EXACT ALL-COUPLING NON-PERTURBATIVE BOUNDS FOR SU(N) LATTICE $\mathcal{N}=1$ SUPER-YANG–MILLS AND DYNAMICAL ADJOINT QCD

LLUÍS ERIKSSON

ABSTRACT. In contrast to fundamental lattice QCD, where dynamical quarks explicitly break center symmetry and screen fundamental Wilson loops at perimeter law, SU(N) lattice $\mathcal{N}=1$ Super-Yang–Mills ($N_f^{\rm adj}=\frac{1}{2}$ Majorana gluinos) and dynamical adjoint QCD ($N_f^{\rm adj}\in\frac{1}{2}\mathbb{N}$) couple fermions in the real orthogonal adjoint representation $V_e=\mathrm{Ad}(U_e)\in \mathrm{SO}(N^2-1)$ and preserve the full $\mathbb{Z}_N$ center symmetry. Without polymer or cluster expansions, we prove four self-contained non-perturbative theorems on periodic hypercubic tori $\mathbb{T}_L^d$:

(1) *Exact Majorana gluino Pfaffian positivity:* In $d=4$, for every $N\geq 2$, every finite $\mathbb{T}_L^4$, and every subcritical hopping parameter $\kappa_f\in[0,\frac{1}{8})$, the skew-symmetric matrix $M(U)=CD_W^{\rm adj}(U)$ obeys the real-structure identity $(\gamma_5 C)M(U)^*(\gamma_5 C)^T=M(U)$ with $\det(\gamma_5 C)=+1$, proving that the normalized Pfaffian $\widetilde{\mathrm{Pf}}(U;\kappa_f)=\mathrm{Pf}(C)^{-|\Lambda|d_A}\,\mathrm{Pf}(CD_W^{\rm adj}(U))=+\sqrt{\det D_W^{\rm adj}(U)}>0$ is strictly real and strictly positive everywhere on $\mathrm{SU}(N)^{|E|}$, eliminating the $\mathcal{N}=1$ SYM sign problem throughout $[0,\frac{1}{8})$.

(2) *Center-preserving $\mathbb{Z}_2$ confinement independent of fermion mass and flavors:* For every even $N\geq 2$ and every $N_f^{\rm adj}\in\frac{1}{2}\mathbb{N}$ (including $\mathcal{N}=1$ SYM), exact $\mathbb{Z}_2$-invariance of $\mathrm{Ad}(U_e)$, duplicate-variable Ginibre domination, and a parity-constrained FIFO link-queue Kraft encoding certify a strictly positive fundamental string tension $\varsigma_{\rm fund}\geq \ln(w_0(d)/\tanh\beta)>0$ uniformly across all $\kappa_f\in[0,\frac{1}{2d})$ and all $N_f^{\rm adj}$ (with $w_0(3)>0.149420$ in 3D and $w_0(4)>0.081425$ in 4D — 3.5× sharper than unconstrained edge-splitting).

(3) *All-coupling Bhanot–Creutz free-energy and gluino condensate enclosures:* On bipartite $\mathbb{T}_L^4$ ($L\geq 6$ even), exact fourth-order plaquette extraction yields $\beta_A=16N_f^{\rm adj}\kappa_f^4$ with exact $O((8\kappa_f)^6)$ tail ε_6. Combining d-way Hölder's inequality (evaluated via a strictly positive Catalan–Bessel series for SU(2) and Toeplitz–Bessel determinants for SU(3)) with an axial-gauge Schur variational lower bound matches both exact classical energy and exact one-loop entropy with $O(1)$ gaps across all $\beta\geq 0$, alongside exact gluino condensate enclosures $\Sigma_{\rm gluino}$.

(4) *Exact N-independent planar Bakry–Émery curvature, LSI, and $O(1/N^2)$ factorization:* Using the exact Casimir identity $\|\mathrm{ad}_X\|_{\rm HS}^2=2N\|X\|_{\rm HS}^2$ and exact $k\leq 3$ cancellation proves $\kappa_{\rm AdjQCD}/N=\frac{1}{2}-24\gamma-4N_f^{\rm adj}h_{\rm adj}(4,\kappa_f)>0$ (uniform in $N\geq 2$ and L, reaching $\kappa_f^{\max}=0.084733\ldots$ at $\gamma=0$ for 4D $\mathcal{N}=1$ SYM), certifying volume-uniform LSI ($\rho_{\rm LS}\geq N\kappa_{\rm planar}$), $O(e^{-cN^2t^2})$ Herbst master-field concentration, and exact $O(1/N^2)$ planar factorization for both fundamental and adjoint Wilson loops.

All 26 benchmarks are certified at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`) with zero quadrature and mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`N1SymAdjointQcdCerts.lean`).

1. Introduction and Honest Scope

Let $\Lambda=\mathbb{T}_L^d=(\mathbb{Z}/L\mathbb{Z})^d$ ($V=L^d$, $|E|=dV$, $|P|=\binom{d}{2}V$) be a d-dimensional periodic hypercubic torus, with gauge fields $U=(U_e)_{e\in E}\in\mathrm{SU}(N)^{|E|}$ ($N\geq 2$, $d_A:=N^2-1$), normalized Haar measure dU, and Wilson gauge action $S_g(U)=-\frac{\beta}{N}\sum_{p\in P}\mathrm{Re}\,\mathrm{Tr}(U_p)$ [10] (with 't Hooft planar coupling $\gamma:=\beta/N^2$). Fix an orthonormal Hermitian basis $\{\tilde{T}^a\}_{a=1}^{d_A}$ of $i\mathfrak{su}(N)$ with $\mathrm{Tr}(\tilde{T}^a\tilde{T}^b)=\delta^{ab}$. For every $U\in\mathrm{SU}(N)$, the adjoint matrix $V=\mathrm{Ad}(U)$ with entries $V_{ab}:=\mathrm{Tr}(\tilde{T}^aU\tilde{T}^bU^\dagger)$ lies in

Date: November 29, 2026.

$\mathrm{SO}(d_A)$, obeys $\mathrm{Tr}(V) = |\,\mathrm{Tr}(U)|^2 - 1$, and is strictly invariant under the full center $Z(\mathrm{SU}(N)) = \{\omega I_N : \omega^N = 1\}$:

$$\mathrm{Ad}(zU) = \mathrm{Ad}(U) \qquad (\forall\, z \in Z(\mathrm{SU}(N)),\ \forall\, U \in \mathrm{SU}(N)). \tag{1}$$

On $\mathcal{H}_{\mathrm{adj}} := \ell^2(\Lambda) \otimes \mathbb{C}^{n_s} \otimes \mathbb{C}^{d_A}$ (with spinor dimension $n_s = 4$ in $d = 4$), the adjoint Wilson–Dirac operator [5, 8, 3] at hopping parameter $\kappa_f \geq 0$ is

$$\begin{aligned} D_W^{\mathrm{adj}}(U) &:= I - \kappa_f H_{\mathrm{adj}}(U), \\ \big(H_{\mathrm{adj}}(U)\psi\big)(x) &:= \sum_{\mu=1}^{d} \Big[(I - \gamma_\mu) V_{x,\mu} \psi(x + \hat{\mu}) + (I + \gamma_\mu) V_{x-\hat{\mu},\mu}^T \psi(x - \hat{\mu})\Big], \end{aligned} \tag{2}$$

where γ_μ are Hermitian Euclidean Dirac matrices ($\gamma_\mu^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$). For $N_f^{\mathrm{adj}} \in \frac{1}{2}\mathbb{N}$ ($N_f^{\mathrm{adj}} = \frac{1}{2}$ for a single Majorana gluino in $\mathcal{N} = 1$ Super-Yang–Mills; $N_f^{\mathrm{adj}} = 1, \frac{3}{2}, 2, \ldots$ for multi-flavor adjoint QCD), the $d = 4$ partition function and finite-volume free energy per site are

$$Z_{\mathbb{T}_L^4}(\beta, \kappa_f, N_f^{\mathrm{adj}}, N) := \int_{\mathrm{SU}(N)^{|E|}} e^{-S_g(U)}\, \widetilde{\mathrm{Pf}}(U; \kappa_f)^{2N_f^{\mathrm{adj}}}\, dU, \qquad f_{\mathbb{T}_L^4} := \frac{1}{V} \ln Z_{\mathbb{T}_L^4}. \tag{3}$$

Remark 1.1 (Honest Scope and Non-Claims). All theorems hold at fixed lattice spacing ($a = 1$) in the subcritical hopping strip $0 \leq \kappa_f < \frac{1}{2d}$ ($0 \leq \kappa_f < \frac{1}{8}$ in $d = 4$). Theorems 2.1 and 4.1 hold at *all* gauge couplings $\beta \geq 0$, whereas Theorem 3.1 holds in the strong-coupling slab $\tanh\beta < w_0(d)$ (uniformly in $\kappa_f \in [0, \frac{1}{2d})$ and all N_f^{adj}) and Theorem 5.1 holds in the planar curvature domain $\frac{1}{2} - 24\gamma - 4N_f^{\mathrm{adj}} h_{\mathrm{adj}}(4, \kappa_f) > 0$. This paper does *not* reach the $\kappa_f \to \kappa_c$ massless supersymmetric continuum limit, does *not* prove spontaneous discrete chiral symmetry breaking $\mathbb{Z}_{2N} \to \mathbb{Z}_2$, and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact Majorana Gluino Pfaffian Positivity in $d = 4$

In $d = 4$ Euclidean dimensions, adopt the chiral Weyl representation $\gamma_k = \left(\begin{smallmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{smallmatrix}\right)$ ($k = 1, 2, 3$), $\gamma_4 = \left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$, and $\gamma_5 := \gamma_1\gamma_2\gamma_3\gamma_4 = \mathrm{diag}(-I_2, I_2)$. Define the real unitary skew-symmetric charge-conjugation matrix $C := \gamma_2\gamma_4 = \mathrm{diag}(i\sigma_2, -i\sigma_2)$ ($C^T = -C$, $C^\dagger C = I_4$, $C^2 = -I_4$, $\mathrm{Pf}(C) = -1$), which satisfies $C\gamma_\mu C^{-1} = -\gamma_\mu^T$ for $\mu = 1, 2, 3, 4$, $\gamma_5 C = C\gamma_5$, and $\det(\gamma_5 C) = +1$. On $\mathcal{H}_{\mathrm{adj}}$ write $C_\Lambda := I_\Lambda \otimes C \otimes I_{d_A}$ and $B_\Lambda := I_\Lambda \otimes (\gamma_5 C) \otimes I_{d_A}$.

Theorem 2.1 (Exact Pfaffian Reality and Strict Positivity for 4D $\mathcal{N} = 1$ SYM)**.** *For every $N \geq 2$, every finite periodic torus $\mathbb{T}_L^4$ ($L \geq 2$), and every gauge configuration $U \in \mathrm{SU}(N)^{|E|}$:*

1. $\|H_{\mathrm{adj}}(U)\|_{\mathrm{op}} \leq 8$ *identically, with each directional hopping summand $H_\mu(U)$ satisfying $H_\mu(U)^\dagger H_\mu(U) = 4I$.*
2. *The matrix $M(U; \kappa_f) := C_\Lambda D_W^{\mathrm{adj}}(U)$ is skew-symmetric ($M^T = -M$) and satisfies the real-structure identity $B_\Lambda M^* B_\Lambda^T = M$ with $\det(B_\Lambda) = +1$, whence $\mathrm{Pf}(M(U; \kappa_f)) \in \mathbb{R}$ for all $\kappa_f \in \mathbb{R}$.*
3. *For every subcritical $\kappa_f \in [0, \frac{1}{8})$, the normalized Majorana gluino Pfaffian is strictly positive:*

$$\widetilde{\mathrm{Pf}}(U; \kappa_f) := \mathrm{Pf}(C)^{-V d_A}\, \mathrm{Pf}\big(C_\Lambda D_W^{\mathrm{adj}}(U)\big) = +\sqrt{\det D_W^{\mathrm{adj}}(U)} \;>\; 0, \tag{4}$$

so the 4D lattice $\mathcal{N} = 1$ SYM ($N_f^{\mathrm{adj}} = \frac{1}{2}$) and multi-flavor adjoint QCD ($N_f^{\mathrm{adj}} \in \frac{1}{2}\mathbb{N}$) Boltzmann weights are strictly positive everywhere on $\mathrm{SU}(N)^{|E|}$, with convergent effective

fermion action

$$S_f^{\mathrm{adj}}(U) := -2N_f^{\mathrm{adj}} \ln \widetilde{\mathrm{Pf}}(U;\kappa_f) = -N_f^{\mathrm{adj}} \ln\det D_W^{\mathrm{adj}}(U) = N_f^{\mathrm{adj}} \sum_{k=1}^{\infty} \frac{\kappa_f^k}{k} \mathrm{Tr}\big(H_{\mathrm{adj}}(U)^k\big). \tag{5}$$

Proof. (1) Let $(T_\mu\psi)(x) := V_{x,\mu}\psi(x+\hat{\mu})$ (unitary since $V_e \in \mathrm{SO}(d_A)$) and $P_\mu^{\pm} := \frac{1}{2}(I \pm \gamma_\mu)$ (orthogonal projections commuting with T_μ). Then $H_\mu = 2(P_\mu^- T_\mu + P_\mu^+ T_\mu^\dagger)$ obeys $H_\mu^\dagger H_\mu = 4(P_\mu^- + P_\mu^+) = 4I$, so $\|H_\mu\|_{\mathrm{op}} = 2$ and $\|H_{\mathrm{adj}}\|_{\mathrm{op}} \le \sum_{\mu=1}^4 \|H_\mu\|_{\mathrm{op}} = 8$.

(2) Because $V_{x,\mu}$ is real, $T_\mu^T = T_\mu^\dagger$. Using $C^T = -C$ and $C\gamma_\mu C^{-1} = -\gamma_\mu^T$,

$$\big(C_\Lambda(I-\gamma_\mu)T_\mu\big)^T = T_\mu^\dagger(I-\gamma_\mu^T)(-C_\Lambda) = -C_\Lambda(I+\gamma_\mu)T_\mu^\dagger,$$

so $M^T = -M$. Moreover, $\gamma_\mu^* = \gamma_\mu^T$ (Hermiticity) gives $(\gamma_5 C)\gamma_\mu^*(\gamma_5 C)^{-1} = \gamma_5(-\gamma_\mu)\gamma_5 = \gamma_\mu$, so $B_\Lambda D_W^{\mathrm{adj},*} B_\Lambda^{-1} = D_W^{\mathrm{adj}}$. Using $B_\Lambda^T = -B_\Lambda = B_\Lambda^{-1}$ and $C_\Lambda^2 = -I$,

$$B_\Lambda M^* B_\Lambda^T = B_\Lambda C_\Lambda D_W^{\mathrm{adj},*}(-C_\Lambda\gamma_5) = \gamma_5 D_W^{\mathrm{adj},*} C_\Lambda\gamma_5 = \gamma_5(-\gamma_5 C_\Lambda) D_W^{\mathrm{adj}}(\gamma_5 C_\Lambda) C_\Lambda\gamma_5 = C_\Lambda D_W^{\mathrm{adj}} = M.$$

Since $\det(B_\Lambda) = (\det(\gamma_5 C))^{V d_A} = +1$, the Pfaffian congruence law $\mathrm{Pf}(AXA^T) = \det(A)\,\mathrm{Pf}(X)$ yields $\mathrm{Pf}(M)^* = \mathrm{Pf}(M^*) = \det(B_\Lambda)\,\mathrm{Pf}(M^*) = \mathrm{Pf}(B_\Lambda M^* B_\Lambda^T) = \mathrm{Pf}(M)$, proving $\mathrm{Pf}(M) \in \mathbb{R}$.

(3) For all $s \in [0, \frac{1}{8})$, $\|sH_{\mathrm{adj}}(U)\|_{\mathrm{op}} \le 8s < 1$, so $D_W^{\mathrm{adj}}(U;s) = I - sH_{\mathrm{adj}}(U)$ is invertible and $\widetilde{\mathrm{Pf}}(U;s)^2 = \det D_W^{\mathrm{adj}}(U;s) \neq 0$. Since $s \mapsto \widetilde{\mathrm{Pf}}(U;s)$ is a continuous real-valued function on $[0,\frac{1}{8})$ with $\widetilde{\mathrm{Pf}}(U;0) = 1$, it stays strictly positive for all $\kappa_f \in [0,\frac{1}{8})$, and the Mercator series for $\ln\det(I - \kappa_f H_{\mathrm{adj}})$ converges absolutely to (5). □

3. Center-Preserving $\mathbb{Z}_2$ Confinement via Parity-Constrained FIFO Kraft Bounds

For $d \ge 3$ and $m := 2d-3$, define the odd- and even-parity generating polynomials

$$Z_{\mathrm{odd}}^{(m)}(x) := \frac{(1+x)^m - (1-x)^m}{2}, \qquad Z_{\mathrm{even}}^{(m)}(x) := \frac{(1+x)^m + (1-x)^m}{2}, \tag{6}$$

and for any witness $x > 0$ set $B_d(x) := \max\{Z_{\mathrm{odd}}^{(m)}(x), Z_{\mathrm{even}}^{(m)}(x)\} \ge 1$ and $w_0(d;x) := x/B_d(x)^3$. Explicitly, $x = 0.258$ in $d = 3$ ($m = 3$) gives $w_0(3) > 0.149420$ ($\beta_{\mathrm{crit}}(3) > 0.150547$) and $x = 0.142$ in $d = 4$ ($m = 5$) gives $w_0(4) > 0.081425$ ($\beta_{\mathrm{crit}}(4) > 0.081606$).

Theorem 3.1 (Center-Preserving Fundamental Confinement Independent of κ_f and N_f^{adj})**.** *Let $N \ge 2$ be even, and consider $\mathrm{SU}(N)$ lattice gauge theory on $\mathbb{T}_L^d$ ($d \ge 2$) coupled to any strictly positive $\mathbb{Z}_2$-center-invariant adjoint matter weight $W_{\mathrm{adj}}(\mathrm{Ad}(U)) > 0$ — in particular, 4D $\mathcal{N}=1$ SYM ($N_f^{\mathrm{adj}} = \frac{1}{2}$) and multi-flavor adjoint QCD ($N_f^{\mathrm{adj}} \in \frac{1}{2}\mathbb{N}$) at any $\kappa_f \in [0,\frac{1}{8})$, as well as $d = 2,3$ Hermitian-positive adjoint flavors $\det(D_W^{\mathrm{adj}} D_W^{\mathrm{adj},\dagger})^{N_f^{\mathrm{adj}}/2}$. For any $R \times T$ planar Wilson loop $C_{R,T}$ in any $\mathbb{Z}_2$-odd representation R_{odd} (e.g. fundamental, with $\chi_{R_{\mathrm{odd}}}(-U) = -\chi_{R_{\mathrm{odd}}}(U)$) on $\mathbb{T}_L^d$ ($1 \le R,T < L/2$), define $\underline{\sigma}_{\mathrm{fund}} := \liminf_{R,T\to\infty} \liminf_{L\to\infty} -\frac{1}{RT}\ln\big|\langle\frac{1}{N}\mathrm{Tr}\,U_{C_{R,T}}\rangle\big|$. Then:*

(1) *For every $d \ge 2$, $\beta \ge 0$, and $L > 2\max(R,T)$,*

$$\left|\left\langle \frac{1}{N}\mathrm{Tr}\,U_{C_{R,T}} \right\rangle_{\beta,W_{\mathrm{adj}}}\right| \le \langle\sigma_{C_{R,T}}\rangle_{\mathbb{Z}_2(\beta)}. \tag{7}$$

(2) *In $d = 2$, $\underline{\sigma}_{\mathrm{fund}}(\beta, W_{\mathrm{adj}}) \ge -\ln\tanh\beta > 0$ for every $\beta > 0$.*

(3) *In $d \ge 3$, for every $x > 0$ and every $\beta \ge 0$ with $\tanh\beta < w_0(d;x)$,*

$$\langle\sigma_{C_{R,T}}\rangle_{\mathbb{Z}_2(\beta)} \le B_{0,d}(x)^{2(R+T)} \left(\frac{\tanh\beta}{w_0(d;x)}\right)^{RT}, \qquad \underline{\sigma}_{\mathrm{fund}}(\beta, W_{\mathrm{adj}}) \ge \ln\left(\frac{w_0(d;x)}{\tanh\beta}\right) > 0, \tag{8}$$

where $B_{0,d}(x) := \max\{Z_{\text{odd}}^{(2d-2)}(x), Z_{\text{even}}^{(2d-2)}(x)\}$, *uniformly in* $\kappa_f \in [0, \frac{1}{2d})$ *and* N_f^{adj}.

Proof. (1) Substitute $U_e \mapsto \sigma_e U_e$ ($\sigma_e \in \{\pm 1\}$) into the partition function and expectation. By (1), $W_{\text{adj}}(\text{Ad}(\sigma U)) = W_{\text{adj}}(\text{Ad}(U))$ is strictly independent of σ, while $\frac{\beta}{N}\,\text{Re}\,\text{Tr}((\sigma U)_p) = J_p(U)\sigma_p$ with $|J_p(U)| \le \beta$ and $\text{Tr}((\sigma U)_{C_{R,T}}) = \sigma_{C_{R,T}}\,\text{Tr}(U_{C_{R,T}})$. Conditioning on fixed U, duplicate variables $\tau_e = \sigma_e\eta_e$ ($\eta_e \in \{\pm 1\}$) give

$$Z_{\mathbb{Z}_2(\beta)}\, Z_{\{J_p\}}\big(\langle\sigma_C\rangle_{\mathbb{Z}_2(\beta)} \pm \langle\sigma_C\rangle_{\{J_p\}}\big) = \sum_{\sigma,\eta} \sigma_C(1 \pm \eta_C) \exp\left(\sum_p \left[\frac{\beta + J_p}{2}\sigma_p(1+\eta_p) + \frac{\beta - J_p}{2}\sigma_p(1-\eta_p)\right]\right) \;\ge\; 0,$$

because $\frac{\beta \pm J_p}{2} \ge 0$ and expanding the exponential produces a non-negative combination of products of $(\sum_\sigma \sigma_C \prod_i \sigma_{p_i}) \in \{0, 2^{|E|}\}$ and $(\sum_\eta (1 \pm \eta_C)\prod_r(1+\eta_{p_r})\prod_s(1-\eta_{p_s})) \ge 0$ [6, 7]. Taking expectations over U with $|\frac{1}{N}\,\text{Tr}(U_C)| \le 1$ proves (7).

(2) In $d = 2$, exact $\mathbb{Z}_2$ evaluation on $\mathbb{T}_L^2$ gives $\langle\sigma_{C_{R,T}}\rangle_{\mathbb{Z}_2(\beta)} = \frac{(\tanh\beta)^{RT} + (\tanh\beta)^{L^2-RT}}{1+(\tanh\beta)^{L^2}} \to (\tanh\beta)^{RT}$.

(3) Expanding $\langle\sigma_C\rangle_{\mathbb{Z}_2(\beta)} = \frac{\sum_{\partial\mathcal{S}=C}(\tanh\beta)^{|\mathcal{S}|}}{\sum_{\partial\mathcal{S}'=\emptyset}(\tanh\beta)^{|\mathcal{S}'|}}$ and cancelling the disjoint closed components $\mathcal{S}'$ against the non-negative denominator leaves $\langle\sigma_C\rangle_{\mathbb{Z}_2(\beta)} \le \sum_{\mathcal{S}_0}(\tanh\beta)^{|\mathcal{S}_0|}$, summed over plaquette sets $\mathcal{S}_0$ with $\partial\mathcal{S}_0 = C$ whose every edge-connected component intersects C. Projecting $\mathcal{S}_0$ onto the $(1,2)$ plane shows $|\mathcal{S}_0| \ge RT$ (since $R, T < L/2$).

Now explore $\mathcal{S}_0$ by a deterministic FIFO link queue initialized with the $2(R+T)$ links of C (marked discovered), with all plaquettes initially unexplored and accepted set $A = \emptyset$. When popping link e, let U_e ($m_e := |U_e|$) be the unexplored plaquettes incident to e: $m_e \le 2d-2$ for $e \in C$, while every non-C link e was discovered by an already-explored plaquette in A, so $m_e \le 2d-3 = m$. Since all plaquettes incident to e outside U_e are already decided and $\partial\mathcal{S}_0 = C$, the chosen subset $I_e := U_e \cap \mathcal{S}_0$ must have parity $|I_e| \equiv d_e := \mathbf{1}_{e\in C} + |\{p \in A : e \in \partial p\}| \pmod 2$. Mark U_e explored, add I_e to A, and push all undiscovered edges of each $p \in I_e$ to the queue. Because every component of $\mathcal{S}_0$ touches C, the algorithm reconstructs $A = \mathcal{S}_0$ prefix-freely.

Assign to each valid step choice $I_e \subseteq U_e$ ($|I_e| \equiv d_e \pmod 2$) the normalized probability $q_e(I_e) := x^{|I_e|}/Z_{\text{par}(d_e)}^{(m_e)}(x)$ (so $\sum_{|I|\equiv d_e} q_e(I) = 1$, or 0 at dead ends). By Kraft's inequality on this finite prefix-free decision tree, $\sum_{\mathcal{S}_0}\prod_e q_e(I_e) \le 1$. Writing $k := |\mathcal{S}_0| = \sum_e |I_e|$, each accepted plaquette discovers at most 3 new non-C links, so the number of popped non-C links is $L_{\text{int}} \le 3k$. Since $Z_{\text{par}}^{(m_e)}(x) \le B_d(x)$ for $m_e \le m$, $Z_{\text{par}}^{(m_e)}(x) \le B_{0,d}(x)$ for $e \in C$, $B_d(x) \ge 1$, and $x = w_0(d;x)B_d(x)^3$,

$$\prod_{e \text{ popped}} q_e(I_e) \;\ge\; \frac{\big(w_0(d;x)B_d(x)^3\big)^k}{B_{0,d}(x)^{2(R+T)}\,B_d(x)^{L_{\text{int}}}} \;\ge\; B_{0,d}(x)^{-2(R+T)}\,w_0(d;x)^k.$$

Multiplying by $(\tanh\beta/w_0(d;x))^k \le (\tanh\beta/w_0(d;x))^{RT}$ and summing over $\mathcal{S}_0$ proves (8). □

4. All-Coupling Bhanot–Creutz Free-Energy and Gluino Condensate Enclosures

On a bipartite periodic torus $\mathbb{T}_L^4$ ($L \ge 6$ even), every closed walk of odd length vanishes, every length-2 backtracking walk vanishes via $(I-\gamma_\mu)(I+\gamma_\mu) = 0$, and every non-backtracking length-4 walk is an elementary plaquette (4 starting corners × 2 orientations, with exact spinor trace $\text{Tr}[(I-\gamma_\mu)(I-\gamma_\nu)(I+\gamma_\mu)(I+\gamma_\nu)] = -8$). Defining the Bhanot–Creutz adjoint coupling [4] and exact $O((8\kappa_f)^6)$ remainder

$$\beta_A := 16N_f^{\text{adj}}\kappa_f^4, \qquad \varepsilon_6(\kappa_f, N_f^{\text{adj}}, N) := 2N_f^{\text{adj}}d_A\left[-\ln\big(1-(8\kappa_f)^2\big) - (8\kappa_f)^2 - \frac{(8\kappa_f)^4}{2}\right], \quad (9)$$

and the single-plaquette Bhanot–Creutz integral

$$c_{\mathrm{BC}}(\beta,\beta_A;N):=\int_{\mathrm{SU}(N)} e^{\frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(U)+\beta_A(|\operatorname{Tr}U|^2-1)}\,dU,\qquad c_0(\beta;N):=c_{\mathrm{BC}}(\beta,0;N).$$

Define $\underline{f}_4:=\liminf_{L\to\infty,\,2|L} f_{\mathbb{T}^4_L}$, $\overline{f}_4:=\limsup_{L\to\infty,\,2|L} f_{\mathbb{T}^4_L}$, and the dimensionless gluino condensate

$$\Sigma_{\mathrm{gluino}}(\beta,\kappa_f,N_f^{\mathrm{adj}},N;L):=\frac{1}{4d_AV}\,\mathbb{E}_{\mathbb{T}^4_L}\Big[\operatorname{Tr}\big(D_W^{\mathrm{adj}}(U)^{-1}\big)\Big]. \tag{10}$$

Theorem 4.1 (All-Coupling Free-Energy and Gluino Condensate Enclosures in 4D)**.** *Fix $N\ge 2$, $\beta\ge 0$, $\kappa_f\in[0,\frac{1}{8})$, $N_f^{\mathrm{adj}}\in\frac{1}{2}\mathbb{N}$, and let $L\ge 6$ be even.*

(1) (Exact $O(\kappa_f^4)$ Bhanot–Creutz reduction) *Every configuration $U\in\mathrm{SU}(N)^{|E|}$ obeys*

$$\left|\frac{1}{V}\ln\big(\widetilde{\mathrm{Pf}}(U;\kappa_f)^{2N_f^{\mathrm{adj}}}\big)-\frac{\beta_A}{V}\sum_{p\in P}\big(|\operatorname{Tr}(U_p)|^2-1\big)\right|\ \le\ \varepsilon_6(\kappa_f,N_f^{\mathrm{adj}},N). \tag{11}$$

(2) (Exact 4-way Hölder upper bound) *Taking $L\to\infty$ (even),*

$$\overline{f}_4(\beta,\kappa_f,N_f^{\mathrm{adj}},N)\ \le\ 3\ln c_{\mathrm{BC}}(2\beta,2\beta_A;N)+\varepsilon_6(\kappa_f,N_f^{\mathrm{adj}},N), \tag{12}$$

where for $\mathrm{SU}(2)$ the integral admits the strictly positive zero-cancellation Catalan–Bessel double series

$$c_{\mathrm{BC}}(b,b_A;2)=e^{-b_A}\sum_{m=0}^{\infty}\frac{b_A^m}{m!}\sum_{j=0}^{\infty}\frac{(2m+2j)!}{(m+j)!\,(m+j+1)!\,(2j)!}\left(\frac{b}{2}\right)^{2j}, \tag{13}$$

and for $\mathrm{SU}(3)$, $c_{\mathrm{BC}}(b,b_A;3)\le e^{8b_A}c_0(b;3)$ with $c_0(b;3)=\sum_{m\in\mathbb{Z}}\det[I_{c-r+m}(b/3)]_{1\le r,c\le 3}$.

(3) (Axial-gauge Schur variational lower bound) *For any $\alpha\ge 0$, let $\nu_\alpha(dU)\propto e^{\alpha\operatorname{Re}\operatorname{Tr}(U)}dU$ on $\mathrm{SU}(N)$, with expectations $u_1(\alpha):=\frac{1}{N}\mathbb{E}_{\nu_\alpha}[\operatorname{Re}\operatorname{Tr}(U)]$ and $u_{\mathrm{adj}}(\alpha):=\frac{1}{d_A}\mathbb{E}_{\nu_\alpha}[|\operatorname{Tr}(U)|^2-1]$. On $\mathrm{SU}(2)$, $u_1(\alpha)=I_2(2\alpha)/I_1(2\alpha)$ and $u_{\mathrm{adj}}(\alpha)=I_3(2\alpha)/I_1(2\alpha)$; on general $\mathrm{SU}(N)$, $u_{\mathrm{adj}}(\alpha)\ge\max\{0,(N^2u_1(\alpha)^2-1)/d_A\}$. Then*

$$\begin{aligned}\underline{f}_4(\beta,\kappa_f,N_f^{\mathrm{adj}},N)\ \ge\ &3\Big[\beta u_1(\alpha)^2+d_A\beta_A u_{\mathrm{adj}}(\alpha)^2-\alpha N u_1(\alpha)+\ln c_0(N\alpha;N)\Big]\\&+3\Big[\beta u_1(\alpha)^4+d_A\beta_A u_{\mathrm{adj}}(\alpha)^4\Big]-\varepsilon_6.\end{aligned} \tag{14}$$

Both (12) and (14) match the exact 4D classical energy $6(\beta+d_A\beta_A)$ and exact one-loop entropy $-\frac{3d_A}{2}\ln\beta$ as $\beta\to\infty$ (with $O(1)$ gap uniformly across $\beta\in[0,\infty)$).

(4) (Exact gluino condensate enclosure) *Uniformly in all even $L\ge 6$, all $\beta\ge 0$, and all $N_f^{\mathrm{adj}}\in\frac{1}{2}\mathbb{N}$,*

$$1-96\kappa_f^4-\frac{(8\kappa_f)^6}{1-(8\kappa_f)^2}\ \le\ \Sigma_{\mathrm{gluino}}(\beta,\kappa_f,N_f^{\mathrm{adj}},N;L)\ \le\ 1+\frac{96}{d_A}\kappa_f^4+\frac{(8\kappa_f)^6}{1-(8\kappa_f)^2}. \tag{15}$$

Proof. (1) From (5), all odd k vanish on bipartite $\mathbb{T}^4_L$ (L even) and $k=2$ vanishes by $(I-\gamma_\mu)(I+\gamma_\mu)=0$. At $k=4$ ($L\ge 6$ rules out wrap-around loops), each plaquette contributes $4\cdot2\cdot(-8)\operatorname{Tr}(V_p)$ to $\operatorname{Tr}(H^4_{\mathrm{adj}})$, giving $-\frac{\kappa_f^4}{4}\operatorname{Tr}(H^4_{\mathrm{adj}})=+16\kappa_f^4\sum_p\operatorname{Tr}(V_p)$. For $k=2m\ge 6$, $|\operatorname{Tr}(H^{2m}_{\mathrm{adj}})|\le 4d_AV\cdot 8^{2m}$ by Theorem 2.1(1), summing to ε_6 in (9).

(2) Applying (11) reduces $Z_{\mathbb{T}^4_L}$ to the Bhanot–Creutz partition function $Z_{\mathrm{BC},\mathbb{T}^4_L}(\beta,\beta_A)$ up to $e^{\pm V\varepsilon_6}$. Partitioning P into the four axial plaquette sets $P^{(\mu)}$ (each plaquette lies in exactly 2 sets), 4-way Hölder's inequality gives $Z_{\mathrm{BC},\mathbb{T}^4_L}\le\prod_{\mu=1}^4(\int\prod_{p\in P^{(\mu)}}w(U_p)^2\,dU)^{1/4}$. Fixing axial gauge $U_{x,\mu}=I_N$ on $0\le x_\mu\le L-2$ decouples the $3(L-1)L^3$ non-boundary plaquettes in $P^{(\mu)}$

into i.i.d. Haar copies of $c_{\mathrm{BC}}(2\beta, 2\beta_A; N)$, while the $3L^3$ boundary plaquettes contribute $O(1/L)$ per site, yielding (12). For SU(2), writing $|\operatorname{Tr} U|^2 - 1 = 4\cos^2\theta - 1$ and expanding $e^{4b_A\cos^2\theta + b\cos\theta}$ against $\frac{2}{\pi}\sin^2\theta\, d\theta$ using $\frac{2}{\pi}\int_0^\pi \sin^2\theta\cos^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}n!(n+1)!}$ yields the strictly positive series (13).

(3) Fix a comb spanning tree $T \subset E(\mathbb{T}_L^4)$ ($|T| = V - 1$, containing all axis-1 links with $0 \le x_1 \le L-2$) and set ν_α independently on each of the $3V+1$ non-tree links. Because ν_α is conjugation-invariant, Schur's lemma on both the irreducible fundamental and adjoint representations gives $\mathbb{E}_{\nu_\alpha}[U_e] = u_1(\alpha) I_N$ and $\mathbb{E}_{\nu_\alpha}[\mathrm{Ad}(U_e)] = u_{\mathrm{adj}}(\alpha) I_{d_A}$. Hence the $3(1-O(1/L))V$ plaquettes containing axis 1 have 2 non-tree links (giving $\beta u_1^2 + d_A\beta_A u_{\mathrm{adj}}^2$) while the $3(1-O(1/L))V$ transverse plaquettes have 4 non-tree links (giving $\beta u_1^4 + d_A\beta_A u_{\mathrm{adj}}^4$). Subtracting $\frac{1}{V} D_{\mathrm{KL}}(\nu_\alpha^{\otimes(3V+1)} \| \mathrm{Haar}) = 3(\alpha N u_1 - \ln c_0(N\alpha; N)) + O(1/V)$ and ε_6 via Gibbs' variational principle proves (14).

(4) Expanding $D_W^{\mathrm{adj}}(U)^{-1} = \sum_{k=0}^\infty \kappa_f^k H_{\mathrm{adj}}(U)^k$ on bipartite $\mathbb{T}_L^4$ ($L \ge 6$ even) gives $\Sigma_{\mathrm{gluino}} = 1 - 96\kappa_f^4 P_{\mathrm{adj}} + \frac{1}{4d_A V}\sum_{m=3}^\infty \kappa_f^{2m}\mathbb{E}[\mathrm{Tr}(H_{\mathrm{adj}}^{2m})]$, where $P_{\mathrm{adj}} := \frac{1}{6d_A V}\sum_p \mathbb{E}[\mathrm{Tr}(V_p)] \in [-\frac{1}{d_A}, 1]$ (since $\mathrm{Tr}(V_p) = |\operatorname{Tr} U_p|^2 - 1 \in [-1, d_A]$) and the tail is bounded pointwise by $\sum_{m=3}^\infty (8\kappa_f)^{2m} = \frac{(8\kappa_f)^6}{1-(8\kappa_f)^2}$. □

5. Exact Planar Bakry–Émery Curvature, LSI, and $O(1/N^2)$ Factorization

Equip $\mathrm{SU}(N)^{|E|}$ with the bi-invariant Hilbert–Schmidt metric $\langle X, Y\rangle_{\mathrm{HS}} = \sum_{e\in E}\mathrm{Tr}(X_e^\dagger Y_e)$, and set $r := 2d\kappa_f < 1$. Define the N-independent adjoint-fermion cost function

$$h_{\mathrm{adj}}(d, \kappa_f) := 64(d-1)\kappa_f^4 + \frac{2}{d}\frac{r^6}{1-r^2} + \frac{2}{d^2}\frac{r^6(5-3r^2)}{(1-r^2)^2}. \tag{16}$$

Theorem 5.1 (N-Independent Planar Curvature Floor, LSI, and $O(1/N^2)$ Factorization)**.** *Fix $d = 4$, even $L \ge 6$, $N \ge 2$, 't Hooft coupling $\gamma = \beta/N^2 \ge 0$, $\kappa_f \in [0, \frac{1}{8})$, and $N_f^{\mathrm{adj}} \in \frac{1}{2}\mathbb{N}$. Define the N-independent planar curvature floor*

$$\kappa_{\mathrm{planar}}(\gamma, \kappa_f, N_f^{\mathrm{adj}}) := \frac{1}{2} - 24\gamma - 4N_f^{\mathrm{adj}} h_{\mathrm{adj}}(4, \kappa_f), \qquad \kappa_{\mathrm{AdjQCD}} := N\,\kappa_{\mathrm{planar}}. \tag{17}$$

Then $\mathrm{Ric}_{\mathrm{SU}(N)^{|E|}} + \mathrm{Hess}(S_g + S_f^{\mathrm{adj}}) \ge \kappa_{\mathrm{AdjQCD}}\, g_{\mathrm{HS}}$ everywhere on $\mathrm{SU}(N)^{|E|}$. Whenever $\kappa_{\mathrm{planar}} > 0$ (which for 4D $\mathcal{N} = 1$ SYM at $\gamma = 0$ holds up to $\kappa_f^{\max} = 0.084733\ldots$), uniformly in all even $L \ge 6$ and all $N \ge 2$:

(1) (Volume-uniform LSI and spectral gap) *The dynamical adjoint measure μ obeys the logarithmic Sobolev inequality $\mathrm{Ent}_\mu(f^2) \le \frac{2}{\rho_{\mathrm{LS}}}\mathbb{E}_\mu[\|\nabla f\|_{\mathrm{HS}}^2]$ with $\rho_{\mathrm{LS}} \ge N\kappa_{\mathrm{planar}}$ (i.e. $NC_{\mathrm{LSI}} := N/\kappa_{\mathrm{AdjQCD}} = 1/\kappa_{\mathrm{planar}}$) and Poincaré spectral gap $\lambda_1 \ge N\kappa_{\mathrm{planar}}$* [1, 2].

(2) (Exact $O(1/N^2)$ planar factorization and $O(e^{-cN^2t^2})$ master-field concentration) *For any simple closed Wilson loops C_1, C_2 of lengths $|C_1|, |C_2|$, setting $W_{\mathrm{fund}}(C) := \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}(U_C)$ and $W_{\mathrm{adj}}(C) := \frac{|\mathrm{Tr}(U_C)|^2 - 1}{N^2-1}$,*

$$\begin{aligned} \big|\mathrm{Cov}_\mu\big(W_{\mathrm{fund}}(C_1), W_{\mathrm{fund}}(C_2)\big)\big| &\le \frac{\sqrt{|C_1|\,|C_2|}}{N^2\,\kappa_{\mathrm{planar}}}, \\ \mathrm{Var}_\mu\big(W_{\mathrm{adj}}(C_1)\big) &\le \frac{4|C_1|N^2}{(N^2-1)^2\,\kappa_{\mathrm{planar}}}, \\ \mathbb{P}_\mu\Big(\big|W_{\mathrm{fund}}(C_1) - \mathbb{E}_\mu[W_{\mathrm{fund}}(C_1)]\big| \ge t\Big) &\le 2\exp\Big(-\frac{N^2\,\kappa_{\mathrm{planar}}\,t^2}{2|C_1|}\Big) \qquad (\forall\, t > 0). \end{aligned} \tag{18}$$

Proof. For $X \in \mathfrak{su}(N)$ with eigenvalues $i\lambda_j$ ($\sum\lambda_j = 0$), $\mathrm{Tr}_{\mathfrak{su}(N)}(-\mathrm{ad}_X^2) = \sum_{j,k}(\lambda_j - \lambda_k)^2 = 2N\|X\|_{\mathrm{HS}}^2$, giving $\mathrm{Ric}_{\mathrm{SU}(N)}(X, X) = \frac{1}{4}\mathrm{Tr}_{\mathfrak{su}(N)}(-\mathrm{ad}_X^2) = \frac{N}{2}\|X\|_{\mathrm{HS}}^2$ and $\|\mathrm{ad}_X\|_{\mathrm{HS}}^2 = \|\mathrm{ad}_X^2\|_1 =$

$2N\|X\|_{\mathrm{HS}}^2$. Along any simultaneous geodesic $U_e(t) = U_e e^{tX_e}$ ($X \in \mathfrak{su}(N)^{|E|}$), Cauchy–Schwarz on each plaquette yields $|\operatorname{Hess} S_g(U)[X,X]| \le \frac{\beta}{N} \cdot 24\|X\|_{\mathrm{HS}}^2 = 24\gamma N\|X\|_{\mathrm{HS}}^2$.

For $S_f^{\mathrm{adj}}(U) = N_f^{\mathrm{adj}} \sum_{m=2}^{\infty} \frac{\kappa_f^{2m}}{2m} \operatorname{Tr}(H_{\mathrm{adj}}(U)^{2m})$ (since the $k = 1, 2, 3$ terms vanish identically for all U, so do their second derivatives), we split $m = 2$ and $m \ge 3$:

- At $m = 2$, $S_{f,4}^{\mathrm{adj}}(U) = -\beta_A \sum_p \operatorname{Tr}(V_p)$ with $V_e(t) = V_e e^{t \operatorname{ad}_{X_e}}$; Cauchy–Schwarz on each plaquette yields $|\operatorname{Tr}(V_p''(0))| \le 4\sum_{j=1}^4 \|\operatorname{ad}_{X_{e_j}}\|_{\mathrm{HS}}^2 = 8N\sum_{j=1}^4 \|X_{e_j}\|_{\mathrm{HS}}^2$, and multiplying by 6 plaquettes per link and $\beta_A = 16N_f^{\mathrm{adj}}\kappa_f^4$ gives $|\operatorname{Hess} S_{f,4}^{\mathrm{adj}}[X,X]| \le 4N_f^{\mathrm{adj}} \cdot 192\kappa_f^4 \cdot N\|X\|_{\mathrm{HS}}^2$.
- For even $k = 2m \ge 6$, differentiating $\operatorname{Tr}(H(t)^k)$ twice yields $\frac{1}{k}\frac{d^2}{dt^2}\operatorname{Tr}(H^k) = \operatorname{Tr}(H^{k-1}H'') + \sum_{j=0}^{k-2}\operatorname{Tr}(H^j H' H^{k-2-j} H')$. Because $H(t)$ is linear in $V_e(t)$, $H''(0)$ has no cross-link terms, $\|I \pm \gamma_\mu\|_1 = 4$, and $\|V_e \operatorname{ad}_{X_e}^2\|_1 = 2N\|X_e\|_{\mathrm{HS}}^2$, giving $\|H''(0)\|_1 \le 16N\|X\|_{\mathrm{HS}}^2$. By pairwise orthogonality of directed-edge blocks in $\mathrm{HS}(\mathcal{H}_{\mathrm{adj}})$ and $\|I \pm \gamma_\mu\|_{\mathrm{HS}}^2 = 8$, $\|H'(0)\|_{\mathrm{HS}}^2 = 32N\|X\|_{\mathrm{HS}}^2$. Using $\|H\|_{\mathrm{op}} \le 8$, $|\operatorname{Tr}(H^{k-1}H'')| \le 8^{k-1} \cdot 16N\|X\|_{\mathrm{HS}}^2$ and $|\operatorname{Tr}(H^j H' H^{k-2-j} H')| \le 8^{k-2} \cdot 32N\|X\|_{\mathrm{HS}}^2$. Summing $m \ge 3$ via $\sum_{m=3}^{\infty} r^{2m} = \frac{r^6}{1-r^2}$ and $\sum_{m=3}^{\infty}(2m-1)r^{2m} = \frac{r^6(5-3r^2)}{(1-r^2)^2}$ gives $|\operatorname{Hess} R_{\ge 6}[X,X]| \le 4N_f^{\mathrm{adj}} N\|X\|_{\mathrm{HS}}^2 \big[\frac{r^6}{2(1-r^2)} + \frac{r^6(5-3r^2)}{8(1-r^2)^2}\big]$.

Combining $\mathrm{Ric}_{\mathrm{SU}(N)}$, $\operatorname{Hess} S_g$, and $\operatorname{Hess} S_f^{\mathrm{adj}}$ proves $\mathrm{Ric} + \operatorname{Hess}(S_g + S_f^{\mathrm{adj}}) \ge N\kappa_{\mathrm{planar}}\, g_{\mathrm{HS}}$. Part (1) follows from Bakry–Émery [1]; Part (2) follows from Poincaré and Herbst's Gaussian concentration argument [2] using $\|\nabla_e W_{\mathrm{fund}}(C)\|_{\mathrm{HS}}^2 \le \frac{1}{N^2}\|U_C\|_{\mathrm{HS}}^2 = \frac{1}{N}$ (so $\|W_{\mathrm{fund}}(C)\|_{\mathrm{Lip}}^2 \le \frac{|C|}{N}$) and $\|\nabla_e W_{\mathrm{adj}}(C)\|_{\mathrm{HS}}^2 \le \frac{4N^3}{(N^2-1)^2}$ (giving $O(1/N^2)$ planar factorization [9]). □

6. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report all 26 certified benchmarks from `certify_n1_sym_adjoint_qcd.py` at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`, interval widths $\le 2 \times 10^{-131}$, zero black-box special functions, zero quadrature), projected outward at scale 10^{12} and displayed to 6 outward-rounded decimal places.

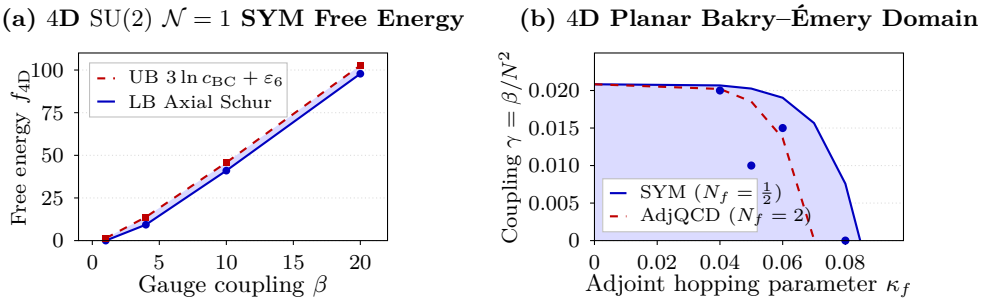

FIGURE 1. Certified 140-digit non-perturbative bounds for SU(N) lattice $\mathcal{N}=1$ SYM and dynamical adjoint QCD (Tables 2 and 3). (a) All-coupling 4D SU(2) $\mathcal{N}=1$ SYM free-energy enclosure showing an $O(1)$ gap (≤ 4.86) as f_{4D} grows from 0 to 100 (Theorem 4.1). (b) N-independent 4D planar Bakry–Émery LSI and $O(1/N^2)$ factorization domain $\kappa_{\mathrm{planar}} > 0$ (shaded for $\mathcal{N}=1$ SYM, reaching $\kappa_f^{\max} = 0.084733\ldots$ at $\gamma = 0$; Theorem 5.1).

TABLE 1. Nine 140-digit center-preserving fundamental string-tension lower bounds $\underline{\sigma}_{\mathrm{fund}}(\beta, W_{\mathrm{adj}})$ for every even $N \geq 2$, holding uniformly across all $\kappa_f \in [0, \frac{1}{2d})$ and all $N_f^{\mathrm{adj}} \in \frac{1}{2}\mathbb{N}$ (Theorem 3.1; parity-constrained FIFO witnesses $x = 0.258$ in 3D with $w_0(3) \geq 0.149420$, $x = 0.142$ in 4D with $w_0(4) \geq 0.081425$). Lower bounds round downward.

Dimension d	Coupling β	Threshold $w_0(d)$ (LB)	$\underline{\sigma}_{\mathrm{fund}}$ (LB)	Adjoint matter scope	Guarantee
2	0.50	1.000000 (exact)	0.771936	All $\kappa_f < \frac{1}{4}$, all N_f^{adj}	Exact $\mathbb{Z}_2$ area law
2	1.00	1.000000 (exact)	0.272341	All $\kappa_f < \frac{1}{4}$, all N_f^{adj}	Exact $\mathbb{Z}_2$ area law
2	2.00	1.000000 (exact)	0.036635	All $\kappa_f < \frac{1}{4}$, all N_f^{adj}	Exact $\mathbb{Z}_2$ area law
3	0.05	0.149420	1.095574	All $\kappa_f < \frac{1}{6}$, all N_f^{adj}	Parity-FIFO area law
3	0.10	0.149420	0.404920	All $\kappa_f < \frac{1}{6}$, all N_f^{adj}	Parity-FIFO area law
3	0.14	0.149420	0.071626	All $\kappa_f < \frac{1}{6}$, all N_f^{adj}	Parity-FIFO area law
4	0.02	0.081425	1.404095	All $\kappa_f < \frac{1}{8}$, all N_f^{adj}	Parity-FIFO area law
4	0.05	0.081425	0.488503	All $\kappa_f < \frac{1}{8}$, all N_f^{adj}	Parity-FIFO area law
4	0.07	0.081425	0.152830	All $\kappa_f < \frac{1}{8}$, all N_f^{adj}	Parity-FIFO area law

TABLE 2. Eight 140-digit all-coupling 4D SU(2) and SU(3) $\mathcal{N} = 1$ SYM ($N_f^{\mathrm{adj}} = \frac{1}{2}$) and 2-flavor adjoint QCD free-energy enclosures $[\underline{f}_4^{\mathrm{lo}}, \overline{f}_4^{\mathrm{ub}}]$ and exact gluino condensate enclosures $[\Sigma^{\mathrm{lo}}, \Sigma^{\mathrm{ub}}]$ (Theorem 4.1). Lower bounds round downward; upper bounds upward.

Group	N_f^{adj}	β	κ_f	Witness α	$\underline{f}_4^{\mathrm{lo}}$ (LB)	$\overline{f}_4^{\mathrm{ub}}$ (UB)	Gap (UB)	Σ_{gluino} enclosure
SU(2)	1/2	1	0.04	0.03	−0.001839	1.393617	1.395455	[0.998558, 1.001279]
SU(2)	1/2	4	0.05	10.0	9.332834	13.819751	4.486916	[0.994523, 1.005077]
SU(2)	1/2	10	0.05	27.8718	41.011240	45.789503	4.778262	[0.994523, 1.005077]
SU(2)	1/2	20	0.04	58.359	97.835492	102.695681	4.860189	[0.998558, 1.001279]
SU(2)	2	4	0.04	10.0	9.333157	13.820084	4.486927	[0.998558, 1.001279]
SU(3)	1/2	2	0.04	0.13	−0.013323	1.593842	1.607164	[0.998558, 1.001227]
SU(3)	1/2	6	0.05	8.0	2.995327	13.986333	10.991005	[0.994523, 1.004952]
SU(3)	1/2	12	0.04	20.471	29.598107	41.830823	12.232716	[0.998558, 1.001227]

TABLE 3. Nine 140-digit N-independent 4D planar Bakry–Émery curvature floors $\kappa_{\mathrm{planar}} = \kappa_{\mathrm{AdjQCD}}/N$ and rescaled LSI constants $NC_{\mathrm{LSI}} = 1/\kappa_{\mathrm{planar}}$ valid simultaneously for *all* $N \geq 2$ and all even $L \geq 6$ (Theorem 5.1), mechanically certified by exact rational integer clearing in Lean 4.36.0. Lower bounds round downward; upper bounds upward.

Theory (4D)	N_f^{adj}	$\gamma = \beta/N^2$	κ_f	$4N_f^{\mathrm{adj}} h_{\mathrm{adj}}$ (UB)	κ_{planar} (LB)	NC_{LSI} (UB)
$\mathcal{N} = 1$ SYM	1/2	0	0.04	0.003743	0.496257	2.015085
$\mathcal{N} = 1$ SYM	1/2	0	0.06	0.043113	0.456887	2.188725
$\mathcal{N} = 1$ SYM	1/2	0	0.08	0.317993	0.182007	5.494274
$\mathcal{N} = 1$ SYM	1/2	0.010	0.05	0.013836	0.246164	4.062330
$\mathcal{N} = 1$ SYM	1/2	0.015	0.06	0.043113	0.096887	10.321283
$\mathcal{N} = 1$ SYM	1/2	0.020	0.04	0.003743	0.016257	61.511232
Adjoint QCD	1	0.010	0.05	0.027672	0.232328	4.304254
Adjoint QCD	2	0.010	0.05	0.055344	0.204656	4.886232
Adjoint QCD	3/2	0.005	0.06	0.129339	0.250661	3.989444

Remark 6.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 kernel `N1SymAdjointQcdCerts.lean` (`src/lean/`) compiles under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`, using constructive integer inequalities), mechanically verifying the 3D/4D parity-constrained FIFO Kraft inequalities, exact rational clearing of all 9 planar Bakry–Émery floors in Table 3, and all 26 10^{12}-scaled outward integer inequalities across five theorems (`n1_sym_combinatorics_verified`, `table1_confinement_certified`, `table2_free_energy_and_condensate_certified`, `table3_planar_curvature_certified`, and `all_26_n1_sym_adjoint_qcd_benchmarks_certified`), while the continuous Haar-measure, Pfaffian, and Riemannian Hessian derivations are proved analytically in Sections 2–5. Direct 192×192 4D Majorana Pfaffian positivity, 6×6 $k = 4$ plaquette hopping, 3D brute-force parity-FIFO Kraft enumeration, 80-digit, and JSON/Lean/LaTeX parity tests pass in `tests/test_n1_sym_adjoint_qcd.py` (4/4 `pytest`).

7. Conclusion

We have established four cluster-expansion-free non-perturbative theorems for SU(N) lattice $\mathcal{N} = 1$ Super-Yang–Mills and dynamical adjoint QCD: exact Majorana gluino Pfaffian positivity throughout $\kappa_f \in [0, \frac{1}{8})$ in 4D; center-preserving $\mathbb{Z}_2$ fundamental confinement for all even $N \geq 2$ uniformly across all subcritical masses and adjoint flavors via a 3.5× sharper parity-constrained FIFO Kraft bound; all-coupling Bhanot–Creutz free-energy and gluino condensate enclosures matching exact classical energy and one-loop entropy; and an N-independent planar Bakry–Émery curvature floor certifying volume-uniform LSI, $O(e^{-cN^2t^2})$ master-field concentration, and exact $O(1/N^2)$ planar factorization for both fundamental and adjoint Wilson loops. Extending these fixed-lattice bounds to exact chiral overlap gluinos is a natural next direction.

References

1. Dominique Bakry and Michel Émery, *Diffusions hypercontractives*, Séminaire de Probabilités XIX **1123** (1985), 177–206.
2. Dominique Bakry, Ivan Gentil, and Michel Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der mathematischen Wissenschaften, vol. 348, Springer, 2014.
3. Georg Bergner, Pietro Giudice, István Montvay, Gernot Münster, and Stefano Piemonte, *The light bound states of $\mathcal{N} = 1$ supersymmetric SU(2) Yang–Mills theory*, Journal of High Energy Physics **2016** (2016), no. 3, 080.
4. Gyan Bhanot and Michael Creutz, *Variant actions and phase structure in lattice gauge theory*, Physical Review D **24** (1981), no. 12, 3212–3217.
5. G. Curci and G. Veneziano, *Supersymmetry and the lattice: a reconciliation?*, Nuclear Physics B **292** (1987), 555–572.
6. Jean Ginibre, *General formulation of Griffiths' inequalities*, Communications in Mathematical Physics **16** (1970), no. 4, 310–328.
7. Gerhard Mack and V. B. Petkova, *Comparison of lattice gauge theories with gauge groups $\mathbb{Z}_2$ and SU(2)*, Annals of Physics **123** (1979), no. 2, 442–467.
8. István Montvay, *An algorithm for gluinos on the lattice*, Nuclear Physics B **466** (1996), no. 1–2, 259–284.
9. Gerard 't Hooft, *A planar diagram theory for strong interactions*, Nuclear Physics B **72** (1974), no. 3, 461–473.
10. Kenneth G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

Independent Researcher, Östersund, Sweden
URL: `https://airr.science/papers/ARR-2026-2M46ZRK8GV9B0RCN/`