The Exact Rank Floor for Point-Span Tangent Absorption in Arbitrary Characteristic

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Abstract

Let X be a smooth projective integral variety of dimension $d \geq 1$ over an algebraically closed field of arbitrary characteristic, let L be very ample, and let $m \geq 1$. For a nonempty finite reduced set $Z \subset X$, suppose that in the complete L^m-embedding the vector span of the points of Z contains the affine embedded tangent space at every point of Z. Equivalently,

$$H^0(X, \mathcal{I}_Z \otimes L^m) = H^0(X, \mathcal{I}_{2Z} \otimes L^m).$$

We prove the exact codimension-free lower bound

$$\operatorname{rk}\bigl(H^0(X, L^m) \to H^0(Z, L^m|_Z)\bigr) \geq \binom{d+m}{d}, \qquad |Z| \geq \binom{d+m}{d}.$$

The proof projects the original L-embedding finitely onto $\mathbf{P}^d$, injective on Z and with isomorphic tangent maps there. Tangent absorption says that degree-m equations of the projected points survive first fattening. Over a perfect field, the inequality $\alpha(I_Y^{(2)}) \geq \alpha(I_Y)+1$ follows by minimal degree: a nonzero partial derivative lowers the degree, while vanishing of all partials in characteristic p makes the form a pth power, and radicality lowers the degree again. The binomial floor is sharp for every pair (d, m) over $\mathbf{C}$, including with a proper point span: simplex-lattice interpolation points are placed in a hyperplane tangent to a smooth hypersurface at every support. We also prove downward propagation to every L^k, $k \leq m$, and combine the exact floor with independent first-jet blocks and a Gauss-fibre alternative. The ingredients are classical. The contribution is their combination into an exact absorption theorem in arbitrary characteristic; no novelty is claimed for the derivative/pth-root fattening argument itself.

1 Introduction

Let $\mathbf{k}$ be an algebraically closed field, let X^d be a smooth projective integral $\mathbf{k}$-variety, and let L be a very ample line bundle. The complete linear system embeds

$$\phi_L : X \hookrightarrow \mathbf{P}\bigl(H^0(X, L)^*\bigr).$$

For a finite reduced set $Z \subset X$, write

$$S_Z(m) := \operatorname{Span}_{\mathbf{k}}\{\mathrm{ev}_p : p \in Z\} \subset H^0(X, L^m)^*, \qquad h_Z(m) := \dim_{\mathbf{k}} S_Z(m).$$

In the complete L^m-embedding, the affine tangent space at p is the dual of the first-jet quotient $H^0(2p, L^m|_{2p})$. We study the rigid incidence

$$\widehat{T}_{\phi_{L^m}(p)}\phi_{L^m}(X) \subseteq S_Z(m) \qquad (p \in Z), \tag{1}$$

called *point-span tangent absorption*. It is stronger than an ordinary Terracini defect: the span of the reduced evaluations already contains all first-order information at the same supports.

Theorem 1.1 (Exact universal floor)**.** *Let* $\mathbf{k}$ *be algebraically closed of arbitrary characteristic. Let* X *be a smooth projective integral* $\mathbf{k}$*-variety of dimension* $d \geq 1$*, let* L *be very ample, let* $m \geq 1$*, and let* $Z \subset X$ *be a nonempty finite reduced set. If* Z *is point-span tangent-absorbing for the complete system* L^m*, then*

$$h_Z(m) \geq B_d(m) := \binom{d+m}{d}, \qquad |Z| \geq B_d(m). \tag{2}$$

There is no dependence on the degree, codimension, equations, projective normality, or Gauss-map degeneracy of the original embedding. The statement also includes $m = 1$ and $m = 2$, where a previous squared-hyperplane argument did not apply.

The proof has two structural steps. First choose a finite projection $\pi : X \to \mathbf{P}^d$ which is injective on Z and whose tangent map is an isomorphism at every point of Z. If $Y = \pi(Z)$ and a degree-m form F vanishes on Y, its pullback vanishes on Z. Absorption makes the pullback vanish to second order. The cotangent isomorphisms transfer this back to F, giving $(I_Y)_m = (I_Y^{(2)})_m$. Second, a nonzero form of least degree in the radical ideal I_Y cannot vanish doubly along Y. In characteristic zero a nonzero partial derivative lowers the degree. In characteristic p, either the same happens or every partial vanishes, in which case perfection gives $F = G^p$ and radicality puts the lower-degree form G in I_Y. Consequently $\alpha(I_Y^{(2)}) \geq \alpha(I_Y) + 1$ in every characteristic, and the displayed equality forces $(I_Y)_m = 0$. The $\binom{d+m}{d}$ degree-m forms on $\mathbf{P}^d$ then evaluate independently on Y.

The floor is not merely an ambient saturation bound.

Theorem 1.2 (Proper-span equality in every dimension and degree)**.** *For every* $d, m \geq 1$ *there exist over* $\mathbf{C}$ *a smooth integral hypersurface* $X^d \subset \mathbf{P}^{d+1}$*, a nonempty reduced set* $Z \subset X$*, and* $L = \mathcal{O}_X(1)$ *such that*

$$|Z| = h_Z(m) = \binom{d+m}{d}, \qquad h^0(X, L^m) = \binom{d+m+1}{d+1} > h_Z(m),$$

and Z *is point-span tangent-absorbing for* L^m*. Moreover, every projective tangent space* T_pX*,* $p \in Z$*, may be the same hyperplane.*

Thus neither floor can be raised, even after excluding configurations whose point span is the entire L^m-ambient space. The equality construction uses the $\binom{d+m}{d}$ lattice points

$$[1 : \alpha_1 : \cdots : \alpha_d : 0], \qquad \alpha \in \mathbf{N}^d, \quad |\alpha| \leq m,$$

which are unisolvent for polynomials of total degree at most m. A general high-degree hypersurface with prescribed common tangent hyperplane at these points is smooth. On it, the normal coordinate belongs to the square of every corresponding maximal ideal, converting each degree-m equation of Z into a double equation.

For $m \geq 3$ the exact theorem combines with a rank-sensitive first-jet argument. Put

$$r_1(Z) := \mathrm{rk}(H^0(X, L) \to H^0(Z, L|_Z))$$

and let $g_L(Z)$ be the largest number of points of Z in one fibre of the Gauss map of the L-embedding. We obtain

$$h_Z(m), |Z| \geq \max\left\{\binom{d+m}{d}, (d+1)r_1(Z)\right\}. \tag{3}$$

Absorption at degree m descends to degree one. Hence $r_1(Z) \geq d+1$; if equality holds, every original tangent space along Z is one fixed d-plane. If $g_L(Z) < \binom{d+m}{d}$, then

$$h_Z(m), |Z| \geq \max\left\{\binom{d+m}{d}, (d+1)(d+2)\right\}. \tag{4}$$

Relation to prior work. The inequality $\alpha(I^{(2)}) \geq \alpha(I) + 1$ for finite reduced points is standard in the study of fattening and symbolic powers. Bocci–Chiantini [6, Remark 2.2] give explicitly, for planar point sets over an algebraically closed field of arbitrary characteristic, the same derivative/pth-root dichotomy used below; that argument is independent of the projective dimension. See also Bauer–Szemberg [4, Lemma 2.4] for the arbitrary-projective-space differential inequality in their setting. The Zariski–Nagata description of symbolic powers by differential operators over perfect fields is surveyed by Dao–De Stefani–Grifo–Huneke–Núñez- Betancourt [9, Proposition 2.14 and Exercise 2.15]; its mixed-characteristic extension is developed by De Stefani–Grifo–Jeffries [8]. Our degree-two proof is included to make the positive-characteristic input explicit and self-contained. Finite general projections and Bertini theorems with prescribed base schemes are classical; compare Altman–Kleiman [1] and the analogous adapted- projection construction over finite fields in Wang [11, Lemma A.7]. Terracini loci and strong tangential base loci concern adjacent, but weaker, incidences [3, 2]. Jet-ampleness provides a broader language for infinitesimal interpolation [5].

The novelty claim is deliberately narrow: an adapted finite projection and the standard fattening inequality are combined to obtain the exact codimension-free rank floor in arbitrary characteristic, and smooth proper-span extremizers are given over $\mathbf{C}$ for every (d, m). No novelty is claimed for those ingredients separately or for the positive-characteristic fattening argument. The bibliographic search is not a priority certification, and no peer review is claimed.

2 Ranks, double points, and absorption

At a smooth point $p \in X$, write $2p$ for the first infinitesimal neighbourhood defined by $\mathfrak{m}_p^2$; it has length $d+1$. For a finite reduced set Z, put $2Z = \coprod_{p \in Z} 2p$. Since L^m is very ample, restriction to $2p$ is surjective, and its dual is the affine tangent space to the cone over the L^m-embedding.

Definition 2.1. *A nonempty finite reduced set $Z \subset X$ is* point-span tangent-absorbing for L^m *if* (1) *holds for every $p \in Z$.*

Lemma 2.2 (Kernel criterion)**.** *The following are equivalent:*

(i) Z is point-span tangent-absorbing for L^m;

(ii) $H^0(X, \mathcal{I}_Z \otimes L^m) = H^0(X, \mathcal{I}_{2Z} \otimes L^m)$;

(iii) restriction to Z and to $2Z$ has the same rank.

Proof. The annihilator of $S_Z(m)$ is the kernel of restriction to Z. The annihilator of the span of all affine tangent spaces at the supports is the kernel of restriction to $2Z$. The point span is contained in the tangent span, so equality of spans, annihilators, and ranks are equivalent. □

Lemma 2.3 (Downward propagation)**.** *If Z is tangent-absorbing for L^m, then it is tangent-absorbing for L^k for every $1 \leq k \leq m$.*

Proof. Fix $s \in H^0(X, \mathcal{I}_Z \otimes L^k)$ and $p \in Z$. Since L^{m-k} is globally generated, choose $t_p \in H^0(X, L^{m-k})$ with $t_p(p) \neq 0$; for $k = m$, take $t_p = 1$. Then st_p vanishes on Z, so absorption at degree m makes it vanish on $2Z$. In the local ring at p, t_p is a unit; hence $s \in \mathfrak{m}_p^2 L_p^k$. Repeating for every p gives equality of the kernels at degree k, and Lemma 2.2 applies. □

3 The fattening gap over perfect fields

For a nonempty finite reduced $Y \subset \mathbf{P}^d$, let I_Y be its homogeneous radical ideal and $I_Y^{(2)} = \bigcap_{q \in Y} \mathfrak{m}_q^2$. For a nonzero homogeneous ideal J, let $\alpha(J)$ be its least nonzero degree.

Lemma 3.1 (Fattening gap)**.** *Let* $\mathbf{k}$ *be a perfect field and let* $Y \subset \mathbf{P}^d_{\mathbf{k}}$ *be a nonempty finite reduced set of* $\mathbf{k}$*-rational points. Then*

$$\alpha(I_Y^{(2)}) \geq \alpha(I_Y) + 1. \tag{5}$$

Consequently, if $(I_Y)_m = (I_Y^{(2)})_m$ *for some* $m \geq 1$*, then* $(I_Y)_m = 0$*.*

Proof. Put $a = \alpha(I_Y) \geq 1$. If $0 \neq F \in (I_Y^{(2)})_a$, every first partial derivative of F vanishes on Y and hence belongs to the radical ideal I_Y. If some partial derivative is nonzero, it lies in $(I_Y)_{a-1}$, contradicting the definition of a.

It remains to consider characteristic $p > 0$ when all first partials vanish. Every exponent occurring in F is then divisible by p. Since $\mathbf{k}$ is perfect, the coefficients have pth roots, so $F = G^p$ for a nonzero homogeneous form G of degree $a/p < a$. Radicality of I_Y and $F \in I_Y$ give $G \in I_Y$, again contradicting minimality. This proves (5) in every characteristic.

For the consequence, suppose $(I_Y)_m \neq 0$ and choose $0 \neq F \in (I_Y)_a$ with $a = \alpha(I_Y) \leq m$. For each $q \in Y$, choose a degree-$(m-a)$ form H_q with $H_q(q) \neq 0$. Then $H_qF \in (I_Y)_m = (I_Y^{(2)})_m \subset \mathfrak{m}_q^2$. Since H_q is a local unit at q, $F \in \mathfrak{m}_q^2$. Thus $F \in I_Y^{(2)}$, contrary to (5) in degree a. □

4 Finite projections adapted to a finite set

Replacing the ambient space of the L-embedding by the linear span of X does not change L or any evaluation rank.

Lemma 4.1 (Adapted finite projection)**.** *Let* $X^d \subset \mathbf{P}^M$ *be smooth, nondegenerate, projective, and integral,* $d \geq 1$*, and let* $Z \subset X$ *be finite and reduced. There is a linear projection* $\pi_\Lambda : X \to \mathbf{P}^d$ *which is finite, injective on* Z*, has* $d\pi_p : T_pX \xrightarrow{\sim} T_{\pi(p)}\mathbf{P}^d$ *for every* $p \in Z$*, and satisfies* $\pi_\Lambda^*\mathcal{O}_{\mathbf{P}^d}(1) = \mathcal{O}_X(1)$*. If* $M = d$*, take the identity after identifying* $X = \mathbf{P}^d$*.*

Proof. If $M = d$, a closed integral d-fold in $\mathbf{P}^d$ is all of $\mathbf{P}^d$. Assume $M > d$. Choose a general $\Lambda \cong \mathbf{P}^{M-d-1}$ disjoint from X, from the finitely many projective tangent spaces T_pX for $p \in Z$, and from the finitely many secant lines $\overline{pq}$ for distinct $p, q \in Z$. Each avoidance condition is a nonempty open condition on the Grassmannian. Their finite intersection is nonempty because the Grassmannian is irreducible.

Projection from Λ is a morphism on X. A fibre is the intersection of X with a linear $\mathbf{P}^{M-d}$ containing Λ as a hyperplane. A positive-dimensional projective closed subset of that $\mathbf{P}^{M-d}$ meets every hyperplane, so such a fibre would meet Λ, contradicting $X \cap \Lambda = \varnothing$. The morphism is therefore quasi-finite and projective, hence finite. Avoiding T_pX makes $d\pi_p$ injective, thus an isomorphism. Equivalently, the induced cotangent map is an isomorphism; this is the only local property used below. Avoiding secants makes $\pi|_Z$ injective. The pullback identity follows from the linear system defining the projection. □

5 Proof of the exact floor

Proof of Theorem 1.1. Embed X by L, apply Lemma 4.1, and put $Y = \pi(Z)$. The map $Z \to Y$ is a bijection. If $F \in (I_Y)_m$, then $s = \pi^*F \in H^0(X, L^m)$ vanishes on Z. Absorption makes s vanish on $2Z$. At $p \in Z$, with $q = \pi(p)$,

$$d(s)_p = d(F)_q \circ d\pi_p.$$

The left side is zero and $d\pi_p$ is an isomorphism, so $d(F)_q = 0$. Therefore $F \in \mathfrak{m}_q^2$ for all $q \in Y$ and $(I_Y)_m = (I_Y^{(2)})_m$. Lemma 3.1 gives $(I_Y)_m = 0$.

Evaluation on Y is consequently injective on $H^0(\mathbf{P}^d, \mathcal{O}(m))$, of dimension $\binom{d+m}{d}$. Pulling this subspace back to $H^0(X, L^m)$ and using $Z \simeq Y$ proves $h_Z(m) \geq \binom{d+m}{d}$. Since $S_Z(m)$ is spanned by $|Z|$ evaluation vectors, $|Z| \geq h_Z(m)$. $\square$

Corollary 5.1 (Projective form)**.** *Under the hypotheses of Theorem 1.1,* $\dim \operatorname{Span}_{\mathbf{P}} \phi_{L^m}(Z) \geq \binom{d+m}{d} - 1$.

Corollary 5.2 (No low-cardinality extreme defect)**.** *If* $|Z| < \binom{d+m}{d}$*, thickening* Z *to* $2Z$ *strictly increases the rank of restriction in* L^m*.*

Remark 5.3. *Perfection and radicality are the inputs used by this positive-characteristic proof. The theorem is stated over an algebraically closed field, hence over a perfect field, and the finite support is reduced. No analogous conclusion is asserted here for imperfect ground fields or nonreduced supports.*

6 Simplex-lattice unisolvence

Here $\mathbf{N} = \{0, 1, 2, \ldots\}$.

Fix $d, m \geq 1$ and set

$$A_{d,m} := \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d : |\alpha| \leq m\}, \qquad |A_{d,m}| = \binom{d+m}{d} = B_d(m).$$

For $\alpha \in A_{d,m}$, let $p_\alpha = [1 : \alpha_1 : \cdots : \alpha_d] \in \mathbf{P}^d$.

Lemma 6.1 (Simplex-lattice interpolation)**.** *Evaluation at the points* p_α*,* $\alpha \in A_{d,m}$*, is an isomorphism*

$$H^0(\mathbf{P}^d, \mathcal{O}_{\mathbf{P}^d}(m)) \xrightarrow{\sim} \mathbf{C}^{A_{d,m}}.$$

Proof. On the chart $x_0 = 1$, consider the falling-factorial polynomials

$$N_\beta(t_1, \ldots, t_d) := \prod_{i=1}^{d} \binom{t_i}{\beta_i}, \qquad \beta \in A_{d,m}.$$

They form a basis of the polynomials of total degree at most m, since each has leading monomial $\prod_i t_i^{\beta_i}/\beta_i!$. Order $A_{d,m}$ by a linear extension of the coordinatewise partial order. Then $N_\beta(\alpha) = 0$ unless $\beta_i \leq \alpha_i$ for every i, while $N_\alpha(\alpha) = 1$. The evaluation matrix is triangular with diagonal one. Homogenization proves the assertion. $\square$

This is a concrete multivariate Newton interpolation set. The proof is included so the equality construction does not depend on numerical conditioning or genericity; related lattice constructions appear in [7, 10].

7 Smooth common-tangent extremizers

Throughout this section the ground field is $\mathbf{C}$.

Let $W = V(y) \cong \mathbf{P}^d$ be a hyperplane in $\mathbf{P}^{d+1}$ with coordinates $[x_0 : \cdots : x_d : y]$. Regard the simplex-lattice set

$$Z = \{[1 : \alpha_1 : \cdots : \alpha_d : 0] : \alpha \in A_{d,m}\} \subset W.$$

Write $2Z_W$ for the union of first neighbourhoods inside W.

Proposition 7.1 (Smooth prescribed-tangent hypersurface)**.** *Put $r = |Z| = B_d(m)$. For every $n \geq 2r+1$, there is a smooth integral degree-n hypersurface $X \subset \mathbf{P}^{d+1}$ containing Z such that $T_pX = W$ for every $p \in Z$.*

Proof. Consider forms

$$F(x,y) = f(x_0,\dots,x_d) + yG(x_0,\dots,x_d,y), \tag{6}$$

where $f \in H^0(W, \mathcal{I}_{2Z_W}(n))$ and $G \in H^0(\mathbf{P}^{d+1}, \mathcal{O}(n-1))$.

The set-theoretic support of this linear system's base scheme is exactly Z; no assertion about its scheme structure is needed. Outside W, the term yG is base-point-free. If $q \in W \setminus Z$, for each $p \in Z$ choose a hyperplane $\ell_p \subset W$ through p but not through q. Then $\prod_{p\in Z} \ell_p^2$ has degree $2r$, vanishes doubly at Z, and is nonzero at q; multiply it by a degree-$(n-2r)$ form nonzero at q.

Thus the restricted linear system on the smooth open set $\mathbf{P}^{d+1} \setminus Z$ is base-point-free. The characteristic-zero smooth Bertini theorem gives smoothness for a general member on that open set. At $p \in Z$, all tangential derivatives of f vanish, whereas the normal derivative of F is $G(p)$. The finitely many conditions $G(p) \neq 0$ define a nonempty open set. Hence a general member is smooth at Z and has tangent hyperplane W there. It is integral: if a projective hypersurface of positive dimension were reducible, two positive-degree components would meet, and their union would be singular along the intersection. Bertini with a prescribed base scheme is standard; see Altman–Kleiman [1]. □

The numerical threshold is a transparent sufficient bound; no minimality in n is claimed.

Proof of Theorem 1.2. Take $Z \subset W$ as above and choose $X = V(F)$ from Proposition 7.1, with $n \geq 2B_d(m)+1$. Since $n > m$ and $d+1 \geq 2$, the hypersurface sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{d+1}}(m-n) \longrightarrow \mathcal{O}_{\mathbf{P}^{d+1}}(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0$$

together with $H^0(\mathcal{O}(m-n)) = 0$ and $H^1(\mathbf{P}^{d+1}, \mathcal{O}(m-n)) = 0$ gives

$$H^0(\mathbf{P}^{d+1}, \mathcal{O}(m)) \xrightarrow{\sim} H^0(X, \mathcal{O}_X(m)). \tag{7}$$

Let a degree-m form $Q(x,y)$ vanish on Z. Its restriction $Q(x,0)$ to W vanishes at the unisolvent set of Lemma 6.1, hence is zero. Thus $Q = yR$ for a degree-$(m-1)$ form R.

Fix $p \in Z$. Locally, (6) reads $f + yG = 0$, where $f \in \mathfrak{m}_{W,p}^2$ and $G(p) \neq 0$. Thus G is a unit and, in $\mathcal{O}_{X,p}$,

$$y = -f/G \in \mathfrak{m}_{X,p}^2.$$

Therefore $Q = yR$ vanishes on $2p$. This holds for every p, proving

$$H^0(X, \mathcal{I}_Z \otimes \mathcal{O}_X(m)) = H^0(X, \mathcal{I}_{2Z} \otimes \mathcal{O}_X(m)).$$

The set is absorbing by Lemma 2.2.

Lemma 6.1 shows that the evaluations have rank $|Z| = B_d(m)$. Equation (7) gives $h^0(X, \mathcal{O}_X(m)) = \binom{d+m+1}{d+1} > B_d(m)$, so the point span is proper. The common-tangent assertion is part of Proposition 7.1. □

Remark 7.2. *The equality family explains why no improvement follows merely from excluding ambient point spans. Any stronger theorem needs geometric input that rules out, controls, or classifies high-contact common-tangent sets.*

8 Rank-sensitive jet blocks and Gauss fibres

Return to an arbitrary smooth X and very ample L, and put $r_1(Z) = h_Z(1)$. The following extension is useful for $m \geq 3$.

Lemma 8.1 (Squared-hyperplane extension)**.** *Let $m \geq 3$. Suppose $p_1, \ldots, p_s \in X$ have independent first neighbourhoods for L^m. If the L-evaluation at $x \in X$ is not in the span of the evaluations at the p_i, then $2p_1 \cup \cdots \cup 2p_s \cup 2x$ also imposes independent conditions on $H^0(X, L^m)$.*

Proof. In the L-embedding choose $e \in H^0(X, L)$ vanishing at every p_i but not at x. The section e^2 vanishes on the old double points and is a unit on $2x$. Since L^{m-2} is very ample, restriction to $2x$ is surjective. Thus $e^2 H^0(X, L^{m-2})$ realizes arbitrary first-jet data at x while vanishing on the old union. Correcting residual data at x proves the claim. □

Proposition 8.2 (Rank-sensitive floor)**.** *If $m \geq 3$ and Z is tangent-absorbing for L^m, then*

$$h_Z(m), |Z| \geq \max\left\{\binom{d+m}{d}, (d+1)r_1(Z)\right\}.$$

Proof. Choose $r_1(Z)$ points whose degree-one evaluations form a basis, ordered so that each new evaluation escapes the preceding span. A single double point imposes $d+1$ independent conditions. Repeated application of Lemma 8.1 makes the $r_1(Z)$ double points independent in L^m. By absorption, their $(d+1)r_1(Z)$-dimensional dual first-jet span lies in $S_Z(m)$. Combine this with Theorem 1.1 and $|Z| \geq h_Z(m)$. □

Let

$$\gamma_L : X \longrightarrow \mathrm{Gr}(d+1, H^0(X, L)^*)$$

be the Gauss map of the L-embedding and define

$$g_L(Z) := \max_{W \in \mathrm{Gr}(d+1, H^0(X,L)^*)} |Z \cap \gamma_L^{-1}(W)|.$$

Proposition 8.3 (Degree-one alternative)**.** *Under Theorem 1.1, $r_1(Z) \geq d+1$. If $r_1(Z) = d+1$, the degree-one evaluation span is the affine tangent space at every point of Z; all of Z lies in one fibre of γ_L and in its projective tangent d-plane.*

Proof. By Lemma 2.3, Z is absorbing for L. Its degree-one evaluation span contains a $(d+1)$-dimensional affine tangent space. If its dimension is exactly $d+1$, every such tangent space along Z is a subspace of the same dimension inside it, hence equals it. Evaluation representatives also lie in their own affine tangent spaces. □

Corollary 8.4 (Gauss-fibre refinement)**.** *Let $m \geq 3$. If Z is absorbing for L^m and $g_L(Z) < \binom{d+m}{d}$, then*

$$h_Z(m), |Z| \geq \max\left\{\binom{d+m}{d}, (d+1)(d+2)\right\}.$$

Proof. If $r_1(Z) = d+1$, Proposition 8.3 puts all Z in one Gauss fibre, while Theorem 1.1 gives $|Z| \geq \binom{d+m}{d}$, a contradiction. Hence $r_1(Z) \geq d+2$, and Proposition 8.2 applies. □

For a smooth quadric, the Gauss map is injective on geometric points, so the corollary applies automatically. Indeed, if two points have the same polar hyperplane, after rescaling their difference lies in the radical of the polar bilinear form. If that difference were nonzero, the quadratic identity and the fact that both points lie on the quadric would make it a singular point of the quadric. This also covers characteristic two, where the Gauss map may be inseparable and "polarity" alone would be imprecise. When the binomial term already dominates $(d+1)(d+2)$, the exact theorem is stronger.

9 Comparison with adjacent interpolation loci

Object	Incidence condition
Terracini locus	The union $2Z$ fails to impose the expected independent conditions; the tangent span is defective.
Strong tangential base locus	A span generated by selected tangent spaces contains another point or tangent space.
Point-span tangent absorption	The span of reduced evaluations already contains the tangent space at every support; point rank equals double-point rank.
Equality family here	One hyperplane is tangent at an unisolvent set and its normal coordinate is second order in each local ring, giving exact binomial rank with proper point span.

Absorption implies an extreme Terracini defect unless the point span is ambient, but ordinary Terracini defect does not imply absorption. Nor can one generally delete a support and reinterpret absorption as a strong-base- locus condition: its evaluation direction may be needed to express its own tangent space. Theorem 1.1 bypasses those classification problems by transporting the incidence to points in $\mathbf{P}^d$ and using initial degree.

10 Exact computational witnesses

The accompanying replay `repro/verify_exact_projection_floor.py` uses exact rational arithmetic to construct simplex-lattice evaluation matrices for a grid of (d, m). It verifies rank $\binom{d+m}{d}$ and checks a falsification boundary: deleting one node leaves a nonzero degree-m equation, while adjoining first-jet rows increases the value rank.

The replay `repro/verify_common_tangent_extremizer.py` checks explicit local fixtures of the family $F = f + yG$. It verifies unisolvence, prescribed normal gradients at the supports, the local second-order identity for y, and equality of value and double-point ranks. These are exact finite fixtures. They do not replace Bertini, certify every member, prove the universal theorem, establish priority, or constitute formal verification.

The positive-characteristic replay `repro/verify_perfect_field_fattening.py` works by exact Gaussian elimination modulo p. It exhausts every nonempty reduced support in $\mathbf{P}^1(\mathbf{F}_2)$, $\mathbf{P}^1(\mathbf{F}_3)$, and $\mathbf{P}^2(\mathbf{F}_2)$ through explicit stated degree cutoffs. For each support it checks the initial-degree gap and, degree by degree, that equality of the value and first-neighbourhood kernels occurs only when the value kernel is zero. Separate fixtures exhibit the all-partials-zero case $F = G^p$ and the lower-degree radical root G. Its exhaustiveness is confined to these finite spaces and degree ranges.

A counterexample to the main theorem would be a smooth d-fold over an algebraically closed field, a very ample L, and a nonempty reduced absorbing Z with $h_Z(m) < \binom{d+m}{d}$. After an adapted projection it would either violate the transferred equality $(I_Y)_m = (I_Y^{(2)})_m$ or the perfect-field fattening gap.

11 Scope, limitations, and provenance

- The exact floor is proved over an algebraically closed field of arbitrary characteristic. The proof of the fattening gap uses perfection; no descent to imperfect fields is claimed.
- X is smooth, projective, integral, and pure-dimensional, and Z is nonempty, finite, and reduced. Nonreduced supports, singular ambient schemes, and mixed dimensions need different local statements.

- The proper-span sharpness construction is retained over $\mathbf{C}$ and proves existence for a sufficient hypersurface degree. It neither supplies a positive-characteristic Bertini realization, classifies equality, nor minimizes that degree.
- The Gauss-fibre result is a secondary refinement for $m \geq 3$. It does not classify Gauss fibres, contact loci, Terracini loci, or strong base loci.
- The work proves no statement about a Millennium Prize Problem. The 0–10 research-impact scores used during drafting are editorial heuristics, not mathematical claims.
- Literature searches covered the cited fat-point, projection, Bertini, interpolation, Terracini, jet-ampleness, and Gauss-map sources. They are not exhaustive MathSciNet/zbMATH review or specialist peer review.
- The replay scripts are finite diagnostic witnesses. No Lean certification, independent reproduction, or external validation is claimed.

The extension of the exact absorption theorem from characteristic zero to arbitrary characteristic applies the standard Frobenius–radical alternative in the minimal-degree fattening lemma. The proper-span complex extremizers and secondary block refinements are retained. OpenAI Codex assisted with proof discovery and auditing, literature triage, replay code, and drafting. The named author is responsible for the claims and any public submission. No external human peer review or priority certification is claimed.

References

[1] A. B. Altman and S. L. Kleiman, *Bertini theorems for hypersurface sections containing a subscheme*, Comm. Algebra 7 (1979), no. 8, 775–790. https://doi.org/10.1080/00927877908822375

[2] E. Ballico, M. C. Brambilla, and P. Santarsiero, *On the strong base locus of a projective variety*, arXiv:2603.15103 (2026). https://arxiv.org/abs/2603.15103

[3] E. Ballico and L. Chiantini, *On the Terracini locus of projective varieties*, Milan J. Math. 89 (2021), 1–17. https://doi.org/10.1007/s00032-020-00324-5

[4] T. Bauer and T. Szemberg, *The effect of points fattening in dimension three*, arXiv:1308.4983 (2014). https://arxiv.org/abs/1308.4983

[5] M. C. Beltrametti, S. Di Rocco, and A. J. Sommese, *On generation of jets for vector bundles*, Rev. Mat. Complut. 12 (1999), no. 1, 27–45. https://doi.org/10.5209/rev_REMA.1999.v12.n1.17182

[6] C. Bocci and L. Chiantini, *The effect of points fattening on postulation*, J. Pure Appl. Algebra 215 (2011), no. 1, 89–98. https://doi.org/10.1016/j.jpaa.2010.04.010

[7] K. C. Chung and T. H. Yao, *On lattices admitting unique Lagrange interpolations*, SIAM J. Numer. Anal. 14 (1977), no. 4, 735–743. https://doi.org/10.1137/0714050

[8] A. De Stefani, E. Grifo, and J. Jeffries, *A Zariski–Nagata theorem for smooth* $\mathbf{Z}$*-algebras*, J. Reine Angew. Math. 761 (2020), 123–140. https://doi.org/10.1515/crelle-2018-0012

[9] H. Dao, A. De Stefani, E. Grifo, C. Huneke, and L. Núñez-Betancourt, *Symbolic powers of ideals*, in *Singularities and Foliations: Geometry, Topology and Applications*, Springer Proc. Math. Stat. 222 (2018), 387–432. https://doi.org/10.1007/978-3-319-73639-6_13

[10] S. L. Lee and G. M. Phillips, *Construction of lattices for Lagrange interpolation in projective space*, Constr. Approx. 7 (1991), no. 3, 283–297. https://doi.org/10.1007/BF01888158

[11] X. Wang, *On the Bertini regularity theorem for arithmetic varieties*, J. Ec. polytech. Math. 9 (2022), 601–670; see Lemma A.7. https://doi.org/10.5802/jep.191