Exact Branch Rigidity and Unique Crossing in Grassmann Matrix–Bingham Free Energies

Finite Certification, All-Field $\mathrm{Gr}_{\mathbb{C}}(3,6)$ and $\mathrm{Gr}_{\mathbb{C}}(4,8)$ Two-Block Order, and a $\mathrm{Gr}_{\mathbb{C}}(2,5)$ Exchange Theorem

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Abstract

Let P be a Haar-distributed complex Grassmann projector and consider the matrix–Bingham normalizer $Z_A(s) = \mathbb{E}\exp\{s\,\mathrm{tr}(AP)\}$ on the trace-zero, Frobenius-unit external sphere. We determine several exact, complementary pieces of its finite-dimensional branch geometry. On every half Grassmannian $\mathrm{Gr}_{\mathbb{C}}(r,2r)$, the balanced rank-$r$ two-block field strictly maximizes the normalized two-block moments of degrees $4,6,8$, and 10. The differences factor through the squared multiplicity defect $(r-k)^2$ and admit explicit positive-coefficient certificates. Together with a matrix-beta endpoint calculation, this proves balanced dominance at both small and large field for every rank and confines any violation of the all-field conjecture to an intermediate-field reentrant island.

At $\mathrm{Gr}_{\mathbb{C}}(3,6)$, the first half rank beyond the exceptional $\mathrm{Gr}_{\mathbb{C}}(2,4)$ model [5], we close that two-block gate for every field. Andreief's identity reduces the three normalizers to elementary Hankel determinants, and an all-degree coefficient argument proves strict Laplace order of the balanced $3+3$ spectrum over the $1+5$ and $2+4$ spectra. Combined with an exact Jacobi–Stein Hessian identity, the two losing branches have an unstable larger-block split at every nonzero field, whereas the balanced branch is Morse–Bott stable. We then prove a fixed-rank finite certification theorem: an explicit event inside the uniform Hermitian matrix cube controls every sufficiently high moment, giving a finite sufficient certificate for coefficientwise two-block order, and hence for all-field order, at each fixed rank. At $\mathrm{Gr}_{\mathbb{C}}(4,8)$, exact Hankel arithmetic through degree 268 plus this analytic tail closes all remaining degrees and closes another half-rank case here. Away from half rank the geometry changes: on $\mathrm{Gr}_{\mathbb{C}}(2,5)$ the one-spike and rank-two canonical branches exchange dominance exactly once. We derive their compactly supported piecewise-polynomial densities, certify four density sign changes by Sturm arithmetic over $\mathbb{Q}(\sqrt{6})$, and use strict total positivity of the Laplace kernel to prove that the crossing is unique and simple. A second exact sign-change argument proves that both positive orientations dominate their negative counterparts, closing the complete oriented two-level family.

These results concern the complete canonical two-block family. We do not prove that an arbitrary multi-level external spectrum is globally dominated, and therefore do not claim an unrestricted all-distortion rate–distortion function. That boundary is stated as part of the theorem ledger.

1 Introduction

For a Haar rank-r projector P on $\mathbb{C}^d$, the orbital Laplace transform

$$Z_A(s) = \mathbb{E}\exp\{s\,\mathrm{tr}(AP)\}, \qquad A = A^*, \quad \mathrm{tr}\,A = 0, \quad \|A\|_F = 1, \tag{1.1}$$

is simultaneously a unitary orbital integral, a hypergeometric function of matrix argument, and the normalizer of a matrix–Bingham exponential family [9, 15, 12, 8]. Its evaluation and asymptotic analysis are classical problems, including recent complete high-dimensional normalizer expansions [2]. The extremal problem in the external spectrum is more delicate: trace and Frobenius constraints do not place the candidate spectra on a majorization chain, so standard Schur convexity does not choose a maximizer [17, 13, 14].

This distinction matters in information theory. An external-spectrum theorem can collapse a Gibbs dual for calibrated Born-probability prediction to a scalar optimization, but only after global optimality over *all* spectra and posterior-mean feasibility have been established. Complete results are known for the exceptional half rank $\mathrm{Gr}_{\mathbb{C}}(2,4)$ and on a high-fidelity segment at arbitrary rank [5, 3]. The preceding paper in this sequence proved all-field local Morse–Bott stability of the balanced half-Grassmann branch, as well as exact metastability thresholds for unbalanced two-block branches [4]. Local stability, however, cannot compare separated values and cannot exclude a first-order exchange.

Here we attack the separated-value problem within the complete canonical two-block family. The unifying tools are finite-dimensional determinantal representations and oscillation control. Schur–Weyl moment identities yield positive polynomial certificates through degree ten at every half rank. At $r = 3$, the first half rank beyond the exceptional $\mathrm{Gr}_{\mathbb{C}}(2,4)$ theorem of [5], matrix-beta compression and Andreief's identity turn the entire Taylor series into three explicit Hankel determinants; a discrete variation argument proves coefficientwise order in every even degree. Off half rank, density sign changes become the useful invariant: a Sturm certificate and the variation-diminishing property of e^{sy} prove a unique canonical exchange on $\mathrm{Gr}_{\mathbb{C}}(2,5)$.

The main results are:

(i) strict balanced two-block moment order in degrees $4, 6, 8, 10$ for every $\mathrm{Gr}_{\mathbb{C}}(r, 2r)$, with exact positive-coefficient certificates;

(ii) exact small- and large-field balanced barriers at every half rank;

(iii) strict all-field, all-degree two-block Laplace order on $\mathrm{Gr}_{\mathbb{C}}(3,6)$;

(iv) an all-field local-maximizer classification of its three two-block critical orbits;

(v) a finite exact certificate theorem at every fixed half rank, closed in full on $\mathrm{Gr}_{\mathbb{C}}(4,8)$ by rational Hankel arithmetic and an analytic tail; and

(vi) a unique and simple canonical branch crossing on $\mathrm{Gr}_{\mathbb{C}}(2,5)$, proved by exact densities, Sturm theory, and strict total positivity.

No numerical optimizer is used in a theorem. The accompanying scripts replay the displayed symbolic identities and exact root counts; the induction and oscillation arguments are contained in the paper.

A targeted primary-literature search found no prior theorem giving the coefficientwise all-field comparisons across every two-level multiplicity on $\mathrm{Gr}_{\mathbb{C}}(3,6)$ or $\mathrm{Gr}_{\mathbb{C}}(4,8)$, nor the complete oriented two-level exchange on $\mathrm{Gr}_{\mathbb{C}}(2,5)$. This is not a claim of exhaustive priority. Jacobi trace laws, Hankel representations, matrix–Bingham normalizers, and total-positivity theory are established ingredients; novelty is claimed only for the normalized cross-multiplicity comparisons, the finite sufficient tail architecture, and the exact crossing count.

2 Canonical two-block fields and matrix-beta compression

Let P be Haar on $\mathrm{Gr}_{\mathbb{C}}(r,d)$, represented by its rank-r orthogonal projector. For $1 \le k \le d-1$, fix a rank-k projector E_k. The trace-zero, Frobenius-unit two-block field with positive multiplicity k is

$$A_{d,k} = \sqrt{\frac{d-k}{dk}}E_k - \sqrt{\frac{k}{d(d-k)}}(I - E_k). \tag{2.1}$$

Its statistic is an affine rescaling of the principal overlap $T_k = \mathrm{tr}(E_k P)$:

$$\mathrm{tr}(A_{d,k}P) = \sqrt{\frac{d}{k(d-k)}}\left(T_k - \frac{kr}{d}\right). \tag{2.2}$$

When $d = 2r$, complementation makes this law symmetric. It is convenient to write

$$Y_k = T_k - \frac{k}{2}, \qquad X_k = \sqrt{\frac{2r}{k(2r-k)}}Y_k, \qquad Z_{r,k}(s) = \mathbb{E}e^{sX_k}. \tag{2.3}$$

The common quadratic moment follows from unitary isotropy:

$$\mathbb{E}X_k^2 = \frac{r}{2(4r^2-1)}, \tag{2.4}$$

independent of k.

For $k \le r$, the nonzero eigenvalues of $E_k P E_k$ have the complex matrix-beta law $B_k(r, r)$. Finite Jacobi-trace densities and their Fourier–Laplace recurrences are classical [7]. With $H_k = 2E_k P E_k - I_k$, the eigenvalue density is proportional to

$$\prod_{i<j}(x_i - x_j)^2 \prod_{i=1}^{k}(1 - x_i^2)^{r-k}, \qquad -1 < x_i < 1. \tag{2.5}$$

Thus the two-block problem is a dimension-shift comparison between Jacobi trace laws. Equivalently, at $k = r$, H_r is uniform on the Hermitian matrix interval $-I < H < I$, and every smaller-k branch is a normalized coordinate compression of this matrix-cube law.

3 All-rank rigidity through degree ten

Put

$$q = k(2r-k), \qquad q_0 = 2r - 1. \tag{3.1}$$

Then $q_0 \le q \le r^2$, and

$$r^2 - q = (r-k)^2, \qquad q - q_0 = (k-1)(2r-k-1). \tag{3.2}$$

For $m \ge 0$, the raw moments of T_k admit the Schur–Weyl expression

$$\mathbb{E}T_k^m = \sum_{\substack{\lambda \vdash m \\ \ell(\lambda) \le \min(k,r)}} f^\lambda s_\lambda(1^k)\frac{(r)_\lambda}{(2r)_\lambda}. \tag{3.3}$$

Here $f^\lambda = m!/\prod_{u\in\lambda} h(u)$, $s_\lambda(1^k) = \prod_{u\in\lambda}(k + c(u))/h(u)$, and $(a)_\lambda = \prod_i (a - i + 1)_{\lambda_i}$. Formula (3.3) follows by expanding $(\mathrm{tr}\, E_k P)^m$, applying Schur–Weyl duality, and integrating the unitary representation matrix coefficients; it is also the standard matrix-beta moment formula [9, 15].

Define the balanced gap

$$\Delta_{2m}(r,k) = \mathbb{E}X_r^{2m} - \mathbb{E}X_k^{2m}. \tag{3.4}$$

Theorem 3.1 (Universal moment order through degree ten)**.** *For every $r \ge 2$, every $1 \le k < r$, and $m = 2, 3, 4, 5$,*

$$\Delta_{2m}(r,k) > 0. \tag{3.5}$$

In particular, the balanced multiplicity uniquely maximizes the fourth, sixth, eighth, and tenth normalized moments throughout the canonical two-block family.

Proof. Center (3.3), rescale by (2.3), and collect in the single invariant $q = k(2r-k)$. For degree four one obtains

$$\Delta_4 = \frac{3(r^2 - q)}{2q(2r-3)(2r-1)(2r+1)(2r+3)}. \tag{3.6}$$

For degree six, set

$$\begin{aligned} N_6(r,q) &= (12r^4 - 27r^2 + 8)q - 32r^4 + 8r^2, \\ D_6 &= (2r-5)(2r-3)(2r-1)^2(2r+1)^2 \\ &\quad \times (2r+3)(2r+5). \end{aligned} \tag{3.7}$$

For $r \geq 3$,

$$\Delta_6 = \frac{15(r^2 - q)N_6(r,q)}{4q^2 r D_6}. \tag{3.8}$$

The coefficient of q is positive, and at $q = q_0$ the resulting polynomial, after $r = y+3$, is

$$24y^5 + 316y^4 + 1578y^3 + 3653y^2 + 3736y + 1165 > 0. \tag{3.9}$$

The remaining case is $\Delta_6(2,1) = 17/5670$.

For degrees eight and ten one finds

$$\Delta_8 = \frac{105(r^2 - q)P_8(r,q)}{4q^3 r^2 D_8}, \tag{3.10}$$

$$\Delta_{10} = \frac{945(r^2 - q)P_{10}(r,q)}{8q^4 r^3 \mathcal{D}_{10}}, \tag{3.11}$$

where the factors are displayed in section A. For $r \geq 4$, substituting $r = y+4$ and $q = 2r-1+x$ makes P_8 a quadratic in x whose three coefficient polynomials have strictly positive coefficients. For $r \geq 5$, the substitution $r = y+5$, $q = 2r-1+x$ does the same for the four coefficients of the cubic P_{10}. The complete integer arrays are given in section B. Since $x, y \geq 0$, both polynomials are positive.

The exceptional exact gaps are

$$\begin{aligned} &\Delta_8(2,1) = \frac{2}{1215}, \quad \Delta_8(3,1) = \frac{967}{3378375}, \quad \Delta_8(3,2) = \frac{8111}{69189120}, \\ &\Delta_{10}(2,1) = \frac{74}{81081}, \quad \Delta_{10}(3,1) = \frac{202877}{1520268750}, \quad \Delta_{10}(3,2) = \frac{17789}{249080832}, \\ &\Delta_{10}(4,1) = \frac{20129}{770461692}, \quad \Delta_{10}(4,2) = \frac{2615}{181945764}, \quad \Delta_{10}(4,3) = \frac{65701}{14578987500}. \end{aligned}$$

All denominators in their stated rank ranges are positive, while $r^2 - q = (r-k)^2 > 0$. This proves every asserted sign. $\square$

The fourth-moment gap has an immediate analytic consequence. Since the laws are symmetric and their variances agree, the first nonzero coefficient of $Z_{r,r} - Z_{r,k}$ is $\Delta_4 s^4/4!$.

Corollary 3.2 (Weak-field barrier)**.** *For every fixed $r \geq 2$ and $k < r$, there is $\varepsilon_{r,k} > 0$ such that*

$$Z_{r,r}(s) > Z_{r,k}(s) \qquad (0 < |s| < \varepsilon_{r,k}). \tag{3.12}$$

4 The exact high-field barrier

The upper endpoint of X_k follows directly from the admissible principal angles:

$$\rho_k := \max X_k = \sqrt{\frac{rk}{2(2r-k)}}. \tag{4.1}$$

It is strictly increasing in k, so $\rho_k < \rho_r = \sqrt{r/2}$ for $k < r$. The endpoint exponent can also be computed exactly.

Proposition 4.1 (Watson endpoint asymptotic). *For fixed $r \geq 2$ and $1 \leq k \leq r$, there is a constant $C_{r,k} > 0$ such that*

$$Z_{r,k}(s) = C_{r,k} e^{s\rho_k} s^{-kr}(1 + O(s^{-1})), \qquad s \to +\infty. \tag{4.2}$$

Consequently, for every $k < r$, $Z_{r,r}(s) > Z_{r,k}(s)$ for all sufficiently large $|s|$.

Proof. In (2.5), write $x_i = 1 - y_i$. At fixed small total deficit $\delta = \sum_i y_i$, the Jacobi weight contributes degree $k(r-k)$ and the squared Vandermonde contributes degree $k(k-1)$. Integration over the $(k-1)$-simplex therefore gives a scalar trace density proportional to δ^{kr-1} with a strictly positive Selberg-type coefficient. The linear rescaling (2.3) preserves the exponent. Watson's lemma gives (4.2). The strict endpoint inequality makes the exponential factor decisive. Symmetry yields the negative-field statement. □

Corollary 4.2 (Reentrant-island obstruction). *Any failure of balanced two-block Laplace order on a half Grassmannian must be confined to a bounded interval separated from zero and infinity. In particular, a competing branch must create at least two additional positive zeros of $Z_{r,r} - Z_{r,k}$.*

The same endpoint idea can be sharpened from an asymptotic statement to an explicit infinite-tail certificate. Set

$$a_r = 1 - \frac{1}{2r}, \qquad B_r = \frac{(2r-1)^2(r+1)}{4r^2(r-1)}, \qquad C_r = 2(4r)^{-r^2} \frac{(r-1)(4r^2-1)}{r+1}. \tag{4.3}$$

Let M_r be the least nonnegative integer for which

$$C_r B_r^{M_r} > 1. \tag{4.4}$$

This is an exact rational definition and requires no logarithm.

Theorem 4.3 (Finite certification at every fixed half rank). *Fix $r \geq 2$. If*

$$\mathbb{E}X_r^{2m} > \mathbb{E}X_k^{2m} \quad (1 \leq k < r,\ 2 \leq m < M_r), \tag{4.5}$$

then the same inequalities hold for every $m \geq 2$, and consequently

$$Z_{r,r}(s) > Z_{r,k}(s) \qquad (s \neq 0,\ 1 \leq k < r). \tag{4.6}$$

Moreover, every sign in the finite hypothesis is exactly decidable by rational Hankel-determinant arithmetic.

Proof. The common variance is $\sigma_r^2 = r/[2(4r^2-1)]$. The largest unbalanced support is attained at $k = r-1$ and has square

$$\rho_*^2 = \frac{r(r-1)}{2(r+1)}.$$

Therefore

$$\mathbb{E}X_k^{2m} \leq \sigma_r^2 \rho_*^{2m-2}. \tag{4.7}$$

For balanced multiplicity, $X_r = \operatorname{tr}(H)/\sqrt{2r}$ with H uniform on $-I < H < I$ in the r^2-real-dimensional Hermitian space. Each event $H > a_r I$ and $H < -a_r I$ has probability exactly $(4r)^{-r^2}$: after $B = (H+I)/2$, it is a translate of the $1/(4r)$-scaled matrix interval. On their union $|X_r| > a_r\sqrt{r/2}$, whence

$$\mathbb{E}X_r^{2m} \geq 2(4r)^{-r^2}\left(a_r^2\frac{r}{2}\right)^m. \tag{4.8}$$

Dividing (4.8) by (4.7) gives

$$\frac{\mathbb{E}X_r^{2m}}{\mathbb{E}X_k^{2m}} \geq C_r B_r^m. \tag{4.9}$$

The identity

$$(2r-1)^2(r+1) - 4r^2(r-1) = 4r^2 - 3r + 1 > 0$$

proves $B_r > 1$, so (4.4) closes every $m \geq M_r$. Hypothesis (4.5) closes the finite prefix, and the two ranges cover every $m \geq 2$. Symmetry and the common variance then prove (4.6) coefficientwise.

Finally, (5.2) in the next section holds at every rank with $a = r-k$. Its even entry coefficients are rational beta integrals satisfying a first-order rational recurrence. A finite Leibniz expansion therefore decides every sign in (4.5) using integers and reduced rational fractions. □

The universal cutoffs are conservative: for $r = 2, \ldots, 10$ they are

$$12, 58, 165, 360, 672, 1131, 1766, 2610, 3695. \tag{4.10}$$

The theorem proves termination at each fixed rank, not positivity of the finite block uniformly in r.

The moment theorem does not by itself exclude such zeros. The next section does exclude them at $r = 3$.

5 All-field two-block rigidity on $\mathrm{Gr}_{\mathbb{C}}(3,6)$

For $k = 1, 2, 3$, put

$$c_k = \sqrt{\frac{6}{k(6-k)}}, \qquad X_k = \frac{c_k}{2}\operatorname{tr} H_k. \tag{5.1}$$

Define the Hankel determinants

$$D_k^{(a)}(u) = \det\left[\int_{-1}^{1} x^{i+j}(1-x^2)^a e^{ux}\,dx\right]_{i,j=0}^{k-1}. \tag{5.2}$$

Andreief's identity [1, 6] applied to (2.5) gives

$$Z_{3,k}(s) = \frac{D_k^{(3-k)}(sc_k/2)}{D_k^{(3-k)}(0)}. \tag{5.3}$$

Differentiating $I_0(u) = 2\sinh(u)/u$ yields

$$\begin{aligned}
D_1^{(2)}(u) &= 16\frac{u^2\sinh u - 3u\cosh u + 3\sinh u}{u^5}, & D_1^{(2)}(0) &= \frac{16}{15},\\
D_2^{(1)}(u) &= 8\frac{2u^4 + 2u^2\cosh(2u) + 4u^2 - 6u\sinh(2u) + 3\cosh(2u) - 3}{u^8}, & D_2^{(1)}(0) &= \frac{16}{45},\\
D_3^{(0)}(u) &= -32\frac{u^4\sinh u - 2u^3\cosh u + 3u^2\sinh u - \sinh^3 u}{u^9}, & D_3^{(0)}(0) &= \frac{32}{135}.
\end{aligned} \tag{5.4}$$

All values at zero are removable limits.

Theorem 5.1 (All-field $\mathrm{Gr}_{\mathbb{C}}(3,6)$ two-block rigidity)**.** *For every real s,*

$$Z_{3,3}(s) \geq Z_{3,2}(s), \qquad Z_{3,3}(s) \geq Z_{3,1}(s). \tag{5.5}$$

Both inequalities are strict for $s \neq 0$. Equivalently, the balanced $3+3$ spectrum uniquely maximizes the matrix–Bingham normalizer among all trace-zero, Frobenius-unit spectra having exactly two levels.

Proof. Write $Z_{3,k}(s) = \sum_{m\geq 0} b_{k,m} s^{2m}$. Expansion of (5.4) gives

$$b_{1,m} = \frac{15(3/10)^m}{(2m+3)(2m+5)(2m+1)!}, \tag{5.6}$$

$$b_{2,m} = \frac{360(3/4)^m}{(m+2)(m+3)(m+4)(2m+3)(2m+5)(2m+1)!}, \tag{5.7}$$

$$b_{3,m} = \frac{135\{19683\,9^m - p(m)\}}{4\,6^m(2m+9)!}, \tag{5.8}$$

where

$$p(m) = 64m^4 + 896m^3 + 4640m^2 + 10552m + 8931. \tag{5.9}$$

The constant coefficients equal one and the quadratic coefficients all equal 3/140.

Put

$$p_1(m) = 9p(m), \qquad q_1(m) = \tfrac{1}{8}(m+1)(m+2)(m+3)(2m+7)(2m+8)(2m+9),$$
$$p_2(m) = 3p(m), \qquad q_2(m) = \tfrac{1}{4}(m+1)(2m+7)(2m+9).$$

Common denominators give the exact identities

$$\frac{4\,6^m(2m+9)!}{15}(b_{3,m} - b_{1,m}) = 177147\,9^m - p_1(m) - 256(9/5)^m q_1(m), \tag{5.10}$$

$$\frac{4\,6^m(2m+9)!}{45}(b_{3,m} - b_{2,m}) = 59049\,9^m - p_2(m) - 2048(9/2)^m q_2(m). \tag{5.11}$$

For $m = n+2$, direct factorization gives

$$\begin{aligned}
5q_1(m) - q_1(m+1) &= (n+4)(n+5)(n+6)(2n+13)(2n^2+14n+15),\\
9p_1(m) - p_1(m+1) &= 2304(n+4)(n+5)(n+6)(2n+13),\\
2q_2(m) - q_2(m+1) &= \tfrac{1}{4}(2n+13)(2n^2+11n+6),\\
9p_2(m) - p_2(m+1) &= 768(n+4)(n+5)(n+6)(2n+13).
\end{aligned} \tag{5.12}$$

Thus $q_1(m)/5^m$, $p_1(m)/9^m$, $q_2(m)/2^m$, and $p_2(m)/9^m$ decrease for $m \geq 2$. Evaluating at $m = 2$ gives

$$\begin{aligned}
9^{-m}\,\mathrm{RHS}(5.10) &\geq 177147 - 256\frac{2574}{5} - \frac{18929}{3} = \frac{585728}{15} > 0,\\
9^{-m}\,\mathrm{RHS}(5.11) &\geq 59049 - 2048\frac{429}{16} - \frac{170361}{81} = \frac{18304}{9} > 0.
\end{aligned} \tag{5.13}$$

Therefore $b_{3,m} > b_{1,m}, b_{2,m}$ for every $m \geq 2$. Summing the coefficient gaps proves (5.5), with strictness away from zero. ☐

The theorem compares every two-level multiplicity, not arbitrary spectra with three or more eigenvalues. A full-spectrum maximizer would require a separate stationary-spectrum reduction.

6 All-field instability classification of the $\mathrm{Gr}_{\mathbb{C}}(3,6)$ branches

The value theorem can be supplemented by exact local geometry. Let

$$\widetilde{Y}_k = \mathrm{tr}(E_k P) - \frac{k}{2}, \qquad z_k(u) = \mathbb{E}e^{u\widetilde{Y}_k}, \qquad m = 6 - k. \tag{6.1}$$

The Jacobi–Stein calculation of [4] shows that the constrained Hessian eigenvalue on a traceless split of the larger block has sign opposite to

$$H_k^-(u) = z_k''(u) + \frac{6(6m-1)}{k}\frac{z_k'(u)}{u} - \frac{m^2}{4} z_k(u). \tag{6.2}$$

Its moment-ratio theorem makes the coefficient of u^{2j} strictly negative whenever

$$2j(m-k) > m(k^2-1). \tag{6.3}$$

Corollary 6.1 (All-field two-block instability classification)**.** *On $\mathrm{Gr}_{\mathbb{C}}(3,6)$, the $1+5$ and $2+4$ two-block critical orbits have an unstable larger-block split for every nonzero field and therefore are not local maxima. The balanced $3+3$ orbit is a strict local maximum modulo conjugation for every nonzero field. Hence it is the unique local maximum within the complete two-level stationary class.*

Proof. For $k=1$, condition (6.3) covers every $j \geq 1$ and

$$H_1^-(u) = -\frac{u^2}{42} - \frac{u^4}{3696} - \cdots < 0 \qquad (u \neq 0). \tag{6.4}$$

For $k=2$, (6.3) covers every $j \geq 4$. The constant and quadratic terms vanish, and exact expansion of (5.4) gives

$$[u^4]H_2^- = -\frac{1}{16170}, \qquad [u^6]H_2^- = -\frac{1}{840840}. \tag{6.5}$$

Thus every nonzero coefficient is negative. The Hessian sign convention in (6.2) supplies a positive constrained-Hessian direction, and hence an instability, for both losing strata. The balanced all-field Morse–Bott sign is the half-Grassmann no-spinodal theorem of [4]. $\square$

This closes two possible loopholes simultaneously: neither losing branch can overtake balanced in value, and neither can restabilize at an intermediate field. It still leaves multi-level stationary spectra outside its scope.

7 A unique canonical exchange on $\mathrm{Gr}_{\mathbb{C}}(2,5)$

The half-rank symmetry is essential. Consider

$$A_1 = \frac{1}{\sqrt{20}}\mathrm{diag}(4,-1,-1,-1,-1), \qquad A_2 = \frac{1}{\sqrt{30}}\mathrm{diag}(3,3,-2,-2,-2), \tag{7.1}$$

and $Z_k(s) = \mathbb{E}e^{s\,\mathrm{tr}(A_k P)}$ for $P \sim \mathrm{Gr}_{\mathbb{C}}(2,5)$.

Theorem 7.1 (Unique canonical crossing)**.** *The difference $Z_2(s) - Z_1(s)$ has exactly one positive zero. The zero is simple. Hence there is a unique $s_\times > 0$ such that*

$$Z_1(s) > Z_2(s) \quad (0 < s < s_\times), \qquad Z_2(s) > Z_1(s) \quad (s > s_\times). \tag{7.2}$$

Moreover, for every $s > 0$ each displayed positive orientation strictly dominates its negative orientation. Since multiplicities 4 and 3 are, up to conjugacy, $-A_1$ and $-A_2$, respectively, (7.2) is the unique exchange of the complete oriented two-level family.

7.1 Exact overlap densities

Set $Y_k = \sqrt{30}\,\mathrm{tr}(A_k P)$. For A_1, $q = P_{11}$ has the Beta$(2,3)$ law and

$$Y_1 = \frac{\sqrt{6}}{2}(5q-2), \qquad f_1(y) = \frac{4\sqrt{6}(\sqrt{6}y-9)^2(\sqrt{6}y+6)}{16875}, \tag{7.3}$$

on $-\sqrt{6} < y < 3\sqrt{6}/2$.

For A_2, let $T = \mathrm{tr}(E_2 P)$. The two principal overlaps have joint density $36(x_1-x_2)^2(1-x_1)(1-x_2)$ on $(0,1)^2$ and $Y_2 = 5T-4$. Integration along $x_1+x_2 = t$ yields

$$f_2(y) = \begin{cases} \dfrac{6(y+4)^3(y^2-42y+66)}{78125}, & -4 < y < 1, \\ \dfrac{6(6-y)^5}{78125}, & 1 < y < 6. \end{cases} \tag{7.4}$$

Direct integration gives

$$\mathbb{E}Y_1 = \mathbb{E}Y_2 = 0, \qquad \mathbb{E}Y_1^2 = \mathbb{E}Y_2^2 = \frac{3}{2}, \qquad \mathbb{E}Y_2^3 - \mathbb{E}Y_1^3 = \frac{2-3\sqrt{6}}{14} < 0. \tag{7.5}$$

7.2 Four sign changes, certified exactly

Let $h = f_2 - f_1$. Outside the support of f_1, $h = f_2 > 0$. On the lower overlap $(-\sqrt{6}, 1)$,

$$h(y) = \frac{6}{78125} p_-(y), \tag{7.6}$$

where

$$\begin{aligned} p_-(y) = y^5 - 30y^4 - \frac{4510}{9}y^3 + \left(-1160 + \frac{2000\sqrt{6}}{9}\right) y^2 \\ + 980y + 4224 - 1500\sqrt{6}. \end{aligned} \tag{7.7}$$

On $(1, 3\sqrt{6}/2)$,

$$h(y) = -\frac{6}{78125} p_+(y), \tag{7.8}$$

where

$$\begin{aligned} p_+(y) = y^5 - 30y^4 + \frac{4240}{9}y^3 + \left(-2160 - \frac{2000\sqrt{6}}{9}\right) y^2 \\ + 5980y - 7776 + 1500\sqrt{6}. \end{aligned} \tag{7.9}$$

Sturm sequences over $\mathbb{Q}(\sqrt{6})$ prove that p_- has one root in each of

$$(-9/4, -11/5), \qquad (-1/2, -9/20), \tag{7.10}$$

and no other root in $(-\sqrt{6}, 1)$; p_+ has one root in each of

$$(5/4, 13/10), \qquad (3, 31/10), \tag{7.11}$$

and no other root in $(1, 3\sqrt{6}/2)$. Their terminal Sturm remainders are nonzero constants, so all four roots are simple. Exact signs at rational probes give

$$\mathrm{sgn}\, h = +, -, +, -, +. \tag{7.12}$$

Thus h has exactly four sign changes.

7.3 Variation diminution

We use the classical strict total positivity of the kernel $K(s, y) = e^{sy}$ [10, 11, 16].

Lemma 7.2 (Continuous generalized Descartes rule)**.** *If a nonzero compactly supported signed density h has m sign changes, then its bilateral Laplace transform*

$$L(s) = \int_{\mathbb{R}} e^{sy} h(y)\, dy \tag{7.13}$$

has at most m real zeros, counted with multiplicity.

Proof. Suppose L had zeros s_i of total multiplicity greater than m. Differentiating (7.13) supplies orthogonality to the corresponding functions $y^j e^{s_i y}$. These functions form an extended complete Chebyshev system because the exponential kernel is strictly totally positive. The standard interpolation theorem then constructs a nonzero linear combination with simple zeros at the sign-change points of h and the same sign as h on every intervening interval. Its integral against h is strictly positive, while the orthogonality makes it zero, a contradiction. □

Lemma 7.3 (Strict orientation dominance)**.** *Let $M_j(t) = \mathbb{E}e^{tY_j}$ for the two rescaled statistics in (7.3)–(7.4). Then*

$$M_j(t) > M_j(-t) \qquad (t > 0,\ j = 1, 2).$$

*Consequently the positive multiplicity-*1 *and multiplicity-*2 *orientations strictly dominate multiplicities* 4 *and* 3*, respectively, at positive field.*

Proof. Put $g_j(y) = f_j(y) - f_j(-y)$ and $O_j(t) = M_j(t) - M_j(-t) = \int e^{ty} g_j(y)\, dy$.

For $0 < y < \sqrt{6}$, direct subtraction in (7.3) gives

$$g_1(y) = \frac{16y(2y^2 - 9)}{1875}.$$

For $\sqrt{6} < y < 3\sqrt{6}/2$, the reflected density vanishes and $g_1(y) > 0$. Since $0 < 9/2 < 6$, the positive half-line has exactly one sign change and the odd density g_1 has the global pattern $-, +, -, +$, hence three sign changes. Moreover, $\mathbb{E}Y_1 = 0$ and $\mathbb{E}Y_1^3 = 3\sqrt{6}/14 > 0$. Thus O_1 has a zero of exactly multiplicity three at the origin and is positive immediately to its right. Lemma 7.2 leaves no budget for another real zero, so $O_1(t) > 0$ for every $t > 0$.

For $0 < y < 1$, (7.4) gives

$$g_2(y) = \frac{12y(y^4 - 390y^2 + 480)}{78125} > 0;$$

the polynomial in y^2 decreases on $[0, 1]$ and has endpoint value 91. For $1 < y < 4$,

$$g_2(y) = \frac{12p(y)}{78125}, \qquad p(y) = 30y^4 - 375y^3 + 1660y^2 - 3000y + 1776.$$

An exact rational Sturm sequence gives exactly two simple roots, one in $(23/20, 7/6)$ and one in $(11/4, 14/5)$, with signs $+, -, +$. For $4 < y < 6$, the reflected density vanishes and $g_2(y) > 0$. Hence the odd density g_2 has exactly five sign changes. Since $\mathbb{E}Y_2 = 0$ and $\mathbb{E}Y_2^3 = 1/7 > 0$, O_2 has a zero of multiplicity three at the origin and is positive just to its right. Lemma 7.2 leaves at most two additional real zeros. Nonzero zeros occur in opposite pairs with equal multiplicity. A simple positive zero would change sign and require a second positive zero to return to the positive large-field sign; an even positive zero and its negative partner would already consume four multiplicities. Both possibilities exceed the remaining budget. Therefore $O_2(t) > 0$ for every $t > 0$. □

Proof of theorem 7.1. After the harmless rescaling $s \mapsto s/\sqrt{30}$, the difference $Z_2 - Z_1$ is the Laplace transform of h. Equations (7.5) show that it has a zero of exactly multiplicity three at the origin and is negative immediately to the right. By (7.12) and theorem 7.2, at most one additional real zero remains, counting multiplicity. The upper support endpoint of Y_2 is 6, strictly larger than $3\sqrt{6}/2$, the endpoint of Y_1; hence $Z_2 - Z_1 > 0$ for all sufficiently large positive s. A positive zero therefore exists. It consumes the remaining zero budget and must be unique and simple. Lemma 7.3 removes the two negative orientations, proving the complete oriented-family statement. □

The diagnostic value $s_\times \approx 4.858579$ is not used in the proof.

8 A finite exact all-field certificate on $\mathrm{Gr}_{\mathbb{C}}(4, 8)$

The fixed-rank theorem becomes a complete all-field result at the next half rank. For $P \sim \mathrm{Gr}_{\mathbb{C}}(4, 8)$, let

$$A_k = \sqrt{\frac{8-k}{8k}} E_k - \sqrt{\frac{k}{8(8-k)}} (I - E_k), \qquad Z_k(s) = \mathbb{E} e^{s \,\mathrm{tr}(A_k P)}, \quad 1 \le k \le 4. \tag{8.1}$$

Theorem 8.1 (Computer-assisted all-field $\mathrm{Gr}_{\mathbb{C}}(4, 8)$ two-block rigidity)**.** *For every real s and $k = 1, 2, 3$,*

$$Z_4(s) \ge Z_k(s), \tag{8.2}$$

with strict inequality for $s \neq 0$. More strongly, after the common constant and quadratic coefficients, every even Taylor coefficient of Z_4 is strictly larger than its counterpart in Z_k.

The proof is computer-assisted only on a finite exact-rational block. Its infinite tail is analytic and independent of the replay.

Proof. Put $c_k^2 = 8/[k(8-k)]$. Matrix-beta compression and Andreief's identity give

$$Z_k(s) = \frac{D_k^{(4-k)}(sc_k/2)}{D_k^{(4-k)}(0)}, \tag{8.3}$$

with D as in (5.2). If

$$J_{a,t} = \int_{-1}^{1} x^{2t} (1 - x^2)^a \, dx,$$

then

$$J_{a,0} = \frac{2^{2a+1}(a!)^2}{(2a+1)!}, \qquad \frac{J_{a,t+1}}{J_{a,t}} = \frac{2t+1}{2t+2a+3}. \tag{8.4}$$

The coefficient of u^ℓ in the (i, j) entry of the determinant is zero when $\ell + i + j$ is odd and otherwise equals

$$\frac{J_{a,(\ell+i+j)/2}}{\ell!}. \tag{8.5}$$

Leibniz expansion and finite convolution of these rational series prove, with no rounding or tolerance,

$$[s^{2m}]Z_4 > [s^{2m}]Z_k, \qquad k = 1, 2, 3, \quad 2 \le m \le 134. \tag{8.6}$$

It remains to close the infinite tail. Every unbalanced branch has $|X_k| \le \rho_3 = \sqrt{6/5}$ and common variance $2/63$, hence

$$\mathbb{E} X_k^{2m} \le \frac{2}{63} \left(\frac{6}{5}\right)^{m-1}. \tag{8.7}$$

For $k = 4$, $X_4 = \operatorname{tr}(H)/\sqrt{8}$ with H uniform on the 16-dimensional Hermitian matrix interval. Each event $H > (19/20)I$ and $H < -(19/20)I$ has probability 40^{-16}, and on their union $|X_4| > (19/20)\sqrt{2}$. Therefore

$$\mathbb{E}X_4^{2m} \geq 2\,40^{-16}\left(\frac{361}{200}\right)^m. \tag{8.8}$$

The ratio of (8.8) to (8.7) is at least

$$\frac{378}{5}\,40^{-16}\left(\frac{361}{240}\right)^m. \tag{8.9}$$

Exact integer comparison shows that (8.9) exceeds one at $m = 134$; its base $361/240$ is greater than one, so every $m \geq 134$ is covered. This overlaps the exact finite block. The laws are symmetric, and summing the strict even coefficient gaps proves (8.2). $\square$

Remark 8.2 (Audit surface)**.** *The finite checker expands each determinant through degree* 268 *using Python* `Fraction`*; it uses neither quadrature nor floating-point sign tests. A hostile audit independently checked normalization, truncation, the* 40^{-16} *event volume, the cutoff overlap, symmetry, and exhaustion of the two-level class. On the constrained reference machine the replay took about* 3.1 *seconds and peaked at* 13.3 *MiB resident memory. These resource figures document the replay; they are not mathematical assumptions.*

Theorem 8.1 is the second half-rank case closed here, not a uniform-in-rank theorem. The general result gives coefficientwise dominance at every fixed rank a terminating exact sufficient certificate; when that certificate passes, it proves all-field order. A failed coefficient sign does not by itself decide the sign of the full Laplace-transform difference, and the required finite block grows with r.

9 What the exact branch theorems do and do not prove

The results separate three distinct levels of control:

(a) *Local orbit stability*: the constrained Hessian near a specified branch, supplied for half rank by [4].

(b) *Canonical branch order*: comparison of all two-level multiplicities. This is closed here for $\mathrm{Gr}_{\mathbb{C}}(3,6)$ and $\mathrm{Gr}_{\mathbb{C}}(4,8)$, and for the complete oriented two-level family on $\mathrm{Gr}_{\mathbb{C}}(2,5)$.

(c) *Full-spectrum global order*: comparison with every trace-zero, Frobenius-unit spectrum, including three or more distinct eigenvalues. This remains open in every example treated here.

For $\mathrm{Gr}_{\mathbb{C}}(3,6)$ or $\mathrm{Gr}_{\mathbb{C}}(4,8)$, a hypothetical counterexample to the balanced global conjecture must have at least three eigenvalues. It cannot lie on a two-block crossing and cannot arise from a spinodal instability of the balanced branch. For $\mathrm{Gr}_{\mathbb{C}}(2,5)$, the unique canonical exchange rules out reentrance between the one-spike and rank-two sources, but a noncanonical multi-level branch could still exceed both.

Accordingly, none of the following is claimed:

(i) that every global stationary spectrum has at most two levels;

(ii) that the balanced $3+3$ field is globally optimal over the full spectral sphere, or that the balanced $4+4$ field is globally optimal over its full spectral sphere;

(iii) that the full-spectrum $\mathrm{Gr}_{\mathbb{C}}(2,5)$ envelope is exhausted by the two displayed positive-orientation branches, or equivalently that no multi-level branch exceeds them; or

(iv) an unrestricted all-distortion rate–distortion function derived from either unproved envelope.

This boundary is not cosmetic. The fixed-norm candidate spectra are typically majorization-incomparable, and the local theory already permits metastable unbalanced half-rank branches. A global theorem needs a new stationary-spectrum reduction, a pair-smoothing inequality, or an exact certificate covering every multiplicity stratum.

10 Reproducibility

Five lightweight exact replay scripts accompany the manuscript, organized into four groups.

(i) `matrix_cube/verify_low_degree_order.py` reconstructs (3.6)–(3.11), verifies every positive-coefficient array in section B, and checks 220 independent Schur–Weyl fixtures.

(ii) `verify_gr36_two_block_laplace.py` reconstructs the Hankel determinants, the coefficient formulas, the discrete factorizations, and the remaining exact Hessian coefficients.

(iii) `finite_global/verify_d5r2_canonical_crossing.py` derives the two densities symbolically and performs every Sturm count and sign decision in exact $\mathbb{Q}(\sqrt{6})$ arithmetic.

(iv) `matrix_cube/verify_finite_tail_cutoffs.py` checks the exact universal cutoffs in (4.10), while `matrix_cube/verify_r4_all_degree.py` checks the complete rational block (8.6) and the exact tail threshold.

The scripts replay finite algebra. They do not substitute a numerical search for an unproved full-spectrum theorem. The release package includes every script and its dependency, the one-command wrapper `replay/run_all_replays.py`, a persisted exact result ledger, an environment-and-command manifest, and `SHA256SUMS`. Verification is explicitly partial: bibliography screening, independent peer review, full-spectrum optimization, and scientific priority are not assessed by the replay.

A Moment-gap polynomials

The degree-eight quantities in (3.10) are

$$\begin{aligned}
A_8 &= 12r^6 - 111r^4 + 163r^2 - 60,\\
B_8 &= -76r^6 + 163r^4 - 60r^2,\\
C_8 &= 240r^6 - 60r^4,\\
P_8(r,q) &= A_8q^2 + B_8q + C_8,\\
D_8 &= (2r-7)(2r-5)(2r-3)(2r-1)^2(2r+1)^2\\
&\quad \times (2r+3)(2r+5)(2r+7).
\end{aligned}$$

For degree ten,

$$\begin{aligned}
A_{10} &= 80r^{10} - 2040r^8 + 14005r^6 - 34367r^4 + 35100r^2 - 12096,\\
B_{10} &= -880r^{10} + 11800r^8 - 34367r^6 + 35100r^4 - 12096r^2,\\
C_{10} &= 6080r^{10} - 28256r^8 + 35100r^6 - 12096r^4,\\
D_{10}^{(0)} &= -21504r^{10} + 53760r^8 - 12096r^6,\\
P_{10}(r,q) &= A_{10}q^3 + B_{10}q^2 + C_{10}q + D_{10}^{(0)},\\
\mathcal{D}_{10} &= (2r-9)(2r-7)(2r-5)(2r-3)^2(2r-1)^2\\
&\quad \times (2r+1)^2(2r+3)^2(2r+5)(2r+7)(2r+9).
\end{aligned}$$

B Positive-coefficient certificates

For $P_8(y+4, 2(y+4)-1+x) = \sum_{j=0}^{2} c_{8,j}(y)x^j$, the coefficient polynomials are

$$\begin{aligned}
c_{8,0} &= 48y^8 + 1336y^7 + 15788y^6 + 102818y^5 + 400038y^4 \\
&\quad + 939548y^3 + 1272831y^2 + 874408y + 214900, \\
c_{8,1} &= 48y^7 + 1244y^6 + 13284y^5 + 75025y^4 + 237852y^3 \\
&\quad + 408430y^2 + 320064y + 55448, \\
c_{8,2} &= 12y^6 + 288y^5 + 2769y^4 + 13584y^3 + 35587y^2 \\
&\quad + 46616y + 23284.
\end{aligned}$$

For $P_{10}(y+5, 2(y+5)-1+x) = \sum_{j=0}^{3} c_{10,j}(y)x^j$,

$$\begin{aligned}
c_{10,0} &= 640y^{13} + 37120y^{12} + 979040y^{11} + 15522336y^{10} + 164732888y^9 \\
&\quad + 1232842288y^8 + 6674149082y^7 + 26363797506y^6 + 75611253136y^5 \\
&\quad + 154443659379y^4 + 216240062046y^3 + 193546929296y^2 \\
&\quad + 96708725184y + 19346224491, \\
c_{10,1} &= 960y^{12} + 53120y^{11} + 1321200y^{10} + 19491680y^9 + 189485684y^8 \\
&\quad + 1274609832y^7 + 6058238765y^6 + 20392541954y^5 + 47930087778y^4 \\
&\quad + 76137334976y^3 + 77150587905y^2 + 45135152502y + 12252504972, \\
c_{10,2} &= 480y^{11} + 25280y^{10} + 591760y^9 + 8107120y^8 + 71984830y^7 \\
&\quad + 432888668y^6 + 1786158088y^5 + 4998471151y^4 + 9107125370y^3 \\
&\quad + 9854406754y^2 + 5054136464y + 430182883, \\
c_{10,3} &= 80y^{10} + 4000y^9 + 87960y^8 + 1118400y^7 + 9086005y^6 \\
&\quad + 49140150y^5 + 178467508y^4 + 427325160y^3 + 639926925y^2 \\
&\quad + 533261250y + 182589154.
\end{aligned}$$

Every displayed coefficient is strictly positive.

C Exact Sturm ledger for the canonical crossing

For reproducibility without decimal roots, the theorem-bearing Sturm counts are

Polynomial	Interval	Number of roots
p_-	$(-\sqrt{6}, 1)$	2
p_-	$(-9/4, -11/5)$	1
p_-	$(-1/2, -9/20)$	1
p_+	$(1, 3\sqrt{6}/2)$	2
p_+	$(5/4, 13/10)$	1
p_+	$(3, 31/10)$	1
p (orientation)	$(1, 4)$	2
p (orientation)	$(23/20, 7/6)$	1
p (orientation)	$(11/4, 14/5)$	1

The exact probe points $-12/5, -2, 0, 6/5, 2, 7/2$ produce the density signs $+, -, +, +, -, +$; accounting for the support-only positive intervals reduces this to the global five-block pattern $+, -, +, -, +$. For the rank-two orientation polynomial p, rational probes in its three root-separated intervals give $+, -, +$. Together with positivity on $(0, 1)$ and $(4, 6)$ and odd reflection, this gives the five sign changes used in Lemma 7.3.

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