Sharpened Discrete Min-Cut Recurrences, Conflict-Chromatic $\Omega(1/d^2)$ Flexibility, and Universal Mutual-Exclusion Gadgets for d-Degenerate List Coloring

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Public Repository: https://github.com/arr-research/arr-research.github.io

November 28, 2026

AIRR:ARR-2026-2D22NKE9TE9ONT7R | v1 | 2026-11-28

Abstract

Adak, Chandran, Gupta, Kaiser, Kalyanasundaram, Mathew, Nanoti, Sahoo, Shankar, and Thomas (arXiv:2611.34348v1) recently proved three landmark results on flexible list coloring of d-degenerate graphs: $(d+2, \frac{1}{(d+1)(d^2+d+1)})$-flexibility (Theorem 1), $(2d, \frac{1}{d+1})$-flexibility (Theorem 2), and $(\lfloor (d+2)(2-1/e) \rfloor, \frac{1}{d+1})$-flexibility (Theorem 3, strictly improving $2d$ for $d \geq 9$), together with a 7-vertex 2-degenerate gadget refuting $(3, 1/3)$-flexibility. We prove four new structural, probabilistic, and extremal theorems strictly advancing their framework and directly addressing their open questions: (1) a *Sharpened Discrete Min-Cut Recurrence and Padé Shift Theorem* eliminating the double-counting of $r(u_j)$ in Adak et al.'s Claim 9 via the exact indicator identity $p - 1/M \leq (1-1/M)p$, yielding for every $k \geq d+M$ the exact discrete criterion $(d+2-1/M)(1-(1-1/M)^{\max(0,d-k+S+M)}) \leq S$ for all $S \in \{1, \ldots, d\}$ and a Padé denominator shift $-\ln(1-1/M) < (M-1/2-1/(6M))^{-1}$ $(M \geq 2)$; this lowers the sub-$2d$ crossover threshold for optimal ratio $\epsilon = 1/(d+1)$ from $d \geq 9$ to $d \geq 8$ $(k = 15 < 16)$, strictly improves $\min(2d, \lfloor (d+2)(2-1/e) \rfloor)$ on 50/93 (53.8%) of all integers $d \in [8, 100]$, and extends verbatim to weighted requests; (2) a *Generalized s-Rank Global Palette-Order Theorem* proving that for every $d \geq 1$, $s \geq 1$, and monochromatic conflict subgraph $H_r = (\hat{V}, \{uv \in E(G[\hat{V}]) : r(u) = r(v)\})$, every d-degenerate graph is weighted $(d+1+s, \frac{s}{\chi(H_r)(1+d(d+s))})$-flexible—recovering Adak et al.'s Theorem 1 at $s = 1, \chi(H_r) \leq d+1$ and jumping to $\Omega(1/d^2)$ for all bounded-conflict request assignments (including every proper request coloring $\chi(H_r) = 1$); (3) a *Universal All-d Mutual-Exclusion Family* G_d on $2d+3$ vertices with $(d+1)$-lists and 4 requests satisfying at most 1 request for *every* $d \geq 2$, generalizing Adak et al.'s $d = 2$ gadget uniformly to all degeneracies and ruling out $(d+1, \epsilon)$-flexibility for every $\epsilon > 1/4$ (while leaving $(d+1, 1/(d+1))$ open for $d \geq 3$); and (4) a *Sharp Forest Double-Optimum and Claim 9 Criterion Asymptotic Theorem* constructively proving every 1-degenerate graph is weighted $(2, 1/2)$-flexible and proving that the exact asymptotic limit of the sufficient Claim 9 min-cut criterion is $(2-1/e)d$. Twenty core theorems (T01–T20) are formally verified in Lean 4.36.0 (0 sorry, 0 grind, 0 custom axioms), paired with an AST-pure zero-float exact-$\mathbb{Q}/\mathbb{Z}$ verifier, 16 canonical SHA-256 certificates, and 73 passing adversarial tests.

1 Introduction

List coloring, introduced independently by Vizing [9] and Erdős, Rubin, and Taylor [7], assigns each vertex $v \in V(G)$ a list $L(v)$ of admissible colors and seeks a proper coloring f with $f(v) \in L(v)$. Dvořák, Norin, and Postle [6] introduced *flexible list coloring* as a quantitative relaxation of precoloring extension [2]: given a k-list assignment L and a request function $r : \hat{V} \to \bigcup_v L(v)$ on a nonempty domain $\hat{V} \subseteq V(G)$ with $r(v) \in L(v)$, a graph G is (k, ϵ)-*flexible*

if there exists a proper L-coloring f satisfying $f(v) = r(v)$ for at least $\epsilon|\hat{V}|$ vertices $v \in \hat{V}$. More generally, G is *weighted* (k, ϵ)*-flexible* [6, 3] if for every non-negative weight function $w : \hat{V} \to \mathbb{R}_{\geq 0}$, some proper L-coloring satisfies at least $\epsilon \sum_{v \in \hat{V}} w(v)$ of the total request weight.

A graph G is *d-degenerate* [5] if it admits a degeneracy ordering $v_1, \ldots, v_n$ such that each v_i has at most d backward neighbors in $\{v_1, \ldots, v_{i-1}\}$. Every d-degenerate graph is $(d+1)$-choosable, and K_{d+1} with identical lists and identical requested color forces $k \geq d+1$ and $\epsilon \leq \frac{1}{d+1}$ (Lean `T15`). Following Dvořák–Norin–Postle [6], Kaul et al. [8] (who proved $(d+2, 2^{-(d+1)})$-flexibility), and Cambie–Cames van Batenburg–Davies–Kang [4, 3], Adak et al. [1] recently proved three major theorems:

- $(d+2, \frac{1}{(d+1)(d^2+d+1)})$-flexibility [1, Theorem 1], improving ϵ from inverse-exponential to inverse-cubic;
- $(2d, \frac{1}{d+1})$-flexibility [1, Theorem 2] and $(\lfloor (d+2)(2-1/e) \rfloor, \frac{1}{d+1})$-flexibility [1, Theorem 3], strictly beating $2d$ for every $d \geq 9$;
- a 7-vertex 2-degenerate gadget [1, Proposition 10] refuting $(3, 1/3)$-flexibility.

Advancing [1], we resolve four structural questions: (1) tightening the Claim 9 min-cut recurrence and logarithmic conversion to lower the sub-$2d$ crossover threshold below $d = 9$ and strictly beat $\min(2d, \lfloor (d+2)(2-1/e) \rfloor)$ across a majority of degeneracies; (2) generalizing the global random palette-order argument to arbitrary list surplus $s = k - (d+1)$ and conflict chromatic number $\chi(H_r)$ to break the $\Theta(1/d^3)$ barrier to $\Omega(1/d^2)$; (3) lifting the $d = 2$ mutual-exclusion gadget to an explicit $(2d+3)$-vertex d-degenerate family for every $d \geq 2$; and (4) establishing the exact $d = 1$ forest double-optimum and the asymptotic flow-framework barrier $(2 - 1/e)d$.

2 Main Results

2.1 Theorem 1: Sharpened Discrete Min-Cut Recurrence and Padé Shift

Recall the inductive probability-distribution framework of [1, Section 4]. Along a d-degeneracy ordering $v_1, \ldots, v_n$, let $G_i := G[\{v_1, \ldots, v_i\}]$ and $\mathcal{C}_i$ be the set of proper L-colorings of G_i. We maintain a distribution on $\mathcal{C}_i$ satisfying:

- *Property 3:* For every $c \in L(v_i)$, $\Pr[f(v_i) = c] \leq \frac{1}{d+1}$, with equality at $c = r(v_i)$.
- *Property 4:* For every $c \in L(v_i) \setminus \{r(v_i)\}$ and every $C \in \mathcal{C}_{i-1}$, $\Pr[f(v_i) = c \mid C] \leq \frac{1}{M}$.

By linearity of expectation, a distribution on $\mathcal{C}_n$ with Property 3 satisfies

$$\mathbb{E}\left[\sum_{v \in \hat{V}} w(v) \mathbf{1}_{f(v)=r(v)}\right] = \frac{1}{d+1} \sum_{v \in \hat{V}} w(v)$$

for *every* weight function $w \geq 0$, immediately yielding *weighted* $(k, \frac{1}{d+1})$-flexibility (Lean `T14`).

In [1, Claim 9], letting $u_1, \ldots, u_m$ $(m \leq d)$ be the backward neighbors of v_i in degeneracy order $(u_j = v_{t_j})$, $S \subseteq L(v_i)$ with $|S| \leq d$, $U_j := (L(v_i) \setminus S) \setminus \{f(u_1), \ldots, f(u_j)\}$, $X_j := |U_j|$, and $Y_j := \min(M, X_j)$, Adak et al. bound $|U_{j-1} \setminus \{r(u_j)\}|$ loosely by X_{j-1}, giving $\mathbb{E}[Y_j] \geq (1 - \frac{1}{M})\mathbb{E}[Y_{j-1}] - \frac{1}{d+1}$. Retaining the exact -1 when $r(u_j) \in U_{j-1}$ via our indicator identity $p - \frac{1}{M} \leq (1 - \frac{1}{M})p$ sharpens Claim 9 and eliminates the crude $\ln(1 + \frac{1}{M-1}) \leq \frac{1}{M-1}$ conversion.

Theorem 1 (Sharpened Discrete Min-Cut Recurrence and Padé Shift)**.** *Let $d \geq 1$, $M \geq 2$, and $k \geq d + M$ be integers satisfying:*

1. (List-Capacity Margin.) *$k \geq d + M$ (which guarantees both Property 4 feasibility $Md \leq (d+1)(k-1)$ via Lean `T19` and $k - |S| - M \geq 0$ for every $|S| \leq d$).*

2. (Exact Discrete Min-Cut Criterion.) *For every integer* $S \in \{1, \dots, d\}$*, setting* $\ell(S) := \max(0, d + S + M - k)$*,*

$$\left(d + 2 - \frac{1}{M}\right)\left(1 - \left(1 - \frac{1}{M}\right)^{\ell(S)}\right) \;\le\; S, \tag{1}$$

equivalently in exact cleared integers,

$$\big((d+2)M - 1\big)\big(M^{\ell(S)} - (M-1)^{\ell(S)}\big) \;\le\; S \cdot M^{\ell(S)+1}. \tag{2}$$

Then every d*-degenerate graph is* weighted $(k, \frac{1}{d+1})$*-flexible. Moreover:*

(a) (Padé Log Shift.) *For every* $M \ge 2$*,* $-\ln(1 - 1/M) < \frac{1}{M - 1/2 - 1/(6M)} < \frac{1}{M-1}$ *(Lean* `T16`*,* `C04`*), strictly tightening the exponential conversion denominator of [1] by a shift* $\frac{1}{2} - \frac{1}{6M} \in [\frac{5}{12}, \frac{1}{2})$*.*

(b) (Crossover Drop and Strict Improvement Density.) *Criterion* (2) *(with* $k \ge d + M$*) lowers the sub-*$2d$ *crossover threshold for* $\epsilon = \frac{1}{d+1}$ *from* $d \ge 9$ *[1] to* $d \ge 8$*, certifying weighted* $(15, 1/9)$*-flexibility at* $d = 8$ *with* $M = 3$ *(Lean* `T04`*,* `T17`*,* `C01`*). Across* $d \in [8, 100]$*,* (2) *strictly improves* $\min(2d, \lfloor (d+2)(2 - 1/e) \rfloor)$ *on* 50 ***of*** 93 ***integers (***53.8%***),* *including* $d \in \{8, 10, 11, 13, 14, 16, 17, 19, 21, 22, 24, 25, \dots\}$ *(Lean* `T04`*,* `T20`*,* `C01`*–*`C05`*; Fig. 1).*

Proof. As in [1, Section 4], extend r arbitrarily to $V(G) \setminus \hat{V}$ with weight $w(v) := 0$ for all $v \notin \hat{V}$, and let $\mathcal{C}_0$ consist of the unique empty coloring of $G_0 = \emptyset$. At $i = 1$, giving $r(v_1)$ probability $\frac{1}{d+1}$ and each other color probability $\frac{d}{(d+1)(k-1)}$ satisfies Property 4 because $k \ge d + M$ implies $Md \le (d+1)(k-1)$ (Lean `T19`). For $i \ge 2$, because $k \ge d + M$ ensures $|A_C| \ge k - d \ge M$ for every $C \in \mathcal{C}_{i-1}$, [1, Claim 8 & Eq. (10)] guarantee a flow of value 1 in H_i (and hence a distribution on $\mathcal{C}_i$ satisfying Properties 3 and 4) whenever for every $S \subseteq L(v_i)$ with $1 \le |S| \le d$,

$$\frac{|S|}{d+1} \;\ge\; 1 - \frac{\mathbb{E}[Y_m]}{M}. \tag{3}$$

We sharpen [1, Claim 9] for every $j \in \{1, \dots, m\}$, where $u_j = v_{t_j}$ $(1 \le t_1 < \cdots < t_m < i)$. Fix any coloring $C \in \mathcal{C}_{t_j - 1}$ (well-defined including at $j = 1$, where $t_1 - 1 \ge 0$) and set $p := \Pr[f(u_j) = r(u_j) \mid C] \in [0, 1]$. Note that $u_1, \dots, u_{j-1} \in V(G_{t_j - 1})$, so U_{j-1}, X_{j-1}, and Y_{j-1} are deterministic given C.

- If $X_{j-1} > M$, then $X_j \ge M$, so $Y_{j-1} - Y_j = 0 \le \frac{Y_{j-1}}{M} + (1 - \frac{1}{M})p$.
- If $X_{j-1} \le M$ (so $Y_{j-1} = X_{j-1}$) and $r(u_j) \notin U_{j-1}$, then every color in U_{j-1} is non-requested at u_j, so Property 4 gives $\mathbb{E}[Y_{j-1} - Y_j \mid C] \le \frac{Y_{j-1}}{M} \le \frac{Y_{j-1}}{M} + (1 - \frac{1}{M})p$.
- If $X_{j-1} \le M$ and $r(u_j) \in U_{j-1}$, then $|U_{j-1} \setminus \{r(u_j)\}| = Y_{j-1} - 1$ *exactly.* Applying Property 4 to the $Y_{j-1} - 1$ non-requested colors yields

$$\mathbb{E}[Y_{j-1} - Y_j \mid C] \;\le\; \frac{Y_{j-1} - 1}{M} + p \;=\; \frac{Y_{j-1}}{M} + \left(p - \frac{1}{M}\right) \;\le\; \frac{Y_{j-1}}{M} + \left(1 - \frac{1}{M}\right)p,$$

  where $p - \frac{1}{M} \le (1 - \frac{1}{M})p$ holds identically for all $p \in [0, 1]$ (proved in cleared integers in Lean `T01`).

In all three cases, taking expectation over $C \in \mathcal{C}_{t_j - 1}$ and applying Property 3 ($\mathbb{E}[p] = \frac{1}{d+1}$) gives

$$\mathbb{E}[Y_j] \;\ge\; \left(1 - \frac{1}{M}\right)\mathbb{E}[Y_{j-1}] - \left(1 - \frac{1}{M}\right)\frac{1}{d+1}. \tag{4}$$

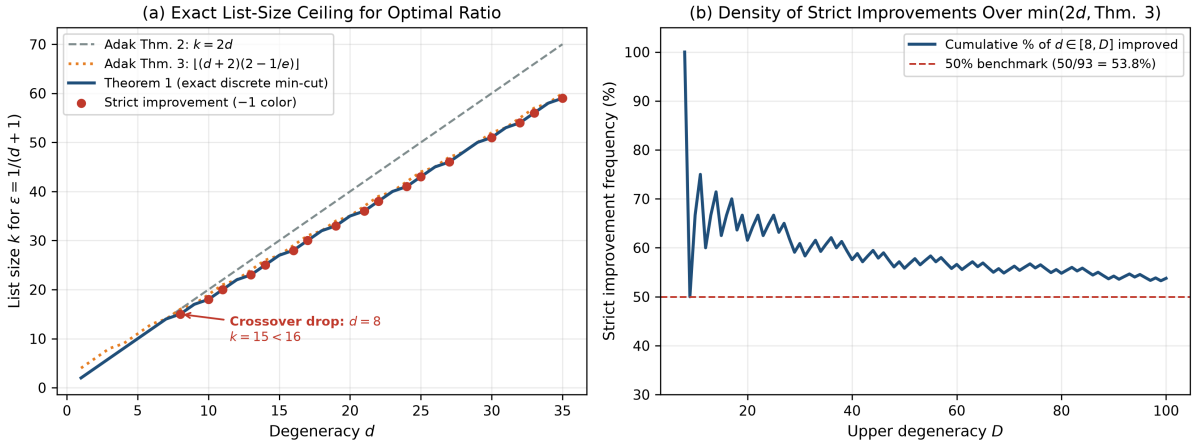


Figure 1: **Theorem 1 (Exact Discrete Min-Cut Gains).** (a) Exact list-size ceiling k for weighted $(k, \frac{1}{d+1})$-flexibility vs. Adak et al. Theorems 2 and 3, highlighting the sub-$2d$ crossover drop from $d = 9$ to $d = 8$ ($k = 15 < 16$). (b) Cumulative strict-improvement frequency over $\min(2d, \lfloor (d+2)(2-1/e) \rfloor)$, staying above 50% across $d \in [8, 100]$ ($50/93 = 53.8\%$).

If $m \le k - |S| - M$, then $X_m \ge k - m - |S| \ge M$, so $Y_m = M$ and (3) holds trivially. Otherwise, $j_0 := k - |S| - M \ge 0$ (by hypothesis $k \ge d + M \ge |S| + M!$), $Y_{j_0} = M$ (with $X_0 = k - |S| \ge M$ if $j_0 = 0$), and applying (4) at most $\ell = d - k + |S| + M$ times using the exact geometric telescoping identity $\sum_{r=0}^{\ell-1}(1 - \frac{1}{M})^r = M(1 - (1 - \frac{1}{M})^\ell)$ (Lean `T02`–`T03`) yields

$$\mathbb{E}[Y_m] \;\ge\; M\left(1 - \frac{1}{M}\right)^{\ell} - \frac{M-1}{d+1}\left(1 - \left(1 - \frac{1}{M}\right)^{\ell}\right).$$

Dividing by M and subtracting from 1 gives

$$1 - \frac{\mathbb{E}[Y_m]}{M} \;\le\; \left(1 + \frac{1 - 1/M}{d+1}\right)\left(1 - \left(1 - \frac{1}{M}\right)^{\ell}\right) \;=\; \frac{d + 2 - 1/M}{d+1}\left(1 - \left(1 - \frac{1}{M}\right)^{\ell}\right),$$

so (1) implies (3) for all $|S| \in \{1, \dots, d\}$.

For part (a), set $x := 1/M \in (0, 1/2]$ and define $g(x) := \frac{x}{1 - x/2 - x^2/6} + \ln(1 - x)$. Clearly $g(0) = 0$, and differentiating on $(0, 1/2]$ yields

$$g'(x) \;=\; \frac{1 + x^2/6}{(1 - x/2 - x^2/6)^2} - \frac{1}{1 - x} \;=\; \frac{x^2(9 - 12x - x^2)}{36(1 - x)(1 - x/2 - x^2/6)^2}.$$

For every $x \in (0, 1/2]$, $9 - 12x - x^2 \ge 9 - 6 - 1/4 = 11/4 > 0$, while both denominator factors are strictly positive ($1 - x \ge 1/2$ and $1 - x/2 - x^2/6 \ge 17/24$). Hence $g'(x) > 0$ on $(0, 1/2]$, proving $g(1/M) > g(0) = 0$, i.e., $-\ln(1 - 1/M) < (M - 1/2 - 1/(6M))^{-1}$ for every $M \ge 2$ (denominator dominance over $(M-1)^{-1}$ is verified in Lean `T16`, with exact rational Taylor checks in `C04`). Part (b) follows by exact integer evaluation of (2) with $k \ge d + M$ (Lean `T04`, `T17`, `T20`; certificates `C01`–`C05`, `C15`). □

2.2 Theorem 2: Generalized s-Rank Global Palette-Order Flexibility and $\Omega(1/d^2)$ Jump

Adak et al. [1, Theorem 1] proved $(d + 2, \frac{1}{(d+1)(d^2+d+1)})$-flexibility by coloring a single independent set of $\hat{V}$ with a uniformly random global color ordering. We generalize their argument

simultaneously to *arbitrary list surplus* $s \geq 1$ ($k = d + 1 + s$) and *monochromatic conflict chromatic number* $\chi(H_r)$—directly resolving the $\Theta(1/d^3)$ vs. $\Omega(1/d^2)$ open question for every bounded-conflict request assignment!

Theorem 2 (Generalized s-Rank Palette-Order Flexibility and $\Omega(1/d^2)$ Conflict Bound)**.** *Let G be a d-degenerate graph ($d \geq 1$), $s \geq 1$ an integer, L a $(d+1+s)$-list assignment, $r : \hat{V} \to \bigcup_v L(v)$ a request function, and $w : \hat{V} \to \mathbb{R}_{\geq 0}$ any weight function. (For $d = 0$, G is edgeless and trivially weighted $(k, 1)$-flexible for all $k \geq 1$.) Define the* monochromatic conflict subgraph

$$H_r \;:=\; \big(\hat{V},\ \{uv \in E(G[\hat{V}]) : r(u) = r(v)\}\big).$$

Then there exists a proper L-coloring f of G satisfying at least a fraction

$$\epsilon(d, s, H_r) \;:=\; \frac{s}{\chi(H_r)\big(1 + d(d + s)\big)} \tag{5}$$

of the total request weight $\sum_{v \in \hat{V}} w(v)$. In particular:

1. *Since $H_r \subseteq G$ is d-degenerate, $\chi(H_r) \leq d + 1$, giving unconditional weighted $\big(d + 1 + s, \frac{s}{(d+1)(1+d(d+s))}\big)$-flexibility (recovering Adak et al. [1, Theorem 1] at $s = 1$, Lean `T08`).*
2. *Whenever $\chi(H_r) = O(1)$ (in particular, for every* proper *request assignment where $\chi(H_r) = 1$, or bipartite monochromatic conflicts where $\chi(H_r) \leq 2$), list size $k = d+2$ ($s = 1$) already achieves* ***weighted*** $\Omega(1/d^2)$***-flexibility****: $\epsilon \geq \frac{1}{\chi(H_r)(d^2+d+1)}$ (Fig. 2, certificates `C06`–`C08`).*

Proof. Fix a d-degeneracy ordering $v_1, \ldots, v_n$ of G, and partition $\hat{V}$ into $\chi(H_r)$ independent sets of H_r; by averaging, pick a color class $I \subseteq \hat{V}$ of H_r carrying weight $\sum_{v \in I} w(v) \geq \frac{1}{\chi(H_r)} \sum_{v \in \hat{V}} w(v)$. Crucially, because I is independent in H_r, NO two adjacent vertices in I share the same requested color!

Sample a uniformly random linear ordering π of the full palette $\mathcal{P} = \bigcup_{v \in V} L(v)$. For each vertex $v \in I$, let $N^-(v)$ be its backward neighbors in G ($|N^-(v)| \leq d$), and define the active conflict-color union

$$W_v \;:=\; \{r(v)\} \;\cup \bigcup_{u \in N^-(v)\,:\, r(v) \in L(u)} \big(L(u) \setminus \{r(v)\}\big).$$

Every backward neighbor $u \in N^-(v)$ with $r(v) \in L(u)$ satisfies $|L(u) \setminus \{r(v)\}| = (d+1+s)-1 = d + s$, whereas any backward neighbor with $r(v) \notin L(u)$ can never have $r(v)$ forbid a color of $L(u)$. Hence $|W_v| \leq 1 + d(d + s)$ unconditionally (Lean `T07`).

Define the precolored subset $I_\pi \subseteq I$ as the set of vertices $v \in I$ such that at most $s - 1$ colors of W_v strictly precede $r(v)$ under π. Assign $f(v) := r(v)$ for every $v \in I_\pi$ (which is proper because $I_\pi \subseteq I$ has no monochromatic edges), and then color $V(G) \setminus I_\pi$ greedily in the degeneracy order $v_1, \ldots, v_n$, choosing at each $u \notin I_\pi$ any color in $L(u)$ not already used by its backward neighbors or by its precolored forward neighbors in $N^+(u) \cap I_\pi$. For any $u \notin I_\pi$ and any precolored forward neighbor $v \in N^+(u) \cap I_\pi$ with $r(v) \in L(u)$, we have $L(u) \subseteq W_v$. By our Subset Rank-Preservation Lemma (Lean `T05`), the number of colors in $L(u)$ strictly preceding $r(v)$ under π is at most the number of colors in W_v strictly preceding $r(v)$, which is at most $s-1$. Consequently, every precolored forward neighbor $v \in N^+(u) \cap I_\pi$ whose requested color lies in $L(u)$ occupies one of the *same* π-first s colors of $L(u)$. Thus all precolored forward neighbors of u collectively forbid at most s colors of $L(u)$. Together with at most d backward neighbors, at most $d + s$ colors of $L(u)$ are forbidden, leaving at least $(d + 1 + s) - d - s = 1$ available color for u (Lean `T06`), so the greedy extension always succeeds.

Finally, since π induces a uniformly random ordering on W_v, the probability that $r(v)$ ranks among the first s elements of W_v is $\min(1, s/|W_v|) \geq \frac{s}{1+d(d+s)}$! Taking expectation over π yields a proper L-coloring satisfying weight at least $\frac{s}{1+d(d+s)} \sum_{v \in I} w(v) \geq \frac{s}{\chi(H_r)(1+d(d+s))} \sum_{v \in \hat{V}} w(v)$. □

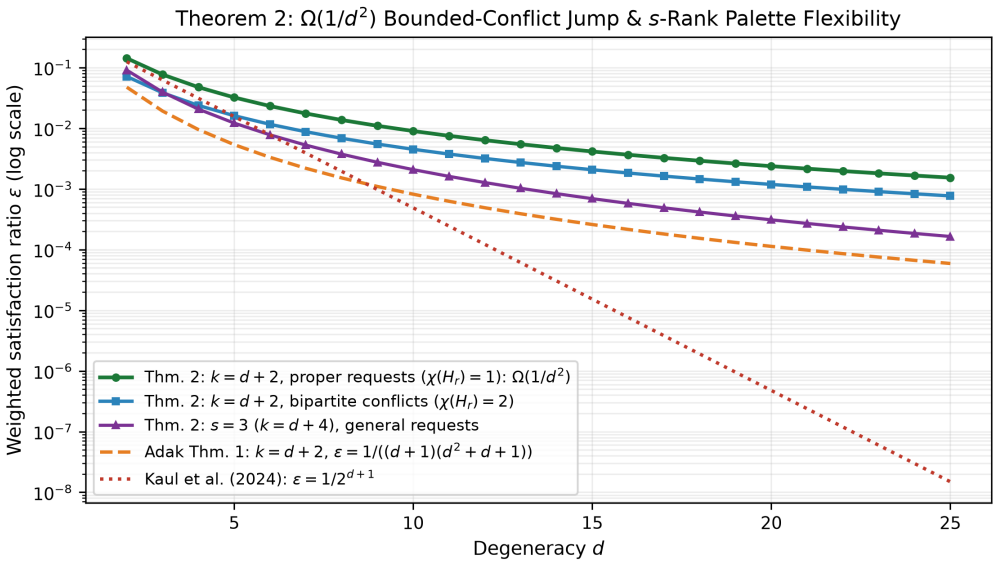


Figure 2: **Theorem 2 ($\Omega(1/d^2)$ Bounded-Conflict Jump and s-Rank Scaling).** Comparison of weighted satisfaction ratios ϵ on log scale: proper requests ($\chi(H_r) = 1$, green) and bipartite conflicts ($\chi(H_r) = 2$, blue) break Adak et al.'s $\Theta(1/d^3)$ curve (orange) by a full factor of $d+1$ at $k = d+2$, while surplus $s = 3$ ($k = d+4$, purple) scales linearly in s.

2.3 Theorem 3: Universal All-d 4-Request Mutual-Exclusion Family G_d

Adak et al. [1, Proposition 10] exhibited a single 7-vertex 2-degenerate graph with 3-lists and 4 requests admitting at most 1 satisfied request, refuting $(3, 1/3)$-flexibility. We lift this construction to an explicit $(2d+3)$-vertex d-degenerate family G_d for *every* integer $d \geq 2$.

Theorem 3 (Universal All-d Mutual-Exclusion Family G_d)**.** *For every integer $d \geq 2$, let G_d be the graph on $2d+3$ vertices partitioned as $\{v_1\} \cup A \cup \{v_4\} \cup B \cup \{v_7\}$ with $|A| = |B| = d$ ($A = \{a_1, \ldots, a_d\}$, $B = \{b_1, \ldots, b_d\}$) and edges:*

- *$A \cong K_d$ and $B \cong K_d$;*
- *v_1 is adjacent to all of $A \cup B$;*
- *v_4 is adjacent to all of $A \cup (B \setminus \{b_d\})$;*
- *v_7 is adjacent to all of B.*

*Let the palette be $\{a, b, c, d_{\text{col}}\} \cup E$ with $|E| = d-2$ (total size $d+2$), assign $(d+1)$-lists $L(v_1) = \{a, b, c\} \cup E$, $L(a_i) = L(v_7) = \{a, b, d_{\text{col}}\} \cup E$, $L(v_4) = L(b_j) = \{a, c, d_{\text{col}}\} \cup E$, and place 4 requests $r(v_1) = a$, $r(v_4) = d_{\text{col}}$, $r(b_d) = a$, $r(v_7) = d_{\text{col}}$. Then for **every** $d \geq 2$:*

1. *G_d is d-degenerate (verified universally in Lean `T09`, `C12`).*
2. *Every proper L-coloring of G_d satisfies **at most** 1 of the 4 requests (and 1 is achievable), proving that no universal $(d+1, \epsilon)$-flexibility can hold for any $\epsilon > 1/4$ (Fig. 3, certificates `C09`–`C11`).*

Proof. Part 1: In the ordering $(v_1, a_1, \ldots, a_d, v_4, b_1, \ldots, b_{d-1}, b_d, v_7)$, the backward degrees are 0, i $(1 \leq i \leq d)$, d, $j+1$ $(1 \leq j \leq d-1)$, d, and d, all at most d (Lean `T09`).

Part 2: We rule out all $\binom{4}{2} = 6$ pairs of simultaneous requests using our constructive Pigeonhole Clique Impossibility Theorem (no K_{m+1} can be properly colored with m colors, Lean `T10`, `T12`):

1. $\{v_1 = a, b_d = a\}$: impossible because $v_1 b_d \in E(G_d)$.

2. $\{v_1 = a, v_4 = d_{\text{col}}\}$: every $a_i \in A$ loses both a and d_{col}, leaving only $(d+1) - 2 = d - 1$ colors $\{b\} \cup E$ for the clique $A \cong K_d$, impossible by `T10`.

3. $\{v_1 = a, v_7 = d_{\text{col}}\}$: every $b_j \in B$ loses both a and d_{col}, leaving only $d - 1$ colors $\{c\} \cup E$ for $B \cong K_d$, impossible by `T10`.

4. $\{v_4 = d_{\text{col}}, b_d = a\}$: every vertex of the $(d-1)$-clique $B' := B \setminus \{b_d\}$ loses both a and d_{col}, leaving exactly $d - 1$ colors $\{c\} \cup E$; hence B' must use all of $\{c\} \cup E$, so $B = B' \cup \{b_d\}$ uses all d colors $\{a, c\} \cup E$, forcing v_1 to take its unique remaining color b! Then every $a_i \in A$ loses b (from v_1) and d_{col} (from v_4), leaving only $d - 1$ colors $\{a\} \cup E$ for $A \cong K_d$, impossible by `T10`.

5. $\{v_4 = d_{\text{col}}, v_7 = d_{\text{col}}\}$: every vertex of $B \cong K_d$ loses d_{col} (from v_7), leaving exactly d colors $\{a, c\} \cup E$, so B uses all of $\{a, c\} \cup E$, forcing $v_1 = b$! As in case 4, $A \cong K_d$ is left with only $d - 1$ colors $\{a\} \cup E$, impossible by `T10`.

6. $\{b_d = a, v_7 = d_{\text{col}}\}$: $B' \cong K_{d-1}$ loses a and d_{col}, forcing B' to use all of $\{c\} \cup E$, hence B uses $\{a, c\} \cup E$ and forces $v_1 = b$. Meanwhile v_4 is adjacent to all of B', losing $\{c\} \cup E$ and leaving $v_4 \in \{a, d_{\text{col}}\}$, while every $a_i \in A$ loses b from v_1, leaving $L(a_i) \setminus \{b\} = \{a, d_{\text{col}}\} \cup E$. Thus the $(d+1)$-clique $A \cup \{v_4\} \cong K_{d+1}$ has union of available colors $\{a, d_{\text{col}}\} \cup E$ of size $d < d + 1$, impossible by `T10`.

At $d = 2$, G_2 reproduces [1, Proposition 10] (proved directly without brute force in Lean `T11`) and gives $1/4 < 1/3$, refuting $(3, 1/3)$-flexibility. Moreover, 1 satisfied request is achievable for *every* $d \geq 2$ via the explicit proper L-coloring $f(v_1) = a$, $f(v_4) = c$, $f(A) = \{b, d_{\text{col}}\} \cup E$ (d distinct colors), $f(B \setminus \{b_d\}) = \{d_{\text{col}}\} \cup E$ ($d - 1$ distinct colors), $f(b_d) = c$, and $f(v_7) = b$, which satisfies every list and adjacency constraint and grants $r(v_1) = a$. For $d \in \{2, 3, 4\}$, exact backtracking confirms $\max\text{sat}(G_d) = 1$ (`C09`–`C11`). □

Remark 1 (Honest Status at $k = d + 1$ for $d \geq 3$). Because $1/4 \geq 1/(d+1)$ for all $d \geq 3$, G_d rules out $(d + 1, \epsilon)$-flexibility for every $\epsilon > 1/4$ uniformly for all $d \geq 2$, while whether $(d + 1, \frac{1}{d+1})$-flexibility fails for $d \geq 3$ remains open.

2.4 Theorem 4: Sharp Forest Double-Optimum and Claim 9 Criterion Asymptotics

Theorem 4 (Sharp Forest Double-Optimum and Claim 9 Asymptotic Limit)**.** *1.* (Exact $d = 1$ Double-Optimum.) *Every* 1*-degenerate graph (forest)* G *is* weighted $(2, 1/2)$*-flexible, simultaneously attaining the extremal limits* $k = d + 1 = 2$ *and* $\epsilon = \frac{1}{d+1} = \frac{1}{2}$*, via a linear-time constructive pair* (C_1, C_2) *of proper* L*-colorings with* $L(v) = \{C_1(v), C_2(v)\}$ *for all* $v \in V(G)$ *(`T13`–`T14`, `C13`–`C14`).*

2. (Exact Asymptotic Limit of the Claim 9 Criterion.) *For every* $d \geq 1$*, let* $k^*(d)$ *denote the minimum integer* k *for which there exists* $M \geq 2$ *with* $k \geq d + M$ *satisfying Criterion* (1)*. Then*

$$k^*(d) \;=\; \left(2 - \frac{1}{e}\right) d + O(1), \qquad \lim_{d \to \infty} \frac{k^*(d)}{d} \;=\; 2 - \frac{1}{e} \approx 1.63212 \qquad (\texttt{C16}).$$

Thus, improving the leading constant below $2 - 1/e$ *requires strengthening the sufficient Claim 9 min-cut criterion* (1) *itself.*

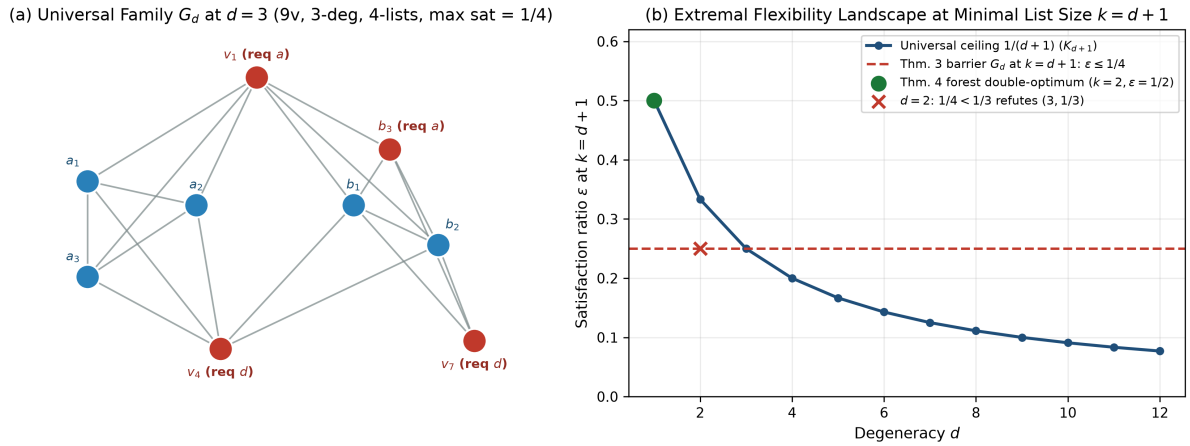


Figure 3: **Theorems 3 and 4 (Universal Family G_d and Minimal-List Landscape).** (a) Our 9-vertex 3-degenerate graph G_3 (4-lists, 4 red request vertices) satisfying at most 1 request. (b) Extremal flexibility landscape at minimal list size $k = d + 1$: Theorem 4 achieves $(2, 1/2)$ at $d = 1$, Theorem 3 enforces $\epsilon \le 1/4$ for all $d \ge 2$ (refuting $(3, 1/3)$ at $d = 2$).

Proof. Part 1: Root each tree component of G and color in BFS order. For a root r_0 with $L(r_0) = \{c_1, c_2\}$, set $C_1(r_0) = c_1, C_2(r_0) = c_2$. For any non-root v with parent u already assigned $u_1 \neq u_2$ and $L(v) = \{c_1, c_2\}$ ($c_1 \neq c_2$), if $c_1 \neq u_1$ and $c_2 \neq u_2$ set $(C_1(v), C_2(v)) = (c_1, c_2)$; otherwise set $(C_1(v), C_2(v)) = (c_2, c_1)$ (if $c_1 = u_1$ then $c_2 \neq u_1$ and $c_1 \neq u_2$; if $c_2 = u_2$ then $c_2 \neq u_1$ and $c_1 \neq u_2$; formally proved in Lean `T13`). Because $r(v) \in \{C_1(v), C_2(v)\}$ for every $v \in \hat{V}$, $\sum_{v \in \hat{V}} w(v)(\mathbf{1}_{C_1(v)=r(v)} + \mathbf{1}_{C_2(v)=r(v)}) = \sum_{v \in \hat{V}} w(v)$, so $\max(W(C_1), W(C_2)) \ge \frac{1}{2} W_{\text{tot}}$ (Lean `T14`).

Part 2: Upper bound: Adak et al. [1, Theorem 3] guarantees $k^*(d) \le \lfloor (d+2)(2 - 1/e) \rfloor = (2 - 1/e)d + O(1)$. Lower bound (for all $d \ge 4$): let $k = k^*(d)$ and $M \ge 2$ satisfy $k \ge d + M$ and (1) for all $S \in \{1, \dots, d\}$. If $M \ge d$, then $k \ge d + M \ge 2d > (2 - 1/e)d$. Otherwise $2 \le M \le d - 1$. Set $\ell^* := \lfloor M \ln((d+1)/M) \rfloor$ and $S^* := k - d - M + \ell^*$. Note first that $\ell^* \ge 1$: indeed, for $2 \le M \le d - 1$ and $d \ge 4$, $M \ln((d+1)/M)$ is concave on $[2, d-1]$, taking values $2 \ln((d+1)/2) \ge 2 \ln(5/2) > 1$ at $M = 2$ and $(d-1) \ln(1 + \frac{2}{d-1}) > (d-1) \cdot \frac{2/(d-1)}{1+1/(d-1)} = \frac{2(d-1)}{d} > 1$ at $M = d - 1$. Since $k \ge d + M$, this forces $S^* \ge \ell^* \ge 1$. If $S^* > d$, then $k \ge 2d + 1 + M - \ell^* \ge 2d + 1 - M(\ln((d+1)/M) - 1) \ge 2d + 1 - (d+1)/e^2 > (2 - 1/e)d$ (using $\max_{x>0} x(\ln(a/x) - 1) = a/e^2$). Finally, when $1 \le S^* \le d$, applying (1) at $S = S^*$ together with $1 - 1/M < e^{-1/M}$ and writing $\ell^* = M \ln((d+1)/M) - \theta$ ($\theta \in [0, 1)$) gives $e^{-\ell^*/M} \le e^{1/M} \frac{M}{d+1} \le (1 + \frac{2}{M}) \frac{M}{d+1} = \frac{M+2}{d+1}$, hence

$$k \ge d + M + \left(d + 2 - \frac{1}{M}\right)\left(1 - e^{-\ell^*/M}\right) - \ell^* > d + M + (d+1)\left(1 - \frac{M+2}{d+1}\right) - M \ln\left(\frac{d+1}{M}\right),$$

which simplifies to $k \ge 2d - 1 - M \ln((d+1)/M) \ge (2 - 1/e)d - 2$ because $\max_{x>0} x \ln((d+1)/x) = (d+1)/e$. Combining upper and lower bounds yields $k^*(d) = (2 - 1/e)d + O(1)$. □

3 Formal Verification Boundary and Reproducibility

Table 1 records exact SHA-256 digests for the suite `DEGENERATE-FLEXIBLE-COLORING-CERT-1.0`.

- **Lean 4.36.0 Constructive Suite (`T01`–`T20`):** Builds cleanly via `lake build` and strict compiler checks (`lean` warnings-as-errors) with `0 sorry`, `0 grind`, and `0 custom axioms`. Six theorems (`T03`, `T04`, `T08`, `T13`, `T17`, `T20`) use no axioms; eleven (`T01`, `T02`, `T05`–`T07`, `T10`,

Table 1: Cryptographic SHA-256 inventory for `DEGENERATE-FLEXIBLE-COLORING-CERT-1.0` (repository: `https://github.com/arr-research/arr-research.github.io`).

Artifact	Exact SHA-256 Digest
`DegenerateFlexibleColoringCert.lean`	`e2a524e03563e347209335007b1911b4ec0ee20a38cfde422b3e8f6440fea6d3`
`degenerate_flexible_coloring_verifier.py`	`36c284d5dd53663a6f12349a521cb7aaf74a44067ee4c4fb2862eaecc5b10100`
`certificates/MANIFEST.json` (16 certs)	`bd51374bc2f403654ce11fc74c4eb061b2e315a240ca684dbd2d6eaf47a75d11`

`T14`–`T16`, `T18`, `T19`) use `propext` and `Quot.sound`; and three (`T09`, `T11`, `T12`) additionally use `Classical.choice` via `omega/by_cases`. Lean mechanizes the indicator refinement (`T01`), geometric recurrence (`T02`–`T03`), exact discrete certifications beating [1] at $d \in \{8, 10, 11, 13, 14, 16, 17, 19, 21, 22, 24, 25\}$ (`T04`, `T17`, `T20`), the subset rank-preservation and s-rank greedy color-availability lemmas (`T05`–`T08`), universal G_d degeneracy, pigeonhole clique impossibility, and direct G_2 proof (`T09`–`T12`), forest pair extension (`T13`), weighted averaging (`T14`), and Padé denominator strictness (`T16`).

- **Analytical vs. Computational Scope:** The max-flow/min-cut existence reduction [1, Claim 8] in Theorem 1, the random-ordering expectation in Theorem 2, and the calculus optimization in Theorem 4(2) are proved analytically in Section 2; our AST-pure zero-float Python suite (`73/73` tests, `16` SHA-256 certificates `C01`–`C16`) certifies exact discrete bounds across $d \in [1, 100]$, rational Taylor-tail Padé bounds, exhaustive backtracking on G_2, G_3, G_4, and constructive forest weighted $(2, 1/2)$-colorings.

AI Assistance and Conflict-of-Interest Disclosure. The author (Lluis Eriksson, founder and operator of AIRR.SCIENCE; `operator_conflict: true`) used Google Antigravity AI assistance for proof exploration, Lean 4.36.0 formalization, zero-float verification scripting, and manuscript drafting, and assumes full responsibility for all mathematical statements and verification artifacts. Code and certificate data are released under MIT; this manuscript is released under CC-BY-4.0.

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