^\vee) &= \sum_{p \in Z} \mu(h_p). \end{aligned}$$

Consequently

$$\operatorname{mult}_\eta(X^\vee) - \operatorname{length}(\Gamma_\eta) = \sum_{p \in Z} (\mu(h_p) - \tau(h_p)) \geq 0. \quad (1)$$

Equality in (1) holds if and only if every h_p is analytically quasihomogeneous. Moreover,

$$\operatorname{Fitt}_0 \Omega_{X/X^\vee, p} = (\det \operatorname{Hess} h_p) \subset \mathcal{O}_{X, p}^{\operatorname{an}}. \quad (2)$$

Theorem 1.3 (Fat-fibre floor forced by absorption). *Assume that Z is point-span s -osculating-absorbing in the complete H^m embedding. For every $p \in Z$ there is a closed immersion*

$$\operatorname{Spec}(\mathcal{O}_{X, p} / \mathfrak{m}_{X, p}^s) \hookrightarrow \Gamma_\eta.$$

In particular, with $N_{d, m} = \binom{d+m}{d}$,

$$\operatorname{length}(\Gamma_\eta) \geq \binom{d+s-1}{d} |Z| \geq \binom{d+s-1}{d} N_{d, m}. \quad (3)$$

The local ramification equation in (2) has order at least $d(s-1)$ at every $p \in Z$. If $s \geq 2$, the fibre scheme is nonreduced at every support.

The lower bound in (3) is not asserted to be optimal for $s > 1$. Our global examples instead show that all integral defects from zero through the number of extremal supports occur while every local Milnor number remains fixed at s^d .

Theorem 1.4 (Global realization of Tjurina–Milnor defects). *Let d, m, s be positive integers with $1 \leq s \leq m$ and*

$$d(s-1) > s+1.$$

Put $N = N_{d,m}$ and suppose

$$D \geq (s^d + 2)N + 1. \quad (4)$$

For every integer q with $0 \leq q \leq N$, there exist a smooth integral degree- D hypersurface $X^d \subset \mathbb{P}^{d+1}$, a hyperplane W , and $\eta = [W] \in X^\vee$ such that:

1. $Z = (\gamma_X^{-1}(\eta))_{\text{red}}$ has exactly N points;
2. $\dim S_Z = N < h^0(X, H^m)$ and Z is point-span s -osculating-absorbing;
3. every tangent-section germ has Milnor number s^d ; exactly q of them have Tjurina number $s^d - 1$, and the others have Tjurina number s^d ;

4.

$$\text{mult}_\eta(X^\vee) = s^d N, \quad \text{length}(\Gamma_\eta) = s^d N - q;$$

5. the ramification equation has order exactly $d(s-1)$ at every point of Z .

The construction is existential up to nonempty Zariski-open choices. The threshold (4) is not claimed to be minimal.

2 The local fibre ideal

Choose analytic coordinates centred at $p \in Z$ in which

$$X : w = h(z_1, \dots, z_d), \quad W : w = 0, \quad h \in \mathfrak{m}^2.$$

The tangent hyperplane at $(z, h(z))$ has equation

$$Y - \sum_i h_i(z) Z_i + \left(\sum_i z_i h_i(z) - h(z) \right) = 0.$$

On the dual chart where the coefficient of Y is one, the Gauss map is therefore, up to harmless signs,

$$z \mapsto \left(h_1(z), \dots, h_d(z), h(z) - \sum_i z_i h_i(z) \right). \quad (5)$$

The extension of the maximal ideal of η is

$$\begin{aligned} I_{\eta,p} &= \left(h_1, \dots, h_d, h - \sum_i z_i h_i \right) \\ &= (h, h_1, \dots, h_d). \end{aligned} \quad (6)$$

The second equality uses only $\sum z_i h_i \in (h_1, \dots, h_d)$; it is not an Euler or quasihomogeneity argument.

The scheme in (6) is the first-jet singularity scheme of the section $X \cap W$. This is the hypersurface specialization of the jet-incidence description of Aluffi–Cukierman [1, §1, pp. 247–249]. We use (6) as a local calculation, not as a new general theorem about discriminants.

Because γ_X is finite, the section singularity is isolated. Passing between the analytic local ring and its completion preserves finite colength. Thus (6) proves the first two identities of Theorem 1.2. The third is Dimca’s classical multiplicity–Milnor formula [3]; see also Parusiński [10] and the modern use in Dimca–Ilardi [5, (1.3)]. Since

$$(\partial h) \subset (h, \partial h),$$

one has $\tau(h) \leq \mu(h)$. Equality is equivalent to $h \in (\partial h)$, and Saito's theorem identifies this with analytic quasihomogeneity [12]. This proves (1).

To prove (2), differentiate (5). The Hessian description of Gauss ramification is classical; compare Piene [11]. Here the exact Fitting equality follows directly from the local presentation. The first d rows are the Hessian matrix $\text{Hess } h$, while the last row is

$$d\left(h - \sum_i z_i h_i\right) = -\sum_i z_i dh_i.$$

It is therefore the $(-z_1, \dots, -z_d)$ -linear combination of the first rows. The maximal minors presenting $\Omega_{X/X^\vee, p}$ generate exactly $(\det \text{Hess } h)$.

Remark 2.1. The fibre length is $\text{length}(R/I_{\eta, p}) = \tau(h)$. It is not the Hilbert–Samuel multiplicity $e(I_{\eta, p}, R)$, and neither invariant should be identified merely from the common ideal of the fibre. In particular, the global multiplicity–Milnor formula does not turn fibre length into Milnor number.

3 Absorption creates infinitesimal fibre thickness

We recall the annihilator argument in the form needed here.

Lemma 3.1 (Local annihilator consequence). *If Z is point-span s -osculating-absorbing and $a \in H^0(X, \mathcal{I}_Z \otimes H^m)$, then*

$$a_p \in \mathfrak{m}_{X, p}^{s+1} H_p^m \quad (p \in Z).$$

Proof. The annihilator of S_Z is $H^0(X, \mathcal{I}_Z \otimes H^m)$. The annihilator of $\widehat{\text{Osc}}_p^s(H^m)$ consists of sections whose germs vanish modulo $\mathfrak{m}_{X, p}^{s+1}$. The defining containment of osculating blocks reverses under annihilators and gives the assertion. $\square$

Let ℓ_W be a linear equation of W . For each $p \in Z$, choose a section $g_p \in H^0(X, H^{m-1})$ nonzero at p , with $g_p = 1$ when $m = 1$. The section $\ell_W g_p$ of H^m vanishes on all of Z . Lemma 3.1 and division by the unit $(g_p)_p$ give

$$(\ell_W|_X)_p \in \mathfrak{m}_{X, p}^{s+1}. \quad (7)$$

In the graph coordinates of Section 2, (7) is precisely $h_p \in \mathfrak{m}_p^{s+1}$. Hence

$$I_{\eta, p} = (h_p, \partial h_p) \subset \mathfrak{m}_p^s. \quad (8)$$

The reversed ideal inclusion in (8) gives the asserted closed immersion $\text{Spec}(\mathcal{O}_{X, p}/\mathfrak{m}_p^s) \hookrightarrow \Gamma_\eta$ and

$$\tau(h_p) \geq \text{length}(\mathcal{O}_{X, p}/\mathfrak{m}_p^s) = \binom{d+s-1}{d}. \quad (9)$$

Order- s absorption contains tangent absorption. The exact support floor from [8, Theorem 1.2], ultimately based on the point-span rank theorem [6], is

$$|Z| \geq N_{d, m}.$$

Summing (9) proves (3). Finally, every Hessian entry belongs to $\mathfrak{m}_p^{s-1}$, so its determinant belongs to $\mathfrak{m}_p^{d(s-1)}$. The determinant is not the zero germ because the finite birational Gauss map is generically separable over $\mathbb{C}$. If $s \geq 2$, (9) gives local fibre length greater than one at every support. This completes the proof of Theorem 1.3.

4 A defect-one isolated germ

The next calculation supplies the two analytic types used in the global construction.

Lemma 4.1 (Fermat and defect-one germs). *Let $s \geq 1$ and*

$$f_0 = \sum_{i=1}^d x_i^{s+1}.$$

Then $\mu(f_0) = \tau(f_0) = s^d$. If $d(s-1) > s+1$ and $\lambda \in \mathbb{C}^$, put*

$$f_1 = \sum_{i=1}^d x_i^{s+1} + \lambda \prod_{i=1}^d x_i^{s-1}. \quad (10)$$

Then

$$\mu(f_1) = s^d, \quad \tau(f_1) = s^d - 1. \quad (11)$$

For both germs the Hessian determinant has order exactly $d(s-1)$.

Proof. For f_0 the gradient ideal is $(x_1^s, \dots, x_d^s)$, and Euler's formula puts f_0 in that ideal. This gives the first assertion.

Put $r = \prod_i x_i^{s-1}$ and $k = d(s-1)$. The hypothesis $k > s+1$ means that (10) has Fermat initial form. The initial forms of its partial derivatives are $(s+1)x_i^s$ and form a homogeneous regular sequence. Therefore the associated graded Milnor algebra has basis

$$x_1^{a_1} \cdots x_d^{a_d}, \quad 0 \leq a_i \leq s-1,$$

and $\mu(f_1) = s^d$. The class of r is nonzero and spans the socle. In the Milnor algebra, Euler's identity reads

$$0 = \sum_i x_i \partial_i f_1 = (s+1)[f_0] + k\lambda[r],$$

so

$$[f_1] = \left(1 - \frac{k}{s+1}\right) \lambda[r] \neq 0.$$

It spans a one-dimensional ideal. Quotienting the Milnor algebra by this class proves $\tau(f_1) = s^d - 1$. The leading Hessian determinant in both cases is a nonzero scalar multiple of r , proving the final assertion. $\square$

For the family (10), the plane case first satisfies $2(s-1) > s+1$ at $s=4$. Its corresponding fixture is

$$f_1 = x^5 + y^5 + x^3 y^3, \quad \mu(f_1) = 16, \quad \tau(f_1) = 15. \quad (12)$$

It is included in the exact replay accompanying the paper. Equation (12) is also a counterexample to the stronger but false bound $\tau \geq s^d$.

5 Simultaneous analytic types from prescribed jets

We give the construction in enough detail to isolate its degree bookkeeping. Its interpolation and extension architecture is adapted from the preceding proper-span construction [7]. Compare also the squared-point independence criterion of Ballico–Chiantini [2, Theorem 3.5]. The general fact that isolated hypersurface singularities can be realized in a hyperplane section of a smooth hypersurface is classical [4, Proposition 11.6]; what is controlled below is the exact support, absorption, two simultaneous analytic types, defect ladder, and an explicit sufficient degree.

Lemma 5.1 (Simultaneous jets from separators). *Let $p_1, \dots, p_t$ be distinct points of $\mathbb{P}^d$ and let $a_i \geq 0$. If*

$$k \geq \sum_{i=1}^t (a_i + 1) - 1,$$

then restriction is surjective:

$$H^0(\mathbb{P}^d, \mathcal{O}(k)) \longrightarrow \bigoplus_{i=1}^t H^0(\mathcal{O}(k) \otimes \mathcal{O}_{\mathbb{P}^d, p_i} / \mathfrak{m}_{p_i}^{a_i+1}).$$

Proof. For $i \neq j$, choose a linear form ℓ_{ij} vanishing at p_j and nonzero at p_i . The product

$$P_i = \prod_{j \neq i} \ell_{ij}^{a_j+1}$$

kills every other target and is a unit at p_i . Forms of degree a_i realize all order- a_i jets at p_i ; multiplying them by P_i isolates that target. The common degree is at most $\sum_j (a_j + 1) - 1$. Multiplication by a form nonzero at all supports raises the degree. $\square$

Choose $N = N_{d,m}$ distinct points $Z \subset W \cong \mathbb{P}^d$ such that

$$H^0(W, \mathcal{O}_W(m)) \xrightarrow{\sim} H^0(Z, \mathcal{O}_Z(m)). \quad (13)$$

Such a set exists because the evaluation lines of all points span the dual of $H^0(W, \mathcal{O}_W(m))$; choose a basis among them. Set

$$a = s^d + 1, \quad E = (a + 1)N = (s^d + 2)N.$$

The binomial theorem gives $d(s-1) \leq s^d - 1 < a$, so the perturbation term in f_1 is visible in the prescribed a -jet. At each point prescribe the a -jet of either f_0 or f_1 from Lemma 4.1, choosing f_1 at exactly q supports. Lemma 5.1 supplies a section $f_* \in H^0(W, \mathcal{O}_W(D))$ with those jets. Let

$$V = H^0(W, \mathcal{I}_{(a+1)Z}(D)), \quad \mathcal{A} = f_* + V.$$

Proposition 5.2 (A section with exactly the prescribed singularities). *If $D \geq E + 1$, there is $f \in \mathcal{A}$ such that $V(f) \subset W$ is smooth away from Z and has at the supports precisely the prescribed contact-equivalence classes.*

Proof. Greuel–Lossen–Shustin prove that an isolated germ is contact $(\tau + 1)$ -determined [9, Corollary 2.24]. Both germs have $\tau \leq s^d$, so their prescribed $a = s^d + 1$ jets determine their contact classes.

Fix $u \in W \setminus Z$. For every $p \in Z$, choose a hyperplane $\ell_{p,u}$ through p but not u . The section

$$A_u = \prod_{p \in Z} \ell_{p,u}^{a+1}$$

has degree E , belongs to $\mathcal{I}_{(a+1)Z}$, and is a unit on the first neighbourhood of u . Since $D - E \geq 1$, multiplication by A_u shows that V surjects onto the first jets at u . Thus the affine condition $j_u^1 f = 0$ has codimension $d + 1$ in $\mathcal{A}$. Its incidence over the d -dimensional base $W \setminus Z$ has dimension at most $\dim \mathcal{A} - 1$. A member outside its closure is smooth away from Z ; finite determinacy gives the claimed germs at Z . $\square$

Embed $W = V(y)$ in $\mathbb{P}^{d+1}$ and lift f independently of y . For $G \in H^0(\mathbb{P}^{d+1}, \mathcal{O}(D - 1))$, put

$$F_G = f + yG.$$

Proposition 5.3 (Smooth extension and exact Gauss support). *There is G , nonzero at every point of Z , such that $X = V(F_G)$ is smooth and integral and*

$$(\gamma_X^{-1}([W]))_{\text{red}} = Z.$$

At each $p \in Z$, the section germ $W|_X$ is contact equivalent to the prescribed germ of f .

Proof. On $W \setminus Z$, smoothness follows from Proposition 5.2. At $p \in Z$, one has $dF_G = G(p) dy$, so $G(p) \neq 0$ makes X smooth. Off W , multiplication by the unit y and the first-jet generation of $\mathcal{O}(D-1)$ show that the singularity condition has codimension $d+2$ in the space of G 's over a $(d+1)$ -dimensional base. The closure of the incidence image is therefore a proper closed subset of the space of G 's. Avoid it together with the finitely many hyperplanes $G(p) = 0$.

The hypersurface sequence and $H^1(\mathbb{P}^{d+1}, \mathcal{O}(-D)) = 0$ give $H^0(X, \mathcal{O}_X) = \mathbb{C}$, so the smooth hypersurface is connected and hence integral. Its section by W is $V(f)$ and has singular locus exactly Z . If a point maps under the Gauss map to $[W]$, it lies in its tangent hyperplane W ; within W , this happens exactly where $V(f)$ is singular. This proves the exact reduced fibre identity. Locally on X ,

$$y = -f/G,$$

which differs from f by a unit and therefore has the prescribed contact class. $\square$

6 Proof of the realization theorem

Apply Propositions 5.2 and 5.3. Since $D > m$, restriction gives

$$H^0(\mathbb{P}^{d+1}, \mathcal{O}(m)) \xrightarrow{\sim} H^0(X, H^m).$$

If a degree- m ambient form Q vanishes on Z , (13) says that $Q|_W = 0$, so $Q = yR$. At every support, $y|_X = -f/G$ has order $s+1$. Therefore every section of H^m vanishing on Z vanishes to order $s+1$ at each support. The reverse-annihilator argument of Lemma 3.1 proves order- s absorption.

Equation (13) also gives

$$\dim S_Z = N.$$

This span is proper because

$$h^0(X, H^m) = \binom{d+m+1}{d+1} > \binom{d+m}{d} = N.$$

Each prescribed germ has Milnor number s^d . Exactly q have Tjurina number $s^d - 1$ and the others have Tjurina number s^d . Theorem 1.2 now gives

$$\begin{aligned} \text{mult}_{[W]}(X^\vee) &= Ns^d, \\ \text{length}(\Gamma_{[W]}) &= (N - q)s^d + q(s^d - 1) = Ns^d - q. \end{aligned}$$

Over $\mathbb{C}$, both μ and τ are invariants of analytic contact equivalence; see [9, Section 2.2]. For the Hessian assertion, the interpolation gives $f - f_i \in \mathfrak{m}^{a+1}$ with $a+1 > s+1$, and on X one has $y|_X = uf$ for the unit $u = -1/G$. Thus the degree- $(s+1)$ initial form of $y|_X$ is the Fermat form of f_i , up to the nonzero scalar $u(0)$ and the invertible linear part of a coordinate change. Its initial Hessian determinant remains nonzero of degree $d(s-1)$. Lemma 4.1 now gives the claimed exact order and completes the proof of Theorem 1.4.

7 Context, reproducibility, and limitations

We use the scheme-theoretic jet-incidence formulation of Aluffi–Cukierman [1]; the multiplicity–Milnor identity is due to Dimca [3], with broader results by Parusiński [10]; and the equality criterion $\mu = \tau$ is Saito’s theorem [12]. The point-span floor and the prescribed-jet extension architecture are authorial antecedents [6, 7, 8], not independent validation. The contribution claimed here is limited to the exact combination of the absorption-forced fat-fibre and ramification bounds with the explicit simultaneous defect realization at the stated sufficient degree.

The accompanying script uses exact rational arithmetic to compute local Macaulay quotients for the defect-one fixtures $(d, s) = (2, 4), (3, 3), (4, 2)$, verify the predicted Milnor and Tjurina numbers, check the Euler and initial Hessian terms in (12), and audit finite grids of the displayed integer formulae. It does not mechanize analytic coordinates, finite determinacy, incidence dimension counts, or literature priority.

The limitations are material:

- Everything is over $\mathbb{C}$. No positive-characteristic extension of the Milnor, Saito, or separability arguments is asserted.
- The fibre is the complete scheme-theoretic fibre of the Gauss normalization over one reduced dual point. Arbitrary subsets of its support are not covered.
- The lower length floor (3) is not proved sharp for $s > 1$. The realization theorem controls a different, near-Milnor range of lengths.
- Formula (2) describes the Fitting ramification scheme on the normal source. No identification of its image with a canonical branch or discriminant scheme on the nonnormal dual is claimed.
- The degree threshold (4) is deliberately sufficient and may be far from minimal. No classification of equality cases or exhaustive priority review is claimed.
- The computational replay certifies only its finite fixtures and arithmetic grids. It is not formal proof or human peer review.

8 Conclusion

Osculating absorption thickens a special Gauss fibre scheme, not merely its reduced support. The thickness is measured locally by Tjurina algebras, while the singularity of the dual is measured by Milnor numbers. Their difference is exactly the total quasihomogeneity defect of the tangent section singularities. The resulting dictionary

$$\text{fat Gauss fibre} \longleftrightarrow \sum \tau_p, \quad \text{dual multiplicity} \longleftrightarrow \sum \mu_p$$

both prevents a common numerical conflation and yields the global floor proved here under the specific absorption hypothesis. Prescribed finite jets show that the two invariants can be separated one unit at a time across an extremal large fibre. The optimal Tjurina floor and the minimal ambient degree are not determined here.

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