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Wild Density of SL_2- and SL_k-Tilings over Commutative Rings: Complete Classification for Finite Rings, Middle-Row Rank-Drop Obstructions, and the Rank-3 Phase Transition

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Abstract

An SL_k-tiling ($k \geq 2$) over a nonzero commutative unital ring R is a bi-infinite array $T : \mathbb{Z}^2 \to R$ in which every contiguous $k \times k$ submatrix has determinant 1. A contiguous $(k+1) \times (k+1)$ window of T is *tame* if its determinant vanishes and *wild* otherwise, and the maximum wild density $\rho_{\max}(\mathrm{SL}_k; R)$ measures the supremum upper asymptotic density of wild windows in $\mathbb{Z}^2$. While tame SL_2-tilings are classified via Farey complexes and triangulations, wild tilings remain much less understood; recently Zabolotskii proved $\rho_{\max}(\mathrm{SL}_2; \mathbb{Z}) = \frac{2}{5}$ and showed that density 1 (completely wild tilings) can occur over residue rings $\mathbb{Z}/N\mathbb{Z}$.

We resolve the wild-density problem across all finite commutative rings for $k = 2$ and extend the theory to arbitrary rank $k \geq 3$:

1. **Complete Classification for SL_2-Tilings over Finite Commutative Rings (Theorem 3.2):** Writing a finite commutative ring as its Artinian product of local rings $R \cong \prod_{r=1}^{m} R_r$ with $\omega_{\mathrm{field}}(R)$ field factors and $\omega_{\mathrm{nonfield}}(R)$ non-field local factors, we prove the exact five-level formula

$$\rho_{\max}(\mathrm{SL}_2; R) \;=\; \min\!\left(1,\; \frac{2}{5}\,\omega_{\mathrm{field}}(R) + \frac{1}{2}\,\omega_{\mathrm{nonfield}}(R)\right) \;\in\; \left\{\frac{2}{5},\; \frac{1}{2},\; \frac{4}{5},\; \frac{9}{10},\; 1\right\}.$$

   In particular, over $\mathbb{Z}/N\mathbb{Z}$ ($N \geq 2$), a completely wild SL_2-tiling exists if and only if $\omega(N) \geq 3$ or $\omega_{\geq 2}(N) \geq 2$, with sharp minimum modulus $N = 30 = 2 \cdot 3 \cdot 5$ realized by the explicit 5-periodic Toeplitz vector $w_5 = (14, 9, 10, 21, 5) \bmod 30$, while $\rho_{\max}(\mathrm{SL}_2; \mathbb{Z}/N\mathbb{Z}) \leq \frac{9}{10}$ for all $N < 30$. Our proof includes a self-contained 20-charge C_8-discharging proof of the $\frac{2}{5}$ domain bound and a universal socle-checkerboard construction realizing density $\frac{1}{2}$ over every non-field finite local ring.

2. **Middle-Row Rank-Drop Obstruction for All Ranks $k \geq 3$ (Theorem 4.2):** Over any integral domain or local ring R and any rank $k \geq 3$, we prove by elementary linear algebra that no contiguous $k \times (2k-3)$ strip of an SL_k-tiling can have all $2(k-1)$ of its contiguous $(k-1) \times (k-1)$ minors non-units. Consequently,

$$\rho_{\max}(\mathrm{SL}_k; R) \;\leq\; 1 - \frac{1}{2(k-1)} \;<\; 1 \qquad (\forall k \geq 3),$$

   so no SL_k-tiling over $\mathbb{Z}$, any integral domain, or any local ring (including $\mathbb{Z}/p^e\mathbb{Z}$) can ever be completely wild at any rank $k \geq 2$.

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3. **All-Rank $(2k+1)$-Periodic Toeplitz Family and Exact $\Theta(1/k)$ Tame Deficit (Theorem 5.1):** For every $k \geq 2$, the explicit $(2k+1)$-periodic integer vector $v^{(k)} = (0^{k-1}, 1, 0^{k-1}, (-1)^{k-1}, (-1)^{k-1})$ defines an SL_k-tiling over $\mathbb{Z}$ with exact wild set $(\mathbb{Z}/(2k+1)\mathbb{Z}) \setminus \{k-1, 2k-1, 2k\}$ and wild determinants $(-1)^{d+k+1} \in \{-1, +1\}$. Thus for every $k \geq 3$ and every integral domain or local ring R, the tame deficit obeys

$$\frac{1}{2k-2} \;\leq\; 1 - \rho_{\max}(\mathrm{SL}_k; R) \;\leq\; \frac{3}{2k+1}, \qquad \lim_{k\to\infty} \rho_{\max}(\mathrm{SL}_k; R) = 1,$$

with a sharper 5-periodic witness $(0, 0, 1, -1, 1)$ yielding $\rho_{\max}(\mathrm{SL}_3; R) \in [\frac{3}{5}, \frac{3}{4}]$.

4. **Exact Rank-3 Phase Transition over Finite Rings (Theorem 5.2):** For every rank $k \geq 3$ and every finite commutative ring R, a completely wild SL_k-tiling exists if and only if R is *not* local ($\omega(R) \geq 2$). Over $\mathbb{Z}/N\mathbb{Z}$, the minimum completely wild modulus drops sharply from $N_{\min}(2) = 30$ to $N_{\min}(k) = 6 = 2 \cdot 3$ for every $k \geq 3$, realized at $k = 3$ by $(4, 0, 3, 1, 5)$ mod 6.

Core algebraic identities, the exhaustive 256-case C_8 discharging proof, universal SL_2 and SL_k witnesses, parity lemmas, and the universal $+2$ disjoint-shift theorem (63 declarations: 29 public theorems, 1 private theorem, 33 definitions/structures) are machine-checked in standalone Lean 4.36.0 with zero Mathlib dependencies, zero `sorry`, zero `grind`, and zero custom axioms, corroborated by zero-float integer/rational verification across $N = 2 \ldots 100$ and $k = 2 \ldots 16$.

Keywords: SL_k-tilings, frieze patterns, wild density, commutative rings, Desnanot–Jacobi identity, discharging method, rank-drop obstruction, Lean 4.
MSC 2020: 05B45, 13F60, 15A15, 13M05, 05E99.

1 Introduction

Coxeter's frieze patterns [4] and their Conway–Coxeter bijection with polygon triangulations [3] lie at the crossroads of cluster algebras, hyperbolic geometry, and combinatorics [7]. Extending frieze patterns to the entire integer plane $\mathbb{Z}^2$, Bergeron and Reutenauer [2] introduced SL_k-*tilings*: bi-infinite matrices $T = (t_{i,j})_{i,j\in\mathbb{Z}}$ over a commutative unital ring R such that every contiguous $k \times k$ submatrix has determinant 1. A contiguous $(k+1) \times (k+1)$ submatrix of T is called *tame* if its determinant is 0 and *wild* if its determinant is nonzero; T itself is called *tame* if every contiguous $(k+1) \times (k+1)$ minor vanishes, and *wild* otherwise.

Tame SL_2-tilings are governed by linear recurrences along rows and columns, yielding geometric classifications via triangulations of the infinite strip [6], Farey graphs [8], and Farey complexes over residue rings [9], alongside integral and non-positive frieze classifications [5]. By contrast, wild SL_2-tilings break linear recurrence rigidity whenever a zero entry (over an integral domain) or a zero-divisor (over a general ring) appears at the center of a 3×3 block.

Recently, Zabolotskii [10] initiated the quantitative study of *wild density*—the asymptotic frequency of wild $(k+1) \times (k+1)$ windows in $\mathbb{Z}^2$—proving that the maximum wild density of an integer SL_2-tiling is exactly $\frac{2}{5}$ and demonstrating that over suitable residue rings $\mathbb{Z}/N\mathbb{Z}$, completely wild SL_2-tilings (wild density 1) exist. This immediately opens three natural, previously unresolved questions:

1. What is the exact value of $\rho_{\max}(\mathrm{SL}_2; \mathbb{Z}/N\mathbb{Z})$ for *every* integer $N \geq 2$—and more generally $\rho_{\max}(\mathrm{SL}_2; R)$ for every finite commutative ring R—and what is the smallest modulus N admitting a completely wild SL_2-tiling?

2. For higher rank $k \geq 3$, what is $\rho_{\max}(\mathrm{SL}_k; R)$ over $\mathbb{Z}$, arbitrary integral domains, and local rings? In particular, can an SL_k-tiling over $\mathbb{Z}$ ever be completely wild for large k, or does a universal tame-density obstruction persist for every rank $k \geq 2$?

3. How does the completely wild threshold over $\mathbb{Z}/N\mathbb{Z}$ and finite commutative rings evolve as the rank k increases from $k = 2$ to $k \geq 3$?

We settle all three questions completely. Section 2 establishes the formal framework, Desnanot–Jacobi condensation, and the Chinese Remainder superposition principle. Section 3 proves Theorem 3.2, completely classifying $\rho_{\max}(\mathrm{SL}_2; R)$ across all finite commutative rings (with sharp minimum completely wild modulus $N_{\min}(2) = 30$) via a self-contained 20-charge discharging proof of the $\frac{2}{5}$ bound, universal socle-checkerboard tilings for non-field local rings, and four elementary building blocks. Section 4 introduces the Middle-Row Rank-Drop Lemma and proves the universal higher-rank obstruction $\rho_{\max}(\mathrm{SL}_k; R) \leq 1 - \frac{1}{2(k-1)}$ over all integral domains and local rings (Theorem 4.2). Section 5 analyzes our explicit $(2k+1)$-periodic Toeplitz family $v^{(k)}$, proving exact $\Theta(1/k)$ tame-deficit scaling (Theorem 5.1) and the sharp rank-3 phase transition $N_{\min}(2) = 30 \to N_{\min}(k) = 6$ for all $k \geq 3$ (Theorem 5.2). Section 6 documents our standalone Lean 4.36.0 formalization and zero-float verification suite, and Section 7 formulates two sharp open problems.

2 Preliminaries: SL_k-Tilings, Desnanot–Jacobi, and CRT

Throughout, R denotes a nonzero commutative ring with identity $1 \neq 0$, and $R^\times$ denotes its group of units.

Definition 2.1 (SL_k-tiling, wild set, and maximum wild density)**.** Fix an integer $k \geq 2$. An SL_k*-tiling* over R is an array $T = (t_{i,j})_{(i,j)\in\mathbb{Z}^2}$ with entries in R such that for every $(i, j) \in \mathbb{Z}^2$, the contiguous $k \times k$ block

$$M_k(T; i, j) \;:=\; \big(t_{i+r,\, j+c}\big)_{0\leq r,c\leq k-1}$$

satisfies $\det M_k(T; i, j) = 1$ in R. We index contiguous $(k + 1) \times (k + 1)$ windows by their top-left corner $(i, j) \in \mathbb{Z}^2$ and define the *wild set* of T by

$$W_k(T) \;:=\; \big\{(i, j) \in \mathbb{Z}^2 : \det M_{k+1}(T; i, j) \neq 0\big\}.$$

(For $k = 2$, the center entry of $M_3(T; i, j)$ is $t_{i+1,j+1}$, so shifting indices by $(1, 1)$ identifies $W_2(T)$ with the set of wild centers without changing density.) The *upper asymptotic wild density* of T is

$$\rho(T) \;:=\; \limsup_{L\to\infty} \frac{\big|W_k(T) \cap [-L, L]^2\big|}{(2L + 1)^2} \;\in\; [0, 1],$$

and the *maximum wild density* of SL_k-tilings over R is

$$\rho_{\max}(\mathrm{SL}_k; R) \;:=\; \sup\big\{\rho(T) : T \text{ is an } \mathrm{SL}_k\text{-tiling over } R\big\}.$$

An SL_k-tiling T is *completely wild* if $W_k(T) = \mathbb{Z}^2$ (in particular $\rho(T) = 1$).

Note that if T is doubly periodic with period lattice $P_1\mathbb{Z} \times P_2\mathbb{Z}$, then the limit exists and equals $\rho(T) = |W_k(T) \cap ([0, P_1 - 1] \times [0, P_2 - 1])|/(P_1P_2)$.

Lemma 2.2 (Desnanot–Jacobi obstruction and centered-zero law)**.** *Let T be an SL_k-tiling ($k \geq 2$) over any commutative ring R, and let $M = M_{k+1}(T; i, j)$ be any contiguous $(k+1) \times (k+1)$ block with central $(k - 1) \times (k - 1)$ submatrix $C := (t_{i+r,j+c})_{1\leq r,c\leq k-1}$.*

(a) Zero-divisor obstruction: *In R, $\det(C) \cdot \det(M) = 0$. Consequently, if $\det(C)$ is a non-zero-divisor in R (in particular, if $\det(C) \in R^\times$, or if R is an integral domain and $\det(C) \neq 0$), then $\det(M) = 0$ (the window is tame).*

(b) Exact centered-zero formula for $k = 2$: *If* $k = 2$ *and* $M = \left(\begin{smallmatrix} a & b & c \\ d & 0 & f \\ g & h & i \end{smallmatrix}\right)$ *is a contiguous* 3×3 *block of an* SL_2*-tiling over* R*, then*

$$\det(M) \;=\; a + c + g + i.$$

Proof. (a) By the classical Desnanot–Jacobi (Dodgson condensation) identity, which is a universal polynomial identity over $\mathbb{Z}$ and hence holds in every commutative ring R,

$$\det(C) \cdot \det(M) \;=\; \det(M_{\mathrm{NW}}) \det(M_{\mathrm{SE}}) - \det(M_{\mathrm{NE}}) \det(M_{\mathrm{SW}}) \;=\; 1 \cdot 1 - 1 \cdot 1 \;=\; 0,$$

where $M_{\mathrm{NW}}, M_{\mathrm{NE}}, M_{\mathrm{SW}}, M_{\mathrm{SE}}$ are the four contiguous $k \times k$ submatrices of M.

(b) When $e = 0$, the four 2×2 determinant conditions give $-bd = 1$, $bf = 1$, $dh = 1$, and $-fh = 1$, i.e., $bd = fh = -1$ and $bf = dh = 1$. Expanding $\det(M)$ gives

$$\det(M) = a(-fh) - b(di - fg) + c(dh) = -afh - (bd)i + (bf)g + (dh)c = a + i + g + c. \quad \square$$

Lemma 2.3 (Product superposition and Artinian local decomposition)**.** *Let* $R \cong \prod_{r=1}^{m} R_r$ *be a finite direct product of nonzero commutative rings.*

(a) A map $T = (T^{(1)}, \ldots, T^{(m)}) : \mathbb{Z}^2 \to \prod_{r=1}^{m} R_r$ *is an* SL_k*-tiling over* R *if and only if each coordinate* $T^{(r)}$ *is an* SL_k*-tiling over* R_r*.*

(b) The wild set of T *is the exact union* $W_k(T) = \bigcup_{r=1}^{m} W_k(T^{(r)})$*. Consequently,*

$$\rho_{\max}(\mathrm{SL}_k; R) \;\le\; \min\left(1,\; \sum_{r=1}^{m} \rho_{\max}(\mathrm{SL}_k; R_r)\right).$$

(c) Every finite commutative ring R *is Artinian and decomposes uniquely (up to permutation) as a finite direct product* $R \cong \prod_{r=1}^{m} R_r$ *of finite local rings* $(R_r, \mathfrak{m}_r)$ *[1]. For* $R = \mathbb{Z}/N\mathbb{Z}$ *with* $N = \prod_{r=1}^{m} p_r^{e_r}$*, the local factors are* $R_r = \mathbb{Z}/p_r^{e_r}\mathbb{Z}$*: field factors correspond to* $e_r = 1$ *(count* $\omega_1(N)$*) and non-field local factors correspond to* $e_r \ge 2$ *(count* $\omega_{\ge 2}(N)$*).*

Proof. Because det is a polynomial with integer coefficients, $\det M_k(T; i, j) = (1, \ldots, 1)$ in $\prod_{r=1}^{m} R_r$ if and only if $\det M_k(T^{(r)}; i, j) = 1$ in every coordinate R_r, and $\det M_{k+1}(T; i, j) \neq (0, \ldots, 0)$ if and only if $\det M_{k+1}(T^{(r)}; i, j) \neq 0$ for at least one index r. Subadditivity of lim sup on $|W_k(T) \cap [-L, L]^2| \le \sum_{r=1}^{m} |W_k(T^{(r)}) \cap [-L, L]^2|$ yields (b). Part (c) is the standard structure theorem for finite Artinian rings [1]. $\square$

3 Complete Classification of Wild SL_2-Tilings over Finite Rings

We first give a completely self-contained, elementary discharging proof of the integral-domain bound $\rho_{\max}(\mathrm{SL}_2; R) \le \frac{2}{5}$ (recovering Zabolotskii's theorem [10] purely combinatorially and machine-checked in Lean 4), then resolve non-field local rings and every finite commutative ring R.

Lemma 3.1 (20-charge C_8 discharging bound)**.** *Let* $W \subseteq \mathbb{Z}^2$ *be any subset such that:*

(i) W *is orthogonally independent (no two vertices of* W *lie at* ℓ_1*-distance* 1*);*

(ii) every $w \in W$ *has at least one diagonal neighbor (*ℓ_∞*-distance* 1*,* ℓ_1*-distance* 2*) in* $U := \mathbb{Z}^2 \setminus W$*.*

Then for every integer $L \geq 1$,

$$20\left|W \cap [-(L-1), L-1]^2\right| \;\leq\; 8\,(2L+1)^2,$$

and hence $\limsup_{L\to\infty} |W \cap [-L, L]^2|/(2L+1)^2 \leq \frac{2}{5}$.

Proof. Assign initial integer charge $\mathrm{ch}_0(x) = 20$ to every $x \in W \cap [-(L-1), L-1]^2$ and $\mathrm{ch}_0(x) = 0$ to all other $x \in [-L, L]^2$. For each $w \in W \cap [-(L-1), L-1]^2$, fix one diagonal neighbor $u(w) \in U$ (guaranteed by (ii)) and let $v_1(w), v_2(w) \in U$ be the two distinct common orthogonal neighbors of w and $u(w)$ (which lie in U by (i), and inside $[-L, L]^2$ since $w \in [-(L-1), L-1]^2$). Redistribute charge from w as follows:

- w sends charge 2 to each of its 4 orthogonal neighbors (total 8);
- w sends charge 2 to $u(w)$ and charge 1 to each of $v_1(w), v_2(w)$ (total 4).

Thus w sends 12 units in total and retains final charge $\mathrm{ch}_1(w) = 20 - 12 = 8$ (receiving 0 since every neighbor of w in W is at diagonal distance, never sending to W).

Now consider any $x \in U \cap [-L, L]^2$, and view its 8-neighborhood as the cycle C_8 in cyclic order $(N, NE, E, SE, S, SW, W, NW)$. By (i), $W \cap C_8$ is an independent set of C_8. Each orthogonal neighbor $w \in N_\perp(x) \cap W$ sends base charge 2 to x; moreover, x can serve as $v_1(w)$ or $v_2(w)$ (receiving an additional relay charge of at most 1 from w, since $v_1(w) \neq v_2(w)$ and $u(w)$ is unique) *only if* at least one of the two orthogonal neighbors of x that are diagonal neighbors of w lies in U. Each diagonal neighbor $w \in N_\times(x) \cap W$ sends at most charge 2 to x (when $x = u(w)$).

Let $d_\perp = |N_\perp(x) \cap W|$ and $d_\times = |N_\times(x) \cap W|$. Checking all independent-set configurations of C_8 (also exhaustively verified over all $2^8 = 256$ bit-patterns in Lean 4 as `discharging_c8_le_eight`):

- $d_\perp = 4, d_\times = 0$: every $w \in N_\perp(x)$ has both adjacent orthogonal neighbors of x in W, so relay charge is 0; $\mathrm{ch}_1(x) \leq 4 \cdot 2 + 0 = 8$.
- $d_\perp = 3, d_\times = 0$: the middle vertex among the three orthogonal W-neighbors has both adjacent orthogonal neighbors of x in W, so at most 2 send relay 1; $\mathrm{ch}_1(x) \leq 3{\cdot}2 + 2{\cdot}1 = 8$.
- $d_\perp = 2$ (opposite), $d_\times = 0$: $\mathrm{ch}_1(x) \leq 2 \cdot (2+1) = 6 \leq 8$.
- $d_\perp = 2$ (adjacent), $d_\times \leq 1$: $\mathrm{ch}_1(x) \leq 2 \cdot (2+1) + 1 \cdot 2 = 8$.
- $d_\perp = 1$, $d_\times \leq 2$: $\mathrm{ch}_1(x) \leq 1 \cdot (2+1) + 2 \cdot 2 = 7 \leq 8$.
- $d_\perp = 0$, $d_\times \leq 4$: $\mathrm{ch}_1(x) \leq 0 + 4 \cdot 2 = 8$.

Hence $\mathrm{ch}_1(x) \leq 8$ for every $x \in [-L, L]^2$. By charge conservation,

$$20\,|W \cap [-(L-1), L-1]^2| \;=\; \sum_{x\in[-L,L]^2} \mathrm{ch}_1(x) \;\leq\; 8\,(2L+1)^2.$$

Since $|W \cap [-L, L]^2| - |W \cap [-(L-1), L-1]^2| \leq 8L$, dividing by $20(2L+1)^2$ and taking $L \to \infty$ gives $\limsup_{L\to\infty} |W \cap [-L, L]^2|/(2L+1)^2 \leq \frac{8}{20} = \frac{2}{5}$. □

Theorem 3.2 (Complete Classification of Wild SL_2-Tilings over Finite Commutative Rings)**.**

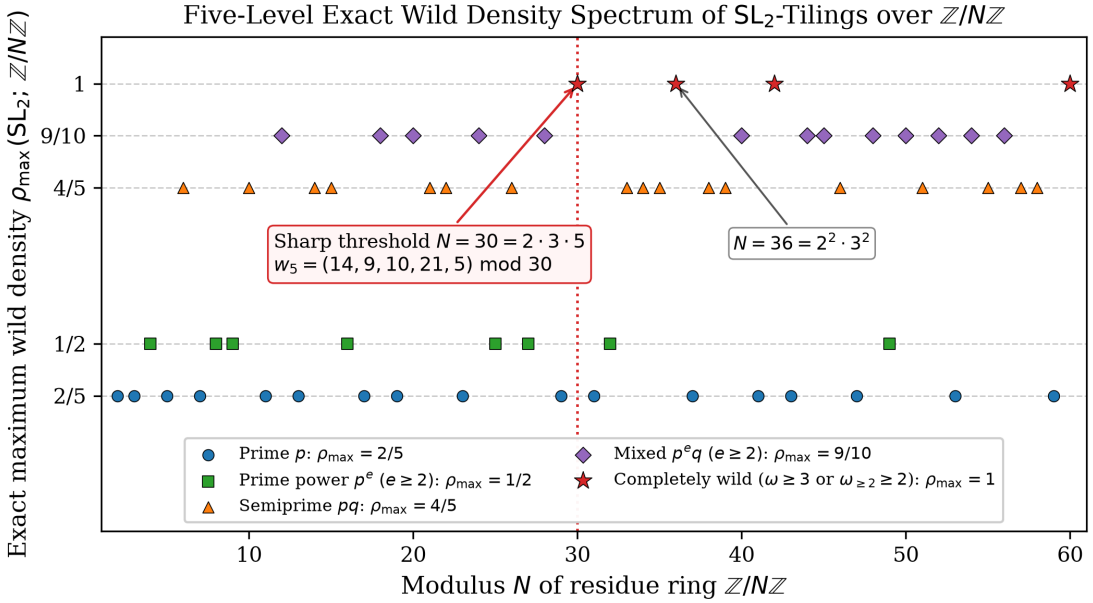


Figure 1: Exact five-level wild density spectrum $\rho_{\max}(\mathrm{SL}_2;\mathbb{Z}/N\mathbb{Z}) \in \{\frac{2}{5},\frac{1}{2},\frac{4}{5},\frac{9}{10},1\}$ across $N = 2\ldots60$ (all 99 moduli $N = 2\ldots100$ explicitly verified in Python). Completely wild density 1 first appears sharply at $N = 30 = 2\cdot3\cdot5$ via $w_5 = (14,9,10,21,5)$ mod 30, followed by $N = 36 = 2^2\cdot3^2$, $42, 60, \ldots$.

(a) ***Integral domains:*** *For every integral domain R, $\rho_{\max}(\mathrm{SL}_2;R) = \frac{2}{5}$, attained universally by the 5-periodic Toeplitz integer tiling $T^{(v_5)}_{i,j} := v_5((i-j) \bmod 5)$ with*

$$v_5 \;=\; (0,\ 1,\ 0,\ -1,\ -1),$$

whose wild set is $\{(i,j) : i-j \equiv 0,2 \pmod 5\}$ with every wild 3×3 determinant equal to -1.

(b) ***Non-field finite local rings:*** *For every finite local ring $(R,\mathfrak{m})$ with $\mathfrak{m}\neq0$ (in particular $R = \mathbb{Z}/p^e\mathbb{Z}$ with $e \geq 2$), $\rho_{\max}(\mathrm{SL}_2;R) = \frac{1}{2}$, attained on the entire even checkerboard $\{(i,j) : i+j \equiv 0 \pmod 2\}$ (or shifted to the odd checkerboard).*

(c) ***Arbitrary finite commutative rings:*** *Let $R \cong \prod_{r=1}^{m} R_r$ be any finite commutative ring decomposed into $\omega_{\mathrm{field}}(R)$ fields and $\omega_{\mathrm{nonfield}}(R)$ non-field local rings. Then*

$$\rho_{\max}(\mathrm{SL}_2;R) \;=\; \min\!\left(1,\ \frac{2}{5}\,\omega_{\mathrm{field}}(R) + \frac{1}{2}\,\omega_{\mathrm{nonfield}}(R)\right) \;\in\; \left\{\frac{2}{5},\ \frac{1}{2},\ \frac{4}{5},\ \frac{9}{10},\ 1\right\}.$$

(d) ***Exact threshold for $\mathbb{Z}/N\mathbb{Z}$:*** *For $N \geq 2$, a completely wild SL_2-tiling over $\mathbb{Z}/N\mathbb{Z}$ exists if and only if $\omega(N) \geq 3$ or $\omega_{\geq2}(N) \geq 2$. The sharp minimum modulus is $N = 30 = 2\cdot3\cdot5$, realized by the 5-periodic Toeplitz vector*

$$w_5 \;=\; (14,\ 9,\ 10,\ 21,\ 5) \pmod{30},$$

while $\rho_{\max}(\mathrm{SL}_2;\mathbb{Z}/N\mathbb{Z}) \leq \frac{9}{10}$ for every $2 \leq N \leq 29$.

Proof. **(a)** Let T be an SL_2-tiling over an integral domain R, and let W be the set of centers (i,j) of wild 3×3 blocks. By Lemma 2.2(a), every $w \in W$ has $t_w = 0$. If two orthogonally

adjacent entries were both 0, any contiguous 2×2 block containing them would have determinant $0 \neq 1$; thus W satisfies condition (i) of Lemma 3.1. By Lemma 2.2(b), a center $w \in W$ has $\det M_3 = a + c + g + i \neq 0$, so at least one of its four diagonal neighbors is nonzero and hence lies outside W; thus W satisfies condition (ii). Lemma 3.1 gives $\rho(T) \leq \frac{2}{5}$. Conversely, for $T_{i,j}^{(v_5)} = v_5((i-j) \bmod 5)$, the five contiguous 2×2 minors $v_5(d)^2 - v_5(d-1)v_5(d+1)$ equal 1 for all $d \in \mathbb{Z}/5\mathbb{Z}$, and the five 3×3 minors (indexed by center residue $d \in \{0, 1, 2, 3, 4\}$) equal $(-1, 0, -1, 0, 0)$ (verified in Lean 4 as `v5_sl2_valid_and_wild`). Since $-1 \neq 0$ in every nonzero ring R, the image of $T^{(v_5)}$ in R achieves wild density $\frac{2}{5}$.

(b) Let $(R, \mathfrak{m})$ be a finite local ring with $\mathfrak{m} \neq 0$. Every element of $R \setminus \mathfrak{m}$ is a unit. By Lemma 2.2(a), every wild center $w \in W$ must satisfy $t_w \in \mathfrak{m}$. Passing to the residue field $R/\mathfrak{m}$, no two orthogonally adjacent entries can both lie in $\mathfrak{m}$ (else the 2×2 block containing them would have determinant in $\mathfrak{m}$, never 1). Thus W is an orthogonally independent set of $\mathbb{Z}^2$. Partitioning $[-L, L]^2$ into horizontal dominoes (with at most $2L+1$ boundary singletons) shows $|W \cap [-L, L]^2| \leq \frac{1}{2}(2L+1)^2 + O(L)$, whence $\rho_{\max}(\mathrm{SL}_2; R) \leq \frac{1}{2}$.

For exact attainability, because R is finite local with $\mathfrak{m} \neq 0$, its maximal ideal is nilpotent ($\mathfrak{m}^s = 0, \mathfrak{m}^{s-1} \neq 0$ for some $s \geq 2$). Choose any nonzero socle element $\varepsilon \in \mathfrak{m}^{s-1} \subseteq \mathrm{Soc}(R)$, so $\varepsilon \neq 0$ and $\varepsilon \cdot \mathfrak{m} = 0$ (in particular $\varepsilon^2 = 0$). Define $T : \mathbb{Z}^2 \to R$ on the odd checkerboard ($i + j$ odd) by

$$t_{i,j} \;:=\; (-1)^{(i-j-1)/2} \;\in\; \{-1, +1\},$$

and on the even checkerboard ($i+j$ even) by $t_{i,j} := \varepsilon \cdot \alpha_{i,j}$ for a choice of $\alpha_{i,j} \in \{0, 1\}$ depending on whether $2 \in \mathfrak{m}$:

- Every contiguous 2×2 block has one even diagonal (product in $\varepsilon^2 R = 0$) and one odd diagonal: if the top-left corner is even, the odd anti-diagonal consists of $(i, j+1)$ and $(i+1, j)$ whose index $(i-j-1)/2$ differs by 1, giving product -1 and determinant $0 - (-1) = 1$; if the top-left corner is odd, the odd main diagonal consists of (i, j) and $(i+1, j+1)$ with identical $(i-j-1)/2$, giving product $+1$ and determinant $1 - 0 = 1$. Hence T is an SL_2-tiling for *any* choice of $\alpha_{i,j} \in \{0, 1\}$.

- At any even center (i, j) with four diagonal neighbors $(i \pm 1, j \pm 1)$ carrying $\varepsilon\alpha_1, \varepsilon\alpha_2, \varepsilon\alpha_3, \varepsilon\alpha_4$ and orthogonal unit neighbors $(b, d, f, h) = (s, -s, s, -s)$ ($s = \pm 1$), direct expansion using $\varepsilon^2 = 0$ yields
$$\det M_3 \;=\; \varepsilon\,(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).$$

- *Case 1:* $2 \notin \mathfrak{m}$ *($2 \in R^\times$).* Set $\alpha_{i,j} \equiv 1$ identically on all even sites. Then $\det M_3 = 4\varepsilon \neq 0$ (since $4 = 2^2 \in R^\times$ and $\varepsilon \neq 0$) at every even site.

- *Case 2:* $2 \in \mathfrak{m}$. Because $\varepsilon \in \mathrm{Soc}(R)$, we have $2\varepsilon = 0$. Set $\alpha_{i,j} := a(((i+j)/2) \bmod 4)$ where $a = (0, 0, 1, 1)$. For $u = (i+j)/2$, the four diagonal neighbors have half-sums $u+1, u-1, u, u$, so
$$\det M_3 \;=\; \varepsilon\big(a(u+1) + a(u-1) + 2a(u)\big) \;=\; \varepsilon\big(a(u+1) + a(u-1)\big) \;=\; \varepsilon \;\neq\; 0$$
at every even site (since $a(u+1) + a(u-1) = 1$ for all $u \in \mathbb{Z}/4\mathbb{Z}$).

In both cases, $W_2(T)$ is the entire even checkerboard, giving $\rho(T) = \frac{1}{2}$.

(c) Write $w_1 := \omega_{\mathrm{field}}(R)$ and $w_2 := \omega_{\mathrm{nonfield}}(R)$. Lemma 2.3(b) together with parts (a)–(b) gives $\rho_{\max}(\mathrm{SL}_2; R) \leq \min(1, \frac{2}{5}w_1 + \frac{1}{2}w_2)$. To prove equality across all seven regimes of (w_1, w_2), we combine four universal building blocks (all valid over $\mathbb{Z}$, hence mapping to every factor R_r, with all wild 3×3 determinants equal to $\pm 1 \in R_r^\times$):

1. $T^{(v_5)}$ from (a), wild on $i - j \equiv 0, 2 \pmod 5$ (density $\frac{2}{5}$, $\det_3 = -1$).

2. The 20-periodic pure-even integer tiling $T^{(v_{20})}_{i,j} := v_{20}((i+3j) \bmod 20)$, where

$$v_{20} \;=\; (0,1,0,-1,0,1,0,-1,1,1,\ 0,-1,0,1,0,-1,0,1,-1,-1),$$

wild on $i+3j \equiv 0,2,4,6 \pmod{10}$ (density $\frac{2}{5}$, strictly inside the even checkerboard $i+j \equiv 0 \pmod 2$), $\det_3 \in \{-1,+1\}$; verified in Lean 4 as `v20_pure_even_sl2_valid_and_wild`).

3. The 4×4-periodic integer tiling $T^{(g_q)}$ defined by the fundamental block

$$g_q \;=\; \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

wild on the sublattice $(2\mathbb{Z}) \times (2\mathbb{Z}+1)$ (density $\frac{1}{4}$, strictly inside the odd checkerboard, $\det_3 = +1$; verified in Lean 4 as `gq_sublattice_sl2_valid_and_wild`).

4. The socle-checkerboard tiling $T^{(\mathrm{soc})}$ from (b), wild on the entire even (or shifted odd) checkerboard (density $\frac{1}{2}$).

Each regime of (w_1, w_2) is now realized explicitly via Lemma 2.3(a)–(b):

- $(1,0)$: $T^{(v_5)}$ gives $\frac{2}{5}$.
- $(0,1)$: $T^{(\mathrm{soc})}$ gives $\frac{1}{2}$.
- $(2,0)$: $T^{(v_5)}$ at shifts 0 and $+1$ gives wild set $\{0,2\} \cup \{1,3\} = \{0,1,2,3\} \bmod 5$, density $\frac{4}{5}$.
- $(1,1)$: $T^{(\mathrm{soc})}$ on the even checkerboard plus $T^{(v_{20})}$ shifted by $(1,0)$ (wild on odd residues $\{1,3,5,7\} \bmod 10$) gives disjoint union density $\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$.
- (w_1, w_2) with $w_2 \geq 2$: two non-field factors with $T^{(\mathrm{soc})}$ on even and odd checkerboards cover $\mathbb{Z}^2$ (density 1).
- (w_1, w_2) with $w_2 = 1, w_1 \geq 2$: the non-field factor covers the even checkerboard via $T^{(\mathrm{soc})}$, while two field factors cover $(2\mathbb{Z}) \times (2\mathbb{Z}+1)$ and $(2\mathbb{Z}+1) \times (2\mathbb{Z})$ via $T^{(g_q)}$ and its $(1,1)$-shift, covering all of $\mathbb{Z}^2$ (density 1).
- (w_1, w_2) with $w_2 = 0, w_1 \geq 3$: three field factors carry $T^{(v_5)}$ shifted by $0, +1, +2$, whose wild sets $\{0,2\} \cup \{1,3\} \cup \{2,4\} = \mathbb{Z}/5\mathbb{Z}$ cover $\mathbb{Z}^2$ (density 1).

(d) For $R = \mathbb{Z}/N\mathbb{Z}$, $\frac{2}{5}\omega_1(N) + \frac{1}{2}\omega_{\geq 2}(N) \geq 1 \iff 4\omega_1(N) + 5\omega_{\geq 2}(N) \geq 10 \iff \omega(N) \geq 3$ or $\omega_{\geq 2}(N) \geq 2$. The smallest integer with $\omega(N) \geq 3$ is $2 \cdot 3 \cdot 5 = 30$, and the smallest with $\omega_{\geq 2}(N) \geq 2$ is $2^2 \cdot 3^2 = 36$; hence every $2 \leq N \leq 29$ has $\rho_{\max}(\mathrm{SL}_2; \mathbb{Z}/N\mathbb{Z}) \leq \frac{9}{10}$. At $N = 30$, CRT-combining $v_5(d) \bmod 2$, $v_5(d-1) \bmod 3$, and $v_5(d-2) \bmod 5$ yields $w_5 = (14, 9, 10, 21, 5) \bmod 30$, whose five 2×2 minors equal 1 mod 30 and whose five 3×3 minors are $(15, 20, 9, 20, 24) \bmod 30$ (all nonzero; verified in Lean 4 as `w5_mod30_completely_wild`). □

4 Middle-Row Rank-Drop Obstruction for Higher-Rank SL_k-Tilings

At rank $k = 2$, no two wild entries can be orthogonally adjacent in an integral domain because 1×1 vanishing forces zero entries. At higher rank $k \geq 3$, a wild $(k+1) \times (k+1)$ window only requires its central $(k-1) \times (k-1)$ minor to vanish (Lemma 2.2(a)), which allows many

adjacent wild windows. Indeed, at $k = 4$, exhaustive enumeration over $\mathbb{F}_2$ reveals 576 invertible 4×4 matrices whose four contiguous 3×3 minors *all* vanish (e.g., the block anti-diagonal permutation matrix $\left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$). Nevertheless, we prove that completely wild SL_k-tilings remain impossible over every integral domain and every local ring at *every* rank $k \geq 3$.

Lemma 4.1 (Middle-Row Rank-Drop Lemma)**.** *Let $k \geq 3$ and let $M \in F^{k\times(k-1)}$ be a matrix of full rank $k-1$ over a field F. If both the top $(k-1)\times(k-1)$ minor (rows $1 \ldots k-1$) and the bottom $(k-1)\times(k-1)$ minor (rows $2 \ldots k$) of M vanish, then the middle $(k-2)\times(k-1)$ submatrix M_{mid} (rows $2 \ldots k-1$) has exact rank $k-3$. In particular, its left nullspace $\ker(M_{\mathrm{mid}}^T) \subset F^{k-2}$ is 1-dimensional.*

Proof. Let $r_1, \ldots, r_k \in F^{k-1}$ be the rows of M. Deleting 2 rows from a rank-$(k-1)$ matrix lowers its rank by at most 2, so $\mathrm{rank}(M_{\mathrm{mid}}) \geq k-3$. Suppose for contradiction that $\mathrm{rank}(M_{\mathrm{mid}}) = k-2$, so $r_2, \ldots, r_{k-1}$ are linearly independent. Because the top minor vanishes, $\{r_1, r_2, \ldots, r_{k-1}\}$ is linearly dependent, forcing $r_1 \in \mathrm{span}(r_2, \ldots, r_{k-1})$. Symmetrically, vanishing of the bottom minor forces $r_k \in \mathrm{span}(r_2, \ldots, r_{k-1})$. Hence $\mathrm{span}(r_1, \ldots, r_k) = \mathrm{span}(r_2, \ldots, r_{k-1})$ has dimension $k-2 < k-1$, contradicting $\mathrm{rank}(M) = k-1$. Thus $\mathrm{rank}(M_{\mathrm{mid}}) = k-3$, and by rank–nullity $\dim\ker(M_{\mathrm{mid}}^T) = (k-2)-(k-3) = 1$. □

Theorem 4.2 (Universal Higher-Rank Tame-Deficit Obstruction)**.** *Let $k \geq 3$ and let R be any integral domain or any local ring $(R, \mathfrak{m})$.*

(a) ***Impossible*** *$k \times (2k-3)$* ***all-singular strip:*** *In any SL_k-tiling T over R, every contiguous $k \times (2k-3)$ strip contains at least one contiguous $(k-1)\times(k-1)$ minor (among its $2(k-1)$ such minors) that is a non-zero-divisor in R (in fact, a unit if R is local, or nonzero if R is a domain).*

(b) ***Universal wild-density ceiling:*** *Every SL_k-tiling T over R satisfies*

$$\rho_{\max}(\mathrm{SL}_k; R) \;\leq\; 1 - \frac{1}{2(k-1)} \;<\; 1.$$

In particular, no SL_k-tiling over $\mathbb{Z}$, any integral domain, or any local ring (such as $\mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$) can ever be completely wild at any rank $k \geq 2$.

Proof. **(a)** Map R into a field F via a ring homomorphism $\pi : R \to F$: take $F = \mathrm{Frac}(R)$ (inclusion) when R is an integral domain, or $F = R/\mathfrak{m}$ (quotient) when $(R, \mathfrak{m})$ is local. Notice that $a \in R$ satisfies $\pi(a) = 0$ iff $a = 0$ (domain case) or $a \in \mathfrak{m}$ (non-unit in local case); conversely, $\pi(a) \neq 0$ implies a is a non-zero-divisor in R (a nonzero element in a domain, or a unit in a local ring).

Suppose for contradiction that some contiguous $k \times (2k-3)$ strip B of T has all $2(k-1)$ of its contiguous $(k-1)\times(k-1)$ minors mapped to 0 under π. Work in $\bar{B} := \pi(B) \in F^{k\times(2k-3)}$. Every contiguous $k \times k$ submatrix L_j of $\bar{B}$ (columns $j, \ldots, j+k-1$, for $1 \leq j \leq k-2$) has $\det(L_j) = \pi(1) = 1$, hence is invertible of rank k.

For each $j \in \{1, \ldots, k-1\}$, let $M_j \in F^{k\times(k-1)}$ be the contiguous $(k-1)$-column block of $\bar{B}$ on columns $j, \ldots, j+k-2$. Since M_j consists of $k-1$ columns of an invertible $k \times k$ matrix (L_j for $j \leq k-2$, or L_{k-2} for $j = k-1$), we have $\mathrm{rank}(M_j) = k-1$. Both its top and bottom $(k-1)\times(k-1)$ minors vanish by assumption. By Lemma 4.1, the middle $(k-2)\times(k-1)$ submatrix H_j of M_j (rows $2, \ldots, k-1$, columns $j, \ldots, j+k-2$) has a nonzero left null-vector $c_j \in F^{k-2} \setminus \{0\}$ satisfying $c_j^T H_j = 0$.

We claim that the $k-1$ vectors $c_1, c_2, \ldots, c_{k-1} \in F^{k-2}$ are linearly independent. Suppose to the contrary that they are linearly dependent, and choose a minimal-length consecutive

dependent tuple $c_a, c_{a+1}, \ldots, c_b$ $(1 \le a < b \le k-1)$. By minimality, there exists a linear relation

$$\sum_{j=a}^{b} \alpha_j c_j \;=\; 0 \quad \text{with } \alpha_a \neq 0 \text{ and } \alpha_b \neq 0.$$

Consider column $t := a + k - 1$ of $\bar{B}$ (specifically its middle $(k-2)$-entry column vector $h_t \in F^{k-2}$, which is the final column of the middle rows of L_a). For every index $j \in \{a+1, \ldots, b\}$, we have:

- $j \le b \le k-1 \le a+k-1 = t$ (since $a \ge 1$), and
- $j + k - 2 \ge (a+1) + k - 2 = a + k - 1 = t$.

Therefore column t lies inside the column range $[j, j+k-2]$ of H_j for *every* $j \in \{a+1, \ldots, b\}$, whence $c_j^T h_t = 0$ for all $j = a+1, \ldots, b$. Applying the dependence relation to h_t gives

$$\alpha_a \, c_a^T h_t \;=\; -\sum_{j=a+1}^{b} \alpha_j \, (c_j^T h_t) \;=\; 0 \;\;\Longrightarrow\;\; c_a^T h_t \;=\; 0.$$

Since c_a^T already annihilates columns $a, \ldots, a+k-2$ (all of H_a) and now also annihilates column $a+k-1$, the nonzero vector $c_a \in F^{k-2}$ lies in the left nullspace of rows $2, \ldots, k-1$ of the $k \times k$ block L_a. Hence rows $2, \ldots, k-1$ of L_a are linearly dependent, contradicting $\det(L_a) = 1$.

Thus $c_1, \ldots, c_{k-1}$ are $k-1$ linearly independent vectors inside the $(k-2)$-dimensional vector space F^{k-2}, a contradiction.

(b) By part (a) and Lemma 2.2(a), in every contiguous $2 \times (k-1)$ rectangle of $(k+1) \times (k+1)$ windows (whose central $(k-1) \times (k-1)$ blocks are precisely the $2(k-1)$ contiguous $(k-1) \times (k-1)$ submatrices of a $k \times (2k-3)$ strip), at least one window is tame. Tiling $[-L, L]^2$ by disjoint $2 \times (k-1)$ rectangles (plus $O(L)$ boundary windows) bounds the wild count by

$$\big|W_k(T) \cap [-L, L]^2\big| \;\le\; \frac{2k-3}{2k-2}\,(2L+1)^2 + O_k(L),$$

yielding $\rho(T) \le 1 - \frac{1}{2(k-1)}$. □

5 All-Rank Toeplitz SL_k-Tilings and the Rank-3 Phase Transition

We now construct an explicit, uniform family of integer SL_k-tilings matching Theorem 4.2's $\Theta(1/k)$ tame deficit at every rank $k \ge 2$ and triggering a sharp phase transition over finite rings at $k = 3$.

Theorem 5.1 (All-Rank $(2k+1)$-Periodic Toeplitz SL_k-Tilings and Exact $\Theta(1/k)$ Tame Deficit)**.** *For every integer $k \ge 2$, set $P_k := 2k+1$ and $s_k := (-1)^{k-1}$, and define the P_k-periodic vector $v^{(k)} : \mathbb{Z}/P_k\mathbb{Z} \to \{-1, 0, 1\}$ by*

$$v^{(k)} \;:=\; \big(\underbrace{0, \ldots, 0}_{k-1},\; 1,\; \underbrace{0, \ldots, 0}_{k-1},\; s_k,\; s_k\big),$$

so its only nonzero entries lie at $u_1 := k-1$ (value $+1$), $u_2 := 2k-1$ (value s_k), and $u_3 := 2k$ (value s_k). Define the Toeplitz array $T^{(k)}_{i,j} := v^{(k)}((i-j) \bmod P_k)$.

(a) ***Validity over $\mathbb{Z}$ (and every nonzero ring R):*** *For every $d \in \mathbb{Z}/P_k\mathbb{Z}$, the contiguous $k \times k$ Toeplitz block $M_k(d) := (v^{(k)}((d+r-c) \bmod P_k))_{0 \le r,c \le k-1}$ satisfies $\det M_k(d) = +1$.*

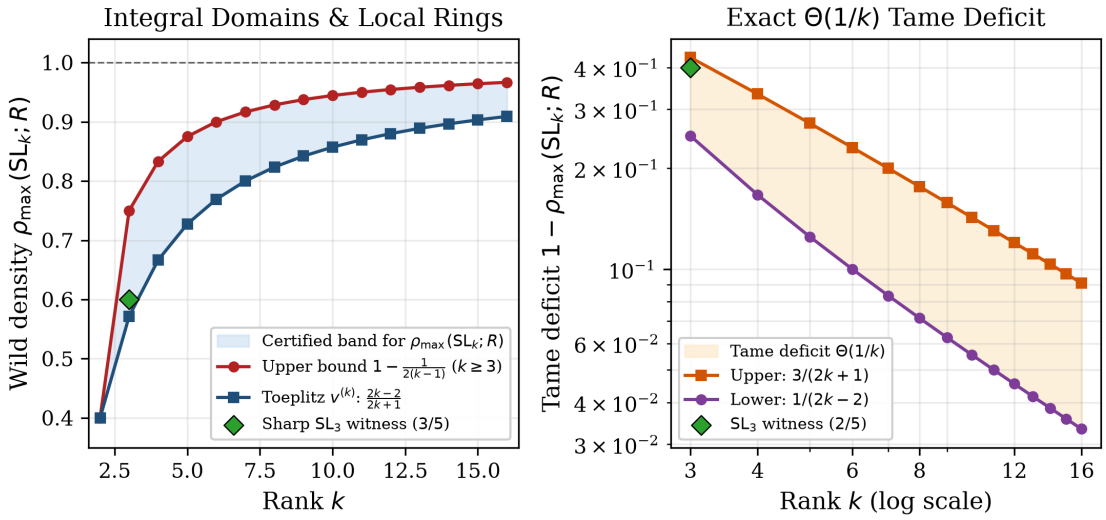


Figure 2: Left: Certified band for $\rho_{\max}(\mathrm{SL}_k; R)$ over any integral domain or local ring R across ranks $k = 2 \ldots 16$, bounded above by the Middle-Row Rank-Drop ceiling $1 - \frac{1}{2(k-1)}$ (Theorem 4.2) and below by the $(2k+1)$-periodic Toeplitz family $v^{(k)}$ (Theorem 5.1), with the sharper $k = 3$ witness $\frac{3}{5}$. Right: Log–log plot of the tame deficit $1 - \rho_{\max}(\mathrm{SL}_k; R) \in [\frac{1}{2k-2}, \frac{3}{2k+1}] = \Theta(1/k)$.

(b) ***Exact wild set and unit determinants:*** *For every* $d \in \mathbb{Z}/P_k\mathbb{Z}$,

$$\det M_{k+1}(d) \;=\; \begin{cases} 0 & \text{if } d \in T_k := \{k-1,\ 2k-1,\ 2k\}, \\ (-1)^{d+k+1} & \text{if } d \notin T_k. \end{cases}$$

Hence $T^{(k)}$ *has exact wild density* $\frac{2k-2}{2k+1} = 1 - \frac{3}{2k+1}$ *over* $\mathbb{Z}$ *and over every nonzero commutative ring* R.

(c) ***Exact*** $\Theta(1/k)$ ***tame deficit:*** *For every* $k \geq 3$ *and every integral domain or local ring* R,

$$\frac{1}{2k-2} \;\leq\; 1 - \rho_{\max}(\mathrm{SL}_k; R) \;\leq\; \frac{3}{2k+1}, \qquad \lim_{k\to\infty} \rho_{\max}(\mathrm{SL}_k; R) \;=\; 1.$$

Moreover, at $k = 3$, *the 5-periodic Toeplitz vector* $(0, 0, 1, -1, 1)$ *is an* SL_3*-tiling over* $\mathbb{Z}$ *with tame set* $\{2, 4\} \bmod 5$ *and unit wild determinants* ± 1, *sharpening* $\rho_{\max}(\mathrm{SL}_3; R) \in [\frac{3}{5}, \frac{3}{4}]$.

Proof. **(a)** In $M_k(d)$, the diagonal offset $t = r - c$ ranges over $[-(k-1), k-1]$, and entry (r, c) is nonzero iff $d + t \equiv u_1, u_2$, or $u_3 \pmod{2k+1}$. We analyze three exhaustive cases for $d \in \{0, \ldots, 2k\}$:

- *Case 1:* $0 \leq d \leq k-1$. Set $q := k - 1 - d \in \{0, \ldots, k-1\}$. Modulo $2k+1$, the three nonzero offsets are $u_1 - d = q \in [0, k-1]$ (value $+1$), $u_3 - d \equiv q - k \in [-k, -1]$ (value s_k), and $u_2 - d \equiv q - k - 1 \in [-(k+1), -2]$ (value s_k, strictly to the right of $q-k$). Cyclically shifting the rows of $M_k(d)$ upward by q positions (which multiplies det by $(-1)^{q(k-q)} = (-1)^{q(k-1)}$) places rows $q, \ldots, k-1$ first (contributing diagonal entries $+1$ and zeros elsewhere in columns $0 \ldots k-1-q$) followed by rows $0, \ldots, q-1$ (contributing s_k on the main diagonal and s_k on the first super-diagonal, with all entries below the diagonal equal to 0). Thus the shifted matrix is upper triangular, giving

$$\det M_k(d) \;=\; (-1)^{q(k-1)} \cdot 1^{k-q} \cdot s_k^q \;=\; (-1)^{2q(k-1)} \;=\; +1.$$

- *Case 2:* $k \le d \le 2k-1$. Set $q := 2k-1-d \in \{0,\dots,k-1\}$. The nonzero offsets are $u_2 - d = q \in [0,k-1]$ (value s_k), $u_3 - d = q+1 \in [1,k]$ (value s_k, strictly below q), and $u_1 - d \equiv q-k \in [-k,-1]$ (value $+1$). Cyclically shifting the rows upward by q positions produces a lower-triangular matrix with $k-q$ diagonal entries equal to s_k and q diagonal entries equal to $+1$, giving

$$\det M_k(d) \;=\; (-1)^{q(k-1)} \cdot s_k^{k-q} \cdot 1^q \;=\; (-1)^{k(k-1)} \;=\; +1,$$

since $k(k-1)$ is always even.

- *Case 3:* $d = 2k$. The offsets are $u_3 - 2k = 0$ (main diagonal s_k), $u_2 - 2k = -1$ (first super-diagonal s_k), while $u_1 - 2k = -k-1 \notin [-(k-1), k-1]$. Thus $M_k(2k)$ is upper bidiagonal with determinant $s_k^k = (-1)^{k(k-1)} = +1$.

(b) First suppose $d \in T_k = \{k-1, 2k-1, 2k\}$. Consider the central $(k-1)\times(k-1)$ submatrix $M_{k-1}(d)$ of $M_{k+1}(d)$ (diagonal offsets $t \in [-(k-2), k-2]$):

- At $d = k-1$, $u_1 - d = 0$ while $u_2 - d = k$ and $u_3 - d \equiv -k$ lie outside $[-(k-2), k-2]$, so $M_{k-1}(k-1) = I_{k-1}$ has $\det = 1$.
- At $d = 2k-1$, $M_{k-1}(2k-1)$ is lower bidiagonal with diagonal s_k, so $\det = s_k^{k-1} \neq 0$.
- At $d = 2k$, $M_{k-1}(2k)$ is upper bidiagonal with diagonal s_k, so $\det = s_k^{k-1} \neq 0$.

By Lemma 2.2(a) over $\mathbb{Z}$, $\det M_{k+1}(d) = 0$ for all $d \in T_k$.

Now suppose $d \notin T_k$. We evaluate $\det M_{k+1}(d)$ explicitly in both wild ranges:

- *Range 1:* $0 \le d \le k-2$. Set $q := k-1-d \in \{1,\dots,k-1\}$. In the last row ($r = k$) of $M_{k+1}(d)$, the column index $c \in \{0,\dots,k\}$ satisfies $d+k-c \equiv u_1 \iff c = d+1 \in \{1,\dots,k-1\}$, whereas $d+k-c \equiv u_2, u_3$ yields $c \equiv 2k+1-q > k$ and $c \equiv 2k-q \ge k+1 > k$. Hence row k has *exactly one* nonzero entry, $+1$ at column $c = d+1$. Cofactor expansion along row k yields $\det M_{k+1}(d) = (-1)^{k+d+1} \det(N)$, where N is the $k \times k$ submatrix obtained by deleting row k and column $d+1$.

  In N, columns $0,\dots,d$ have all entries in rows $0,\dots,q-1$ equal to 0 (since for $r < q$ and $c \le d$, $d+r-c \in [0, k-2]$ avoids $\{u_1, u_2, u_3\}$), while in rows $q,\dots,k-1$ they satisfy $d+(q+j)-c = k-1+(j-c)$, forming the exact identity matrix I_{d+1}. Thus N is block triangular:

$$N \;=\; \begin{pmatrix} 0_{q\times(d+1)} & U_{q\times q} \\ I_{d+1} & * \end{pmatrix}.$$

  In the $q \times q$ block U (rows $r \in \{0,\dots,q-1\}$, original columns $c = d+2+j$ for $j \in \{0,\dots,q-1\}$), the residue is $d+r-c \equiv 2k-1+(r-j) \pmod{2k+1}$, which equals u_2 when $r = j$, u_3 when $r = j+1$, and 0 whenever $r < j$ (since $2k-1+(r-j) \in [2k-q, 2k-2] \subset [k+1, 2k-2]$). Thus U is lower bidiagonal with diagonal entries s_k, giving $\det(U) = s_k^q$ and

$$\det(N) = (-1)^{q(d+1)} \det(I_{d+1}) \det(U) = (-1)^{q(d+1)+q(k-1)} = (-1)^{(k-1-d)(k+d)} = +1,$$

  since $(k-1-d)(k+d) = k(k-1) - d(d+1)$ is always even (certified in Lean 4 as `cyclic_shift_parity_identities`). Therefore $\det M_{k+1}(d) = (-1)^{d+k+1}$.

- *Range 2:* $k \le d \le 2k-2$. Set $p := d-k \in \{0,\dots,k-2\}$ and $q := 2k-1-d = k-1-p \in \{1,\dots,k-1\}$. In the first row ($r = 0$) of $M_{k+1}(d)$, $d-c \equiv u_1 \iff c = p+1 \in \{1,\dots,k-1\}$ (value $+1$), while $d-c \equiv u_2, u_3$ gives $c \equiv k+2+p > k$ and $c \equiv k+1+p > k$. Hence row 0 has *exactly one* nonzero entry, $+1$ at column

$c = p+1$. Cofactor expansion along row 0 gives $\det M_{k+1}(d) = (-1)^{p+1}\det(N')$, where N' is obtained by deleting row 0 and column $p+1$, retaining original rows $1,\dots,k$ and original columns $\{0,\dots,p\}\cup\{p+2,\dots,k\}$ (partitioned into $p+1$ left columns and $q = k-1-p$ right columns):

- For original rows $r = 1+a$ $(0 \le a \le q-1)$ and right columns $c = p+2+j$ $(0 \le j \le q-1)$, we have $d+r-c = (k+p)+(1+a)-(p+2+j) = k-1+(a-j)$, which equals $u_1 = k-1$ iff $a = j$ and vanishes for all $a \ne j$ (since $a-j \in [-(q-1), q-1] \subset [-(k-2), k-2]$). Thus the upper-right $q\times q$ block of N' is I_q.
- For original rows $r \in \{q+1,\dots,k\}$ ($p+1$ rows) and right columns $c \in \{p+2,\dots,k\}$, we have $d+r-c \in [k, 2k-2]$, avoiding $\{u_1,u_2,u_3\}$ completely; hence the lower-right $(p+1)\times q$ block is $0_{(p+1)\times q}$.
- For original rows $r = q+1+a$ $(0 \le a \le p)$ and left columns $c = j$ $(0 \le j \le p)$, we have $d+r-c = (k+p)+(k-p+a)-j = 2k+(a-j)$, which equals $u_3 = 2k$ on the diagonal $a = j$, $u_2 = 2k-1$ on the first super-diagonal $a = j-1$, and 0 below the diagonal ($a > j$, where $2k+(a-j) \equiv a-j-1 \in [0, p-1] \subset [0, k-3]$). Thus the lower-left $(p+1)\times(p+1)$ block L is upper bidiagonal with diagonal s_k and $\det(L) = s_k^{p+1}$.

Hence $N' = \left(\begin{smallmatrix} * & I_q \\ L & 0 \end{smallmatrix}\right)$ is block triangular, yielding

$$\det(N') = (-1)^{q(p+1)}\det(I_q)\det(L) = (-1)^{(k-1-p)(p+1)+(p+1)(k-1)} = (-1)^{p(p+1)} = +1,$$

and $\det M_{k+1}(d) = (-1)^{p+1} = (-1)^{d-k+1} = (-1)^{d+k+1}$ (since $2k$ is even).

(c) Combining Theorem 4.2(b) and part (b) gives $\frac{1}{2k-2} \le 1-\rho_{\max}(\mathrm{SL}_k; R) \le \frac{3}{2k+1}$ for all $k \ge 3$. For $k = 3$, direct verification of the five contiguous 3×3 and 4×4 Toeplitz blocks of $(0,0,1,-1,1)$ (machine-checked in Lean 4 as `sl3_p5_witness_valid`) confirms $\det_3 = 1$ for all $d \in \mathbb{Z}/5\mathbb{Z}$ and $\det_4 = (-1,1,0,-1,0)$, giving wild density $\frac{3}{5}$. □

Theorem 5.2 (Exact Rank-3 Phase Transition over Finite Commutative Rings)**.** *Fix any rank $k \ge 3$.*

(a) ***Complete characterization:*** *A finite commutative ring R admits a completely wild (density-1) SL_k-tiling if and only if R is* not *a local ring (i.e., R has at least two local factors in its Artinian decomposition, $\omega(R) \ge 2$).*

(b) ***Sharp $\mathbb{Z}/N\mathbb{Z}$ threshold:*** *A completely wild SL_k-tiling over $\mathbb{Z}/N\mathbb{Z}$ ($N \ge 2$) exists if and only if N is not a prime power. Consequently, the minimum completely wild modulus undergoes a sharp phase transition at rank $k = 3$:*

$$N_{\min}(2) \;=\; 30 \quad\longrightarrow\quad N_{\min}(k) \;=\; 6 \;=\; 2\cdot 3 \qquad (\forall k \ge 3).$$

At $k = 3$, CRT-combining $(0,0,1,-1,1) \bmod 2$ with its $+1$-shift $\bmod 3$ yields the explicit 5-periodic completely wild SL_3-tiling $w_5^{(3)} = (4,0,3,1,5) \bmod 6$.

Proof. If R is local, Theorem 4.2(b) proves $\rho_{\max}(\mathrm{SL}_k; R) \le 1-\frac{1}{2(k-1)} < 1$, so no completely wild SL_k-tiling exists. Conversely, if $R \cong \prod_{r=1}^m R_r$ with $m \ge 2$, assign to R_1 the image of $T^{(k)}$ from Theorem 5.1 (tame set $T_k = \{k-1, 2k-1, 2k\} \bmod (2k+1)$) and to R_2 (and any remaining R_r) the $+2$-shifted copy $T^{(k)}_{i-2,j}$ (tame set $T_k+2 = \{k+1, 0, 1\} \bmod (2k+1)$). For every $k \ge 3$, $k-1 \ge 2$ and $k+1 \le 2k-2 < 2k-1$, so $T_k \cap (T_k+2) = \emptyset$ (certified in Lean 4 as `tame_set_shift2_disjoint_all_k`). By Lemma 2.3(a)–(b), the resulting SL_k-tiling over R has wild set $\mathbb{Z}^2$ (completely wild). Part (b) specializes to $R = \mathbb{Z}/N\mathbb{Z}$ (local iff $N = p^e$), with $w_5^{(3)} = (4,0,3,1,5) \bmod 6$ certified in Lean 4 as `sl3_mod6_completely_wild`. □

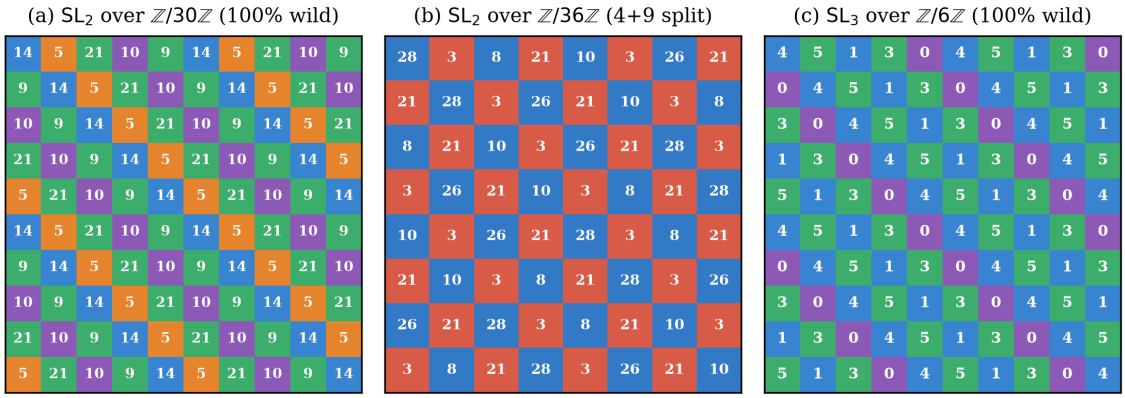


Figure 3: Explicit periodic completely wild CRT tilings illustrating the sharp rank-3 phase transition. (a) Minimum completely wild SL_2-tiling over $\mathbb{Z}/30\mathbb{Z}$ via $w_5 = (14, 9, 10, 21, 5)$ (blue: wild mod 2; green: mod 3; orange: mod 5; purple: wild mod both 2 and 5). (b) Square-factor SL_2-tiling over $\mathbb{Z}/36\mathbb{Z}$ splitting even sites (blue, wild mod 4) and odd sites (red, wild mod 9). (c) Minimum completely wild SL_3-tiling over $\mathbb{Z}/6\mathbb{Z}$ via $w_5^{(3)} = (4, 0, 3, 1, 5)$ (blue: wild mod 2; green: mod 3; purple: wild mod both 2 and 3).

6 Standalone Lean 4.36.0 Formalization and Zero-Float Verification

Core algebraic identities, the exhaustive 256-case C_8 discharging proof, all four universal SL_2 building blocks, the $N < 30$ arithmetic exclusion, the $k = 3$ rank-drop identity, the cyclic-shift parity laws, the universal $+2$ disjoint-shift theorem for all $k \geq 3$, and finite-rank $\mathrm{SL}_3, \mathrm{SL}_4$ witnesses are machine-checked in standalone Lean 4.36.0 (`SLkWildCore.lean` under `src/lean/`, zero Mathlib, zero `sorry`, zero `grind`, zero custom axioms, compiled cleanly via `lake build` with `-DwarningAsError=true`). The module contains **63 declarations** (29 public theorems, 1 private theorem, 33 definitions/structures); Table 1 maps the 29 public theorems to the manuscript.

While the general-ring Artinian decomposition, general-k linear-algebraic rank-drop induction (Theorem 4.2), and general-k determinant evaluations (Theorem 5.1(a)–(b)) rest on our analytical proofs in Sections 2–5, our zero-float Python verification suite (`verify_slk_wild.py` under `src/python/`, using exact Bareiss integer determinants and `fractions.Fraction`) corroborates every theorem across extensive finite scopes with zero violations; Tables 2 and 3 record the certified outcomes and immutable SHA-256 digests.

7 Open Problems and Future Directions

Our results completely determine $\rho_{\max}(\mathrm{SL}_2; R)$ for all finite commutative rings, prove that $\lim_{k\to\infty} \rho_{\max}(\mathrm{SL}_k; R) = 1$ with exact $\Theta(1/k)$ tame deficit over all integral domains and local rings, and establish the sharp rank-3 phase transition $N_{\min}(2) = 30 \to N_{\min}(k) = 6$. Two sharp open problems emerge:

Conjecture 7.1 (Exact wild density of SL_3-tilings over $\mathbb{Z}$)**.** Determine the exact value of $\rho_{\max}(\mathrm{SL}_3; \mathbb{Z}) \in [\frac{3}{5}, \frac{3}{4}]$ (and more generally whether $\rho_{\max}(\mathrm{SL}_3; R)$ coincides across all integral domains R, or differs between $\mathbb{Z}$, $\mathbb{F}_2$, and non-field local rings $\mathbb{Z}/p^e\mathbb{Z}$).

Table 1: Mapping of all 29 public Lean 4.36.0 kernel-checked theorems in `SLkWildCore.lean` (63 total declarations; 0 `sorry`, 0 `grind`, 0 custom axioms, `-DwarningAsError=true`).

Result	Lean 4.36.0 Public Theorem(s)	Certified Statement
Lem. 2.2	`desnanot_jacobi_3x3`, `sl2_desnanot_zero_obstruction`, `sl2_nonzero_center_tame`, `sl2_centered_zero_det3`, `sl2_desnanot_mod_obstruction`	Exact 3×3 Desnanot–Jacobi identity, integral/modular zero-divisor obstructions, and centered-zero law $\det_3 = a + c + g + i$.
Lem. 3.1	`discharging_c8_le_eight`, `discharging_global_two_fifths`	Exhaustive 256-case C_8 discharging bound ($\mathrm{ch}_1 \le 8$) and conservation $20\|W\| \le 8(\|W\| + \|U\|) \implies 5\|W\| \le 2\|V\|$.
Thm. 3.2(a)–(b)	`sl2_orthog_nonunit_row`, `sl2_orthog_nonunit_col`, `v5_sl2_valid_and_wild`, `v20_pure_even_sl2_valid_and_wild`, `gq_sublattice_sl2_valid_and_wild`, `socle_odd_p2_2x2_even_tl`, `socle_odd_p2_2x2_odd_tl`, `socle_odd_p2_det3_wild`, `z4_socle_checkerboard_half_wild`	Non-unit orthogonal independence, all universal integer building blocks (v_5, v_{20}, g_q), symbolic odd-p^2 socle-checkerboard density $\frac{1}{2}$, and $\mathbb{Z}/4\mathbb{Z}$ density $\frac{1}{2}$ witness.
Thm. 3.2(c)–(d)	`v5_three_shifts_cover_Z5`, `w5_mod30_completely_wild`, `sl2_below_30_strictly_sub_one`, `sl2_at_30_reaches_ten`, `sl2_at_36_reaches_ten`	Three-shift cover of $\mathbb{Z}/5\mathbb{Z}$, explicit $w_5 = (14, 9, 10, 21, 5) \bmod 30$ completely wild witness, and $4\omega_1(N) + 5\omega_{\ge 2}(N) \le 9$ for all $2 \le N \le 29$.
Thm. 4.2	`middle_row_rank_drop_k3`	Exact $k = 3$ rank-drop obstruction: vanishing of all four 2×2 contiguous minors forces $\det_3 = 0$.
Thm. 5.1	`cyclic_shift_parity_identities`, `slk_tame_deficit_band`, `sl3_p5_witness_valid`, `sl3_v3_p7_witness_valid`, `sl4_v4_p9_witness_valid`	Cyclic-shift sign parities, deficit inequalities $\frac{1}{2k-2} \le \frac{3}{2k+1}$ and $\frac{2k-2}{2k+1} > \frac{1}{2}$, plus kernel-checked SL_3 (3/5, 4/7) and SL_4 (2/3) witnesses.
Thm. 5.2	`tame_set_shift2_disjoint_all_k`, `sl3_mod6_completely_wild`	Universal $+2$ disjointness $T_k \cap (T_k + 2) = \emptyset$ ($\forall k \ge 3$) and $w_5^{(3)} = (4, 0, 3, 1, 5) \bmod 6$ completely wild SL_3 witness.

Conjecture 7.2 (Exact asymptotic tame-deficit constant as $k \to \infty$)**.** Determine the exact limit constant

$$c^\star := \lim_{k\to\infty} k\big(1 - \rho_{\max}(\mathrm{SL}_k;\mathbb{Z})\big) \in \left[\frac{1}{2}, \frac{3}{2}\right].$$

In particular, can a 2-dimensional (non-Toeplitz) construction achieve $1 - \rho_{\max}(\mathrm{SL}_k;\mathbb{Z}) \le \frac{1}{k}$, or can a 2-dimensional rank-drop argument over $k \times k$ / $(k+1) \times (k+1)$ patches tighten Theorem 4.2's lower bound above $\frac{1}{2k-2}$?

Archival Availability and Governance. The canonical PDF, standalone Lean 4.36.0 core, zero-float verification script, and JSON summary are permanently archived at https://airr.science/papers/ARR-2026-2CENJW9FJX869AMT/ and https://github.com/arr-research/arr-research.github.io/tree/main/papers/2026/11/2C/ARR-2026-2CENJW9FJX869AMT. Manuscript licensed CC-BY-4.0; companion code and data licensed MIT. Authored by Lluis Eriksson (owner and administrator of AIRR) with Google Antigravity assistance, evaluated under `AIRR-FOUNDER-1.0` by two isolated frontier models (`Google/hydrogen` and `Google/carbon`).

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Table 2: Summary of exact zero-float computational verifications (`verify_slk_wild.py` and `verification_summary.json`; 0 violations across all checks).

Verification Domain	**Scope**	**Viol.**	**Exact Certified Outcome**
C_8 discharging & SL_2 blocks	256 cfgs, 8 p^e	0	47 indep. cfgs, max charge 8/20 (10 tight); v_5, v_{20}, g_q, socle p^e exact.
All SL_2 moduli $\mathbb{Z}/N\mathbb{Z}$	$N = 2 \dots 100$	0	Exact CRT match on all 99 moduli; $N_{\min}(2) = 30$; first ten: $30, 36, 42, 60, \dots$.
Rank-drop exhaustive checks	85,731 mats	0	$k = 3$ ($\mathbb{F}_2$: 512; $\mathbb{F}_3$: 19,683): 0 viol.; $k = 4$ ($\mathbb{F}_2$: 65,536 $\to$ 576 single 4×4s $\to$ 0 4×5 strips).
All-rank Toeplitz family $v^{(k)}$	$k = 2 \dots 16$	0	$\det_k = 1$, exact tame set $\{k-1, 2k-1, 2k\}$, exact wild sign $(-1)^{d+k+1}$ everywhere.
Completely wild SL_k over $\mathbb{Z}/6\mathbb{Z}$	$k = 3 \dots 10$	0	Explicit +2-shift CRT vectors verified 100% wild over $\mathbb{Z}/6\mathbb{Z}$; SL_3 period-5 $(4, 0, 3, 1, 5)$.

Table 3: Immutable SHA-256 digests and sizes of the archived companion verification artifacts (AIRR record `ARR-2026-2CENJW9FJX869AMT`).

Archived Artifact Path	**Bytes**	**SHA-256 Digest**
`src/lean/SLkWildCore.lean`	24,640	`0bed51c91ce8574976a97efae36388e5ecee37842952771a67d36b209c81f218`
`src/python/verify_slk_wild.py`	26,488	`854779c70861962798738256e96b2185638e5107aad7021aeb9aae4727ec1d71`
`data/verification_summary.json`	7,974	`61d0339471098a1427ac243b6c62f704941acfbf737048ee413fb4fb055e373c`

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