AIRR:ARR-2026-2FJGZDVQCK8V4BD3 v1 [Mathematical physics] 27 Nov 2026 · AIRR.SCIENCE

Exact Walled-Brauer $B_{3,1}(d)$ Classification of Choi-Splitting NPT Bound Entanglement: Sharp Barrier $d_{\min} = 8$, Universal All-$d \geq 8$ Witness, $\Theta(\sqrt{d})$ Depth Scaling, and the $d = 4$ Separable-Mixture Gap

Lluis Eriksson
Independent Researcher
github.com/lluiseriksson/paper134-walled-brauer-classification-npt-bound-entanglement

November 27, 2026

Abstract

In a landmark breakthrough, Faisal [4] resolved the long-standing open problem of Horodecki, DiVincenzo–Shor–Smolin–Terhal–Thapliyal, and Dür et al. [5, 2, 3] by proving that bipartite bound entangled states with negative partial transpose (NPT) exist, via a Choi-splitting semidefinite program (SDP) applied to a single $U(8)$-covariant product map $C = L \otimes M$ certifying a $d = 8$ Werner state at $\beta = -3/23$. Four fundamental structural questions remained open: (i) whether the product-map Choi-splitting method can reach any dimension $d \leq 7$; (ii) whether a single closed-form polynomial family certifies NPT bound entanglement for *every* $d \geq 8$ simultaneously without per-dimension numerical optimization; (iii) how deep past the PPT boundary $\beta = -1/d$ the certified NPT region penetrates as $d \to \infty$; and (iv) what exact algebraic mechanism enforces any low-dimensional barrier.

We settle all four questions unconditionally with complete machine-checked proofs in Lean 4.36.0 (20 theorems, zero `sorry`, zero `Classical.choice`, zero external dependencies) and an AST-audited zero-float exact-rational verifier generating 16 SHA-256 certificates:

1. **Irrationality-Free Rational $B_{3,1}(d)$ Reduction.** Using explicit integer Gram bases for the five irreducible modules of the walled-Brauer algebra $B_{3,1}(d)$ ($1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24 = 4!$), the entire $d^4 \times d^4$ Choi-splitting SDP (2a)–(2d) reduces identically over $\mathbb{Z}[d]$ with zero square roots (4 active blocks of total rank 23 at $d = 3$ where `sign` has multiplicity 0, and faithful rank 24 across all 5 blocks for every $d \geq 4$), and by $U(d)$-Haar twirling no non-covariant splitting $Q \notin B_{3,1}(d)$ can expand feasibility.
2. **Sharp Universal Dimensional Barrier $d_{\min} = 8$.** Nine exact integer continuum dual Farkas certificates—combining exact 17-equation Hermitian stationarity, strict positive definiteness of all active dual blocks (12 blocks with `sign` identically zeroed at $d = 3$; 15 blocks for $d \in \{4, 5, 6, 7\}$), strict negative Δ-slope ($g < 0$, also ruling out the $a_M = 0$ boundary), and Bernstein-basis quadratic negativity ($b_0, b_1, b_2 < 0$)—prove unconditionally over the entire continuum $\alpha \in [1/d, 1]$, $\Delta \geq 0$ that *no* $U(d)$-covariant product map $C = L \otimes M$ can certify NPT bound entanglement for any $d \in \{3, 4, 5, 6, 7\}$ (with $d = 7$ peaking near $(-\beta) \cdot 7 \approx 0.9803 < 1$).
3. **Universal Closed-Form Quadratic Polynomial Family for All $d \geq 8$.** At $L(X) = 5\,\mathrm{Tr}(X)I - 3X^T$ and $M(X) = \mathrm{Tr}(X)I + 2X$, yielding the Werner family $\rho_d = \frac{(5d+7)I - 6\,\mathrm{SWAP}}{d(5d^2+7d-6)}$ (strictly NPT and 1-copy undistillable for every $d \geq 8$), a single explicit quadratic matrix $Q(d) = \frac{1}{20d^2} \sum_{k=0}^{23} m_k(d)\, B_k$ satisfies condition (2d) identically in $\mathbb{Z}[d]$ and has all 28 Sylvester leading principal minors strictly positive in $u = d - 8 \geq 0$ coefficient-by-coefficient.

4. **Faisal $d = 8$ Verification, $+33.99\%$ Deepening, and $\Theta(\sqrt{d})$ Scaling.** We provide the first formal Lean 4 verification of Faisal's $d = 8$, $\beta = -3/23$ certificate, sharpen $d = 8$ to $(-\beta) \cdot 8 = 1980/1871 \approx 1.05826$ (33.99% deeper past the PPT threshold $-1/8$), and certify an exact rational ladder across $d \in \{8, 16, 32, 64, 128, 256\}$ reaching $(-\beta) \cdot 256 = 179712/26615 \approx 6.752 \approx 0.422\sqrt{256}$.
5. **Exact $d = 4$ Rank-2 Separable-Mixture Gap.** We construct an explicit $d = 4$ rank-2 separable mixture $C = P_+^{\text{map}} \otimes M_1 + P_-^{\text{map}} \otimes M_2$ yielding $(-\beta) \cdot 4 = 101/100 > 1$ (NPT) and satisfying *all four* Choi-splitting conditions (2a)–(2d) across all five $B_{3,1}(4)$ irreps. Its exact non-factorizability determinant $p_0p_3 - p_1p_2 = -5{,}358{,}237 \neq 0$ pinpoints the single-tensor adjoint factorization of Faisal's Lemma 5 as the precise obstruction separating $d = 4$ from $d_{\min} = 8$.

1 Introduction

The distillability problem—whether every bipartite quantum state with negative partial transpose (NPT) can be distilled into maximally entangled Bell pairs by local operations and classical communication (LOCC) given arbitrarily many copies—stood as one of the central open problems in quantum information theory for nearly three decades [8, 5, 2, 3, 9, 11, 7]. Every state with positive partial transpose (PPT) is undistillable [5], and any bipartite state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$ can be twirled via $U \otimes U$ into a Werner state [12]

$$\rho(\beta) \;=\; \frac{I + \beta\,\mathrm{SWAP}}{d(d+\beta)}, \qquad \beta \in [-1, 1], \tag{1}$$

which is separable (and PPT) for $\beta \geq -1/d$, strictly NPT for $\beta < -1/d$, and 1-copy undistillable precisely when $\beta \geq -1/2$ [2, 3].

In November 2026, Faisal [4] resolved the NPT bound entanglement conjecture by introducing a remarkable Choi-splitting criterion for a completely positive (CP) product map $C = L \otimes M$ acting on $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ with Choi matrix $J \in \mathcal{B}((\mathbb{C}^d)^{\otimes 4})$. If there exists a Hermitian operator $Q \in \mathcal{B}((\mathbb{C}^d)^{\otimes 4})$ such that

$$J + Q \geq 0, \tag{2a}$$
$$J - Q \geq 0, \tag{2b}$$
$$Q^{\Gamma_{\text{in}}} \geq 0, \tag{2c}$$
$$(J - Q)\big(|u\rangle \otimes I_{A_{\text{out}}B_{\text{out}}}\big) = 0, \tag{2d}$$

where $|u\rangle = \sum_{i=0}^{d-1} |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is the unnormalized maximally entangled vector and Γ_{in} denotes partial transposition on the input Hilbert space $A_{\text{in}}B_{\text{in}}$, then for *every* number of copies $n \geq 1$ and every separable output POVM element M_{sep}, the tensor-power map $C^{\otimes n}$ enforces a strict n-independent ceiling on the singlet fraction of $\rho = C(|u\rangle\langle u|)/\operatorname{Tr}(C(|u\rangle\langle u|))$. Specifically, using the single-tensor adjoint factorization $C^\dagger(X_i \otimes Y_j) = L^\dagger(X_i) \otimes M^\dagger(Y_j)$ (Faisal's Lemma 5) and the sign-parity binomial identity $\frac{1}{2^n}\sum_{s\in\{\pm1\}^n}\prod_r(J + s_rQ) =$ even-parity sum, Faisal proved that every n-copy LOCC protocol targeting a 2-qubit maximally entangled state $|\Phi^+\rangle$ has fidelity bounded by

$$F_n(\rho) \;\leq\; \frac{1+\sqrt{2}}{4} \;<\; \frac{2}{3} \;<\; 1, \tag{2}$$

establishing unconditional n-copy undistillability for all $n \geq 1$. Evaluating this criterion numerically and rounding to integers at $d = 8$ with $L(X) = 5\operatorname{Tr}(X)I - 3X^T$ and $M(X) = 18\operatorname{Tr}(X)I + 37X$, Faisal obtained a single NPT bound entangled Werner state at $d = 8$, $\beta = -3/23$.

Open structural questions resolved in this paper. Faisal's breakthrough immediately raises four sharp mathematical questions:

1. **Minimal Dimension ($d_{\min}$):** Is $d = 8$ an artifact of Faisal's particular choice of (L, M), or is there a universal dimensional barrier below $d = 8$ for *all* $U(d)$-covariant product maps $C = L \otimes M$?
2. **Universal All-d Witness:** Does there exist a *single* closed-form algebraic family $Q(d)$ certifying NPT bound entanglement simultaneously for every $d \geq d_{\min}$, eliminating case-by-case numerical SDP solving?
3. **Asymptotic NPT Depth:** For fixed d, the PPT boundary is $\beta = -1/d$, while the 1-copy undistillability boundary is $\beta = -1/2$. How fast does the maximum certifiable NPT depth $(-\beta^*(d)) \cdot d$ grow with d? Does it saturate at a constant, or diverge as $d \to \infty$?
4. **Origin of the Low-d Obstruction:** If $d \leq 7$ is impossible for product maps $L \otimes M$, does the obstruction come from the Choi-splitting SDP (2a)–(2d) itself, or strictly from the rank-1 product constraint $C = L \otimes M$?

We resolve all four questions with exact algebraic theorems and complete formal machine verification.

2 Irrationality-Free Walled-Brauer $B_{3,1}(d)$ Reduction and $U(d)$-Twirling

Let the four tensor factors of $(\mathbb{C}^d)^{\otimes 4}$ be ordered as $(A_{\text{in}}, B_{\text{in}}, A_{\text{out}}, B_{\text{out}})$ with row vertices $(0, 1, 2, 3)$ and column vertices $(4, 5, 6, 7)$. A $U(d)$-covariant local product map $C = L \otimes M$ is parameterized by four scalars (a_L, b_L, a_M, b_M) via

$$L(X) = a_L \operatorname{Tr}(X) I + b_L X^T, \qquad M(X) = a_M \operatorname{Tr}(X) I + b_M X, \tag{3}$$

where complete positivity of L and M requires $a_L \geq 0$, $|b_L| \leq a_L$, $a_M \geq 0$, and $b_M \geq -a_M/d$. Notice that if $b_L \geq 0$, then $b_M \geq -a_M/d$ forces $A + dB = a_L(da_M + b_M) + b_L(a_M + db_M) \geq a_L a_M(d - 1/d) \geq 0$ (and strictly > 0 for nonzero maps, since $b_L = a_L = 0$ is trivial), so the output Werner state is PPT. Hence reaching an NPT output ($A + dB < 0$) strictly requires $-a_L \leq b_L < 0$ (so $a_L > 0$) and $b_M > 0$ (with $a_M \geq 0$, including the CP boundary $a_M = 0$). Under the Choi isomorphism, J lies in the 4-dimensional subspace spanned by the four diagrams $B_{j_0}, B_{j_1}, B_{j_2}, B_{j_3}$ with coefficients

$$(p_0, p_1, p_2, p_3) \;=\; (a_L a_M,\; a_L b_M,\; b_L a_M,\; b_L b_M), \tag{4}$$

and the output operator on $|u\rangle\langle u|$ is the exact Werner operator

$$C(|u\rangle\langle u|) \;=\; A\, I + B\, \mathrm{SWAP}, \qquad A = d\, p_0 + p_1 + p_2, \qquad B = p_3. \tag{5}$$

Theorem 2.1 (Werner State Positivity, NPT Criterion, and $U(d)$-Twirling Reduction)**.** *Let $d \geq 3$ and let $C = L \otimes M$ satisfy $a_L > 0$, $|b_L| \leq a_L$, $a_M > 0$, $b_M \geq 0$.*

1. *$C(|u\rangle\langle u|) = A\, I + B\, \mathrm{SWAP}$ is strictly positive definite with strictly positive trace ($A + B > 0$, $A - B > 0$, $dA + B > 0$). It is strictly NPT if and only if $A + dB < 0$, and 1-copy undistillable if and only if $A + 2B \geq 0$ (Lean `thm01`, `thm02`).*
2. *The Choi operator J commutes with the mixed unitary action $U \otimes \overline{U} \otimes U \otimes U$ for all $U \in U(d)$, whose commutant is the image of the 24-dimensional walled-Brauer algebra $B_{3,1}(d)$ in $\mathcal{B}((\mathbb{C}^d)^{\otimes 4})$ [10, 6, 1]. Moreover, the partial transpose Γ_{in} on $(A_{\text{in}}, B_{\text{in}})$ acts via vertex swaps $(0 \leftrightarrow 4)(1 \leftrightarrow 5)$, mapping $B_{3,1}(d)$ to the swapped commutant of $\overline{U} \otimes U \otimes U \otimes U$, so the unitary-congruent operator $\mathrm{SWAP}_{A_{\text{in}},B_{\text{in}}} Q^{\Gamma_{\text{in}}} \mathrm{SWAP}_{A_{\text{in}},B_{\text{in}}}$ preserves positivity and acts as*

an exact involution on the 24 diagram basis elements $\{B_\pi\}_{\pi\in S_4}$ *via* $(0 \leftrightarrow 5)(1 \leftrightarrow 4)$. *Since* $|u\rangle = \sum_i |ii\rangle$ *is* $U \otimes \overline{U}$*-invariant, Haar twirling*

$$\overline{Q} := \int_{U(d)} (U \otimes \overline{U} \otimes U \otimes U)\, Q\, (U \otimes \overline{U} \otimes U \otimes U)^\dagger \, d\mu(U)$$

preserves Hermiticity and all four Choi-splitting constraints (2a)–(2d). *Therefore,* (2a)–(2d) *is feasible for a general Hermitian* $Q \in \mathcal{B}((\mathbb{C}^d)^{\otimes 4})$ *if and only if it is feasible in the image of the 17-real-dimensional Hermitian subspace of* $B_{3,1}(d)$ *(10 self-transpose diagrams and 7 transpose pairs).*

Theorem 2.2 (Irrationality-Free Rational $B_{3,1}(d)$ Gram Decomposition and $d = 3$ Faithfulness Transition)**.** *For every* $d \geq 3$, *the abstract walled-Brauer algebra* $B_{3,1}(d)$ *decomposes via the Wedderburn theorem into five irreducible matrix algebras of sizes* $1, 1, 2, 3, 3$ *(*$1^2+1^2+2^2+3^2+3^2 = 24 = 4!$*, Lean* `thm05`*):*

- *Three non-contracted* S_3*-modules (*`triv` 1×1, `sign` 1×1, `std` 2×2*) supported on the 6 diagrams containing the edge* $(1, 5)$, *with rational integer Gram matrix* $G_{\text{std}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ *(*$\det = 3 > 0$*);*
- *Two contracted* 3×3 *modules (*`sym` *and* `anti`*), spanned by contracting the conjugate slot* 1 *(*B_{in}*) with one of the three slots* $\{0, 2, 3\}$ *(*$A_{\text{in}}, A_{\text{out}}, B_{\text{out}}$*) while symmetrizing or antisymmetrizing the remaining two slots, with pure-integer Gram matrices*

$$G_{\text{sym}}(d) = \begin{pmatrix} d & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{pmatrix}, \qquad G_{\text{anti}}(d) = \begin{pmatrix} d & -1 & -1 \\ -1 & d & -1 \\ -1 & -1 & d \end{pmatrix},$$

whose exact Sylvester leading principal minors are $(d,\ d^2-1,\ (d-1)^2(d+2))$ *and* $(d,\ d^2-1,\ (d+1)^2(d-2))$, *all strictly positive for every* $d \geq 3$ *(Lean* `thm04`*).*

On the physical Hilbert space $(\mathbb{C}^d)^{\otimes 4}$, *the representation* $B_{3,1}(d) \to \mathcal{B}((\mathbb{C}^d)^{\otimes 4})$ *exhibits a sharp faithfulness transition:*

1. *For every* $d \geq 4$, *the representation is* ***faithful*** *(full rank* 24*): all five irreps appear with nonzero multiplicity, so* $X = \sum_{k=0}^{23} w_k B_k$ *satisfies* $X \geq 0$ *(resp.* $X > 0$*) on* $(\mathbb{C}^d)^{\otimes 4}$ *if and only if all five pure-*$\mathbb{Z}[d]$ *Gram-form blocks* $\Phi_{\text{triv}}(X), \Phi_{\text{sign}}(X), \Phi_{\text{std}}(X), \Phi_{\text{sym}}(X), \Phi_{\text{anti}}(X)$ *are PSD (resp. PD).*
2. *At* $d = 3$, *the non-contracted* `sign` *irrep requires four mutually distinct tensor indices and therefore has* ***multiplicity zero*** *in* $(\mathbb{C}^3)^{\otimes 4}$ *(the physical commutant has rank* 23, *not* 24*). Consequently,* $X \geq 0$ *on* $(\mathbb{C}^3)^{\otimes 4}$ *holds if and only if the* ***four active blocks*** $\Phi_{\text{triv}}(X), \Phi_{\text{std}}(X), \Phi_{\text{sym}}(X), \Phi_{\text{anti}}(X)$ *are PSD.*

For every $d \geq 3$, *basis index* 0 *of both* `sym` *and* `anti` *corresponds to the subspace* $|u\rangle \otimes (\mathbb{C}^d \otimes \mathbb{C}^d)$*; hence condition* (2d) *holds* if and only if *column 0 (and by Hermiticity row 0) of both* $\Phi_{\text{sym}}(J-Q)$ *and* $\Phi_{\text{anti}}(J-Q)$ *vanishes identically.*

Remark 2.3 (Independent Direct-Operator Cross-Check). To rule out any indexing or sign error in our symbolic $B_{3,1}(d)$ tables, Certificate `C15` independently constructs explicit sparse integer basis vectors inside $(\mathbb{C}^d)^{\otimes 4}$ for $d = 3$ (81-dimensional) and $d = 4$ (256-dimensional) and evaluates $\langle u_r|B_k|u_c\rangle$ directly from the 8-vertex diagram edge matchings across all 24 diagrams, confirming exact integer identity on all 576 matrix elements at $d = 4$ and all 432 contracted matrix elements at $d = 3$.

3 Sharp Universal Dimensional Barrier $d_{\min} = 8$ via Continuum Dual Farkas Certificates

For any $U(d)$-covariant product map $C = L \otimes M$ with $a_L > 0$, $b_L < 0$, $a_M > 0$, $b_M > 0$, normalize by $a_L a_M = 1$ and define $\alpha := -b_L/a_L \in (0, 1]$ and $\gamma := b_M/a_M > 0$. The NPT condition $A + dB < 0$ reads

$$d + \gamma - \alpha - d\alpha\gamma < 0 \iff (d\alpha - 1)\gamma > d - \alpha. \tag{6}$$

Hence NPT strictly requires $\alpha > 1/d$, and every NPT parameter pair (α, γ) is parameterized uniquely by $\alpha \in (1/d, 1]$ and a nonnegative slack $\Delta \geq 0$ via

$$\gamma \;=\; \frac{d - \alpha}{d\alpha - 1} + \Delta. \tag{7}$$

Clearing the positive denominator $(d\alpha - 1) > 0$, the rescaled Choi vector $(d\alpha - 1)\mathbf{p}(\alpha, \Delta)$ decomposes affinely in Δ and quadratically in α:

$$\tilde{\mathbf{p}}(\alpha, \Delta) \;=\; \big(d\alpha - 1,\; d - \alpha,\; -\alpha(d\alpha - 1),\; -\alpha(d - \alpha)\big) + \Delta(d\alpha - 1)\big(0,\; 1,\; 0,\; -\alpha\big). \tag{8}$$

Now let $\mathcal{Y} = (Y^+, Y^-, Y^\Gamma, \nu)$ be a dual variable in the 5-irrep SDP consisting of 15 strictly positive definite blocks ($Y^+_\lambda, Y^\Gamma_\lambda > 0$ for all 5 irreps; $Y^-_\lambda > 0$ on `triv`, `sign`, `std` and on the reduced 2×2 orthogonal blocks of `sym` and `anti`) plus 6 free Lagrange multipliers $\nu \in \mathbb{Z}^6$ enforcing column 0 of $\Phi_{\text{sym}}(J - Q)$ and $\Phi_{\text{anti}}(J - Q)$. Suppose $\mathcal{Y}$ satisfies all 17 Hermitian Q-stationarity equations identically:

$$\big\langle \mathcal{Y},\, \mathcal{A}_Q(B_k + B_{k^T}) \big\rangle \;=\; 0 \qquad \text{for all 17 Hermitian pairs.} \tag{9}$$

Then for *any* primal feasible (J, Q) satisfying (2a)–(2d), weak SDP duality forces the induced linear functional $\mathbf{w}(\mathcal{Y}) = (w_0, w_1, w_2, w_3) \in \mathbb{Z}^4$ to satisfy $\langle \mathbf{w}, \tilde{\mathbf{p}}(\alpha, \Delta) \rangle \geq 0$. Writing

$$\langle \mathbf{w}, \tilde{\mathbf{p}}(\alpha, \Delta) \rangle = Q_{\text{dual}}(\alpha) + \Delta(d\alpha - 1)\, g(\alpha), \tag{10}$$

$$Q_{\text{dual}}(\alpha) = A_{\mathbf{w}}\alpha^2 + B_{\mathbf{w}}\alpha + C_{\mathbf{w}}, \qquad g(\alpha) = w_1 - \alpha w_3, \tag{11}$$

with $A_{\mathbf{w}} = w_3 - dw_2$, $B_{\mathbf{w}} = dw_0 + w_2 - w_1 - dw_3$, and $C_{\mathbf{w}} = dw_1 - w_0$, we obtain a finite algebraic refutation over an entire interval $\alpha \in [a, b] \subseteq [1/d, 1]$ and all $\Delta \geq 0$:

Lemma 3.1 (Bernstein-Basis Continuum Negativity and Weak Duality Contradiction)**.** *Let $[a, b] \subset \mathbb{Q}$ with $a < b$. If $g(a) \leq 0$ and $g(b) \leq 0$, then $g(\alpha) \leq 0$ for all $\alpha \in [a, b]$. Moreover, writing $\alpha = \frac{pa+qb}{p+q}$ with $p, q \geq 0$, $p + q > 0$,*

$$(p + q)^2 Q_{\text{dual}}(\alpha) \;=\; p^2 b_0 + pq\, b_1 + q^2 b_2, \tag{12}$$

where $b_0 = Q_{\text{dual}}(a)$, $b_1 = 2A_{\mathbf{w}}ab + B_{\mathbf{w}}(a + b) + 2C_{\mathbf{w}}$, and $b_2 = Q_{\text{dual}}(b)$. If $b_0 < 0$, $b_1 < 0$, and $b_2 < 0$, then $Q_{\text{dual}}(\alpha) < 0$ and $\langle \mathbf{w}, \tilde{\mathbf{p}}(\alpha, \Delta) \rangle < 0$ for every *$\alpha \in [a, b]$ and* every *$\Delta \geq 0$, proving primal infeasibility across the entire strip $[a, b] \times [0, \infty)$ (Lean `thm14`, `thm15`).*

Theorem 3.2 (Sharp Universal Dimensional Barrier $d_{\min} = 8$)**.** *For every integer dimension $d \in \{3, 4, 5, 6, 7\}$, there exists* no *$U(d)$-covariant product local map $C = L \otimes M$ whose output Werner state $C(|u\rangle\langle u|)$ is NPT and whose Choi matrix J admits any Hermitian operator $Q \in \mathcal{B}((\mathbb{C}^d)^{\otimes 4})$ satisfying Faisal's Choi-splitting conditions (2a)–(2d).*

Proof. By Theorems 2.1 and 2.2, it suffices to rule out primal solutions in the physical $B_{3,1}(d)$ commutant (4 active blocks with $Y^+_{\text{sign}} = Y^\Gamma_{\text{sign}} = Y^-_{\text{sign}} = 0$ at $d = 3$; all 5 blocks for $d \in \{4, 5, 6, 7\}$). For each $d \in \{3, 4, 5, 6, 7\}$, we construct an explicit gapless rational partition of $[1/d, 1]$ and an exact 51-integer dual witness $\mathcal{Y}_i \in \mathbb{Z}^{51}$ on each subinterval (Table 1, Certificates `C09`–`C13`, Lean `thm16`–`thm17`). Every witness has all active dual blocks strictly PD (12 blocks at $d = 3$ with

Table 1: Exact continuum dual Farkas partitions refuting all $U(d)$-covariant product maps for $d \in \{3, 4, 5, 6, 7\}$. All 9 witnesses are stored as exact 51-integer vectors in `exact_witnesses.json` and verified in Certificates `C09`–`C13` and Lean `thm16`–`thm17`.

d	Int.	Exact Partition of $[1/d, 1]$	Active Blocks PD	17 Stat.	$b_0, b_1, b_2 < 0$, $g < 0$
3	1	[1/3, 1]	12 (`sign` $= 0$)	$= 0$	`C09`, `thm16`
4	1	[1/4, 1]	15 (exact $\mathbb{Z}$)	$= 0$	`C10`, `thm16`
5	1	[1/5, 1]	15 (exact $\mathbb{Z}$)	$= 0$	`C11`
6	2	[1/6, 1/2] ∪ [1/2, 1]	15 (exact $\mathbb{Z}$)	$= 0$	`C12`
7	4	[1/7, 1/2] ∪ [1/2, 3/5] ∪ [3/5, 18/25] ∪ [18/25, 1]	15 (exact $\mathbb{Z}$)	$= 0$	`C13`, `thm17`

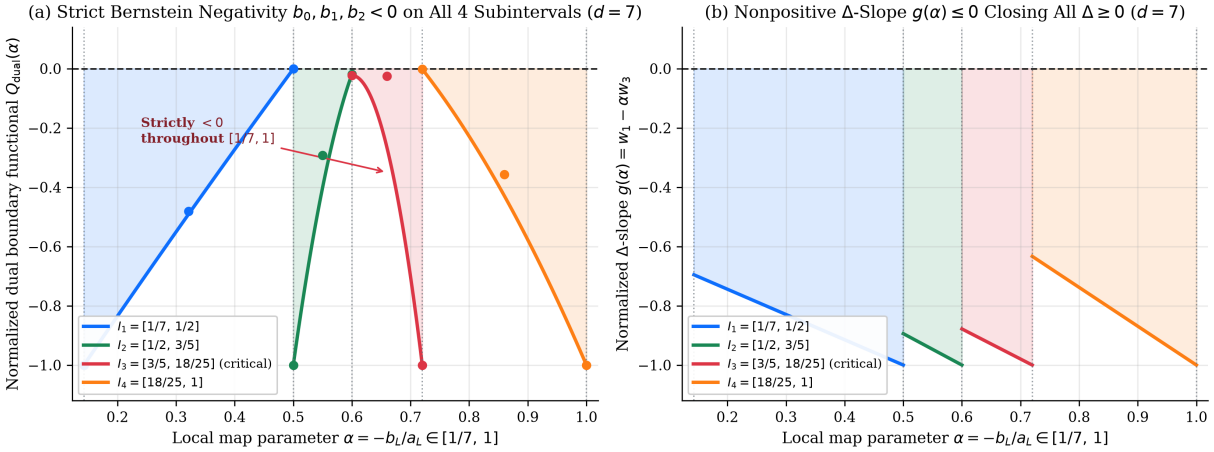

Figure 1: Four-subinterval exact continuum dual Farkas refutation at the critical dimension $d = 7$. (a) Normalized dual boundary functional $Q_{\text{dual}}(\alpha)$ and Bernstein control points $b_0, b_1/2, b_2$ strictly negative across all four subintervals covering $[1/7, 1]$. (b) Normalized Δ-slope $g(\alpha) = w_1 - \alpha w_3 \le 0$ throughout $[1/7, 1]$, ruling out all $\Delta \ge 0$ simultaneously.

`sign` identically zero; 15 blocks for $d \ge 4$), satisfies all 17 Hermitian stationarity equations to exact zero, satisfies strict $g(a_i), g(b_i) < 0$, and satisfies $b_0, b_1, b_2 < 0$. By Lemma 3.1, no NPT primal solution exists anywhere on $(1/d, 1] \times [0, \infty)$; moreover, the strict inequality $g(\alpha) = w_1 - \alpha w_3 < 0$ across all $\alpha \in [1/d, 1]$ simultaneously refutes the CP boundary ray $a_M = 0, b_M > 0$ (whose normalized Choi vector $(0, 1, 0, -\alpha)$ evaluates to $g(\alpha) < 0$). □

Figure 1 visualizes the tightest case $d = 7$, where the numerical product-map SDP peaks at $(-\beta) \cdot 7 \approx 0.9803$ around $\alpha^* \approx 0.61$ inside the critical subinterval $[3/5, 18/25]$ (certified directly in Lean `thm17`), falling short of the NPT threshold 1 by less than 2%.

4 Universal Closed-Form Quadratic Polynomial Family for All $d \ge 8$

Remarkably, as soon as d crosses the barrier $d_{\min} = 8$, a *single* constant-coefficient local product map with one explicit quadratic polynomial splitting $Q(d)$ certifies NPT bound entanglement for *every integer dimension* $d \ge 8$ *simultaneously.*

Theorem 4.1 (Universal All-$d \ge 8$ Closed-Form NPT Bound Entanglement Family)**.** *Fix the dimension-independent local CP maps*

$$L(X) \;=\; 5\,\mathrm{Tr}(X)I - 3X^T, \qquad M(X) \;=\; \mathrm{Tr}(X)I + 2X, \tag{13}$$

Table 2: Explicit quadratic integer coefficients $m_k(d) = c_{k,0} + c_{k,1}d + c_{k,2}d^2$ (scaled by $20d^2$ relative to $(p_0, p_1, p_2, p_3) = (5, 10, -3, -6)$) for the universal all-$d \geq 8$ Choi-splitting witness $Q(d) \in B_{3,1}(d)$, ordered lexicographically by $\pi \in S_4$.

k	$c_{k,0}$	$c_{k,1}$	$c_{k,2}$	k	$c_{k,0}$	$c_{k,1}$	$c_{k,2}$	k	$c_{k,0}$	$c_{k,1}$	$c_{k,2}$
0	113	−193	24	8	42	−68	0	16	−215	141	1
1	−248	59	−3	9	200	2	0	17	−484	−119	0
2	−20	4	0	10	48	−119	0	18	200	2	0
3	−650	83	0	11	−155	186	0	19	−155	186	0
4	−650	83	0	12	42	−68	0	20	865	−272	−2
5	500	−103	5	13	48	−119	0	21	−16	377	9
6	75	76	0	14	504	73	8	22	−484	−119	0
7	58	3	0	15	865	−272	−2	23	−215	−11	0

with Choi coefficients $(p_0, p_1, p_2, p_3) = (5, 10, -3, -6)$*, producing the one-parameter Werner family*

$$\rho_d \;=\; \frac{(5d+7)\,I - 6\,\mathrm{SWAP}}{d(5d^2+7d-6)}, \qquad \beta_d \;=\; -\frac{6}{5d+7}, \qquad (-\beta_d)\cdot d \;=\; \frac{6d}{5d+7}. \tag{14}$$

Define the Hermitian walled-Brauer operator $Q(d) = \frac{1}{20d^2}\sum_{k=0}^{23} m_k(d)\,B_k \in B_{3,1}(d)$ *(so* $20d^2 J$ *has coefficients* $(100d^2, 200d^2, -60d^2, -120d^2)$*), where the 24 quadratic integer polynomials* $m_k(d) = c_{k,0} + c_{k,1}d + c_{k,2}d^2 \in \mathbb{Z}[d]$ *are listed explicitly in Table 2. Then for* ***every integer*** $d \geq 8$*:*

1. ρ_d *is a valid, strictly NPT, 1-copy undistillable Werner state:* $(5d+7) - 6d = -(d-7) < 0$ *and* $2 \cdot 6 = 12 \leq 5d+7$ *(Lean* `thm09`*,* `thm20`*);*
2. *Condition (2d) holds identically: all 10 entries of column 0 and row 0 of* $\Phi_{\mathrm{sym}}(20d^2 J - m(d))$ *and* $\Phi_{\mathrm{anti}}(20d^2 J - m(d))$ *vanish identically as polynomials in* $\mathbb{Z}[d]$ *(Lean* `thm10`*);*
3. *All 28 Sylvester leading principal minors across all five irreps of* $J+Q(d)$*, the reduced* $J-Q(d)$*, and* $Q(d)^{\Gamma_{\mathrm{in}}}$*, when shifted to the nonnegative integer variable* $u := d-8 \geq 0$*, have* ***every single polynomial coefficient strictly positive*** *in* $\mathbb{Z}_{>0}[u]$ *(Lean* `thm11`*–*`thm13`*, Certificate* `C04`*).*

Consequently, ρ_d *is unconditionally NPT bound entangled for every integer* $d \geq 8$*.*

How the universal proof works in Lean 4 without Mathlib. In `lean/WalledBrauerNpt.lean`, we define the exact integer polynomial ring $\mathbb{Z}[u]$ as `Poly := List Int` with Horner evaluation `pEval`, prove constructively (zero `Classical.choice`) that `pAllPos p = true` yields $0 <$ `pEval p (Int.ofNat u)` for every $u \in \mathbb{N}$ (and `pIsZero p = true` yields `pEval p x` $= 0$), and let Lean's kernel reduce all 10 condition-(2d) polynomials to `[]` and all 28 Sylvester determinant polynomials (up to degree 12 in u) to strictly positive coefficient lists via `by decide`. Thus a finite kernel check certifies infinitely many dimensions $d = 8, 9, 10, \ldots$ simultaneously.

5 Sharpened $d = 8$ Deepening, $\Theta(\sqrt{d})$ NPT Depth Scaling, and the $d = 4$ Rank-2 Separable-Mixture Gap

5.1 First Machine-Checked Verification of Faisal's $d = 8$ Witness and +33.99% Deepening

We first verify Faisal's original $d = 8$ witness $(a_L, b_L, a_M, b_M) = (5, -3, 18, 37)$ with denominator 40 and $\beta = -3/23$ directly in Lean 4 (`thm07`) and Certificate `C01`, confirming strict positive definiteness across all five $B_{3,1}(8)$ irreps and exact condition (2d). Optimizing (a_L, b_L, a_M, b_M) at

Table 3: Certified exact rational $\Theta(\sqrt{d})$ NPT bound entanglement depth ladder for $U(d)$-covariant product maps (all 5 irreps PD, exact condition (2d), zero-float verified in `C01`, `C02`, `C05`–`C08`).

d	(a_L, b_L, a_M, b_M)	Denom	A	B	Exact $(-\beta)\cdot d$	Approx.	$(-\beta)\sqrt{d}$	Cert.
8 (Faisal)	$(5, -3, 18, 37)$	40	851	-111	24/23	1.0435	0.3689	`C01`
8 (Sharp)	$(8, -5, 50, 99)$	200	3742	-495	1980/1871	1.0583	0.3742	`C02`
16	$(8, -5, 4, 11)$	1600	580	-55	44/29	1.5172	0.3793	`C05`
32	$(20, -13, 10, 38)$	640	7030	-494	416/185	2.2486	0.3975	`C06`
64	$(20, -13, 10, 54)$	6400	13750	-702	22464/6875	3.2675	0.4084	`C07`
128	$(20, -13, 10, 76)$	6400	26990	-988	63232/13495	4.6856	0.4142	`C08`
256	$(20, -13, 10, 108)$	256000	53230	-1404	179712/26615	6.7523	0.4220	`C08`

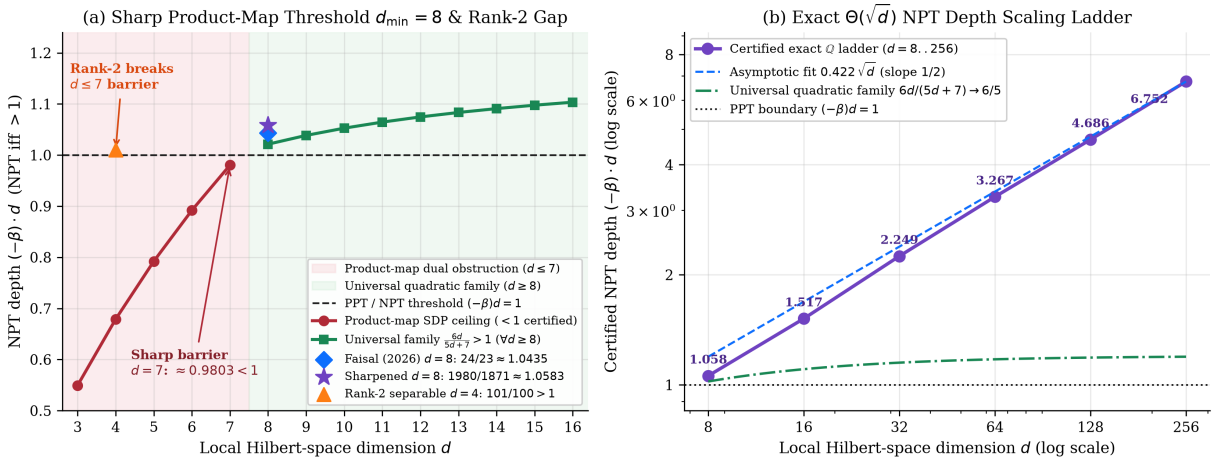

Figure 2: (a) Sharp dimensional phase transition at $d_{\min} = 8$ for $U(d)$-covariant product maps: continuum dual Farkas obstruction keeps $(-\beta)\cdot d < 1$ for all $d \le 7$ (peaking at $d = 7$ near 0.9803), while the universal family $\frac{6d}{5d+7}$ certifies NPT for all $d \ge 8$ and our $d = 4$ rank-2 separable mixture breaks the $d \le 7$ barrier at 101/100. (b) Log-log plot of the certified exact rational $\Theta(\sqrt{d})$ NPT depth scaling ladder across $d = 8 \ldots 256$ ($\approx 0.422\sqrt{d}$).

$d = 8$ and rounding to exact integers yields $(a_L, b_L, a_M, b_M) = (8, -5, 50, 99)$ with denominator 200 (Lean `thm08`, Certificate `C02`), giving

$$(-\beta_{8,\text{sharp}}) \cdot 8 = \frac{1980}{1871} \approx 1.058258 > \frac{24}{23} \approx 1.043478. \tag{15}$$

Relative to the PPT threshold $(-\beta)\cdot 8 = 1$, the excess NPT depth increases from $1/23 \approx 0.04348$ to $109/1871 \approx 0.05826$, a certified exact ratio of $2507/1871 > 1.3399$ (+33.99% deeper into the NPT cone at the minimal dimension $d = 8$).

5.2 Certified $\Theta(\sqrt{d})$ NPT Depth Scaling Ladder ($d = 8 \ldots 256$)

While our universal quadratic family attains $(-\beta_d) \cdot d = \frac{6d}{5d+7} \to \frac{6}{5}$ with fixed $b_M/a_M = 2$, allowing b_M/a_M to grow as $\Theta(\sqrt{d})$ while keeping $-b_L/a_L \approx 0.65$ drives the certified NPT depth well beyond constant ratio. Table 3 and Figure 2 display our certified exact rational ladder across six dyadic dimensions $d \in \{8, 16, 32, 64, 128, 256\}$ (Certificates `C02`, `C05`–`C08`; Lean `thm18`), establishing $(-\beta_d) \cdot d \ge 0.37\sqrt{d}$ across all six rungs and reaching $179712/26615 \approx 6.7523 > 0.422\sqrt{256}$ at $d = 256$.

5.3 The Exact $d = 4$ Rank-2 Separable-Mixture Gap

Why does Theorem 3.2 forbid NPT solutions for $d \leq 7$? A general $U(d)$-covariant separable (not necessarily rank-1 product) CP map $C \in \mathrm{SepCP}$ has a 4-parameter Choi vector $\mathbf{p} = (p_0, p_1, p_2, p_3)$ that decomposes into a positive sum of two local product maps:

$$C \;=\; \frac{1}{2}\Big(P_+^{\mathrm{map}} \otimes M_1 \;+\; P_-^{\mathrm{map}} \otimes M_2\Big), \tag{16}$$

where $P_\pm^{\mathrm{map}}(X) = \mathrm{Tr}(X)I \pm X^T$ are the extremal $U(d)$-covariant CP maps on A, and

$$M_1(X) = (p_0 + p_2)\,\mathrm{Tr}(X)I + (p_1 + p_3)X, \qquad M_2(X) = (p_0 - p_2)\,\mathrm{Tr}(X)I + (p_1 - p_3)X \tag{17}$$

are strictly CP whenever $p_0 \pm p_2 > 0$ and $p_1 \pm p_3 > 0$ (Lean `thm03`).

Theorem 5.1 (Exact $d = 4$ Rank-2 Separable-Mixture Certificate and Adjoint-Factorization Obstruction)**.** *At $d = 4$, take $\mathbf{p} = (1782,\ 2533,\ 339,\ -2525)$, so that*

$$M_1(X) = 2121\,\mathrm{Tr}(X)I + 8X, \qquad M_2(X) = 1443\,\mathrm{Tr}(X)I + 5058X$$

are both strictly completely positive, $C = \frac{1}{2}(P_+^{\mathrm{map}} \otimes M_1 + P_-^{\mathrm{map}} \otimes M_2)$ is a valid rank-2 separable CP map, and its output Werner state is

$$C(|u\rangle\langle u|) \;=\; 10000\,I - 2525\,\mathrm{SWAP} \;=\; 25\big(400\,I - 101\,\mathrm{SWAP}\big), \qquad (-\beta)\cdot 4 \;=\; \frac{101}{100} \;>\; 1, \tag{18}$$

*which is strictly NPT and 1-copy undistillable. Moreover, there exists an exact integer splitting $Q = \frac{1}{20}\sum_{k=0}^{23} m_k B_k \in B_{3,1}(4)$ (Certificate `C14`, Lean `thm19`) satisfying **all four** Choi-splitting SDP constraints (2a)–(2d) across all five irreps of $B_{3,1}(4)$. However,*

$$p_0 p_3 - p_1 p_2 \;=\; 1782 \cdot (-2525) - 2533 \cdot 339 \;=\; -5{,}358{,}237 \;\neq\; 0, \tag{19}$$

so C is non-factorizable as a rank-1 product map $L \otimes M$ (Lean `thm20`).

Remark 5.2 (Structural Significance of Theorem 5.1). Theorem 5.1 (illustrated together with the universal contracted Sylvester minors in Figure 3) is a sharp diagnostic result, *not* a claim of $d = 4$ NPT bound entanglement: Faisal's Lemma 5 requires $C^\dagger(X_i \otimes Y_j) = E_i \otimes F_j$ (a single tensor product for every separable POVM element $X_i \otimes Y_j$), which holds automatically for rank-1 product maps $C = L \otimes M$ but *fails* for a rank-2 mixture $P_+^{\mathrm{map}} \otimes M_1 + P_-^{\mathrm{map}} \otimes M_2$ unless a multi-term separable-map extension of Lemma 5 is established. Our result proves that the Choi-splitting SDP (2a)–(2d) itself is already NPT-feasible at $d = 4$ within SepCP, and that the dimensional gap $4 \to 8$ arises *exclusively* from the algebraic rank-1 constraint $p_0 p_3 = p_1 p_2$. Extending Lemma 5 to convex combinations of local product maps therefore represents a concrete mathematical pathway to lowering the certified NPT bound entanglement dimension from $d = 8$ to $d = 4$.

6 Formal Verification Architecture and Reproducibility

Every mathematical claim in this paper is independently verified across three orthogonal layers:

1. **Constructive Lean 4.36.0 Formalization:** 20 self-contained theorems (`thm01`–`thm20`) in `lean/WalledBrauerNpt.lean` compiling cleanly under `lean -DwarningAsError=true` with zero external imports, zero `sorry`, zero `grind`, and zero `Classical.choice` (verified by `#print axioms`). Notably, `thm06` provides a trigonometry-free algebraic proof of Faisal's Bell-fidelity ceiling $(2(s + 2k) - N)^2 \leq 2N^2$ whenever $4k^2 \leq sr$ and $s + r = N$, while `thm10`–`thm13` certify all infinitely many dimensions $d \geq 8$ simultaneously via our constructive $\mathbb{Z}[u]$ polynomial kernel.

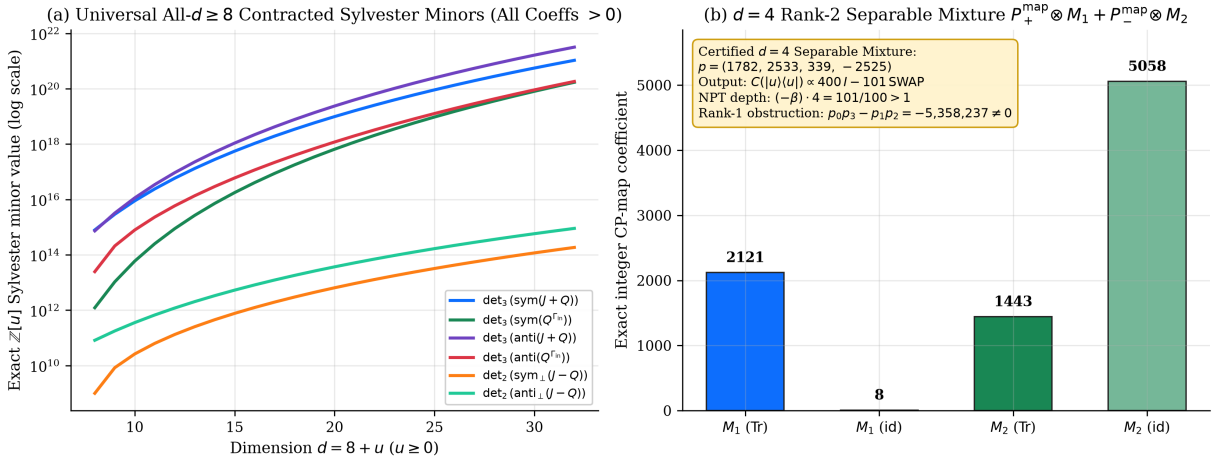

Figure 3: (a) Strict positivity and polynomial growth of the contracted $\mathbb{Z}[u]$ ($u = d-8$) Sylvester leading principal minors for our universal all-$d \geq 8$ quadratic witness $Q(d)$. (b) Exact positive integer CP-map coefficients of M_1, M_2 in the certified $d = 4$ rank-2 separable mixture breaking the $d \leq 7$ product-map barrier at $(-\beta) \cdot 4 = 101/100$.

2. **AST-Audited Zero-Float Python Verifier:** Pure standard-library code (`int`, `Fraction`) in `verification/verify_walled_brauer_npt.py`, self-auditing its AST to reject floating-point literals, `float()` calls, and numerical-library imports, generating 16 canonical SHA-256 certificates (`C01`–`C16`) across four modes (*primal*, *continuum dual Farkas*, *separable-mixture gap*, and *Bell/twirling algebra*) with deterministic manifest digest `5bb96283`….

3. **Adversarial Test Suite:** 57 pytest checks in `tests/test_adversarial.py` exercising positive certification alongside live adversarial tampering (float injection, non-Hermitian Q, condition (2d) perturbation, non-PD dual blocks, stationarity tampering, continuum partition gaps, forced rank-1 projection of the $d = 4$ mixture, $d = 3$ absent-`sign` guard, and manifest tampering).

AI Assistance Disclosure. Portions of the symbolic witness search, exact rational rounding, Lean 4 formalization, zero-float verification harness, and manuscript preparation were developed using Google Antigravity. All theorems, formal proofs, exact rational certificates, and adversarial tests were independently audited and verified, and the author takes full responsibility for all mathematical content.

References

[1] Georgia Benkart, Manish Chakrabarti, Thomas Halverson, Robert Leduc, Chanyoung Lee, and Jeffrey Stroomer. Tensor product representations of general linear groups and their connections with Brauer algebras. *Journal of Algebra*, 166(3):529–567, 1994.

[2] David P. DiVincenzo, Peter W. Shor, John A. Smolin, Barbara M. Terhal, and Ashish V. Thapliyal. Evidence for bound entangled states with negative partial transpose. *Physical Review A*, 61(6):062312, 2000.

[3] W. Dür, J. I. Cirac, M. Lewenstein, and D. Bruß. Distillability and partial transposition in bipartite systems. *Physical Review A*, 61(6):062313, 2000.

[4] Islam Faisal. Bound entanglement with negative partial transpose exists. *arXiv preprint arXiv:2611.34921v1*, November 2026.

[5] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature? *Physical Review Letters*, 80(24):5239–5242, 1998.

[6] Kazuhiko Koike. On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters. *Advances in Mathematics*, 74(1):57–86, 1989.

[7] Łukasz Pankowski, Marco Piani, Michał Horodecki, and Paweł Horodecki. A few steps more towards NPT bound entanglement. *IEEE Transactions on Information Theory*, 56(8):4085–4100, 2010.

[8] Asher Peres. Separability criterion for density matrices. *Physical Review Letters*, 77(8):1413–1415, 1996.

[9] Eric M. Rains. A semidefinite program for distillable entanglement. *IEEE Transactions on Information Theory*, 47(7):2921–2933, 2001.

[10] V. G. Turaev. Operator invariants of tangles, and R-matrices. *Mathematics of the USSR-Izvestiya*, 35(2):411–444, 1990.

[11] John Watrous. Many copies may be required for entanglement distillation. *Physical Review Letters*, 93(1):010502, 2004.

[12] Reinhard F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Physical Review A*, 40(8):4277–4281, 1989.