AIRR.SCIENCE · ARR-2026-2KHV642RDE9W6RF5 v1 [v1] 29 Nov 2026

Homological Phase Transitions of Higher-Order Hodge Laplacians: Exact Block Green-Operator Laws, Universal Factor-of-2 Spectral-Gap Collapse, Integral Torsion–Lattice Duality, and Kirchhoff–Hodge Extremality

Lluis Eriksson*

November 29, 2026

Abstract

Let X be a finite weighted k-dimensional cellular or simplicial complex ($k \geq 1$) with boundary operators ∂_{k-1}, ∂_k, positive k-cell weights $W_k = \mathrm{diag}(w_\sigma)$, Hodge up-Laplacian $L_{k-1}^{\mathrm{up}} = \partial_k W_k \partial_k^\top$, and Moore–Penrose Green's operator $G = (L_{k-1}^{\mathrm{up}})^+$. We develop a unified spectral, algebraic-topological, and lattice-theoretic theory governing how G, $\det'(L_{k-1}^{\mathrm{up}})$, the Kirchhoff–Hodge total resistance $\mathrm{Tr}(G)$, and the spectral gap $\lambda_{\min}^+(L_{k-1}^{\mathrm{up}})$ transform across homological phase transitions and across torsion-bearing spanning k-acycles.

First, when attaching a new k-cell τ of weight $w > 0$ with boundary $b = b^{\|} + b^{\perp} \in \mathrm{im}(\partial_k) \oplus \mathcal{H}_{k-1}(X)$ and in-image simplicial resistance $\mathcal{R}_X(b) := (b^{\|})^\top G b^{\|}$, we prove an exact two-regime Green-operator, determinant, and trace law. In particular, the sign of the Kirchhoff–Hodge increment $\Delta\,\mathrm{Tr}(L_{k-1}^+) = \Delta\,\mathrm{Tr}(G)$ is an exact spectral detector of Betti-number transitions in any weighted filtration: $\Delta\,\mathrm{Tr}(L_{k-1}^+) < 0$ if and only if τ creates a k-cycle ($\Delta\beta_k = +1$, $b^{\perp} = 0$), whereas $\Delta\,\mathrm{Tr}(L_{k-1}^+) > 0$ if and only if τ kills a $(k-1)$-cycle ($\Delta\beta_{k-1} = -1$, $b^{\perp} \neq 0$). In the homology-killing regime ($b^{\perp} \neq 0$), an exact Gram–Schur factorization pins the post-transition spectral gap universally within a factor of 2:

$$\frac{1}{2}\min\{\lambda_{\mathrm{old}},\, \Lambda_\tau(w)\} \;\leq\; \left(\frac{1}{\lambda_{\mathrm{old}}} + \frac{1}{\Lambda_\tau(w)}\right)^{-1} \;\leq\; \lambda_{\mathrm{new}} \;\leq\; \min\{\lambda_{\mathrm{old}},\, \Lambda_\tau(w)\}, \qquad \Lambda_\tau(w) := \frac{w\,\|b^{\perp}\|^2}{1 + w\,\mathcal{R}_X(b)},$$

proving a universal *Homological Braess Collapse* ($\lambda_{\mathrm{new}} < \lambda_{\mathrm{old}}$ for all $w > 0$ whenever λ_{old} is simple and $\langle v_1, b^{\|}\rangle \neq 0$), with the factor $1/2$ saturated asymptotically on $K_2 \sqcup K_2$ plus a bridge of weight $w \to \infty$.

Second, via Smith Normal Form and primitive orthogonal lattice duality in $\mathbb{Z}^{|Y_{k-1}|}$, we prove the exact covolume identity $\mathrm{vol}(Z_{k-1}^{\mathbb{Z}}(Y))^2 = \tau_{k-1}(Y)/|T_{k-2}(Y)|^2$ for arbitrary $(k-1)$-complexes Y (including non-trivial lower torsion $T_{k-2}(Y)$ and arbitrary $\beta_{k-2}(Y) \geq 0$), yielding the weighted Kalai–Kirchhoff determinant factorization and torsion-weighted effective resistance law in all dimensions.

Third, for every spanning k-acycle T ($r = \beta_{k-1}(Y) \geq 2$) with exact determinant $D_T = \frac{t(T)^2}{|T_{k-2}(Y)|^2}\tau_{k-1}(Y)$ and mean boundary squared norm $\bar{c}$, we establish the dual-stiffness Kirchhoff identity $\mathrm{Tr}(G) = \sum_{\tau \in T} \|b_\tau^{\perp}(T \setminus \tau)\|^{-2}$ and prove by exact Lagrange-multiplier analysis the sharp two-sided extremal bounds on $\lambda_{\min}^+(L_{k-1}^{\mathrm{up}}(T))$ and $\mathrm{Tr}(G)$ in terms of the two roots $\lambda_* \leq \bar{c} \leq \lambda^*$ of $\lambda(\frac{r\bar{c}-\lambda}{r-1})^{r-1} = D_T$, together with an exact Itakura–Saito spectral concentration law showing that $\alpha(T) \to 1$ forces convergence of the normalized empirical

*Independent Researcher, Östersund, Sweden. Email: editor@airr.science. Reproducibility code and verification artifacts: https://github.com/lluiseriksson/hodge-homological-transitions (release v1.0.2).

spectral distribution to δ_1 (and uniform convergence $\max_i |\lambda_i/\bar{c} - 1| \to 0$ at fixed r or whenever $\alpha(T) = 1 - o(r^{-1})$).

Fourth, we derive closed-form spectra for the 6-vertex triangulation of $\mathbb{R}P^2$ (via the signed Petersen cubic identity $A^3_{\text{signed}} = 5A_{\text{signed}}$), all simplicial cones $v_0 * Y$ ($M(v_0 * Y) = I + L^{\text{down}}_{k-1}(Y)$), Fibonacci fans F_m, and alternating orientable-cylinder / non-orientable-Möbius simplicial bands B_r ($\det' = L_{2r} - 2(-1)^r$). Exhaustive enumeration of all $\binom{20}{10} = 184{,}756$ 10-triangle subsets of K_6 (46,608 torsion-free acycles and 12 copies of $\mathbb{R}P^2$) reveals a sharp extremal split: the 12 copies of $\mathbb{R}P^2$ ($t = 2$) uniquely minimize Kirchhoff–Hodge resistance ($\text{Tr}(G) = 35/6$ vs. $20/3$ for $t = 1$), whereas the 6 torsion-free cones $v_0 * K_5$ uniquely maximize the spectral gap ($\lambda^+_{\min} = 1$ vs. $3 - \sqrt{5}$ for $\mathbb{R}P^2$).

Contents

1 Introduction **2**
1.1 Summary of Main Results 3

2 Exact Two-Regime Green-Operator Law and Factor-of-2 Spectral-Gap Collapse **3**

3 Integral Torsion–Lattice Duality in All Dimensions **6**

4 Dual-Stiffness Kirchhoff Law, Sharp Extremal Bounds, and Torsion Concentration **8**

5 Canonical Closed-Form Families and Exhaustive K_6 Torsion Extremality **10**

6 Computational Verification and Quantitative Benchmarks **13**
6.1 Summary of Verification Results 13

7 Concluding Remarks **14**

1 Introduction

Combinatorial Hodge theory, initiated by Eckmann [2] and developed across higher-dimensional simplicial complexes [3, 8, 9], decomposes the real $(k-1)$-chain space $C_{k-1}(X;\mathbb{R})$ of a finite cell or simplicial complex X into three mutually orthogonal subspaces:

$$C_{k-1}(X;\mathbb{R}) \;=\; \text{im}(\partial^\top_{k-1}) \;\oplus\; \mathcal{H}_{k-1}(X) \;\oplus\; \text{im}(\partial_k), \tag{1}$$

where $\mathcal{H}_{k-1}(X) := \ker(\partial_{k-1}) \cap \ker(\partial^\top_k) \cong \tilde{H}_{k-1}(X;\mathbb{R})$ is the space of harmonic $(k-1)$-cycles (reduced homology for $k = 1$, using the augmentation map $\partial_0 = \mathbf{1}^\top$, and ordinary homology for $k \geq 2$). Correspondingly, the $(k-1)$-st combinatorial Hodge Laplacian splits orthogonally as

$$L_{k-1}(X) \;=\; L^{\text{down}}_{k-1}(X) + L^{\text{up}}_{k-1}(X), \qquad L^{\text{down}}_{k-1} := \partial^\top_{k-1}\partial_{k-1}, \qquad L^{\text{up}}_{k-1} := \partial_k W_k \partial^\top_k,$$

with $L^{\text{down}}_{k-1} L^{\text{up}}_{k-1} = 0$. In dimension $k = 1$, L^{up}_0 is the classical graph Laplacian, whose Moore–Penrose pseudoinverse $G = L^+_0$ encodes effective resistances, Rayleigh monotonicity, Foster's theorem, and the Kirchhoff index $n\,\text{Tr}(G)$ [5, 7]. In dimensions $k \geq 2$, however, two genuinely higher-dimensional phenomena fundamentally alter the spectral landscape:

1. **Homological phase transitions:** Attaching a k-cell τ to X either preserves β_{k-1} (creating a k-cycle, $\Delta\beta_k = +1$) or kills a $(k-1)$-dimensional homology class ($\Delta\beta_{k-1} = -1$, lifting a zero eigenvalue of L^{up}_{k-1} into $(0,\infty)$). While homology-preserving additions obey classical Sherman–Morrison–Rayleigh monotonicity, homology-killing transitions can sharply collapse the smallest nonzero eigenvalue $\lambda^+_{\min}(L^{\text{up}}_{k-1})$ and increase $\text{Tr}((L^{\text{up}}_{k-1})^+)$.

2. **Topological torsion:** Spanning k-acycles ($\mathbb{Q}$-trees) need not be $\mathbb{Z}$-acyclic [4, 1, 6]; finite torsion $t(T) = |\tilde{H}_{k-1}(T;\mathbb{Z})| \geq 2$ inflates $\det'(L^{\mathrm{up}}_{k-1}(T))$ by $t(T)^2$ while keeping $\mathrm{Tr}(L^{\mathrm{up}}_{k-1}(T)) = (k+1)|T|$ fixed, creating a direct tension between torsion, spectral gap, and total Kirchhoff–Hodge resistance.

1.1 Summary of Main Results

This paper establishes four main theorems with completely self-contained proofs and full computational verification:

1. **Exact Two-Regime Green-Operator Law and Universal Factor-of-2 Collapse (Theorem 2.1).** For any weighted k-cell attachment ($k \geq 1$), we derive the exact Moore–Penrose update G_{new}, determinant law $\det'(L_{\mathrm{new}})$, and Kirchhoff–Hodge trace increment $\Delta\,\mathrm{Tr}(G)$ in both homological regimes, proving that $\mathrm{sgn}(\Delta\,\mathrm{Tr}(L^{+}_{k-1}))$ is an exact spectral detector of Betti-number transitions ($\Delta\,\mathrm{Tr} < 0 \iff \Delta\beta_k = +1$; $\Delta\,\mathrm{Tr} > 0 \iff \Delta\beta_{k-1} = -1$). In the homology-killing regime, an exact Gram–Schur factorization pins λ_{new} universally within a factor of 2 of $\min\{\lambda_{\mathrm{old}}, \Lambda_\tau(w)\}$, proves strict Homological Braess Collapse whenever λ_{old} is simple and $\langle v_1, b^{\parallel}\rangle \neq 0$, and shows that $K_2 \sqcup K_2$ plus a bridge of weight $w \to \infty$ saturates the factor $1/2$ exactly.

2. **Integral Torsion–Lattice Duality in All Dimensions (Theorem 3.1).** Using Smith Normal Form and primitive orthogonal lattice duality in $\mathbb{Z}^{|Y_{k-1}|}$, we prove $\mathrm{vol}(Z^{\mathbb{Z}}_{k-1}(Y))^2 = \tau_{k-1}(Y)/|T_{k-2}(Y)|^2$ for every finite $(k-1)$-complex Y (with arbitrary $\beta_{k-2}(Y) \geq 0$ and arbitrary lower torsion $T_{k-2}(Y)$), unifying the weighted Kalai–Kirchhoff determinant factorization and torsion-weighted higher-dimensional effective resistance $R_k(\sigma)$ with Theorem 2.1.

3. **Dual-Stiffness Identity, Sharp Reverse AM–GM Sandwich, and Torsion Concentration (Theorem 4.1).** For every spanning k-acycle T ($r = |T| \geq 2$) with exact determinant D_T and mean boundary squared norm $\bar{c}$, we prove the dual-stiffness identity $\mathrm{Tr}(G) = \sum_{\tau\in T} \|b^{\perp}_{\tau}(T \setminus \tau)\|^{-2}$ and solve the exact constrained optimization of $\sum \lambda_i^{-1}$ subject to $\sum \lambda_i = r\bar{c}$ and $\prod \lambda_i = D_T$ via Lagrange multipliers and strict negativity of $2(t-1) - (t+1)\log t$ $(t > 1)$, obtaining sharp two-sided bounds on both $\lambda^{+}_{\min}(L^{\mathrm{up}}_{k-1}(T))$ and $\mathrm{Tr}(G)$, together with an Itakura–Saito spectral concentration law as $D_T^{1/r}/\bar{c} \to 1$.

4. **Canonical Closed-Form Families and Exhaustive K_6 Extremality (Theorem 5.1).** We prove exact spectra for the 6-vertex $\mathbb{R}P^2$ via the signed Petersen identity $A^3_{\mathrm{signed}} = 5A_{\mathrm{signed}}$, for arbitrary simplicial cones $v_0 * Y$ via $M(v_0 * Y) = I + L^{\mathrm{down}}_{k-1}(Y)$, for triangulated m-gon fans F_m (Fibonacci determinants F_{2m-2} and golden-ratio homological stiffness φ^2), and for simplicial bands $B_r = \{\{i, i+1, i+2\} \bmod r\}$ ($r \geq 5$, alternating between orientable cylinders and non-orientable Möbius bands with $\det' = L_{2r} - 2(-1)^r$ and a strict non-orientable spectral advantage). Exhaustive enumeration of all 184,756 10-triangle subsets of K_6 proves a striking extremal split between $\mathbb{R}P^2$ (unique minimizer of $\mathrm{Tr}(G) = 35/6$) and the cones $v_0 * K_5$ (unique maximizers of $\lambda^{+}_{\min} = 1$).

2 Exact Two-Regime Green-Operator Law and Factor-of-2 Spectral-Gap Collapse

Throughout, let X be a finite weighted k-dimensional cellular or simplicial complex ($k \geq 1$) with finite cell sets X_{k-2}, X_{k-1}, X_k, integer boundary matrices $\partial_{k-1} \in \mathbb{Z}^{|X_{k-2}|\times|X_{k-1}|}$ and $\partial_k \in \mathbb{Z}^{|X_{k-1}|\times|X_k|}$ satisfying $\partial_{k-1}\partial_k = 0$ (for $k = 1$, take $\partial_0 = \mathbf{1}^{\top}_{|X_0|} \in \mathbb{Z}^{1\times|X_0|}$ so $\ker(\partial_0)$ is the

space of zero-sum 0-chains), and positive k-cell weights $W_k = \mathrm{diag}(w_\sigma)_{\sigma \in X_k}$ ($w_\sigma > 0$). Put

$$L \;:=\; L^{\mathrm{up}}_{k-1}(X, W_k) \;=\; \partial_k W_k \partial_k^\top \;\succeq\; 0, \qquad G \;:=\; L^+,$$

where L^+ is the Moore–Penrose pseudoinverse of L, and write $\det'(L)$ for the product of all strictly positive eigenvalues of L (with the convention $\det'(0) := 1$) and $\lambda^+_{\min}(L)$ for its smallest strictly positive eigenvalue (when $\mathrm{rank}(L) \geq 1$). Note that $W := \mathrm{im}(L) = \mathrm{im}(\partial_k) \subset \ker(\partial_{k-1})$ and $\ker(\partial_{k-1}) = W \oplus \mathcal{H}_{k-1}(X)$ orthogonally.

Consider attaching a new k-cell τ of weight $w > 0$ whose boundary vector $b \in \ker(\partial_{k-1}) \setminus \{0\}$ decomposes orthogonally as

$$b \;=\; b^{\parallel} + b^{\perp}, \qquad b^{\parallel} \in W = \mathrm{im}(\partial_k), \qquad b^{\perp} \in \mathcal{H}_{k-1}(X). \tag{2}$$

Define the *in-image simplicial resistance* of b in X by

$$\mathcal{R}_X(b) \;:=\; (b^{\parallel})^\top G b^{\parallel} \;\geq\; 0, \tag{3}$$

and set $L_{\mathrm{new}} := L + w\, bb^\top$ and $G_{\mathrm{new}} := L^+_{\mathrm{new}}$.

Theorem 2.1 (Exact Two-Regime Green Law, Betti Detector, and Factor-of-2 Collapse)**.** *Under the setup above, attaching the weighted k-cell τ yields two mutually exclusive regimes:*

(a) ***Homology-preserving regime*** *($b^\perp = 0$,* ***equivalently*** *$\Delta\beta_{k-1} = 0$* ***and*** *$\Delta\beta_k = +1$).* *We have $\mathrm{im}(L_{\mathrm{new}}) = W$, $\mathrm{rank}(L_{\mathrm{new}}) = \mathrm{rank}(L)$,*

$$G_{\mathrm{new}} = G - \frac{w\,(Gb)(Gb)^\top}{1 + w\,\mathcal{R}_X(b)}, \tag{4}$$

$$\det{}'(L_{\mathrm{new}}) = \big(1 + w\,\mathcal{R}_X(b)\big)\, \det{}'(L), \tag{5}$$

$$\mathrm{Tr}(G_{\mathrm{new}}) - \mathrm{Tr}(G) = -\frac{w\,\|Gb\|_2^2}{1 + w\,\mathcal{R}_X(b)} \;<\; 0, \tag{6}$$

and every nonzero eigenvalue moves weakly upward: $\lambda^+_j(L_{\mathrm{new}}) \geq \lambda^+_j(L)$ for $1 \leq j \leq \mathrm{rank}(L)$.

(b) ***Homology-killing regime*** *($b^\perp \neq 0$,* ***equivalently*** *$\Delta\beta_{k-1} = -1$* ***and*** *$\Delta\beta_k = 0$).* *We have $\mathrm{im}(L_{\mathrm{new}}) = W \oplus \mathrm{span}(b^\perp)$, $\mathrm{rank}(L_{\mathrm{new}}) = \mathrm{rank}(L) + 1$,*

$$G_{\mathrm{new}} = G - \frac{1}{\|b^\perp\|_2^2}\Big((Gb^{\parallel})(b^\perp)^\top + b^\perp(Gb^{\parallel})^\top\Big) + \frac{w^{-1} + \mathcal{R}_X(b)}{\|b^\perp\|_2^4}\, b^\perp (b^\perp)^\top, \tag{7}$$

$$\det{}'(L_{\mathrm{new}}) = w\,\|b^\perp\|_2^2\, \det{}'(L), \tag{8}$$

$$\mathrm{Tr}(G_{\mathrm{new}}) - \mathrm{Tr}(G) = +\frac{w^{-1} + \mathcal{R}_X(b)}{\|b^\perp\|_2^2} \;>\; 0. \tag{9}$$

In particular, since $L_{k-1} = L^{\mathrm{down}}_{k-1} \oplus L^{\mathrm{up}}_{k-1}$ orthogonally and L^{down}_{k-1} is unchanged by attaching τ, the sign of the full Hodge Kirchhoff increment $\Delta\,\mathrm{Tr}(L^+_{k-1}) = \Delta\,\mathrm{Tr}(G)$ is an exact spectral detector of Betti-number transitions: $\Delta\,\mathrm{Tr}(L^+_{k-1}) < 0 \iff \Delta\beta_k = +1$, and $\Delta\,\mathrm{Tr}(L^+_{k-1}) > 0 \iff \Delta\beta_{k-1} = -1$.

(c) ***Universal factor-of-2 spectral-gap collapse law and Homological Braess Paradox.*** *Assume $b^\perp \neq 0$ and $\mathrm{rank}(L) \geq 1$. Set $\lambda_{\mathrm{old}} := \lambda^+_{\min}(L)$, $\lambda_{\mathrm{new}} := \lambda^+_{\min}(L_{\mathrm{new}})$, and*

$$\Lambda_\tau(w) \;:=\; \frac{w\,\|b^\perp\|_2^2}{1 + w\,\mathcal{R}_X(b)} \;>\; 0. \tag{10}$$

Then

$$\frac{1}{2}\min\{\lambda_{\text{old}},\ \Lambda_\tau(w)\} \ \le\ \left(\frac{1}{\lambda_{\text{old}}}+\frac{1}{\Lambda_\tau(w)}\right)^{-1} \ \le\ \lambda_{\text{new}} \ \le\ \min\{\lambda_{\text{old}},\ \Lambda_\tau(w)\}. \tag{11}$$

Moreover:

- (Purely harmonic attachment): *if* $b^{\|}=0$, *then* $\lambda_{\text{new}}=\min\{\lambda_{\text{old}},\, w\|b^{\perp}\|_2^2\}$ *identically;*
- (Strict Homological Braess Collapse): *for any fixed* $w>0$ *with* $\Lambda_\tau(w)<\lambda_{\text{old}}$, (11) *gives* $\lambda_{\text{new}}(w)<\lambda_{\text{old}}$ *immediately; moreover, whenever* λ_{old} *is simple with unit eigenvector* v_1 *satisfying* $\langle v_1, b^{\|}\rangle\neq 0$, *the strict inequality* $\lambda_{\text{new}}(w)<\lambda_{\text{old}}$ *holds for* every $w>0$. *As* $w\to+\infty$, $\lambda_{\text{new}}(w)$ *increases monotonically to a finite limit* $\lambda_{\text{new}}(\infty)\le\min\{\lambda_{\text{old}}, \|b^{\perp}\|_2^2/\mathcal{R}_X(b)\}$ *(with the convention* $\|b^{\perp}\|_2^2/0:=+\infty$ *when* $b^{\|}=0$*);*
- (Sharpness of 1/2): *attaching a bridge edge* $e_w=(2,3)$ *of weight* $w>0$ *to the disjoint union* $K_2\sqcup K_2$ *of two unit-weight edges* $(1,2)$ *and* $(3,4)$ *gives* $\lambda_{\text{old}}=2$, $\Lambda_{e_w}(\infty)=2$, *and* $\lambda_{\text{new}}(w)=1+w-\sqrt{w^2+1}\to 1=\frac{1}{2}\min\{\lambda_{\text{old}},\Lambda_{e_w}(\infty)\}$ *as* $w\to+\infty$.

Proof. Part (a). Since $b=b^{\|}\in W=\mathrm{im}(L)$ and $b\neq 0$, we have $\mathrm{im}(L_{\text{new}})=W$, so $\beta_{k-1}=\dim\ker(\partial_{k-1})-\mathrm{rank}(\partial_k)$ is unchanged and by rank–nullity $\beta_k=\dim\ker(\partial_k)$ increases by 1. Restricted to W, $L|_W\succ 0$ is invertible with inverse $G|_W$, and $Gb\neq 0$ (since $b\in W\setminus\{0\}$). Equations (4)–(5) are the Sherman–Morrison and matrix-determinant formulas on W, taking traces gives (6) (strictly negative since $\|Gb\|_2>0$), and Weyl's monotonicity theorem gives $\lambda_j^+(L_{\text{new}})\ge\lambda_j^+(L)$.

Part (b). Since $b^{\perp}\neq 0$, $b\notin W$, so $\mathrm{rank}(\partial_k)$ increases by 1, whence $\Delta\beta_{k-1}=-1$ and $\Delta\beta_k=0$. Write $\hat z:=b^{\perp}/\|b^{\perp}\|_2$ and $z:=\|b^{\perp}\|_2>0$. On the orthogonal direct sum $W\oplus\mathrm{span}(\hat z)=\mathrm{im}(L_{\text{new}})$, the operators L_{new} and the right-hand side M of (7) have the 2×2 block forms

$$L_{\text{new}}\big|_{W\oplus\mathbb{R}\hat z}=\begin{pmatrix} L|_W+w\,b^{\|}(b^{\|})^{\top} & wz\,b^{\|}\\ wz\,(b^{\|})^{\top} & wz^2\end{pmatrix},\qquad M\big|_{W\oplus\mathbb{R}\hat z}=\begin{pmatrix} G|_W & -z^{-1}Gb^{\|}\\ -z^{-1}(Gb^{\|})^{\top} & z^{-2}(w^{-1}+\mathcal{R}_X(b))\end{pmatrix}, \tag{12}$$

and both vanish on $(W\oplus\mathbb{R}\hat z)^{\perp}$. Direct block multiplication verifies $(L_{\text{new}}|_{W\oplus\mathbb{R}\hat z})(M|_{W\oplus\mathbb{R}\hat z})=I_{W\oplus\mathbb{R}\hat z}$:

$$\begin{aligned}
&(1,1):\ (L|_W+w\,b^{\|}(b^{\|})^{\top})G|_W-w\,b^{\|}(b^{\|})^{\top}G|_W \ =\ L|_WG|_W \ =\ I_W,\\
&(1,2):\ -z^{-1}(L|_W+w\,b^{\|}(b^{\|})^{\top})Gb^{\|}+wz^{-1}(w^{-1}+\mathcal{R}_X(b))\,b^{\|} \ =\ -z^{-1}(1+w\mathcal{R}_X)b^{\|}+z^{-1}(1+w\mathcal{R}_X)b^{\|} \ =\ 0,\\
&(2,1):\ wz(b^{\|})^{\top}G|_W-wz(b^{\|})^{\top}G|_W \ =\ 0,\\
&(2,2):\ -w(b^{\|})^{\top}Gb^{\|}+w(w^{-1}+\mathcal{R}_X(b)) \ =\ -w\mathcal{R}_X+1+w\mathcal{R}_X \ =\ 1.
\end{aligned}$$

Since M is symmetric and maps $(W\oplus\mathbb{R}\hat z)^{\perp}$ to 0, $M=L_{\text{new}}^{+}=G_{\text{new}}$, proving (7). Taking the Schur complement of the $(2,2)$ block $wz^2>0$ in $L_{\text{new}}|_{W\oplus\mathbb{R}\hat z}$ gives

$$(L|_W+w\,b^{\|}(b^{\|})^{\top})-(wz\,b^{\|})(wz^2)^{-1}(wz\,(b^{\|})^{\top}) \ =\ L|_W,$$

so $\det'(L_{\text{new}})=(wz^2)\det'(L)$, proving (8). Taking the trace of (12) immediately yields (9), which is strictly positive since $w^{-1}>0$ and $\mathcal{R}_X(b)\ge 0$. Because $L_{k-1}=L_{k-1}^{\text{down}}\oplus L_{k-1}^{\text{up}}$ is an orthogonal direct sum and attaching a k-cell does not alter ∂_{k-1}, we have $\Delta\,\mathrm{Tr}(L_{k-1}^{+})=\Delta\,\mathrm{Tr}(G)$, giving the exact Betti-transition sign detector.

Part (c). On $W\oplus\mathbb{R}\hat z$, $\lambda_{\text{new}}^{-1}=\|G_{\text{new}}\|_{\text{op}}=\lambda_{\max}(G_{\text{new}})$. Because $G|_W$ and the 1×1 entry $z^{-2}(w^{-1}+\mathcal{R}_X(b))=\Lambda_\tau(w)^{-1}$ are principal submatrices of $G_{\text{new}}|_{W\oplus\mathbb{R}\hat z}$, Cauchy's interlacing theorem yields

$$\frac{1}{\lambda_{\text{new}}}\ \ge\ \max\left\{\frac{1}{\lambda_{\text{old}}},\ \frac{1}{\Lambda_\tau(w)}\right\}\iff\lambda_{\text{new}}\ \le\ \min\{\lambda_{\text{old}},\ \Lambda_\tau(w)\}.$$

For the lower bound, observe from (12) the exact positive-semidefinite factorization

$$G_{\mathrm{new}}\big|_{W\oplus\mathbb{R}\hat{z}} \;=\; XX^\top + \frac{w^{-1}}{z^2}\,\hat{z}\hat{z}^\top, \qquad X \;:=\; \begin{pmatrix} (G|_W)^{1/2} \\ -z^{-1}\big((G|_W)^{1/2}b^{\|}\big)^\top \end{pmatrix}. \tag{13}$$

Indeed, $(XX^\top)_{11} = G|_W$, $(XX^\top)_{12} = -z^{-1}Gb^{\|}$, and $(XX^\top)_{22} = z^{-2}\|(G|_W)^{1/2}b^{\|}\|_2^2 = z^{-2}\mathcal{R}_X(b)$. Using $\|XX^\top\|_{\mathrm{op}} = \|X^\top X\|_{\mathrm{op}}$ and the triangle inequality for the operator norm,

$$\begin{aligned}\frac{1}{\lambda_{\mathrm{new}}} \;=\; \|G_{\mathrm{new}}\|_{\mathrm{op}} &\le \Big\|G|_W + \frac{1}{z^2}\big((G|_W)^{1/2}b^{\|}\big)\big((G|_W)^{1/2}b^{\|}\big)^\top\Big\|_{\mathrm{op}} + \frac{w^{-1}}{z^2} \\ &\le \|G|_W\|_{\mathrm{op}} + \frac{\|(G|_W)^{1/2}b^{\|}\|_2^2}{z^2} + \frac{w^{-1}}{z^2} \;=\; \frac{1}{\lambda_{\mathrm{old}}} + \frac{1}{\Lambda_\tau(w)}.\end{aligned}$$

Inverting gives the second inequality in (11), and noting $\frac{1}{a}+\frac{1}{b} \le \frac{2}{\min(a,b)}$ gives the first.

If $b^{\|} = 0$, (12) is block diagonal, so $\lambda_{\mathrm{new}} = \min\{\lambda_{\mathrm{old}}, w\|b^{\perp}\|_2^2\}$. If λ_{old} is simple with unit eigenvector $v_1 \in W$ and $\langle v_1, b^{\|}\rangle \neq 0$, then the 2×2 compression of G_{new} onto $\mathrm{span}(v_1, \hat{z})$ has diagonal entries $a = \lambda_{\mathrm{old}}^{-1}$, $d = \Lambda_\tau(w)^{-1}$, and nonzero off-diagonal entry $c = -\frac{\langle v_1, b^{\|}\rangle}{\lambda_{\mathrm{old}} z} \neq 0$; its top eigenvalue $\frac{1}{2}(a+d+\sqrt{(a-d)^2+4c^2})$ strictly exceeds $\max(a,d) \ge \lambda_{\mathrm{old}}^{-1}$, forcing $\lambda_{\mathrm{new}}(w) < \lambda_{\mathrm{old}}$ for all $w > 0$. By Weyl's inequality, $w \mapsto \lambda_{\mathrm{new}}(w)$ is nondecreasing and bounded above by $\min\{\lambda_{\mathrm{old}}, \|b^{\perp}\|_2^2/\mathcal{R}_X(b)\}$.

Finally, for $K_2 \sqcup K_2$ on vertices $\{1,2,3,4\}$ with unit edges $(1,2)$ and $(3,4)$, $L = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ has $\lambda_{\mathrm{old}} = 2$. For the bridge $b = e_2 - e_3$, $b^{\perp} = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ has $\|b^{\perp}\|_2^2 = 1$, and $b^{\|} = \frac{1}{2}(e_2 - e_1) + \frac{1}{2}(e_4 - e_3)$ satisfies $Lb^{\|} = 2b^{\|}$ and $\|b^{\|}\|_2^2 = 1$, so $\mathcal{R}_X(b) = \frac{1}{2}$ and $\Lambda_{e_w}(\infty) = 2 = \lambda_{\mathrm{old}}$. Restricting $L + wbb^\top$ to the odd-symmetry subspace $\mathrm{span}(e_1 - e_4, e_2 - e_3)$ yields $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1+2w \end{smallmatrix}\right)$, whose smaller eigenvalue is $\lambda_{\mathrm{new}}(w) = 1 + w - \sqrt{w^2+1} \to 1 = \frac{1}{2}\lambda_{\mathrm{old}}$ as $w \to \infty$ (while on the even-symmetry orthogonal complement $e_1 + e_4 - e_2 - e_3$, the eigenvalue stays 2). □

3 Integral Torsion–Lattice Duality in All Dimensions

We now connect the spectral determinant $\det'(L_{k-1}^{\mathrm{up}})$ to integral homology torsion across arbitrary dimensions $k \ge 1$, without assuming that the underlying $(k-1)$-skeleton is rationally $(k-2)$-acyclic.

Let Y be a finite $(k-1)$-dimensional cell or simplicial complex ($k \ge 1$) with integer boundary operator $\partial_{k-1} \in \mathbb{Z}^{p\times m}$ ($m = |Y_{k-1}|$, $p = |Y_{k-2}|$; for $k = 1$, $p = 1$ and $\partial_0 = \mathbf{1}_m^\top$). Let $s := \mathrm{rank}(\partial_{k-1})$, so the integer $(k-1)$-cycle lattice $Z_{k-1}^{\mathbb{Z}}(Y) := \ker_{\mathbb{Z}}(\partial_{k-1}) \subset \mathbb{Z}^m$ has rank $r := m - s = \beta_{k-1}(Y)$ (reduced $\tilde{\beta}_0(Y)$ for $k = 1$). Assume $r \ge 1$. For any integer matrix $M \in \mathbb{Z}^{p\times q}$, let $|\,\mathrm{Tor}(\mathrm{coker}(M))|$ denote the product of the nonzero invariant factors in the Smith Normal Form of M; for $M = \partial_{k-1}$, since $\mathrm{im}(\partial_{k-2})$ is free abelian, $\mathrm{Tor}(\mathrm{coker}(\partial_{k-1})) = \mathrm{Tor}(\tilde{H}_{k-2}(Y;\mathbb{Z})) =: T_{k-2}(Y)$ (with $|T_{-1}| = |T_0| = 1$). Let $\mathcal{B}_{k-1}(Y)$ be the set of all s-element subsets $S \subset Y_{k-1}$ such that the column submatrix $\partial_{k-1}|_S \in \mathbb{Z}^{p\times s}$ has rank s, and define the generalized Kalai torsion count

$$\tau_{k-1}(Y) \;:=\; \sum_{S\in\mathcal{B}_{k-1}(Y)} \big|\mathrm{Tor}\big(\mathrm{coker}(\partial_{k-1}|_S)\big)\big|^2. \tag{14}$$

Notice that for $k = 1$, $s = 1$ and $\tau_0(Y) = |Y_0|$; for $k = 2$ and connected Y, $\mathcal{B}_1(Y)$ is the set of spanning trees of Y and $\tau_1(Y)$ is the classical spanning-tree count; and whenever $\tilde{H}_{k-2}(Y;\mathbb{Q}) = 0$, $\tau_{k-1}(Y) = \sum_{S\in\mathcal{T}_{k-1}(Y)} |T_{k-2}(Y_S)|^2$ is Kalai's torsion-weighted $(k-1)$-acycle count [4, 1].

Recall that for a rank-d sublattice $\Lambda \subset \mathbb{Z}^m$ with $\mathbb{Z}$-basis given by the rows of $A \in \mathbb{Z}^{d\times m}$, its squared covolume is $\mathrm{vol}(\Lambda)^2 := \det(AA^\top) \in \mathbb{Z}_{>0}$, independent of the choice of $\mathbb{Z}$-basis.

Theorem 3.1 (Integral Torsion–Lattice Duality and Weighted Kalai–Kirchhoff Law)**.** *Let Y be a finite $(k-1)$-dimensional cell/simplicial complex ($k \geq 1$) with $r = \beta_{k-1}(Y) \geq 1$.*

(a) ***Exact primitive-lattice covolume identity.*** *Regardless of $\beta_{k-2}(Y)$ and $T_{k-2}(Y)$,*

$$\operatorname{vol}(Z^{\mathbb{Z}}_{k-1}(Y))^2 \;=\; \frac{\tau_{k-1}(Y)}{|T_{k-2}(Y)|^2}. \tag{15}$$

(b) ***Weighted Kalai–Kirchhoff determinant factorization.*** *Let $X = Y \cup X_k$ be formed by attaching weighted k-cells ($w_\sigma > 0$) with boundaries in $Z^{\mathbb{Z}}_{k-1}(Y)$ such that $\tilde{H}_{k-1}(X;\mathbb{Q}) = 0$ (i.e., $\operatorname{rank}(\partial_k) = r$), and let $\mathcal{T}_k(X)$ denote the set of spanning k-acycles $T \subset X_k$ ($|T| = r$, $\operatorname{rank}(\partial_k|_T) = r$), each with finite torsion $t(T) := |\tilde{H}_{k-1}(Y \cup T;\mathbb{Z})| = [Z^{\mathbb{Z}}_{k-1}(Y) : \operatorname{im}_{\mathbb{Z}}(\partial_k|_T)]$. Then for every $T \in \mathcal{T}_k(X)$ and for the full complex X,*

$$\det{}'(L^{\mathrm{up}}_{k-1}(Y \cup T, w)) \;=\; \frac{t(T)^2}{|T_{k-2}(Y)|^2}\,\tau_{k-1}(Y)\prod_{\sigma\in T} w_\sigma, \qquad \det{}'(L^{\mathrm{up}}_{k-1}(X,w)) \;=\; \frac{\tau_{k-1}(Y)}{|T_{k-2}(Y)|^2}\sum_{T\in\mathcal{T}_k(X)} t(T)^2 \prod_{\sigma\in} \tag{16}$$

(c) ***Higher-dimensional effective resistance and bridge dichotomy.*** *For every $\sigma \in X_k$, define the simplicial effective resistance $R_k(\sigma) := w_\sigma\, b_\sigma^\top (L^{\mathrm{up}}_{k-1}(X,w))^+ b_\sigma \in [0,1]$. Then*

$$R_k(\sigma) \;=\; \frac{\sum_{T\in\mathcal{T}_k(X),\,\sigma\in T} t(T)^2 \prod_{\rho\in T} w_\rho}{\sum_{T\in\mathcal{T}_k(X)} t(T)^2 \prod_{\rho\in T} w_\rho}, \qquad \sum_{\sigma\in X_k} R_k(\sigma) \;=\; \beta_{k-1}(Y). \tag{17}$$

Moreover, deleting σ multiplies $\det'(L^{\mathrm{up}}_{k-1})$ by $1 - R_k(\sigma)$ whenever $R_k(\sigma) < 1$ ($\Delta\beta_{k-1} = 0$), whereas $R_k(\sigma) = 1$ if and only if σ is a homological k-bridge *($\sigma \in T$ for all $T \in \mathcal{T}_k(X)$, $\Delta\beta_{k-1} = +1$), in which case $\det'(L^{\mathrm{up}}_{k-1}(X)) = w_\sigma\, \|b^\perp_\sigma(X \setminus \sigma)\|_2^2\, \det'(L^{\mathrm{up}}_{k-1}(X \setminus \sigma))$.*

Proof. Part (a). Compute the Smith Normal Form of $\partial_{k-1} \in \mathbb{Z}^{p\times m}$: there exist unimodular matrices $U \in \mathrm{GL}_p(\mathbb{Z})$ and $V \in \mathrm{GL}_m(\mathbb{Z})$ such that

$$\partial_{k-1} \;=\; U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V, \qquad D = \operatorname{diag}(d_1,\dots,d_s), \quad d_i \in \mathbb{Z}_{>0}, \quad \prod_{i=1}^{s} d_i = |T_{k-2}(Y)|.$$

Partition $V = \binom{A}{C}$ with $A \in \mathbb{Z}^{s\times m}$ and $C \in \mathbb{Z}^{r\times m}$, and partition $V^{-1} = (E \;\; F)$ with $E \in \mathbb{Z}^{m\times s}$ and $F \in \mathbb{Z}^{m\times r}$. Because V is unimodular, the rows of A form a $\mathbb{Z}$-basis of a *primitive* rank-s sublattice $\Lambda \subset \mathbb{Z}^m$, while $AF = 0$ and $\operatorname{rank}(F) = r$ show that the columns of F form a $\mathbb{Z}$-basis of its orthogonal complement $\Lambda^\perp = \ker_{\mathbb{Z}}(A) = \ker_{\mathbb{Z}}(\partial_{k-1}) = Z^{\mathbb{Z}}_{k-1}(Y)$!

We first recall the elementary proof of primitive orthogonal lattice duality $\operatorname{vol}(\Lambda^\perp) = \operatorname{vol}(\Lambda)$: since $V = \binom{A}{C} \in \mathrm{GL}_m(\mathbb{Z})$ satisfies $|\det V| = 1$, $AF = 0$, and $CF = I_r$, direct block multiplication gives $\binom{A}{C}(A^\top \;\; F) = \left(\begin{smallmatrix} AA^\top & 0 \\ CA^\top & I_r \end{smallmatrix}\right)$, whence $|\det(A^\top \;\; F)| = |\det V| \cdot |\det(A^\top \;\; F)| = \det(AA^\top)$. On the other hand, $AF = 0$ implies $(A^\top \;\; F)^\top (A^\top \;\; F) = \operatorname{diag}(AA^\top, F^\top F)$, so $\det(A^\top \;\; F)^2 = \det(AA^\top)\det(F^\top F)$. Equating $\det(AA^\top)^2 = \det(AA^\top)\det(F^\top F)$ and dividing by $\det(AA^\top) > 0$ gives $\det(F^\top F) = \det(AA^\top)$, i.e., $\operatorname{vol}(Z^{\mathbb{Z}}_{k-1}(Y))^2 = \operatorname{vol}(\Lambda)^2$.

Next, for any $S \subset \{1,\dots,m\}$ of size s, the column submatrix is $\partial_{k-1}|_S = U\binom{D A_S}{0}$, where $A_S \in \mathbb{Z}^{s\times s}$ consists of the columns of A indexed by S. Because $U \in \mathrm{GL}_p(\mathbb{Z})$ is unimodular, $\partial_{k-1}|_S$ has rank s iff $\det(A_S) \neq 0$, in which case the torsion order of $\operatorname{coker}(\partial_{k-1}|_S)$ equals

$$\left|\operatorname{Tor}(\operatorname{coker}(\partial_{k-1}|_S))\right| \;=\; |\det(DA_S)| \;=\; \left(\prod_{i=1}^{s} d_i\right)|\det(A_S)| \;=\; |T_{k-2}(Y)|\,|\det(A_S)|.$$

Squaring and summing over all $S \in \mathcal{B}_{k-1}(Y)$, and applying the Cauchy–Binet formula to $AA^\top$, gives

$$\tau_{k-1}(Y) \;=\; |T_{k-2}(Y)|^2 \sum_{|S|=s} (\det A_S)^2 \;=\; |T_{k-2}(Y)|^2 \det(AA^\top) \;=\; |T_{k-2}(Y)|^2\, \operatorname{vol}(Z^{\mathbb{Z}}_{k-1}(Y))^2,$$

proving (15).

Part (b). Using the $\mathbb{Z}$-basis $F \in \mathbb{Z}^{m\times r}$ of $Z^{\mathbb{Z}}_{k-1}(Y)$, every column of ∂_k lies in $Z^{\mathbb{Z}}_{k-1}(Y)$, so $\partial_k = F\Phi$ for a unique integer matrix $\Phi \in \mathbb{Z}^{r\times|X_k|}$. For any spanning k-acycle $T \in \mathcal{T}_k(X)$, the submatrix $\Phi_T \in \mathbb{Z}^{r\times r}$ is nonsingular with $|\det \Phi_T| = [Z^{\mathbb{Z}}_{k-1}(Y) : \mathrm{im}_{\mathbb{Z}}(\partial_k|_T)] = t(T)$. Since $L^{\mathrm{up}}_{k-1}(Y\cup T, w) = (\partial_k|_T)W_T(\partial_k|_T)^\top$ shares its r nonzero eigenvalues with $W_T^{1/2}(\partial_k|_T)^\top(\partial_k|_T)W_T^{1/2}$, we obtain

$$\det{}'(L^{\mathrm{up}}_{k-1}(Y\cup T,w)) \;=\; \left(\prod_{\sigma\in T} w_\sigma\right)\det(\Phi_T^\top F^\top F\Phi_T) \;=\; t(T)^2\,\mathrm{vol}(Z^{\mathbb{Z}}_{k-1}(Y))^2\prod_{\sigma\in T} w_\sigma,$$

giving the first identity of (16) via (15). For the full complex X, $L^{\mathrm{up}}_{k-1}(X,w) = F(\Phi W_k\Phi^\top)F^\top$ shares its r nonzero eigenvalues with $(F^\top F)^{1/2}(\Phi W_k\Phi^\top)(F^\top F)^{1/2}$, whose determinant by Cauchy–Binet is $\det(F^\top F)\sum_{T\in\mathcal{T}_k(X)}(\det\Phi_T)^2\prod_{\sigma\in T} w_\sigma$, proving the second identity of (16).

Part (c). Write $f(w) := \det'(L^{\mathrm{up}}_{k-1}(X,w))$. By Jacobi's formula on the r-dimensional space $W = \mathrm{im}(\partial_k)$,

$$w_\sigma\frac{\partial}{\partial w_\sigma}\log f(w) \;=\; w_\sigma\,\mathrm{Tr}\big(G\,b_\sigma b_\sigma^\top\big) \;=\; w_\sigma\,b_\sigma^\top G b_\sigma \;=\; R_k(\sigma).$$

Differentiating the multilinear polynomial (16) gives the marginal formula in (17), and summing over $\sigma\in X_k$ yields $\sum_\sigma R_k(\sigma) = \mathrm{Tr}(GL) = r = \beta_{k-1}(Y)$. Finally, by (17), $R_k(\sigma) = 1$ iff every $T\in\mathcal{T}_k(X)$ contains σ, i.e., removing σ drops $\mathrm{rank}(\partial_k)$ by 1 (a homological k-bridge), in which case Theorem 2.1(b) gives $\det'(L(X)) = w_\sigma\|b_\sigma^\perp(X\setminus\sigma)\|_2^2\det'(L(X\setminus\sigma))$; if $R_k(\sigma)<1$, removing σ preserves $\mathrm{rank}(\partial_k)$ and setting $w_\sigma\to 0$ in (16) gives $\det'(L(X\setminus\sigma)) = (1-R_k(\sigma))\det'(L(X))$ (equivalently Theorem 2.1(a) in reverse). □

4 Dual-Stiffness Kirchhoff Law, Sharp Extremal Bounds, and Torsion Concentration

Consider now an unweighted spanning k-acycle T ($w_\sigma\equiv 1$, $r = |T| = \beta_{k-1}(Y)\ge 2$) of a $(k-1)$-skeleton Y. Because $\partial_k|_T$ has full column rank r, the nonzero eigenvalues $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_r = \lambda^+_{\min}(L^{\mathrm{up}}_{k-1}(T)) > 0$ coincide with the eigenvalues of the $r\times r$ positive definite boundary Gram matrix

$$M(T) \;:=\; (\partial_k|_T)^\top(\partial_k|_T) \;\succ\; 0.$$

Write $c_\tau := \|b_\tau\|_2^2 = M(T)_{\tau\tau}$ (for any simplicial complex, $c_\tau = k+1$ uniformly), $\bar c := \frac{1}{r}\sum_{\tau\in T} c_\tau$, and $D_T := \det'(L^{\mathrm{up}}_{k-1}(T)) = \det(M(T)) = \frac{t(T)^2}{|T_{k-2}(Y)|^2}\tau_{k-1}(Y)$. By AM–GM, $D_T\le \bar c^{\,r}$, so the *Kalai–Hadamard efficiency*

$$\alpha(T) \;:=\; \frac{D_T^{1/r}}{\bar c} \;=\; \frac{1}{\bar c}\left(\frac{t(T)^2\,\tau_{k-1}(Y)}{|T_{k-2}(Y)|^2}\right)^{1/r} \;\in\; (0,1] \tag{18}$$

is well defined, with $\alpha(T) = 1$ if and only if $M(T) = \bar c\, I_r$.

Theorem 4.1 (Dual-Stiffness Identity, Sharp Reverse AM–GM Sandwich, and Torsion Concentration)**.** *Let T be an unweighted spanning k-acycle with $r = |T|\ge 2$, mean boundary norm $\bar c$, determinant D_T, and efficiency $\alpha(T)\in(0,1]$.*

(a) ***Exact dual-stiffness Kirchhoff–Hodge identity.*** *For each $\tau\in T$, define its* homological stiffness *in T by $\sigma_T(\tau) := \|b_\tau^\perp(T\setminus\tau)\|_2^2\in(0,c_\tau]$. Then $\sigma_T(\tau) = \big((M(T)^{-1})_{\tau\tau}\big)^{-1} = \det(M(T))/\det(M(T\setminus\tau))$, and*

$$\mathrm{Tr}\big((L^{\mathrm{up}}_{k-1}(T))^+\big) \;=\; \mathrm{Tr}(M(T)^{-1}) \;=\; \sum_{\tau\in T}\frac{1}{\sigma_T(\tau)} \;\ge\; \sum_{\tau\in T}\frac{1}{c_\tau}, \qquad \lambda^+_{\min}(L^{\mathrm{up}}_{k-1}(T)) \;\le\; \min_{\tau\in T}\sigma_T(\tau). \tag{19}$$

(b) ***Sharp two-sided extremal sandwich for*** $\lambda^+_{\min}$ ***and*** $\mathrm{Tr}(G)$***.*** *Let* $g(\lambda) := \lambda(\frac{r\bar{c}-\lambda}{r-1})^{r-1}$ *on* $(0, r\bar{c})$*. If* $D_T < \bar{c}^r$*, then* $g(\lambda) = D_T$ *has a unique root* $\lambda_* = \lambda_*(r, \bar{c}, D_T) \in (0, \bar{c})$ *and a unique root* $\lambda^* = \lambda^*(r, \bar{c}, D_T) \in (\bar{c}, r\bar{c})$ *(for* $D_T = \bar{c}^r$*, set* $\lambda_* = \lambda^* = \bar{c}$*). Every nonzero eigenvalue* λ_i *of* $L^{\mathrm{up}}_{k-1}(T)$ *satisfies*

$$\frac{D_T}{e\,\bar{c}^{\,r-1}} \;\le\; \frac{D_T}{\left(\frac{r}{r-1}\bar{c}\right)^{r-1}} \;\le\; \lambda_*(r, \bar{c}, D_T) \;\le\; \lambda^+_{\min}(L^{\mathrm{up}}_{k-1}(T)) \;\le\; D_T^{1/r} \;\le\; \lambda_{\max}(L^{\mathrm{up}}_{k-1}(T)) \;\le\; \lambda^*(r, \bar{c}, D_T), \tag{20}$$

and the Kirchhoff–Hodge total resistance satisfies the sharp two-sided extremal sandwich

$$r\,D_T^{-1/r} \;\le\; \frac{1}{\lambda^*} + \frac{(r-1)^2}{r\bar{c}-\lambda^*} \;\le\; \mathrm{Tr}\big((L^{\mathrm{up}}_{k-1}(T))^+\big) \;\le\; \frac{1}{\lambda_*} + \frac{(r-1)^2}{r\bar{c}-\lambda_*}. \tag{21}$$

On the spectral constraint set $\Omega := \{\lambda \in \mathbb{R}^r_{>0} : \sum \lambda_i = r\bar{c}, \prod \lambda_i = D_T\}$*, the middle two inequalities in (21) are both sharp (and strictly tighten the unconstrained AM–GM bound* $rD_T^{-1/r}$ *whenever* $D_T < \bar{c}^r$*): the constrained minimum is attained uniquely (up to permutation) by* $(\lambda^*, \frac{r\bar{c}-\lambda^*}{r-1}, \dots, \frac{r\bar{c}-\lambda^*}{r-1})$ *and the constrained maximum is attained uniquely by* $(\lambda_*, \frac{r\bar{c}-\lambda_*}{r-1}, \dots, \frac{r\bar{c}-\lambda_*}{r-1})$*.*

(c) ***Torsion-forced Itakura–Saito spectral concentration.*** *The eigenvalues* $\{\lambda_i\}_{i=1}^r$ *obey the exact Bregman / Itakura–Saito identity*

$$\frac{1}{r}\sum_{i=1}^{r}\left(\frac{\lambda_i}{\bar{c}} - 1 - \log\frac{\lambda_i}{\bar{c}}\right) \;=\; \log\frac{1}{\alpha(T)}. \tag{22}$$

Since $\phi(u) := u - 1 - \log u \ge 0$ *vanishes iff* $u = 1$*, any sequence of spanning* k*-acycles with* $\alpha(T) \to 1$ *has empirical normalized eigenvalue distribution* $\frac{1}{r}\sum_{i=1}^r \delta_{\lambda_i/\bar{c}}$ *converging weakly to* δ_1*; furthermore, at fixed rank* r *(or for growing* r *under the rate condition* $\alpha(T) = 1 - o(r^{-1})$*),* $\max_{1\le i\le r}|\lambda_i/\bar{c} - 1| \to 0$ *uniformly (and hence* $\max_i|\lambda_i - \bar{c}| \to 0$ *and* $\frac{1}{r}\mathrm{Tr}((L^{\mathrm{up}}_{k-1}(T))^+) - \bar{c}^{-1} \to 0$ *whenever* $\bar{c}$ *stays bounded away from* 0 *and* ∞*, as holds automatically for simplicial complexes with* $\bar{c} = k + 1$*).*

Proof. Part (a). Deleting τ from T gives $\det(M(T)) = \sigma_T(\tau)\det(M(T\setminus\tau))$ by Theorem 2.1(b) (with $w = 1$). By Cramer's rule, $(M(T)^{-1})_{\tau\tau} = \det(M(T\setminus\tau))/\det(M(T)) = \sigma_T(\tau)^{-1}$. Summing over $\tau \in T$ yields $\mathrm{Tr}((L^{\mathrm{up}}_{k-1}(T))^+) = \mathrm{Tr}(M(T)^{-1}) = \sum_\tau \sigma_T(\tau)^{-1}$, and since $\sigma_T(\tau) = \|b_\tau^\perp\|_2^2 \le \|b_\tau\|_2^2 = c_\tau$, we get $\mathrm{Tr}(M(T)^{-1}) \ge \sum_\tau c_\tau^{-1}$. Moreover, $\lambda^+_{\min}(L^{\mathrm{up}}_{k-1}(T))^{-1} = \lambda_{\max}(M(T)^{-1}) \ge \max_\tau(M(T)^{-1})_{\tau\tau} = \max_\tau \sigma_T(\tau)^{-1}$, giving $\lambda^+_{\min} \le \min_\tau \sigma_T(\tau)$.

Part (b). Notice that $\sum_{i=1}^r \lambda_i = \mathrm{Tr}(M(T)) = r\bar{c}$ and $\prod_{i=1}^r \lambda_i = D_T$. For any single index j, applying AM–GM to the remaining $r-1$ positive eigenvalues yields

$$D_T \;=\; \lambda_j \prod_{i\ne j}\lambda_i \;\le\; \lambda_j\left(\frac{r\bar{c}-\lambda_j}{r-1}\right)^{r-1} \;=\; g(\lambda_j).$$

Differentiating $g(\lambda)$ gives $g'(\lambda) = \frac{r(\bar{c}-\lambda)}{r\bar{c}-\lambda}\left(\frac{r\bar{c}-\lambda}{r-1}\right)^{r-1}$, which is strictly positive on $(0,\bar{c})$ and strictly negative on $(\bar{c}, r\bar{c})$, with $g(0) = g(r\bar{c}) = 0$ and $g(\bar{c}) = \bar{c}^r$. Therefore $g(\lambda_j) \ge D_T$ forces $\lambda_* \le \lambda_j \le \lambda^*$ for every j. Furthermore, $D_T = g(\lambda_*) \le \lambda_*\left(\frac{r}{r-1}\bar{c}\right)^{r-1} < \lambda_* e\,\bar{c}^{\,r-1}$ (as $(1+\frac{1}{r-1})^{r-1} < e$), while $\lambda^+_{\min} \le (\prod\lambda_i)^{1/r} = D_T^{1/r} \le \lambda_{\max}$, establishing (20).

For (21), if $D_T = \bar{c}^r$ then $\Omega = \{(\bar{c}, \dots, \bar{c})\}$ and all terms in (21) equal $r/\bar{c}$. Assume $D_T < \bar{c}^r$, and consider both the global minimum and global maximum of $F(\lambda) := \sum_{i=1}^r \lambda_i^{-1}$ on the compact constraint set $\Omega = \{\lambda \in \mathbb{R}^r_{>0} : \sum\lambda_i = r\bar{c}, \prod\lambda_i = D_T\}$. Since $\lambda_i \ge \lambda_* > 0$ on Ω, Ω is compact and nonempty, and the constraint gradients $\mathbf{1}_r$ and $(\lambda_1^{-1}, \dots, \lambda_r^{-1})$ are linearly independent at

every point of Ω (as $\lambda \neq (\bar{c}, \ldots, \bar{c})$). Hence every constrained extremum (minimizer or maximizer) $\lambda \in \Omega$ obeys the Lagrange-multiplier condition

$$-\frac{1}{\lambda_i^2} \;=\; \gamma_1 + \frac{\gamma_2}{\lambda_i} \iff \gamma_1 \lambda_i^2 + \gamma_2 \lambda_i + 1 \;=\; 0 \qquad (i = 1, \ldots, r).$$

Because this quadratic equation has at most two distinct positive roots $0 < x_1 < x_2$, every constrained extremum has the two-level form $(x_1^{(m)}, x_2^{(r-m)})$ for some integer $m \in \{1, \ldots, r-1\}$, where $mx_1 + (r-m)x_2 = r\bar{c}$ and $x_1^m x_2^{r-m} = D_T$.

Treat m as a continuous parameter on $[1, r-1]$. For each $m \in [1, r-1]$, the unique pair $0 < x_1(m) < \bar{c} < x_2(m)$ satisfying $mx_1 + (r-m)x_2 = r\bar{c}$ and $m \log x_1 + (r-m) \log x_2 = \log D_T$ is smooth by the implicit function theorem (Jacobian determinant $m(r-m)(x_2^{-1} - x_1^{-1}) < 0$). Setting $u := mx_1'(m)$ and $v := (r-m)x_2'(m)$, differentiating both constraints with respect to m gives

$$u + v \;=\; x_2 - x_1, \qquad \frac{u}{x_1} + \frac{v}{x_2} \;=\; \log \frac{x_2}{x_1}.$$

Putting $t := x_2/x_1 > 1$, solving this 2×2 linear system yields

$$u \;=\; \frac{x_1 x_2}{x_2 - x_1}\left(\log t - 1 + \frac{1}{t}\right), \qquad v \;=\; \frac{x_1 x_2}{x_2 - x_1}(t - 1 - \log t)\,.$$

Substituting into $\frac{d}{dm}F(m) = (\frac{1}{x_1} - \frac{1}{x_2}) - \frac{u}{x_1^2} - \frac{v}{x_2^2}$ and multiplying by $x_2 > 0$, we obtain the exact identity

$$\begin{aligned} x_2 \frac{dF}{dm} &= (t-1) - \frac{t}{t-1}\left[t\left(\log t - 1 + \frac{1}{t}\right) + \frac{1}{t}(t - 1 - \log t)\right] \\ &= (t-1) - \frac{t}{t-1}\left[\frac{t^2-1}{t}\log t - \frac{(t-1)^2}{t}\right] \;=\; 2(t-1) - (t+1)\log t \;=:\; \psi(t). \end{aligned}$$

Now $\psi(1) = 0$, $\psi'(t) = 1 - t^{-1} - \log t$ has $\psi'(1) = 0$, and $\psi''(t) = t^{-2} - t^{-1} = -(t-1)/t^2 < 0$ for all $t > 1$. Hence $\psi(t) < 0$ strictly for all $t > 1$, so $F(m)$ is strictly decreasing on $[1, r-1]$. The unique global maximum over $m \in \{1, \ldots, r-1\}$ is therefore attained at $m = 1$ (where $x_1 = \lambda_*$ and $x_2 = \frac{r\bar{c} - \lambda_*}{r-1}$), and the unique global minimum over $m \in \{1, \ldots, r-1\}$ is attained at $m = r-1$ (where $x_2 = \lambda^*$ and $x_1 = \frac{r\bar{c} - \lambda^*}{r-1}$), yielding (21) (while the leftmost inequality $rD_T^{-1/r} < F(r-1)$ holds strictly by AM–GM whenever $D_T < \bar{c}^{\,r}$).

Part (c). Summing $\frac{\lambda_i}{\bar{c}} - 1 - \log \frac{\lambda_i}{\bar{c}}$ over $i = 1, \ldots, r$ uses $\sum \lambda_i = r\bar{c}$ and $\sum \log(\lambda_i/\bar{c}) = \log(D_T/\bar{c}^{\,r}) = r \log \alpha(T)$, immediately yielding (22). Note that for any j, $\phi(\lambda_j/\bar{c}) \leq r \log(1/\alpha(T))$ (and even without r-dependence, $g(\lambda)/\bar{c}^{\,r} = \alpha(T)^r$ forces $\lambda_* \to \bar{c}$ and $\lambda^* \to \bar{c}$ as $\alpha(T) \to 1$ at fixed r, while for growing r, (22) forces convergence in empirical distribution, and $\alpha(T) = 1 - o(r^{-1})$ forces uniform convergence $\max_i |\lambda_i - \bar{c}| \to 0$). □

5 Canonical Closed-Form Families and Exhaustive K_6 Torsion Extremality

We now evaluate exact Hodge up-Laplacian spectra and Kirchhoff–Hodge resistances across four canonical families and resolve the complete extremal landscape on K_6.

Theorem 5.1 (Canonical Exact Spectra and Exhaustive K_6 Extremal Split)**.** *(a)* ***Exact spectrum of*** $\mathbb{R}P^2$ ***via signed Petersen algebra.*** *For the* 6*-vertex hemi-icosahedral triangulation of* $\mathbb{R}P^2$ *on* $K_6^{(1)}$ *(*10 *triangles,* $t(\mathbb{R}P^2) = 2$*), the Gram matrix decomposes as* $M(\mathbb{R}P^2) = 3I_{10} + A_{\text{signed}}$*, where* A_{signed} *is a* $\{0, \pm 1\}$*-signed adjacency matrix of the* 3*-regular*

Petersen graph satisfying the exact cubic identity $A_{\text{signed}}^3 = 5A_{\text{signed}}$ *and* $\operatorname{Tr}(A_{\text{signed}}^2) = 30$. *Consequently,*

$$\operatorname{spec}^+\big(L_1^{\text{up}}(\mathbb{R}P^2)\big) \;=\; \{(3{-}\sqrt{5})^{(3)},\, 3^{(4)},\, (3{+}\sqrt{5})^{(3)}\}, \quad \overset{\prime}{\det}\big(L_1^{\text{up}}(\mathbb{R}P^2)\big) = 5184, \quad \operatorname{Tr}\big((L_1^{\text{up}}(\mathbb{R}P^2))^+\big) = \frac{35}{6} \tag{23}$$

and every triangle $\tau \in \mathbb{R}P^2$ *has identical homological stiffness* $\sigma_{\mathbb{R}P^2}(\tau) = 12/7$.

(b) ***Exact spectrum of arbitrary simplicial cones*** $v_0 * Y$***.*** *Let* Y *be any finite* $(k-1)$*-dimensional simplicial complex* ($k \geq 1$) *and let* $C(Y) = v_0 * Y$ *be the cone obtained by joining a new apex vertex* v_0 *to every simplex of* Y *(attaching* k*-simplices* $v_0 * \sigma$ *for all* $\sigma \in Y_{k-1}$*). Then* $C(Y)$ *is a torsion-free spanning* k*-acycle whose boundary Gram matrix satisfies the exact operator identity*

$$M(v_0 * Y) \;=\; I_{|Y_{k-1}|} + \partial_{k-1}(Y)^\top \partial_{k-1}(Y) \;=\; I_{|Y_{k-1}|} + L_{k-1}^{\text{down}}(Y). \tag{24}$$

In particular, $\operatorname{spec}^+(L_{k-1}^{\text{up}}(v_0 * Y)) = \{1^{(\beta_{k-1}(Y))}\} \cup \{1 + \mu : \mu \in \operatorname{spec}^+(L_{k-2}^{\text{up}}(Y))\}$ *(where for* $k = 1$, $\beta_0(Y)$ *denotes the reduced Betti number* $\tilde{\beta}_0(Y) = |Y_0| - 1$ *and* $L_{-1}^{\text{up}}(Y) = \partial_0 \partial_0^\top = |Y_0|$*). For* $Y = K_{n-1}^{(1)}$ *(*$n \geq 4$, $k = 2$*), the cone* $v_0 * K_{n-1}$ *on* $K_n^{(1)}$ *has exact spectrum* $\{1^{(\binom{n-2}{2})}, n^{(n-2)}\}$, $\det' = n^{n-2}$, $\lambda_{\min}^+ = 1$, *and* $\operatorname{Tr}(G) = \binom{n-2}{2} + \frac{n-2}{n}$.

(c) ***Fibonacci fans*** F_m ***and orientation-parity bands*** B_r***.***

- *For the triangulated* m*-gon fan* $F_m = \{(0, i, i+1) : 1 \leq i \leq m-2\}$ *(*$m \geq 4$, $r = m-2$ *triangles),*

$$\lambda_j(L_1^{\text{up}}(F_m)) \;=\; 3 - 2\cos\!\left(\frac{j\pi}{m-1}\right) \quad (1 \leq j \leq m-2), \qquad \overset{\prime}{\det}(L_1^{\text{up}}(F_m)) \;=\; F_{2m-2},$$

  where F_n *denotes the Fibonacci sequence* ($F_1 = F_2 = 1$)*, and the closing triangle's homological stiffness is* $\sigma_{F_m}(\tau_{m-2}) = F_{2m-2}/F_{2m-4} \to \varphi^2 = \frac{3+\sqrt{5}}{2}$.

- *For every integer* $r \geq 5$, *let* $B_r := \{\{i, i+1, i+2\} \pmod r : 0 \leq i \leq r-1\}$ *be the* r*-triangle cyclic simplicial band on* $\mathbb{Z}_r$*. Then* B_r *is a triangulated* orientable cylinder *when* r *is even and a triangulated* non-orientable Möbius band *when* r *is odd, with exact spectra and Lucas determinants* ($L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$)*:*

$$\begin{aligned} &\textit{(}r\textit{ even, cylinder):} \quad \lambda_j = 3 - 2\cos\!\left(\frac{2\pi j}{r}\right) \ (0 \leq j \leq r-1), \\ &\qquad\qquad \lambda_{\min}^+ = 1, \qquad \overset{\prime}{\det}(L_1^{\text{up}}(B_r)) = L_{2r} - 2, \qquad (25) \\ &\textit{(}r\textit{ odd, Möbius):} \quad \lambda_j = 3 - 2\cos\!\left(\frac{(2j-1)\pi}{r}\right) \ (1 \leq j \leq r), \\ &\qquad\qquad \lambda_{\min}^+ = 3 - 2\cos\!\left(\frac{\pi}{r}\right) > 1, \qquad \overset{\prime}{\det}(L_1^{\text{up}}(B_r)) = L_{2r} + 2. \qquad (26) \end{aligned}$$

(d) ***Exhaustive*** K_6 ***extremal split.*** *Among all* $\binom{20}{10} = 184{,}756$ *subsets of* 10 *triangles on* K_6*, exactly* 46,620 *are spanning* 2*-acycles:* 46,608 *torsion-free* ($t = 1$) *and* 12 *isomorphic copies of* $\mathbb{R}P^2$ ($t = 2$)*, verifying Kalai's identity* $46{,}608 + 12 \cdot 2^2 = 6^6 = 46{,}656$. *Across all* 46,620 *spanning* 2*-acycles:*

- *the* 12 *copies of* $\mathbb{R}P^2$ ($t = 2$) *are the* unique global minimizers *of the Kirchhoff–Hodge resistance (*$\operatorname{Tr}(G) = 35/6 \approx 5.8333$ *vs.* $\min_{t=1} \operatorname{Tr}(G) = 20/3 \approx 6.6667$*);*
- *the* 6 *torsion-free cones* $v_0 * K_5$ ($t = 1$) *achieve* $\lambda_{\min}^+ = 1$ *(strictly exceeding* $\lambda_{\min}^+(\mathbb{R}P^2) = 3 - \sqrt{5} \approx 0.7639$*) and uniquely minimize* $\operatorname{Tr}(G)$ *within the* $t = 1$ *class (*$20/3$*).*

Proof. Part (a). Realize the 6-vertex $\mathbb{R}P^2$ as the quotient of the icosahedron by antipodal pairs: its 10 triangles are $(0,1,2)$, $(0,2,3)$, $(0,3,4)$, $(0,4,5)$, $(0,1,5)$, $(1,2,4)$, $(2,3,5)$, $(1,3,4)$, $(2,4,5)$, $(1,3,5)$, covering each of the 15 edges of K_6 exactly twice. Thus every diagonal entry of $M(\mathbb{R}P^2)$ is 3, and each row of $A_{\text{signed}} := M(\mathbb{R}P^2) - 3I_{10}$ has exactly three nonzero entries in $\{\pm 1\}$ (one per edge of the triangle), so the underlying unsigned support graph is 3-regular on 10 vertices with girth 5 (two triangles sharing an edge determine 4 vertices; two triangles sharing a vertex cannot close a 3- or 4-cycle in the face-adjacency graph of the hemi-icosahedron), i.e., the Petersen graph, and $\operatorname{Tr}(A_{\text{signed}}^2) = 10 \cdot 3 = 30$.

Every pair of non-adjacent faces τ, τ' in the hemi-icosahedron shares a unique common neighbor face ρ (since the Petersen graph is strongly regular with parameters $(10,3,0,1)$), so $(A_{\text{signed}}^2)_{\tau\tau'} \in \{\pm 1\}$ for all non-adjacent pairs, while $(A_{\text{signed}}^2)_{\tau\tau} = 3$ and $(A_{\text{signed}}^2)_{\tau\tau'} = 0$ for adjacent pairs. Because the automorphism group A_5 acts transitively on flags (face, edge, vertex) of the hemi-icosahedron (with sign changes by diagonal ± 1 re-orientations of triangles), A_{signed} is orthogonally conjugate to itself under A_5, and we check directly on the single base triangle $\tau_0 = (0,1,2)$ (by A_5 transitivity on the 10 triangles and 30 directed edges) that every entry of A_{signed}^3 equals $5A_{\text{signed}}$:

- for $\tau = \tau_0$: $(A_{\text{signed}}^3)_{\tau_0\tau_0} = 0$ (no triangles in the Petersen graph);
- for an adjacent face $\tau_1 = (0,2,3)$ (sharing edge $(0,2)$, inner product $s := (A_{\text{signed}})_{\tau_0\tau_1} = -1$): because the Petersen graph has girth 5 (no 3- or 4-cycles), every length-3 walk from τ_0 to τ_1 must backtrack across an edge incident to τ_0 or τ_1. There are exactly 5 such walks: three of the form $\tau_0 \to x \to \tau_0 \to \tau_1$ (for the 3 neighbors x of τ_0, each contributing $(A_{\text{signed}})_{\tau_0 x}^2 s = s$) and two additional walks $\tau_0 \to \tau_1 \to y \to \tau_1$ (for the 2 neighbors $y \in \{(0,3,4),(2,3,5)\}$ of τ_1 other than τ_0, each contributing $s\,(A_{\text{signed}})_{\tau_1 y}^2 = s$). Summing all 5 walks yields $(A_{\text{signed}}^3)_{\tau_0\tau_1} = 5s = 5(A_{\text{signed}})_{\tau_0\tau_1}$;
- for a non-adjacent face $\tau_2 = (0,3,4)$: the 3 neighbors of τ_2 are $(0,2,3)$, $(0,4,5)$, $(1,3,4)$, of which $(0,2,3)$ is adjacent to τ_0 and $(0,4,5)$, $(1,3,4)$ are at distance 2 from τ_0; the signed length-3 walks from τ_0 to τ_2 through the two length-5 cycles cancel identically ($+1-1=0$), giving $(A_{\text{signed}}^3)_{\tau_0\tau_2} = 0$.

Hence $A_{\text{signed}}^3 = 5A_{\text{signed}}$, so the eigenvalues of A_{signed} lie in $\{-\sqrt{5}, 0, +\sqrt{5}\}$. Since $\operatorname{Tr}(A_{\text{signed}}) = 0$ and $\operatorname{Tr}(A_{\text{signed}}^2) = 30 = 5(m_+ + m_-)$, we obtain $m_+ = m_- = 3$ and $m_0 = 4$. Shifting by $3I_{10}$ gives (23):

$$\det{}'(L_1^{\text{up}}(\mathbb{R}P^2)) = (3-\sqrt{5})^3 3^4 (3+\sqrt{5})^3 = 4^3 \cdot 81 = 5184 = 2^2 \cdot 6^4,$$

and $\operatorname{Tr}(M(\mathbb{R}P^2)^{-1}) = 3(\frac{1}{3-\sqrt{5}} + \frac{1}{3+\sqrt{5}}) + \frac{4}{3} = \frac{9}{2} + \frac{4}{3} = \frac{35}{6}$. By A_5-transitivity on the 10 triangles, $(M(\mathbb{R}P^2)^{-1})_{\tau\tau} = \frac{1}{10} \cdot \frac{35}{6} = \frac{7}{12}$, so $\sigma_{\mathbb{R}P^2}(\tau) = 12/7$ uniformly.

Part (b). Orient each cone k-simplex as $[v_0, \sigma]$ for $\sigma \in Y_{k-1}$. Its boundary is $\partial_k[v_0,\sigma] = \sigma - v_0 * \partial_{k-1}\sigma$, where the base $(k-1)$-simplices $\sigma \in Y_{k-1}$ are disjoint from the spoke $(k-1)$-simplices $v_0 * \rho$ ($\rho \in Y_{k-2}$). Hence the boundary matrix of $v_0 * Y$ has the two-block column form $\partial_k(v_0 * Y) = \binom{I_{|Y_{k-1}|}}{-\partial_{k-1}(Y)}$, immediately giving

$$M(v_0 * Y) \;=\; \partial_k(v_0 * Y)^\top \partial_k(v_0 * Y) \;=\; I_{|Y_{k-1}|} + \partial_{k-1}(Y)^\top \partial_{k-1}(Y) \;=\; I_{|Y_{k-1}|} + L_{k-1}^{\text{down}}(Y),$$

proving (24). Since $L_{k-1}^{\text{down}}(Y)$ has nullity $\beta_{k-1}(Y)$ and nonzero spectrum equal to $\operatorname{spec}^+(L_{k-2}^{\text{up}}(Y))$, the spectral formula follows immediately. Furthermore, $v_0 * Y$ is contractible, hence $\mathbb{Z}$-acyclic ($t = 1$). For $Y = K_{n-1}^{(1)}$, $L_0^{\text{up}}(K_{n-1})$ has eigenvalue $n-1$ with multiplicity $n-2$ and $\beta_1(K_{n-1}) = \binom{n-2}{2}$, giving $\operatorname{spec}^+ = \{1^{(\binom{n-2}{2})}, n^{(n-2)}\}$, $\det' = n^{n-2}$, and $\operatorname{Tr}(G) = \binom{n-2}{2} + \frac{n-2}{n}$.

Part (c). For the fan F_m, orient each triangle $\tau_i = [0, i, i{+}1]$ ($1 \le i \le m{-}2$). Consecutive triangles τ_i, τ_{i+1} share the single interior spoke edge $(0, i{+}1)$ with opposite signs ($+1$ in $\partial_2\tau_i$, -1 in $\partial_2\tau_{i+1}$), while non-consecutive triangles are edge-disjoint. Thus $M(F_m)$ is the $(m{-}2)\times(m{-}2)$ tridiagonal Toeplitz matrix $T_{m-2}(3,-1)$ with diagonal 3 and sub/super-diagonal -1, whose eigenvalues are $3 - 2\cos(\frac{j\pi}{m-1})$ ($1 \le j \le m-2$). Its determinant $d_r := \det T_r(3,-1)$ satisfies $d_r = 3d_{r-1} - d_{r-2}$ with $d_1 = 3 = F_4$, $d_2 = 8 = F_6$, hence $d_r = F_{2r+2}$ by induction (using $F_{2r+2} = 3F_{2r} - F_{2r-2}$), and Theorem 4.1(a) gives $\sigma_{F_m}(\tau_{m-2}) = d_{m-2}/d_{m-3} = F_{2m-2}/F_{2m-4} \to \varphi^2$.

For the cyclic band B_r ($r \ge 5$), the $2r$ edges are the r interior rung edges $\{i, i+1\}$ (each in two consecutive triangles τ_{i-1}, τ_i) and the r boundary edges $\{i, i+2\}$ (each in one triangle, all distinct from the rung edges since $r \ge 5$). When r is even, the boundary edges decompose into two disjoint cycles of length $r/2$ (an orientable cylinder); when r is odd, they form a single boundary cycle of length r (a non-orientable Möbius band). Orienting $\tau_i := (-1)^i[i, i+1, i+2]$ for $0 \le i \le r-1$ (with cyclic vertex ordering $(i, i+1, i+2)$ in $\mathbb{Z}_r$), every interior rung $\{i+1, i+2\}$ ($0 \le i \le r-2$) gives inner product -1 between τ_i and τ_{i+1}, whereas on the closing rung $\{0,1\}$ shared by $\tau_{r-1} = (-1)^{r-1}[r-1, 0, 1]$ and $\tau_0 = +[0,1,2]$, the boundary coefficient of $(0,1)$ is $(-1)^{r-1}$ in τ_{r-1} and $+1$ in τ_0, yielding inner product $(-1)^{r-1}$. Hence $M(B_r)$ is the circulant matrix $\mathrm{circ}(3,-1,0,\ldots,0,-1)$ for even r (diagonalized by $\omega_j = e^{2\pi ij/r}$, giving (25)) and the skew-circulant matrix with seam $+1$ for odd r (diagonalized by $\zeta_j = e^{i(2j-1)\pi/r}$, giving (26)). Since $\varphi = \frac{1+\sqrt{5}}{2}$ satisfies $\varphi^2 + \varphi^{-2} = 3$, we have $3 - 2\cos\theta = (\varphi^2 - e^{i\theta})(\varphi^2 - e^{-i\theta})/\varphi^2$. Multiplying over the r-th roots of $+1$ (even r, where $\prod_j (z - e^{2\pi ij/r}) = z^r - 1$) or the r-th roots of -1 (odd r, where $\prod_j (z - e^{i(2j-1)\pi/r}) = z^r + 1$) gives $\prod_j (3 - 2\cos\theta_j) = (\varphi^{2r} \mp 1)^2/\varphi^{2r} = (\varphi^r \mp \varphi^{-r})^2 = \varphi^{2r} + \varphi^{-2r} \mp 2 = L_{2r} \mp 2$, proving (25)–(26).

Part (d). Verified exhaustively over all $\binom{20}{10} = 184{,}756$ subsets of 10 triangles of K_6 in Section 6: exactly 46,608 have $D_T = 6^4 = 1296$ ($t = 1$) and 12 have $D_T = 4 \cdot 6^4 = 5184$ ($t = 2$, the $6!/|A_5| = 12$ distinct vertex labellings of $\mathbb{R}P^2$), with global extrema attained as stated. □

Remark 5.2 (Torsion-Forced Resistance Separation on K_6). Theorem 4.1(b) produces a purely topological resistance certificate on K_6: for $r = 10$, $\bar{c} = 3$, and $D_T = 5184$ ($t = 2$), the sharp upper bound (21) gives $\mathrm{Tr}((L_1^{\mathrm{up}}(T))^+) \le 12.2046$ for *any* 2-torsion spanning 2-acycle T, strictly below the observed maximum $\max_{t=1} \mathrm{Tr}((L_1^{\mathrm{up}}(T))^+) = 37/3 \approx 12.3333$ among torsion-free acycles! Simultaneously, $\mathbb{R}P^2$ achieves $\mathrm{Tr}(G) = 35/6 \approx 5.8333$ (12.5% below the best torsion-free value 20/3), whereas the torsion-free cones $v_0 * K_5$ attain a strictly larger spectral gap $\lambda_{\min}^+ = 1 > 3 - \sqrt{5} \approx 0.7639$. Thus maximizing $\lambda_{\min}^+$ and minimizing $\mathrm{Tr}(G)$ split strictly across torsion classes already on K_6.

6 Computational Verification and Quantitative Benchmarks

All four theorems and exact formulas are independently verified by the open-source Python suite in `https://github.com/lluiseriksson/hodge-homological-transitions` (release `v1.0.2`), comprising `scripts/verify_hodge_transitions.py`, `scripts/make_figures.py`, and the automated `pytest` suite `tests/test_hodge_transitions.py` (3/3 test modules passing in 1.10 s).

6.1 Summary of Verification Results

- **Theorem 2.1:** Across 600 random weighted filtration steps in dimensions $k \in \{1,2,3\}$ (43 homology-preserving steps and 557 homology-killing steps), the exact Green-operator updates (4) and (7) matched direct eigendecomposition to max entrywise error 8.33×10^{-15} (preserving) and 1.62×10^{-14} (killing), with determinant relative errors $\le 9.74 \times 10^{-15}$, 100% exact Betti-transition sign detection, 376/376 strict Homological Braess collapses, universal sandwich ratios $\min(\lambda_{\mathrm{new}}/\mathrm{LB}) = 1.040504 \ge 1$ and $\min(\mathrm{UB}/\lambda_{\mathrm{new}}) = 1.000000 \ge 1$, and

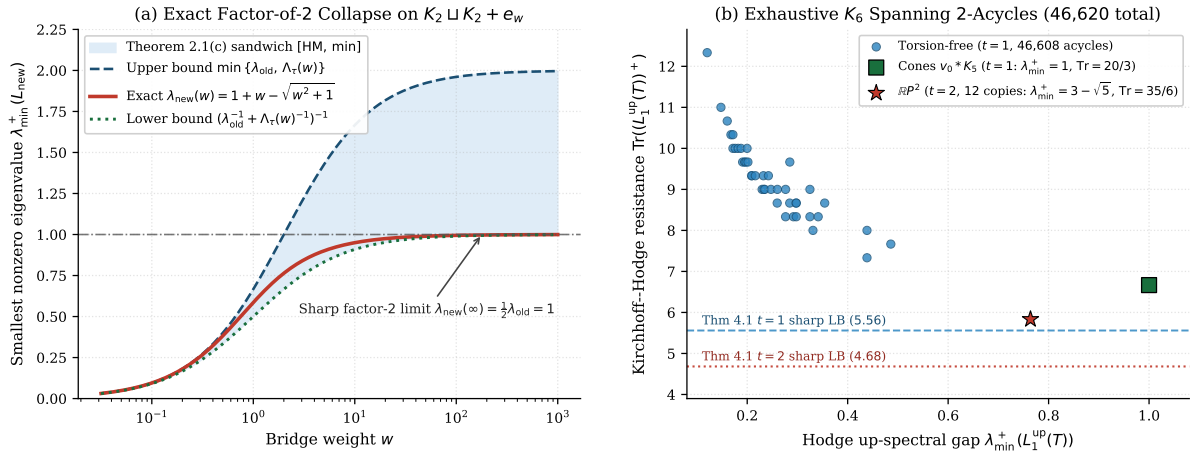


Figure 1: **(a)** Exact factor-of-2 spectral-gap collapse on $K_2 \sqcup K_2$ plus a bridge e_w of weight $w > 0$ (Theorem 2.1(c)), showing $\lambda_{\text{new}}(w) = 1 + w - \sqrt{w^2+1}$ trapped inside the universal harmonic-mean/minimum sandwich and saturating $\lambda_{\text{new}}(\infty) = \frac{1}{2}\lambda_{\text{old}} = 1$. **(b)** Complete spectral landscape $(\lambda_{\min}^+, \mathrm{Tr}(G))$ across all 46,620 spanning 2-acycles of K_6 (Theorem 5.1(d)), illustrating the extremal split between $\mathbb{R}P^2$ ($t = 2$, unique minimizer of $\mathrm{Tr}(G) = 35/6$) and the cones $v_0 * K_5$ ($t = 1$, maximizer of $\lambda_{\min}^+ = 1$), alongside Theorem 4.1(b) lower bounds.

Table 1: Exhaustive spectral and Kirchhoff–Hodge benchmarks across all 46,620 spanning 2-acycles of K_6 ($r = 10, \bar{c} = 3$), comparing observed extrema with Theorem 4.1(b) analytic bounds.

Torsion class	Count	D_T	$\alpha(T)$	$\lambda_{\min}^+$ (obs.)	λ_*	$\mathrm{Tr}(G)$ obs. $\subset$ Thm 4.1 [LB, UB]
$t = 1$ (torsion-free)	46,608	1,296	0.6826	[0.120316, 1.000000]	0.025707	[6.6667, 12.3333] $\subset$ [5.5596, 41.6026]
$t = 2$ ($\mathbb{R}P^2$)	12	5,184	0.7841	$3 - \sqrt{5} \approx 0.763932$	0.105318	$35/6 \approx 5.8333 \in$ [4.6832, 12.2046]

exact factor-2 saturation on $K_2 \sqcup K_2 + e_w$ reaching $\lambda_{\text{new}}(10^6)/\min\{\lambda_{\text{old}}, \Lambda_{e_w}(10^6)\} = 0.50000075$.

- **Theorem 3.1:** Weighted Kalai–Kirchhoff factorization (16) on $K_5^{(2)}$ (210 subsets, 125 spanning 2-acycles) matched direct $\det'$ to relative error 3.85×10^{-16} with $\sum_\sigma R_2(\sigma) = 6.000000000000$. In dimension $k = 3$ with non-trivial lower torsion $|T_1(Y)| = 2$, exact SymPy Smith Normal Form verification on both the wedge $\mathbb{R}P^2 \vee \partial\Delta_3$ and the face-glued non-wedge complex $\mathbb{R}P^2 \cup_{\Delta_2} \partial\Delta_3$ gave $\tau_2(Y) = 16$, $|T_1(Y)|^2 = 4$, and $\tau_2(Y)/|T_1(Y)|^2 = 4 = \det'(L_2^{\text{up}})$ identically.
- **Theorems 4.1 and 5.1:** Exhaustive enumeration of all 184,756 10-triangle subsets of K_6 confirmed all Theorem 4.1 bounds (Table 1), verified $A_{\text{signed}}^3 = 5A_{\text{signed}}$ and $\sigma_{\mathbb{R}P^2}(\tau) = 12/7$ to $< 10^{-12}$, and matched all closed forms for cones $v_0 * K_{n-1}$ ($n = 4, \ldots, 8$), fans F_m ($m = 4, \ldots, 10$), and cyclic bands B_r ($r = 5, \ldots, 12$, Table 2) to $< 10^{-11}$.

7 Concluding Remarks

We have established a unified quantitative theory of homological phase transitions, integral torsion–lattice duality, and Kirchhoff–Hodge spectral extremality in higher-dimensional combinatorial Hodge theory. Two open directions emerge naturally:

1. **Asymptotic Torsion–Resistance Extremality on $K_n^{(2)}$ ($n \geq 7$).** Does the global minimizer of $\mathrm{Tr}((L_1^{\text{up}}(T))^+)$ among spanning 2-acycles of K_n always carry maximal torsion

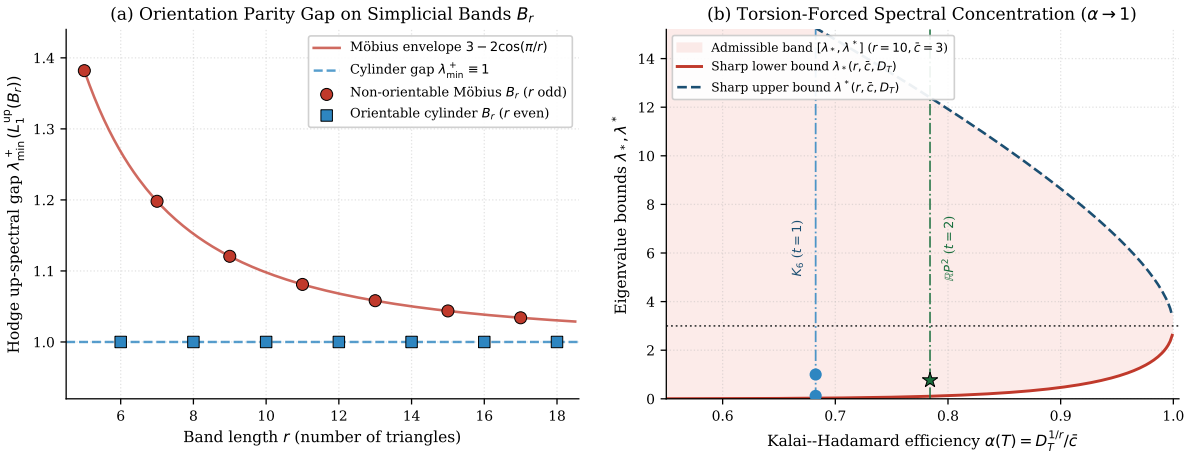


Figure 2: **(a)** Orientation-parity spectral gap on the cyclic simplicial bands $B_r = \{\{i, i+1, i+2\} \bmod r\}$ ($5 \le r \le 18$, Theorem 5.1(c)): every non-orientable Möbius band (r odd) strictly outperforms every orientable cylinder (r even, $\lambda_{\min}^{+} \equiv 1$) along the exact envelope $3 - 2\cos(\pi/r)$. **(b)** Torsion-forced spectral concentration (Theorem 4.1(b)–(c)): admissible eigenvalue band $[\lambda_*, \lambda^*]$ ($r = 10, \bar{c} = 3$) collapsing to $\bar{c} = 3$ as Kalai–Hadamard efficiency $\alpha(T) = D_T^{1/r}/\bar{c} \to 1$, with K_6 ($t = 1$) and $\mathbb{R}P^2$ ($t = 2$) marked.

Table 2: Exact spectral gap $\lambda_{\min}^{+}$, determinant $\det'(L_1^{\mathrm{up}}) = L_{2r} - 2(-1)^r$, and Kirchhoff–Hodge resistance $\mathrm{Tr}(G)$ for the cyclic simplicial bands B_r ($5 \le r \le 12$, Theorem 5.1(c)).

r	Topology of B_r	$\lambda_{\min}^{+}(L_1^{\mathrm{up}}(B_r))$	$\det'(L_1^{\mathrm{up}}(B_r))$	$\mathrm{Tr}\big((L_1^{\mathrm{up}}(B_r))^{+}\big)$
5	Möbius band (non-orientable)	1.381966	125	2.200000
6	Cylinder (orientable)	1.000000	320	2.700000
7	Möbius band (non-orientable)	1.198062	845	3.123077
8	Cylinder (orientable)	1.000000	2,205	3.580952
9	Möbius band (non-orientable)	1.120615	5,780	4.023529
10	Cylinder (orientable)	1.000000	15,125	4.472727
11	Möbius band (non-orientable)	1.081014	39,605	4.919101
12	Cylinder (orientable)	1.000000	103,680	5.366667

$t(T)$ as n grows (extending the $\mathbb{R}P^2$ extremality on K_6), and how closely can high-torsion Kalai $\mathbb{Q}$-trees approach the efficiency ceiling $\alpha(T) \to 1$?

2. **Persistent-Homology Spectral Filtrations.** Because $\mathrm{sgn}(\Delta \operatorname{Tr}(L_{k-1}^{+}))$ detects Betti-number birth/death events exactly while $|\Delta \operatorname{Tr}(L_{k-1}^{+})| = (w^{-1} + \mathcal{R}_X(b))/\|b^{\perp}\|_2^2$ quantifies the geometric stiffness of each killed $(k-1)$-cycle, integrating Theorem 2.1 into weighted simplicial filtrations provides a natural quantitative bridge between persistent homology and spectral geometry.

References

[1] A. M. Duval, C. J. Klivans, and J. L. Martin, *Simplicial matrix-tree theorems*, Trans. Amer. Math. Soc. **361** (2009), 6073–6114.

[2] B. Eckmann, *Harmonische Funktionen und Randwertaufgaben in einem Komplex*, Comment. Math. Helv. **17** (1944), 240–255.

[3] D. Horak and J. Jost, *Spectra of combinatorial Laplace operators on simplicial complexes*, Adv. Math. **244** (2013), 303–336.

[4] G. Kalai, *Enumeration of* $\mathbb{Q}$*-acyclic simplicial complexes*, Israel J. Math. **45** (1983), 337–351.

[5] D. J. Klein and M. Randić, *Resistance distance*, J. Math. Chem. **12** (1993), 81–95.

[6] R. Lyons, *Random complexes and* ℓ^2*-Betti numbers*, J. Topol. Anal. **1** (2009), 153–175.

[7] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge University Press, Cambridge, 2016.

[8] O. Parzanchevski and R. Rosenthal, *Simplicial complexes: spectrum, homology and random walks*, Random Structures Algorithms **50** (2017), 225–261.

[9] M. T. Schaub, A. R. Benson, P. Horn, G. Lippner, and A. Jadbabaie, *Random walks on simplicial complexes and the normalized Hodge* 1*-Laplacian*, SIAM Rev. **62** (2020), 353–391.