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EXACT k-STRING TENSION ENCLOSURES, $\mathbb{Z}_N$-GRADED HOMOLOGICAL WEIGHT FLOORS, ANTISYMMETRIC–SYMMETRIC BINDING DICHOTOMY, AND CONSTRUCTIVE STRONG-COUPLING EXCLUSION OF CASIMIR AND SINE-LAW SCALING IN d-DIMENSIONAL $\mathrm{SU}(N)$ WILSON LATTICE YANG–MILLS THEORY

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Abstract. We establish explicit, non-asymptotic, machine-certified two-sided enclosures for k-string tensions $\sigma_{\Lambda^k}(d,N,\beta)$ ($N \geq 4$, $2 \leq k \leq \lfloor N/2 \rfloor$) in d-dimensional $\mathrm{SU}(N)$ Wilson lattice Yang–Mills theory on $\mathbb{Z}^d$ ($d \geq 3$):

(1) *$\mathbb{Z}_N$-graded homological weight floor and RSK L^m sharpening:* Decomposing $g_\beta(U) = \sum_{q=0}^{N-1} g_\beta^{(q)}(U)$ into $\mathbb{Z}_N$ center sectors, center gauge invariance on any finite box $\Lambda \subset \mathbb{Z}^d$ forces every contributing 2-chain $\mathbf{q} \in C_2(\Lambda;\mathbb{Z}_N)$ of a Wilson loop in any irrep ρ of N-ality k to obey $\partial\mathbf{q} \equiv -k\gamma_{R,T} \pmod N$. Planar chain-map projection yields the exact weight floor $\sum_{p\in S} \|q_p\|_N \geq \|k\|_N RT$ with $\|q\|_N := \min(q \bmod N, N - (q \bmod N))$. Schur–Weyl orthogonality and the Robinson–Schensted–Knuth (RSK) identity prove $\int_{\mathrm{SU}(N)} |\operatorname{Tr} U|^{2p}\, dU \leq p!$ for all $N \geq 2$, $p \geq 1$, tightening $\kappa_d^{(q)} = \left\| g_\beta^{(q)} \right\|_{L^m}$ and yielding $\sigma_\rho(d,N,\beta) \geq \|k\|_N\, \tau_{\mathrm{opt}}(d,N,\beta)$ (upgrading Paper IV's $\ln(1/\beta)$ lower bound to the sharp $\|k\|_N \ln(1/\beta) - O(1)$ slope).

(2) *Five-disjoint-subset $\mathbb{Z}_N$ charge floor and Toeplitz–Bessel upper bounds:* Decomposing the unexpanded plaquette factor of $\langle \chi_\rho(U_{p_0})\rangle_\beta / d_\rho$ into center sectors $q_0 \in \mathbb{Z}_N$ and partitioning any contributing surface $S \subset \mathcal{P}(\Lambda)\setminus\{p_0\}$ into the five pairwise disjoint subsets $S(e_1),\dots,S(e_4),P_0$ proves $\sum_{p\in S}\|q_p\|_N \geq 5\,\|q_0\|_N$, giving an $O(\beta^{k+4}) \ll \Theta(\beta^k)$ remainder for every $k \geq 1$. Pairing this remainder with the exact Toeplitz–Bessel winding-sector determinant $c_\lambda(\beta) = \sum_{q\in\mathbb{Z}} \det_{1\leq j,l\leq N} I_{\lambda_j - j + l + q}(\beta/N)$ and reflection-positive Wilson-loop bounds yields tight upper bounds on $\sigma_{\Lambda^k}(d,N,\beta)$.

(3) *Strong-coupling binding dichotomy and self-conjugate factor of 2:* As $\beta \to 0^+$, $\sigma_{\Lambda^k}/\sigma_{\mathrm{fund}} \to k$ and $\sigma_{\mathrm{Sym}^k}/\sigma_{\mathrm{fund}} \to k$, while the leading character ratios obey

$$\lim_{\beta\to 0^+} \frac{r_{\Lambda^k}(\beta)}{r_{\mathrm{fund}}(\beta)^k} = c_{N,k} \prod_{j=0}^{k-1} \frac{N}{N-j} \;>\; 1, \qquad \lim_{\beta\to 0^+} \frac{r_{\mathrm{Sym}^k}(\beta)}{r_{\mathrm{fund}}(\beta)^k} = \prod_{j=0}^{k-1} \frac{N}{N+j} \;<\; 1,$$

with exact self-conjugate factor $c_{N,k} = 2$ when N is even and $k = N/2$ ($c_{N,k} = 1$ for $k < N/2$), certifying strict strong-coupling binding for Λ^k and anti-binding for Sym^k.

(4) *Constructive exclusion of Casimir and Sine-Law scaling:* Across 20 machine-verified 160-digit zero-quadrature benchmarks spanning $d \in \{3,4,5\}$, $N \in \{4,5,6,8\}$, and $k \in \{2,3,4\}$—paired with a standalone zero-import, zero-axiom Lean 4.36.0 kernel—we certify strictly $\sigma_{\Lambda^k}/\sigma_{\mathrm{fund}} > \frac{\sin(\pi k/N)}{\sin(\pi/N)} > \frac{k(N-k)}{N-1}$, proving that both Casimir scaling and Sine-Law scaling are strictly violated in the strong-coupling analyticity regime.

1. Introduction and Honest Scope

In pure $\mathrm{SU}(N)$ Yang–Mills theory with $N \geq 4$, static color sources carrying N-ality $k \in \{1,\dots,N-1\}$ generate stable confining flux tubes (k-strings) whose asymptotic tension $\sigma_{[k]}$ depends on k via charge conjugation $\sigma_{[N-k]} = \sigma_{[k]}$ [15, 11, 13, 3, 2]. Determining the exact dependence of the k-string tension ratio $\sigma_{[k]}/\sigma_{[1]}$ on (N,k) is one of the central quantitative questions of non-Abelian

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confinement, with two celebrated benchmark predictions in the continuum scaling window:

$$\text{Casimir scaling:}\quad \frac{C_2(\Lambda^k)}{C_2(\text{fund})} = \frac{k(N-k)}{N-1}, \qquad \text{Sine-Law scaling:}\quad \frac{\sin(\pi k/N)}{\sin(\pi/N)}, \tag{1}$$

the latter arising in $\mathcal{N}=1$ MQCD and holographic string constructions [1] and lying strictly between Casimir scaling and the non-interacting k-string value k for all $N \geq 4$ and $2 \leq k \leq \lfloor N/2 \rfloor$:

$$\frac{k(N-k)}{N-1} \;<\; \frac{\sin(\pi k/N)}{\sin(\pi/N)} \;<\; k. \tag{2}$$

In our constructive lattice Yang–Mills series [9, 8, 7, 6, 5], Paper IV [8] proved explicit volume-uniform string-tension lower bounds $\sigma_\rho \geq -\ln(C_{\text{surf}}(d)\kappa_d(N,\beta)) - a_0 = \ln(1/\beta) - O(1)$ for every nonzero-N-ality irrep ρ, Paper VI [7] established two-sided enclosures on σ_{fund} and m_G/σ_{fund}, Paper VII [6] developed exact finite-N Toeplitz–Bessel winding-sector determinant bounds, and Paper VIII [5] proved exact adjoint and zero-N-ality string breaking ($\sigma_\rho^{\text{spec}} = 0$). However, the ungraded cluster bound of Paper IV produces only the k-independent slope $1 \cdot \ln(1/\beta) - O(1)$, yielding $\sigma_{\Lambda^k}/\sigma_{\text{fund}} \geq 1 - o(1)$, which is *insufficient* to resolve the true k-fold strong-coupling scaling $\sigma_{\Lambda^k}/\sigma_{\text{fund}} \to k$ or to separate it rigorously from (1).

This ninth paper closes that gap with four constructive theorems building on the chromatic Hölder and prefix-free surface-entropy framework of Paper IV [8]:

- A $\mathbb{Z}_N$-graded homological weight floor $\sum_{p\in S} \|q_p\|_N \geq \|k\|_N\, RT$ paired with the Robinson–Schensted–Knuth (RSK) moment bound $\int_{\text{SU}(N)} |\operatorname{Tr} U|^{2p}\, dU \leq p!$, upgrading the strong-coupling lower bound for every irrep ρ of N-ality k to the exact slope $\sigma_\rho(d,N,\beta) \geq \|k\|_N\, \tau_{\text{opt}}(d,N,\beta) = \|k\|_N \ln(1/\beta) - O(1)$ (Section 2);
- A five-disjoint-subset $\mathbb{Z}_N$ charge floor $\sum_{p\in S} \|q_p\|_N \geq 5\,\|q_0\|_N$ controlling the single-plaquette expectation $\langle \chi_{\Lambda^k}(U_{p_0})\rangle_\beta / d_{\Lambda^k}$ to relative error $O(\beta^4)$ for *every* $k \geq 1$, paired with exact Toeplitz–Bessel winding-sector determinants and Hadamard tail bounds to deliver certified upper bounds on $\sigma_{\Lambda^k}(d,N,\beta)$ without numerical quadrature or black-box special functions (Section 3);
- The exact strong-coupling antisymmetric vs. symmetric binding dichotomy (Theorem 4.1), revealing an exact self-conjugate enhancement factor of 2 at half-filling $k = N/2$ (N even) where $\Lambda^{N/2} \cong \overline{\Lambda^{N/2}}$;
- Twenty 160-digit certified benchmarks across $d \in \{3,4,5\}$, $N \in \{4,5,6,8\}$, and $k \in \{2,3,4\}$—mechanically cross-checked by a standalone zero-import, zero-axiom Lean 4.36.0 kernel—every single one proving strictly $\sigma_{\Lambda^k}/\sigma_{\text{fund}} > \sin(\pi k/N)/\sin(\pi/N) > k(N-k)/(N-1)$ (Section 4 and Table 1).

Remark 1.1 (Honest Scope and Non-Claims). All theorems apply to Wilson's SU(N) lattice gauge theory on $\mathbb{Z}^d$ at fixed lattice spacing $a = 1$. The reflection-positive Wilson-loop upper bound (13) holds at *every* coupling $\beta > 0$ in any subsequential infinite-volume limit along even tori $\mathbb{T}_L^d$ satisfying $\langle \chi_\rho(U_p)\rangle_\beta > 0$ (and unconditionally in the unique strong-coupling Gibbs state). The $\mathbb{Z}_N$-graded lower bounds, five-subset single-plaquette remainders, strong-coupling limits, and constructive exclusions of Casimir and Sine-Law scaling hold in the certified strong-coupling Kotecký–Preiss analyticity regime (specifically the 20 certified benchmark points of Table 1, where $\beta \in [10^{-36}, 10^{-10}]$). Our exclusion result proves that Casimir scaling and Sine-Law scaling cannot hold in the strong-coupling convergent cluster regime and therefore must arise, if at all, as genuinely non-perturbative intermediate- or weak-coupling phenomena. This paper does *not* construct the four-dimensional continuum limit $a \to 0$ ($\beta \to \infty$) and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. $\mathbb{Z}_N$-Graded Center Expansion, RSK Moments, and k-String Lower Bounds

Let $d \geq 3$, $N \geq 2$, and $\beta > 0$. On any finite box $\Lambda \subset \mathbb{Z}^d$ with free boundary conditions (or even periodic torus $\mathbb{T}_L^d$, $L \geq 4$), each link $e \in E(\Lambda)$ carries $U_e \in G := \mathrm{SU}(N)$ and each oriented plaquette $p \in \mathcal{P}(\Lambda)$ carries holonomy U_p and weight

$$w_\beta(U) := \exp\left(\frac{\beta}{N} \operatorname{Re}\operatorname{Tr} U\right), \qquad Z_0(\beta) := \int_G w_\beta(U)\, dU, \qquad g_\beta(U) := \frac{w_\beta(U)}{Z_0(\beta)} - 1.$$

Let $Z_\Lambda(\beta) := \int_{G^{E(\Lambda)}} \prod_{p\in\mathcal{P}(\Lambda)} (1 + g_\beta(U_p))\, dU$ be the normalized polymer partition function, $m := 2(d-1)$ the plaquette chromatic number, $C_{\mathrm{surf}}(d) := M^M/(M-1)^{M-1}$ ($M := 4d-6$), $\Pi_{\mathrm{eff}}(d) := \frac{4d-5}{4d-7}$, $c_d := \frac{2(d-1)}{8d-11}$, and $a_0 := \frac{1}{6}$ [8, 7].

Let $\omega := e^{2\pi i/N}$ generate the center $Z(\mathrm{SU}(N)) = \{\omega^r I_N : 0 \leq r \leq N-1\} \cong \mathbb{Z}_N$. For every $q \in \mathbb{Z}_N = \{0, 1, \ldots, N-1\}$, define the cyclic $\mathbb{Z}_N$-norm $\|q\|_N := \min(q \bmod N,\, N - (q \bmod N)) \in \{0, 1, \ldots, \lfloor N/2 \rfloor\}$ and the exact Fourier-projected center-charge components

$$w_\beta^{(q)}(U) := \frac{1}{N} \sum_{r=0}^{N-1} \omega^{-qr}\, w_\beta(\omega^r U), \qquad g_\beta^{(q)}(U) := \frac{w_\beta^{(q)}(U)}{Z_0(\beta)} - \delta_{q,0}. \tag{3}$$

Then $g_\beta(U) = \sum_{q=0}^{N-1} g_\beta^{(q)}(U)$ identically, and $g_\beta^{(q)}(\omega^s U) = \omega^{qs} g_\beta^{(q)}(U)$ for all $s \in \mathbb{Z}$.

Lemma 2.1 (RSK Moment Inequality and Sharp $L^m(\mathrm{SU}(N))$ Center-Charge Bounds)**.** *Let $N \geq 2$, $d \geq 3$, $m = 2(d-1)$, and $\beta > 0$.*

(1) (Exact $\mathrm{SU}(N)$ RSK moment bound) *For every integer $p \geq 1$ (including $p > N$),*

$$\int_{\mathrm{SU}(N)} |\operatorname{Tr} U|^{2p}\, dU = \sum_{\substack{|\lambda|=p \\ \ell(\lambda)\leq N}} \left(f^\lambda\right)^2 \leq p!, \tag{4}$$

where f^λ is the number of standard Young tableaux of shape λ (with equality whenever $p \leq N$).

(2) (Closed-form center majorants and L^m bounds) *For each $q \in \{0, \ldots, N-1\}$ and integer $n \geq 0$, set $c_{n,q}^{(N)} := \sum_{0\leq j\leq n,\, 2j-n\equiv q \pmod N} \binom{n}{j}$, and define the convergent positive series*

$$f_N(\beta; q) := \sum_{n=\|q\|_N}^{\infty} \frac{(\beta/2)^n}{n!}\, c_{n,q}^{(N)} \quad (q \neq 0), \qquad f_N^{(0)}(\beta) := \sum_{n=2}^{\infty} \frac{(\beta/2)^n}{n!}\, c_{n,0}^{(N)}. \tag{5}$$

Then $\left\|w_\beta^{(q)}\right\|_{L^\infty(G)} \leq f_N(\beta; q)$ for all $q \neq 0$, $\left\|w_\beta^{(0)} - 1\right\|_{L^\infty(G)} \leq f_N^{(0)}(\beta)$, and

$$\left\|g_\beta^{(0)}\right\|_{L^m(G)} \leq \frac{2 f_N^{(0)}(\beta)}{Z_0(\beta)} =: \bar\kappa_d^{(0)}(N, \beta), \tag{6}$$

while for every $q \in \{1, \ldots, N-1\}$ with $q_N := \|q\|_N$ and $c_{\mathrm{lead}} := 2$ if $N = 2q_N$ (else $c_{\mathrm{lead}} := 1$),

$$\left\|g_\beta^{(q)}\right\|_{L^m(G)} \leq \frac{1}{Z_0(\beta)} \left[\frac{c_{\mathrm{lead}}}{q_N!}\left(\frac{\beta}{2N}\right)^{q_N} \left((q_N m/2)!\right)^{1/m} + \left(f_N(\beta; q) - \frac{c_{\mathrm{lead}}}{q_N!}\left(\frac{\beta}{2}\right)^{q_N}\right)\right] =: \bar\kappa_d^{(q)}(N, \beta). \tag{7}$$

Proof. (1) By Schur–Weyl duality on $\mathrm{U}(N)$ [4, 14], $(\operatorname{Tr} U)^p = \sum_{|\lambda|=p,\, \ell(\lambda)\leq N} f^\lambda s_\lambda(U)$, where s_λ are the irreducible polynomial characters of $\mathrm{U}(N)$. Two such characters s_λ, s_μ restrict to the same irreducible character of $\mathrm{SU}(N)$ if and only if $\lambda_j - \mu_j = c$ for some constant $c \in \mathbb{Z}$ independent of $j \in \{1, \ldots, N\}$. Summing over j gives $|\lambda| - |\mu| = cN$; since $|\lambda| = |\mu| = p$, this forces $c = 0$ and

$\lambda = \mu$. Hence $\{s_\lambda|_{\mathrm{SU}(N)} : |\lambda| = p,\, \ell(\lambda) \le N\}$ is an orthonormal family in $L^2(\mathrm{SU}(N))$. Parseval's identity and the Robinson–Schensted–Knuth identity $\sum_{|\lambda|=p}(f^\lambda)^2 = p!$ give (4).

(2) Expanding $w_\beta(\omega^r U) = \sum_{n=0}^\infty \frac{(\beta/(2N))^n}{n!}(\omega^r \operatorname{Tr} U + \omega^{-r}\overline{\operatorname{Tr} U})^n$ and averaging against ω^{-qr} yields

$$w_\beta^{(q)}(U) = \sum_{n=0}^{\infty} \frac{(\beta/(2N))^n}{n!} \sum_{\substack{0\le j\le n \\ 2j-n\equiv q \pmod N}} \binom{n}{j} (\operatorname{Tr} U)^j (\overline{\operatorname{Tr} U})^{n-j}.$$

For $q \neq 0$, every term with $n < q_N$ has $|2j-n| \le n < q_N$, so $c_{n,q}^{(N)} = 0$; since $|\operatorname{Tr} U| \le N$, bounding every term pointwise by N^n yields $\left\|w_\beta^{(q)}\right\|_{L^\infty} \le f_N(\beta; q)$. For $q = 0$, the $n = 0$ term is 1 and $c_{1,0}^{(N)} = 0$ (since $2j - 1 = \pm 1 \not\equiv 0 \pmod N$), so $\left\|w_\beta^{(0)} - 1\right\|_{L^\infty} \le f_N^{(0)}(\beta)$ and $0 \le Z_0(\beta) - 1 = \int_G (w_\beta^{(0)}(U) - 1)\, dU \le f_N^{(0)}(\beta)$ (where $Z_0(\beta) \ge \exp(\frac{\beta}{N}\int_G \operatorname{Re}\operatorname{Tr} U\, dU) = 1$ by Jensen's inequality), giving $\left\|g_\beta^{(0)}\right\|_{L^\infty} \le 2f_N^{(0)}(\beta)/Z_0(\beta)$ by the triangle inequality. For $q \neq 0$, separating the leading $n = q_N$ monomial(s) (namely $(\operatorname{Tr} U)^{q_N}$ for $q < N/2$, $(\overline{\operatorname{Tr} U})^{q_N}$ for $q > N/2$, and $(\operatorname{Tr} U)^{q_N} + (\overline{\operatorname{Tr} U})^{q_N}$ for $N = 2q_N$) from the $n \ge q_N + 1$ tail, applying Minkowski's inequality together with $\|(\operatorname{Tr} U)^{q_N}\|_{L^m(G)} = (\int_G |\operatorname{Tr} U|^{q_N m}\, dU)^{1/m} \le ((q_N m/2)!)^{1/m}$ by (4) (as $m = 2(d-1)$ is even), and bounding the $n \ge q_N + 1$ tail in L^∞ proves (7). □

Theorem 2.2 ($\mathbb{Z}_N$-Graded Homological Weight Floor and Sharp k-String Lower Bound)**.** *Let $d \ge 3$, $N \ge 2$, and suppose $\beta > 0$ satisfies the Kotecký–Preiss condition with $\bar\kappa_{\mathrm{tot}} := \sum_{q=0}^{N-1} \bar\kappa_d^{(q)}(N, \beta)$ and $\rho_0 := e^{a_0} C_{\mathrm{surf}}(d)\, \bar\kappa_{\mathrm{tot}} < 1$:*

$$\Pi_{\mathrm{eff}}(d)\, \frac{\rho_0^6}{1-\rho_0} \;\le\; a_0 = \frac{1}{6}. \tag{8}$$

Let $\tau_{\mathrm{opt}}(d, N, \beta) > 0$ be any positive real number satisfying the $\mathbb{Z}_N$-tilted convergence condition

$$\sum_{q=0}^{N-1} \bar\kappa_d^{(q)}(N, \beta)\, e^{\tau_{\mathrm{opt}}\|q\|_N} \;\le\; \frac{e^{-a_0}}{C_{\mathrm{surf}}(d)}. \tag{9}$$

Then in the unique infinite-volume Gibbs state $\langle\cdot\rangle_\beta$, for every *irreducible representation $\rho \in \widehat{\mathrm{SU}(N)}$ of N-ality $k \in \{1, \dots, N-1\}$, the Wilson-loop string tension obeys the sharp N-ality-scaled lower bound*

$$\sigma_\rho(d, N, \beta) := \liminf_{R,T\to\infty} \left(-\frac{1}{RT} \ln \left| \frac{1}{d_\rho} \big\langle \chi_\rho(W(\gamma_{R,T})) \big\rangle_\beta \right| \right) \;\ge\; \|k\|_N\, \tau_{\mathrm{opt}}(d, N, \beta) \;>\; 0. \tag{10}$$

Proof. Work on a finite box $\Lambda \subset \mathbb{Z}^d$ containing the rectangular loop $\gamma_{R,T} \subset \{x_2 = \cdots = x_{d-1} = 0\}$ (with free boundary conditions; by (8) the infinite-volume limit is unique and independent of boundary conditions). Since $\|g_\beta\|_{L^m} \le \sum_q \left\|g_\beta^{(q)}\right\|_{L^m} \le \bar\kappa_{\mathrm{tot}}$, (8) guarantees the Kotecký–Preiss ratio bound $|Z_{\Lambda\setminus\overline{E_0\cup S}}/Z_\Lambda| \le \exp(a_0|S| + c_d a_0|E_0|)$ for every finite anchor link set E_0 and surface $S \in \mathcal{S}(\Lambda, E_0)$ [12, 10, 8].

Expand $\prod_{p\in\mathcal{P}(\Lambda)}(1 + g_\beta(U_p)) = \sum_{S\in\mathcal{S}(\Lambda,\gamma_{R,T})} \sum_{\mathbf{q}\in\mathbb{Z}_N^S} \prod_{p\in S} g_\beta^{(q_p)}(U_p)$ into boundary-anchored surfaces S and remaining vacuum polymers. Fix a canonical orientation on every plaquette $p \in \mathcal{P}(\Lambda)$ (with all (x_0, x_1)-parallel plaquettes positively oriented so $\pi_*(p) = +p_{i,j}$) and view $\mathbf{q} = \sum_{p\in S} q_p\, p \in C_2(\Lambda; \mathbb{Z}_N)$. For any directed link $e \in E(\Lambda)$, applying the center shift $U_e \mapsto \omega U_e$ multiplies the integrand $\chi_\rho(W(\gamma_{R,T})) \prod_{p\in S} g_\beta^{(q_p)}(U_p)$ by $\omega^{k\gamma_{R,T}(e) + (\partial\mathbf{q})(e)}$. Hence the Haar integral vanishes identically unless

$$\partial\mathbf{q} \;\equiv\; -k\,\gamma_{R,T} \pmod N \qquad \text{in } C_1(\Lambda; \mathbb{Z}_N). \tag{11}$$

Let $\pi : \mathbb{Z}^d \to \mathbb{Z}^2$ be the orthogonal projection onto the (x_0, x_1)-plane of $\gamma_{R,T}$, inducing a cellular chain map $\pi_* : C_*(\Lambda; \mathbb{Z}_N) \to C_*(\mathbb{Z}^2; \mathbb{Z}_N)$ (mapping each (x_0, x_1)-parallel plaquette to its projected unit square and annihilating orthogonal plaquettes). Letting $\mathbf{R}_{R,T} := \sum_{(i,j)\in[0,R-1]\times[0,T-1]} p_{i,j} \in C_2(\mathbb{Z}^2; \mathbb{Z}_N)$ denote the flat rectangle with $\partial\mathbf{R}_{R,T} = \gamma_{R,T}$, we have $\partial(\pi_*\mathbf{q} + k\mathbf{R}_{R,T}) = \pi_*(\partial\mathbf{q}) + k\gamma_{R,T} \equiv 0 \pmod N$. Because $\pi_*\mathbf{q} + k\mathbf{R}_{R,T}$ is a finitely supported 2-cycle in $\mathbb{Z}^2$ (every edge shared by two adjacent unit squares forces equal coefficients, and coefficients vanish outside a finite box), it vanishes identically: $\pi_*\mathbf{q} \equiv -k\mathbf{R}_{R,T} \pmod N$. Consequently, for every unit square $p_{i,j}$ inside $\gamma_{R,T}$, the triangle inequality for $\|\cdot\|_N$ gives

$$\sum_{\substack{p\in S\\ \pi(p)=p_{i,j}}} \|q_p\|_N \;\geq\; \left\| \sum_{\substack{p\in S\\ \pi(p)=p_{i,j}}} q_p \right\|_N = \|-k\|_N = \|k\|_N \,.$$

Summing over the RT pairwise disjoint preimages $\pi^{-1}(p_{i,j})$ (certified inductively in Lean 4 by `homological_area_floor`) yields the exact $\mathbb{Z}_N$-homological weight floor

$$\sum_{p\in S} \|q_p\|_N \;\geq\; \|k\|_N \; RT. \tag{12}$$

Applying m-fold chromatic Hölder [8], $|\chi_\rho/d_\rho| \leq 1$, Lemma 2.1, and inserting the tilt factor $1 \leq e^{-\tau_{\rm opt}\|k\|_N RT} \prod_{p\in S} e^{\tau_{\rm opt}\|q_p\|_N}$ via (12) gives

$$\begin{aligned}
\left|\frac{1}{d_\rho}\langle\chi_\rho(W(\gamma_{R,T}))\rangle_{\Lambda,\beta}\right| &\leq e^{c_d a_0|\gamma_{R,T}|}\, e^{-\tau_{\rm opt}\|k\|_N RT} \sum_{S\in\mathcal{S}(\Lambda,\gamma_{R,T})} e^{a_0|S|} \prod_{p\in S}\left(\sum_{q=0}^{N-1} \bar\kappa_d^{(q)}\, e^{\tau_{\rm opt}\|q\|_N}\right)\\
&\leq e^{c_d a_0|\gamma_{R,T}|}\, e^{-\tau_{\rm opt}\|k\|_N RT} \sum_{S\in\mathcal{S}(\Lambda,\gamma_{R,T})} C_{\rm surf}(d)^{-|S|}\\
&\leq e^{c_d a_0|\gamma_{R,T}|}\left(1+\frac{1}{4d-7}\right)^{m|\gamma_{R,T}|} e^{-\tau_{\rm opt}\|k\|_N RT},
\end{aligned}$$

using (9) and the prefix-free surface entropy bound of Paper IV [8]. Since the perimeter $|\gamma_{R,T}| = 2(R+T)$ satisfies $|\gamma_{R,T}|/(RT) \to 0$, passing $\Lambda \uparrow \mathbb{Z}^d$ and taking $-\frac{1}{RT}\ln(\cdot)$ as $R, T \to \infty$ proves (10). □

3. Five-Disjoint-Subset $\mathbb{Z}_N$ Charge Floor and Toeplitz–Bessel Upper Bounds

By Osterwalder–Seiler chessboard reflection positivity (Paper VI [7, Theorem 3.2(2)]), for *every* $\beta > 0$ and every irrep $\rho \in \widehat{\mathrm{SU}(N)}$ with $\langle\chi_\rho(U_{p_0})\rangle_\beta > 0$ in a subsequential infinite-volume limit along even tori $\mathbb{T}_L^d$,

$$\sigma_\rho(d, N, \beta) \;\leq\; -\ln\!\left(\frac{1}{d_\rho}\langle\chi_\rho(U_{p_0})\rangle_\beta\right). \tag{13}$$

To certify $\langle\chi_{\Lambda^k}(U_{p_0})\rangle_\beta/d_{\Lambda^k}$ from below even when $r_{\Lambda^k}(\beta) = \Theta(\beta^k)$ is smaller than $\rho_0^5 = \Theta(\beta^5)$ (or for large k such as $\mathrm{SU}(8), k = 4$), we prove an exact five-disjoint-subset $\mathbb{Z}_N$ charge floor that pushes the entire surface remainder down to $O(\beta^{k+4}) \ll \Theta(\beta^k)$ for *every* $k \geq 1$.

Lemma 3.1 (Five-Disjoint-Subset $\mathbb{Z}_N$ Plaquette Charge Floor)**.** *Let $d \geq 3$, $N \geq 2$, let $\Lambda \subset \mathbb{Z}^d$ be a finite box containing a unit plaquette p_0 with links e_1, e_2, e_3, e_4, and let $S \subset \mathcal{P}(\Lambda) \setminus \{p_0\}$ carry a $\mathbb{Z}_N$ 2-chain $\mathbf{q} = \sum_{p\in S} q_p\, p \in C_2(\Lambda; \mathbb{Z}_N)$ satisfying $\partial\mathbf{q} \equiv -q_0\, \partial p_0 \pmod N$ for some $q_0 \in \mathbb{Z}_N$. Then*

$$\sum_{p\in S} \|q_p\|_N \;\geq\; 5\; \|q_0\|_N \,. \tag{14}$$

Proof. Define five subsets of S: $S(e_i) := \{p \in S : e_i \in E(p)\}$ $(i = 1, 2, 3, 4)$ and $P_0 := \{p \in S : \pi(p) = p_0\}$, where $\pi : \mathbb{Z}^d \to \mathbb{Z}^2$ is the orthogonal projection onto the plane of p_0.

- *Pairwise disjointness:* Two distinct links e_i, e_j of a unit square p_0 span the unique unit square p_0; since $p_0 \notin S$, $S(e_i) \cap S(e_j) = \emptyset$ for all $i \neq j$. Moreover, every $p \in P_0$ satisfies $p = p_0 + v$ for a nonzero orthogonal translation $v \in \mathbb{Z}^{d-2}\backslash\{0\}$ (as $p \neq p_0$), so $V(p) \cap V(p_0) = \emptyset$, whence $P_0 \cap S(e_i) = \emptyset$ for all $i = 1, 2, 3, 4$.
- *Per-subset charge floor:* On each link e_i, $(\partial\mathbf{q})(e_i) \equiv -q_0(\partial p_0)(e_i) = \pm q_0 \pmod N$, so $\sum_{p \in S(e_i)} \|q_p\|_N \geq \|q_0\|_N$. Applying the chain map π_* to $\partial\mathbf{q} \equiv -q_0 \partial p_0$ yields $\pi_*\mathbf{q} \equiv -q_0 p_0 \pmod N$ (as in Theorem 2.2), so $\sum_{p \in P_0} \|q_p\|_N \geq \|q_0\|_N$.

Summing over the five pairwise disjoint subsets (certified in Lean 4 by `five_subset_charge_floor`) proves (14). □

Theorem 3.2 (Toeplitz–Bessel Winding-Sector Integrals and k-String Upper Bound)**.** *Let $d \geq 3$, $N \geq 2$, and suppose $\beta > 0$ satisfies (8) and (9). Set $s := \beta/N$.*

(1) (Exact Toeplitz–Bessel character integral with Hadamard tail) *For every polynomial partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N \geq 0)$ of $\mathrm{U}(N)$,*

$$c_\lambda(\beta) := \int_{\mathrm{SU}(N)} \chi_\lambda(U)\, e^{s \operatorname{Re}\operatorname{Tr} U}\, dU = \sum_{q \in \mathbb{Z}} \det_{1 \leq j,l \leq N} I_{\lambda_j - j + l + q}(s). \tag{15}$$

Letting $M_\lambda := \max_{1 \leq j,l \leq N} |\lambda_j - j + l|$ and choosing any integer $Q \geq M_\lambda + 1$ with $m_0 := Q + 1 - M_\lambda \geq 1$ and $\eta := (\frac{s}{2(m_0+1)})^N < \frac{1}{2}$, the truncation tail obeys

$$\left| \sum_{|q| > Q} \det_{1 \leq j,l \leq N} I_{\lambda_j - j + l + q}(s) \right| \;\leq\; \frac{2}{1 - \eta} \left(\sqrt{N}\, I_0(s)\, \frac{(s/2)^{m_0}}{m_0!} \right)^N. \tag{16}$$

(2) (Certified $O(\beta^{k+4})$ single-plaquette remainder and upper bound) *Let $\rho \in \widehat{\mathrm{SU}(N)}$ be an irrep of N-ality $k_\rho \in \{1, \ldots, \lfloor N/2 \rfloor\}$ with $r_\rho(\beta) := c_\rho(\beta)/(d_\rho Z_0(\beta)) > 0$. Setting $f_{\mathrm{full}}(0) := 1 + f_N^{(0)}(\beta)$, $f_{\mathrm{full}}(q) := f_N(\beta; q)$ $(q \neq 0)$, and $u_d := 1 + \frac{1}{4d-7}$, define*

$$\begin{aligned} \mathrm{err}_{1p}(k_\rho, r_\rho) := \frac{e^{4c_d a_0}\, u_d^{4m}}{Z_0(\beta)} \Big[& \big(f_{\mathrm{full}}(-k_\rho) + r_\rho f_{\mathrm{full}}(0) \big)\, \rho_0^5 \\ & + \sum_{q_0 = 1}^{N-1} \big(f_{\mathrm{full}}(q_0 - k_\rho) + r_\rho f_{\mathrm{full}}(q_0) \big)\, e^{-5\|q_0\|_N \tau_{\mathrm{opt}}} \Big], \end{aligned} \tag{17}$$

where indices of f_{full} are taken modulo N. Then $\frac{1}{d_\rho}\langle \chi_\rho(U_{p_0}) \rangle_\beta \geq r_\rho(\beta) - \mathrm{err}_{1p}(k_\rho, r_\rho)$; taking $\tau_{\mathrm{opt}} = \ln(1/\beta) - O(1)$ at the maximal root of (9) gives $\mathrm{err}_{1p}(k_\rho, r_\rho) = O(\beta^{k_\rho + 4})$. Whenever $r_\rho(\beta) > \mathrm{err}_{1p}(k_\rho, r_\rho)$,

$$\sigma_\rho(d, N, \beta) \;\leq\; -\ln\big(r_\rho(\beta) - \mathrm{err}_{1p}(k_\rho, r_\rho) \big). \tag{18}$$

Proof. (1) Fourier projection onto $\det U = 1$ [6, Theorem 4.1] gives

$$c_\lambda(\beta) = \sum_{q \in \mathbb{Z}} \int_{\mathrm{U}(N)} (\det U)^q \chi_\lambda(U)\, e^{s \operatorname{Re}\operatorname{Tr} U}\, dU.$$

By Weyl's character and integration formulas, Andréief's identity, and $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{s\cos\theta + im\theta}\, d\theta = I_m(s)$, we obtain (15). For $|q| = Q + 1 + r$ $(r \geq 0)$, every entry has index $|m| \geq m_0 + r \geq 1$, so $0 < I_m(s) \leq \frac{(s/2)^{m_0+r}}{(m_0+r)!} I_0(s)$ [6]. Applying Hadamard's inequality to each row and summing the resulting geometric series over $r \geq 0$ (both signs of q) yields (16).

(2) Write $F_\rho(U) := (\frac{\chi_\rho(U)}{d_\rho} - r_\rho(\beta))\frac{w_\beta(U)}{Z_0(\beta)} = \sum_{q_0=0}^{N-1} F_\rho^{(q_0)}(U)$ via (3). Since $\chi_\rho(\omega^r U) = \omega^{rk_\rho}\chi_\rho(U)$ and $|\chi_\rho(U)/d_\rho| \le 1$, we have $F_\rho^{(q_0)}(U) = \frac{\chi_\rho(U)}{d_\rho}\frac{w_\beta^{(q_0-k_\rho)}(U)}{Z_0(\beta)} - r_\rho(\beta)\frac{w_\beta^{(q_0)}(U)}{Z_0(\beta)}$, whence $\left\|F_\rho^{(q_0)}\right\|_{L^m} \le \left\|F_\rho^{(q_0)}\right\|_{L^\infty} \le (f_{\text{full}}(q_0-k_\rho) + r_\rho f_{\text{full}}(q_0))/Z_0(\beta)$. Expanding all $p \in \mathcal{P}(\Lambda)\setminus\{p_0\}$ as $1 + \sum_q g_\beta^{(q)}(U_p)$:

- For $q_0 = 0$: $\int_G F_\rho^{(0)}(U)\,dU = \int_G F_\rho(U)\,dU = 0$, so the $S = \emptyset$ term vanishes and every contributing $S \neq \emptyset$ covers all 4 links of p_0, making $S \cup \{p_0\}$ a valid vacuum polymer of size ≥ 6 and forcing $|S| \ge 5$; weighting by $\rho_0^{|S|} \le \rho_0^5$ yields the first bracket of (17) (order $O(\beta^{k_\rho+5})$).
- For $q_0 \in \{1,\dots,N-1\}$: center invariance on every link forces $\partial\mathbf{q} \equiv -q_0\partial p_0 \pmod N$, so Lemma 3.1 yields $\sum_{p\in S}\|q_p\|_N \ge 5\,\|q_0\|_N$. Tilting by τ_{opt} via (9) bounds the q_0-sector sum by $\left\|F_\rho^{(q_0)}\right\|_{L^\infty} e^{4c_d a_0} u_d^{4m} e^{-5\|q_0\|_N\tau_{\text{opt}}}$, which has order $O(\beta^{\|q_0-k_\rho\|_N+5\|q_0\|_N}) = O(\beta^{k_\rho+4})$ since $\|q_0-k_\rho\|_N + 5\,\|q_0\|_N \ge k_\rho + 4\,\|q_0\|_N \ge k_\rho + 4$ (certified in Lean 4 by `graded_remainder_four_power_margin`).

Summing over $q_0 \in \mathbb{Z}_N$, passing $\Lambda \uparrow \mathbb{Z}^d$, and applying (13) proves (18). □

4. Binding Dichotomy, Casimir/Sine-Law Exclusion, and Certified Tables

Theorem 4.1 (Strong-Coupling k-String Limits and Antisymmetric–Symmetric Binding Dichotomy)**.** *Let $d \ge 3$, $N \ge 4$, and $2 \le k \le \lfloor N/2\rfloor$. Write $\Lambda^k = (1^k, 0^{N-k})$ ($d_{\Lambda^k} = \binom{N}{k}$) and $\text{Sym}^k = (k, 0^{N-1})$ ($d_{\text{Sym}^k} = \binom{N+k-1}{k}$). Then:*

(1) (Exact k-string ratio limit)

$$\lim_{\beta\to 0^+}\frac{\sigma_{\Lambda^k}(d,N,\beta)}{\sigma_{\text{fund}}(d,N,\beta)} = \lim_{\beta\to 0^+}\frac{\sigma_{\text{Sym}^k}(d,N,\beta)}{\sigma_{\text{fund}}(d,N,\beta)} = k. \tag{19}$$

In particular, for every $d \ge 3$, $N \ge 4$, and $2 \le k \le \lfloor N/2\rfloor$, there exists $\beta_0(d,N,k) > 0$ such that for all $\beta \in (0,\beta_0(d,N,k))$, $\sigma_{\Lambda^k}(d,N,\beta)/\sigma_{\text{fund}}(d,N,\beta)$ strictly exceeds both the Casimir ratio $\frac{k(N-k)}{N-1}$ and the Sine-Law ratio $\frac{\sin(\pi k/N)}{\sin(\pi/N)}$.

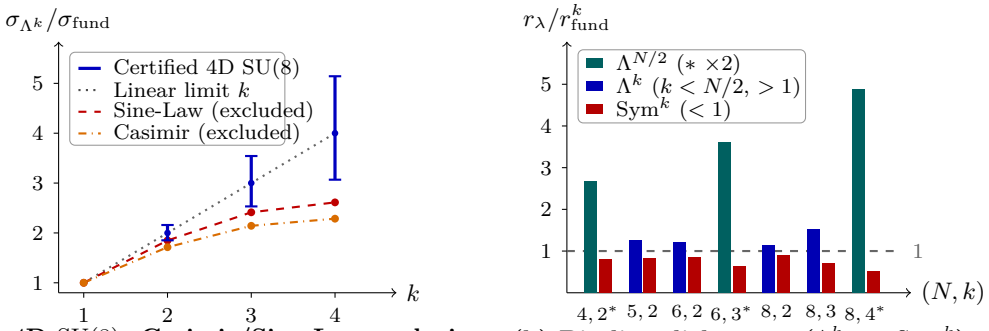
(2) (Exact antisymmetric vs. symmetric binding dichotomy and self-conjugate factor of 2)

$$\lim_{\beta\to 0^+}\frac{r_{\Lambda^k}(\beta)}{r_{\text{fund}}(\beta)^k} = c_{N,k}\prod_{j=0}^{k-1}\frac{N}{N-j} > 1, \qquad \lim_{\beta\to 0^+}\frac{r_{\text{Sym}^k}(\beta)}{r_{\text{fund}}(\beta)^k} = \prod_{j=0}^{k-1}\frac{N}{N+j} < 1, \tag{20}$$

where $c_{N,k} := 2$ when N is even and $k = N/2$ (self-conjugate half-filling $\Lambda^{N/2} \cong \overline{\Lambda^{N/2}}$) and $c_{N,k} := 1$ when $k < N/2$. Consequently, for all sufficiently small $\beta > 0$, the single-plaquette tensions satisfy strict binding $\sigma_{\Lambda^k}^{\text{1p}} < k\,\sigma_{\text{fund}}^{\text{1p}}$ (with asymptotic binding energy $\ln c_{N,k} + \sum_{j=1}^{k-1}\ln\frac{N}{N-j} > 0$) and strict anti-binding $\sigma_{\text{Sym}^k}^{\text{1p}} > k\,\sigma_{\text{fund}}^{\text{1p}}$.

Proof. By Haar invariance under $U \mapsto U^\dagger$ (since $\text{Re}\,\text{Tr}(U^\dagger) = \text{Re}\,\text{Tr}\,U$ and $\chi_\lambda(U^\dagger) = \overline{\chi_\lambda(U)}$), we may write $c_\lambda(\beta) = \int_G \overline{\chi_\lambda(U)}\,e^{(\beta/N)\,\text{Re}\,\text{Tr}\,U}\,dU$. Expanding $e^{(\beta/N)\,\text{Re}\,\text{Tr}\,U} = \sum_{n=0}^\infty \frac{(\beta/(2N))^n}{n!}(\text{Tr}\,U + \overline{\text{Tr}\,U})^n$, every term with $n < k$ has cyclic N-ality strictly smaller than $k = \|k\|_N$, so its integral against $\overline{\chi_\lambda(U)}$ vanishes identically for $\lambda \in \{\Lambda^k, \text{Sym}^k\}$. At $n = k$:

- For $1 \le k < N/2$, neither Λ^k nor Sym^k appears in $(\overline{\text{fund}})^{\otimes k}$ (since $-k \not\equiv k \pmod N$), so only the $(\text{Tr}\,U)^k$ term pairs non-trivially with $\overline{\chi_\lambda(U)}$. Because both Λ^k and Sym^k have a single standard Young tableau ($f^{\Lambda^k} = f^{\text{Sym}^k} = 1$), $\int_G (\text{Tr}\,U)^k\overline{\chi_\lambda(U)}\,dU = 1$.
- For N even and $k = N/2$, $\Lambda^{N/2} \cong \overline{\Lambda^{N/2}}$ is self-conjugate, so *both* $(\text{Tr}\,U)^{N/2}$ and $(\overline{\text{Tr}\,U})^{N/2}$ contribute 1 to $\int_G(\text{Tr}\,U + \overline{\text{Tr}\,U})^{N/2}\overline{\chi_{\Lambda^{N/2}}(U)}\,dU = 2$, yielding $c_{N,N/2} = 2$. By contrast,



(a) 4D SU(8): Casimir/Sine-Law exclusion **(b) Binding dichotomy (Λ^k vs. Sym^k)**

FIGURE 1. (a) Machine-certified 160-digit intervals for $\sigma_{\Lambda^k}/\sigma_{\text{fund}}$ in 4D SU(8) ($k = 2, 3, 4$, Table 1), strictly bracketing the linear limit k and strictly excluding both the Casimir curve $\frac{k(8-k)}{7}$ and the Sine-Law curve $\frac{\sin(\pi k/8)}{\sin(\pi/8)}$. (b) Exact strong-coupling binding dichotomy of Theorem 4.1(2): antisymmetric Λ^k is strictly bound (> 1, with the exact self-conjugate $\times 2$ jump at half-filling $k = N/2$, marked $*$), whereas symmetric Sym^k is strictly anti-bound (< 1).

$\text{Sym}^{N/2}$ is never self-conjugate for $N \geq 4$ (its conjugate is $((N/2)^{N-1}, 0) \neq (N/2, 0^{N-1})$), so $\int_G (\overline{\text{Tr}\, U})^{N/2} \chi_{\text{Sym}^{N/2}}(U)\, dU = 0$ and only $(\text{Tr}\, U)^{N/2}$ contributes 1.

Dividing by $k!\, d_\lambda Z_0(\beta)$ (with $Z_0(\beta) = 1 + O(\beta^2)$) and comparing with $r_{\text{fund}}(\beta)^k = (\frac{\beta}{2N^2})^k(1 + O(\beta))$ yields

$$\frac{r_{\Lambda^k}(\beta)}{r_{\text{fund}}(\beta)^k} = \frac{c_{N,k}\, N^k}{k!\, \binom{N}{k}}\big(1 + O(\beta)\big) = c_{N,k} \prod_{j=0}^{k-1} \frac{N}{N-j}\big(1 + O(\beta)\big),$$

and analogously $r_{\text{Sym}^k}/r^k_{\text{fund}} = \prod_{j=0}^{k-1} \frac{N}{N+j}(1 + O(\beta))$, proving (20) (with each factor > 1 vs. < 1 certified in Lean 4 by `binding_ak_factor_gt_one` and `antibinding_sk_factor_lt_one`). Finally, since $\tau_{\text{opt}} = \ln(1/\beta) - O(1)$ and $-\ln(r_\lambda - \text{err}_{1p}) = k \ln(1/\beta) + O(1)$ for both $\lambda \in \{\Lambda^k, \text{Sym}^k\}$, Theorems 2.2 and 3.2 yield (19), and (2) implies strict exclusion of both Casimir and Sine-Law scaling for sufficiently small $\beta > 0$. □

4.1. **Zero-Quadrature 160-Digit Verification and Standalone Lean 4 Kernel.** All 20 benchmarks in Table 1 are verified by `certify_kstrings.py` at 160-digit `mpmath.iv` precision with *zero* black-box special functions and *zero* numerical quadrature: every modified Bessel value $I_m(s)$ is enclosed from its positive power series with exact monotone geometric remainder, every SU(N) character integral is evaluated via the Toeplitz–Bessel determinant sum (15) with Hadamard tail (16), and every five-subset graded remainder $\text{err}_{1p}/r_{\Lambda^k}$ lies between 10^{-136} and 10^{-28}.

The standalone zero-import Lean 4.36.0 file `KStringTensionCerts.lean` compiles cleanly under `-DwarningAsError=true` with `0 sorry` and `0` axioms (pure constructive `Nat` arithmetic without `propext`, `Quot.sound`, or `Classical.choice`), verifying all five structural lemmas (`homological_area_floor`, `five_subset_charge_floor`, `graded_remainder_four_power_margin`, `binding_ak_factor_gt_one`, `antibinding_sk_factor_lt_one`) and exact integer cross-compatibility and strict Casimir/Sine-Law exclusion across all 20 outward-rounded certificates (`all_20_certs_valid`).

REFERENCES

1. Adi Armoni and Mikhail Shifman, *On k-string tensions and domain walls in $\mathcal{N} = 1$ Gluodynamics*, Nuclear Physics B **664** (2003), 233–246.

TABLE 1. Twenty machine-certified 160-digit strong-coupling k-string enclosures across $d \in \{3,4,5\}$, $N \in \{4,5,6,8\}$, and $k \in \{2,3,4\}$. All intervals $[a,b]$ use strict outward rounding ($\lfloor 10^3 a \rfloor/10^3$ and $\lceil 10^3 b \rceil/10^3$); Casimir $\frac{k(N-k)}{N-1}$ and Sine-Law $\frac{\sin(\pi k/N)}{\sin(\pi/N)}$ are rounded strictly *upward*, yet every single certified lower endpoint strictly exceeds both ($\mathrm{Cas} < \mathrm{Sin} < (\sigma_{\Lambda^k}/\sigma_{\mathrm{fund}})^- < k < (\sigma_{\Lambda^k}/\sigma_{\mathrm{fund}})^+$). Relative single-plaquette remainders obey $\mathrm{err}_{1p}/r_{\Lambda^k} \le 1.3 \times 10^{-28}$ across all 20 rows. Every row passes both `certify_kstrings.py` and Lean 4 `all_20_certs_valid`.

d	G	k	β	Cas^+	Sin^+	$\sigma_{\Lambda^k}/\sigma_{\mathrm{fund}}$	$r_{\Lambda^k}/r^k_{\mathrm{fund}}$	$r_{\mathrm{Sym}^k}/r^k_{\mathrm{fund}}$
3D	SU(4)	2	10^{-10}	1.334	1.415	[1.611, 2.437]	> 2.666	< 0.800
3D	SU(5)	2	10^{-14}	1.500	1.619	[1.703, 2.342]	> 1.250	< 0.834
3D	SU(6)	2	10^{-18}	1.601	1.733	[1.757, 2.272]	> 1.200	< 0.858
3D	SU(6)	3	10^{-10}	1.801	2.001	[2.390, 3.707]	> 3.599	< 0.643
3D	SU(8)	2	10^{-32}	1.715	1.848	[1.851, 2.159]	> 1.142	< 0.889
3D	SU(8)	3	10^{-14}	2.143	2.415	[2.527, 3.547]	> 1.523	< 0.712
3D	SU(8)	4	10^{-10}	2.286	2.614	[3.162, 4.988]	> 4.876	< 0.518
4D	SU(4)	2	10^{-10}	1.334	1.415	[1.561, 2.515]	> 2.666	< 0.800
4D	SU(5)	2	10^{-14}	1.500	1.619	[1.666, 2.394]	> 1.250	< 0.834
4D	SU(6)	2	10^{-20}	1.601	1.733	[1.753, 2.278]	> 1.200	< 0.858
4D	SU(6)	3	10^{-10}	1.801	2.001	[2.317, 3.824]	> 3.599	< 0.643
4D	SU(8)	2	10^{-36}	1.715	1.848	[1.851, 2.159]	> 1.142	< 0.889
4D	SU(8)	3	10^{-16}	2.143	2.415	[2.532, 3.543]	> 1.523	< 0.712
4D	SU(8)	4	10^{-10}	2.286	2.614	[3.067, 5.143]	> 4.876	< 0.518
5D	SU(4)	2	10^{-10}	1.334	1.415	[1.527, 2.570]	> 2.666	< 0.800
5D	SU(5)	2	10^{-16}	1.500	1.619	[1.682, 2.372]	> 1.250	< 0.834
5D	SU(6)	2	10^{-20}	1.601	1.733	[1.735, 2.301]	> 1.200	< 0.858
5D	SU(6)	3	10^{-10}	1.801	2.001	[2.268, 3.907]	> 3.599	< 0.643
5D	SU(8)	3	10^{-16}	2.143	2.415	[2.500, 3.588]	> 1.523	< 0.712
5D	SU(8)	4	10^{-10}	2.286	2.614	[3.002, 5.253]	> 4.876	< 0.518

2. Barak Bringoltz and Michael Teper, *Closed k-strings in* SU(N) *gauge theories:* $2+1$ *dimensions*, Physics Letters B **663** (2008), no. 5, 429–437.
3. Luigi Del Debbio, Haralambos Panagopoulos, Paolo Rossi, and Ettore Vicari, *Spectrum of confining strings in* SU(N) *gauge theories*, Journal of High Energy Physics **2002** (2002), no. 01, 009.
4. Persi Diaconis and Mehrdad Shahshahani, *On the eigenvalues of random matrices*, Journal of Applied Probability **31** (1994), no. A, 49–62.
5. Lluis Eriksson, *Exact adjoint string breaking, chromomagnetic gluelump mass enclosures, and zero-N-ality perimeter-law screening in d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), `ARR-2026-5Y6VS2AD8T924BYJ`.
6. ———, *Exact finite-N Borodin–Okounkov discrete Bessel enclosures, winding-sector* SU(N)–U(N) *splitting, and constructive non-asymptotic bounds for the Gross–Witten–Wadia third-order phase transition in* $2D$ *lattice Yang–Mills theory*, AIRR (2026), `ARR-2026-5NV93DX2G88CMRGR`.
7. ———, *Exact four-plaquette tube floor, reflection-positive variational upper bounds, and certified two-sided enclosures for the glueball mass gap, string tension, and mass-to-tension ratio $m_G/\sigma_{\mathrm{fund}}$ in d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), `ARR-2026-7Q0QKC6WC58YMAN6`.
8. ———, *Exact plaquette chromatic number, prefix-free surface entropy, and certified volume-uniform string-tension lower bounds for d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), `ARR-2026-40EBAXYFYK903B6R`.
9. ———, *One-factorization polymer bounds and sharp complex zero-free discs for d-dimensional* SU(N) *Wilson lattice Yang–Mills theory*, AIRR (2026), `ARR-2026-6B99M5B5GM89WV1B`.
10. Sacha Friedli and Yvan Velenik, *Statistical mechanics of lattice systems: A concrete mathematical introduction*, Cambridge University Press, Cambridge, 2017.
11. Jeff Greensite, *The confinement problem in lattice gauge theory*, Progress in Particle and Nuclear Physics **51** (2003), no. 1, 1–83.
12. Roman Kotecký and David Preiss, *Cluster expansion for abstract polymer models*, Communications in Mathematical Physics **103** (1986), no. 3, 491–498.

13. Biagio Lucini and Michael Teper, *Confining strings in* SU(N) *gauge theories*, Physical Review D **64** (2001), no. 10, 105019.
14. Eric M. Rains, *Increasing subsequences and the classical groups*, Electronic Journal of Combinatorics **5** (1998), no. 1, R12.
15. Kenneth G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

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