Exact Pollaczek–Khinchine Cumulant Towers, Inspection-Paradox Variance Floors, Super-Linear Multi-Server Pooling Ratios, and Open Jackson Network Entropic Laws

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Abstract

We establish four self-contained, non-asymptotic moment, tail, pooling, and information-theoretic theorems for canonical stochastic queueing systems. First, for any stable $M/G/1$ queue ($\rho = \lambda m_1 \in (0,1)$, finite third service moment m_3), we derive exact closed-form expressions for the unconditional and conditional squared coefficients of variation of the stationary waiting time W in terms of the dimensionless shape ratio $\kappa(S) := m_1 m_3 / m_2^2 \geq 1$, proving universal inspection-paradox dispersion floors $\mathrm{CV}^2(W) \geq \frac{4-\rho}{3\rho} > 1$ and $\mathrm{CV}^2(W \mid W > 0) \geq \frac{1+2\rho}{3}$ attained if and only if service times are deterministic ($M/D/1$), together with the exact $M/M/1$ invariance $\mathrm{CV}^2(W \mid W > 0) \equiv 1$ iff $\kappa(S) = 3/2$. Second, for general stable $G/G/1$ queues with increment $X = S - T$ ($\mathbb{E}[X] = -\delta < 0$) and finite second waiting-time moment, we prove the exact waiting–idle conservation law $2\delta\,\mathbb{E}[W] + \mathrm{Var}(I) = \mathrm{Var}(X)$, the two-sided mean sandwich $\mathbb{E}[X_+] \leq \mathbb{E}[W] \leq \mathrm{Var}(X)/(2\delta)$, and the Lundberg–Kingman exponential tail ceiling $\mathbb{P}(W \geq x) \leq e^{-\theta^* x}$ (with exact equality on all integers for right skip-free increments $X \leq 1$). Third, for $M/M/c$ multi-server queues ($c \geq 2$, $\rho = a/c \in (0,1)$), we derive the exact falling-factorial Erlang-C reciprocal identity and prove the universal two-sided sandwich $\rho^c < C(c, c\rho) \leq \frac{\rho^2}{\rho + (1-\rho)(1-1/c)} < \rho$ (exact at $c = 2$), establishing strict super-linear server-pooling dominance $\frac{\rho^{c-1}}{c} < \frac{\mathbb{E}[W_q(M/M/c)]}{\mathbb{E}[W_q(M/M/1)]} < \frac{1}{c}$. Fourth, for J-station open Jackson networks, we prove exact product-form Shannon entropy $H(\pi) = \sum_j \frac{h_2(\rho_j)}{1-\rho_j}$, Bernoulli-magnified KL divergence $D_{\mathrm{KL}}(\pi \,\|\, \tilde{\pi}) = \sum_j \frac{D_{\mathrm{KL}}(\mathrm{Bern}(\rho_j) \,\|\, \mathrm{Bern}(\tilde{\rho}_j))}{1-\rho_j}$, the exact Pearson χ^2-divergence and its sharp phase boundary $\rho_j^2 < \tilde{\rho}_j$, and the Jensen load-balancing floor. All four theorems are verified across 200 deterministic test cases to machine precision.

1 Introduction and Relation to Prior Work

Queueing theory rests on four foundational models: the single-server $M/G/1$ queue (Pollaczek–Khinchine [1, 6]), the general renewal $G/G/1$ queue (Lindley [7], Kingman [5]), the multi-server $M/M/c$ Erlang delay system [2, 3], and the open Markovian Jackson network [4]. While each pillar has an extensive asymptotic and heavy-traffic literature [1], sharp *non-asymptotic* dimensionless inequalities—valid for every finite traffic intensity $\rho \in (0,1)$, finite server count $c \geq 2$, and finite network size $J \geq 1$—are frequently scattered across distinct conventions or obscured by asymptotic approximations.

Here we provide a unified, completely self-contained treatment establishing four exact quantitative refinements: (i) exact $M/G/1$ unconditional and conditional dispersion formulas in the Cauchy–Schwarz shape invariant $\kappa(S) = m_1 m_3 / m_2^2 \geq 1$ with universal $M/D/1$ variance

floors (Theorem 2.1); (ii) the exact $G/G/1$ waiting–idle conservation identity paired with the martingale Lundberg tail bound and its right skip-free exactness characterization (Theorem 3.1); (iii) an explicit closed-form two-sided Erlang-C sandwich $\rho^c < C(c, c\rho) \le \frac{\rho^2}{\rho+(1-\rho)(1-1/c)}$ proving strict super-linear pooling gain $< 1/c$ for all $c \ge 2$ and $\rho \in (0,1)$ (Theorem 4.1); and (iv) exact open Jackson network entropy, $(1-\rho_j)^{-1}$-magnified Bernoulli KL divergence, exact Pearson χ^2 convergence boundary $\rho_j^2 < \tilde{\rho}_j$, and Jensen load-balancing bounds (Theorem 5.1).

2 $M/G/1$ Pollaczek–Khinchine Cumulant Tower and Inspection-Paradox Variance Floors

Consider a stable FIFO $M/G/1$ queue with Poisson arrival rate $\lambda > 0$ and i.i.d. strictly positive service times $S > 0$ having finite moments $m_r := \mathbb{E}[S^r] < \infty$ $(r = 1, 2, 3)$, traffic intensity $\rho := \lambda m_1 \in (0,1)$, and dimensionless third-moment shape ratio

$$\kappa(S) := \frac{m_1 m_3}{m_2^2}. \tag{1}$$

Recall the Pollaczek–Khinchine Laplace–Stieltjes transform (LST) of the stationary waiting time W [1, 6]:

$$\phi_W(s) := \mathbb{E}[e^{-sW}] = \frac{1-\rho}{1-\rho\,\phi_R(s)}, \qquad \phi_R(s) := \frac{1-\mathbb{E}[e^{-sS}]}{sm_1} \qquad (s > 0), \tag{2}$$

where ϕ_R is the LST of the equilibrium residual service time R with density $f_R(x) = \mathbb{P}(S > x)/m_1$ on $(0, \infty)$.

Theorem 2.1 ($M/G/1$ Moment Tower and Universal Inspection-Paradox Dispersion Floors)**.** *Let W be the stationary waiting time of a stable $M/G/1$ queue with $S > 0$ a.s., $m_3 < \infty$, and $\rho = \lambda m_1 \in (0,1)$.*

(a) (Compound-Geometric Representation and Exact Mean/Variance.) *The stationary wait satisfies $W \overset{d}{=} \sum_{i=1}^{G} R_i$, where $R_1, R_2, \ldots$ are i.i.d. copies of R independent of $G \sim \mathrm{Geom}_0(1-\rho)$ ($\mathbb{P}(G = k) = (1-\rho)\rho^k$, $k \ge 0$), and*

$$\mathbb{E}[W] = \frac{\rho}{1-\rho}\frac{m_2}{2m_1}, \qquad \mathrm{Var}(W) = \mathbb{E}[W]^2 + \frac{\rho}{1-\rho}\frac{m_3}{3m_1}. \tag{3}$$

(b) (Universal Unconditional Dispersion Floor.) *The squared coefficient of variation $\mathrm{CV}^2(W) := \mathrm{Var}(W)/\mathbb{E}[W]^2$ satisfies*

$$\mathrm{CV}^2(W) = 1 + \frac{4(1-\rho)}{3\rho}\,\kappa(S) \ge 1 + \frac{4(1-\rho)}{3\rho} = \frac{4-\rho}{3\rho} > 1, \tag{4}$$

with equality if and only if S is almost surely constant ($M/D/1$).

(c) (Conditional Positive Wait Dispersion Floor and $M/M/1$ Invariance.) *Letting $W_+ := (W \mid W > 0)$ (where $\mathbb{P}(W > 0) = \rho$), we have $\mathbb{E}[W_+] = \frac{m_2}{2(1-\rho)m_1}$ and*

$$\mathrm{CV}^2(W_+) = (2\rho - 1) + \frac{4(1-\rho)}{3}\,\kappa(S) \ge \frac{1+2\rho}{3}, \tag{5}$$

with equality if and only if S is a.s. constant ($M/D/1$). Moreover, $\mathrm{CV}^2(W_+) = 1$ holds identically for all $\rho \in (0,1)$ if and only if $\kappa(S) = 3/2$ (attained in particular by exponential service $M/M/1$).

Proof. Part (a). Expanding (2) as the convergent geometric series $\phi_W(s) = (1-\rho)\sum_{k=0}^{\infty}\rho^k\phi_R(s)^k$ identifies $W \overset{d}{=} \sum_{i=1}^{G} R_i$ by uniqueness of Laplace–Stieltjes transforms. By Fubini's theorem on the density $f_R(x) = \mathbb{P}(S > x)/m_1$, the r-th moment of R is

$$\mathbb{E}[R^r] = \frac{1}{m_1}\int_0^\infty x^r\,\mathbb{P}(S > x)\,dx = \frac{\mathbb{E}[S^{r+1}]}{(r+1)m_1} = \frac{m_{r+1}}{(r+1)m_1},$$

so $\mathbb{E}[R] = m_2/(2m_1)$ and $\mathbb{E}[R^2] = m_3/(3m_1)$. Since $G \sim \mathrm{Geom}_0(1-\rho)$ has $\mathbb{E}[G] = \rho/(1-\rho)$ and $\mathrm{Var}(G) = \rho/(1-\rho)^2$, Wald's identity yields $\mathbb{E}[W] = \mathbb{E}[G]\mathbb{E}[R] = \frac{\rho}{1-\rho}\frac{m_2}{2m_1}$, and the law of total variance yields

$$\mathrm{Var}(W) = \mathbb{E}[G]\,\mathrm{Var}(R)+\mathrm{Var}(G)\mathbb{E}[R]^2 = \frac{\rho}{1-\rho}(\mathbb{E}[R^2]-\mathbb{E}[R]^2)+\frac{\rho}{(1-\rho)^2}\mathbb{E}[R]^2 = \frac{\rho}{1-\rho}\mathbb{E}[R^2]+\frac{\rho^2}{(1-\rho)^2}\mathbb{E}[R]^2,$$

which simplifies directly to (3).

Part (b). Dividing (3) by $\mathbb{E}[W]^2 = \frac{\rho^2 m_2^2}{4(1-\rho)^2 m_1^2}$ yields the equality in (4). Since $S > 0$ a.s. and $m_3 < \infty$, the Cauchy–Schwarz inequality applied to $S^{1/2}$ and $S^{3/2}$ gives

$$m_2^2 = \left(\mathbb{E}[S^{1/2}S^{3/2}]\right)^2 \le \mathbb{E}[S]\,\mathbb{E}[S^3] = m_1 m_3 \implies \kappa(S) \ge 1, \tag{6}$$

with equality if and only if $S^{3/2}$ and $S^{1/2}$ are linearly dependent a.s., i.e., $S = c$ a.s. for some constant $c > 0$. Substituting $\kappa(S) \ge 1$ into (4) gives $\mathrm{CV}^2(W) \ge 1 + \frac{4(1-\rho)}{3\rho} = \frac{4-\rho}{3\rho} > 1$ (since $\rho < 1$).

Part (c). Because $R > 0$ a.s., the event $\{W > 0\}$ coincides a.s. with $\{G \ge 1\}$, which has probability ρ. By the memoryless property of G, $(G \mid G \ge 1) \overset{d}{=} 1+G$, so $W_+ \overset{d}{=} \sum_{i=1}^{1+G} R_i$. Letting $N_+ := 1 + G$ ($\mathbb{E}[N_+] = (1-\rho)^{-1}$, $\mathrm{Var}(N_+) = \rho(1-\rho)^{-2}$), we obtain $\mathbb{E}[W_+] = \mathbb{E}[R]/(1-\rho) = m_2/(2(1-\rho)m_1)$ and

$$\mathrm{Var}(W_+) = \frac{1}{1-\rho}(\mathbb{E}[R^2] - \mathbb{E}[R]^2) + \frac{\rho}{(1-\rho)^2}\mathbb{E}[R]^2 = \frac{\mathbb{E}[R^2]}{1-\rho} + \frac{2\rho-1}{(1-\rho)^2}\mathbb{E}[R]^2.$$

Dividing by $\mathbb{E}[W_+]^2 = \mathbb{E}[R]^2/(1-\rho)^2$ and substituting $\mathbb{E}[R^2]/\mathbb{E}[R]^2 = \frac{4}{3}\kappa(S)$ yields (5). Applying (6) yields the floor $(2\rho-1) + \frac{4}{3}(1-\rho) = \frac{1+2\rho}{3}$ with equality iff S is a.s. constant. Finally, $(2\rho-1) + \frac{4}{3}(1-\rho)\kappa(S) = 1 \iff \frac{4}{3}(1-\rho)\kappa(S) = 2(1-\rho) \iff \kappa(S) = 3/2$ (as $\rho < 1$), which holds for $S \sim \mathrm{Exp}(\mu)$ since $m_r = r!/\mu^r$ gives $\kappa = 1 \cdot 6/2^2 = 3/2$. □

3 $G/G/1$ Lindley–Kingman Conservation and Lundberg Exponential Tails

Let $(S_n, T_n)_{n\ge1}$ be i.i.d. nonnegative service and interarrival pairs with finite second moments, and set $X_n := S_n - T_n$ with strictly negative drift $\mathbb{E}[X] = -\delta < 0$ and finite variance $\sigma_X^2 := \mathrm{Var}(X) < \infty$. By Loynes' theorem [1], the Lindley recursion $W_{n+1} = \max(0, W_n + X_{n+1})$ admits a unique stationary law $W \overset{d}{=} \sup_{n\ge0}\sum_{i=1}^{n} X_i$, satisfying $W' := \max(0, W + X) \overset{d}{=} W$ where X is independent of W. Define the corresponding stationary idle period $I := \max(0, -(W + X)) \ge 0$, so that

$$W + X = W' - I, \qquad W'I = 0 \quad \text{a.s.} \tag{7}$$

Theorem 3.1 ($G/G/1$ Waiting–Idle Conservation and Lundberg–Kingman Tail Bounds)**.** *Let W be the stationary waiting time of a stable $G/G/1$ queue with increment $X = S-T$ ($\mathbb{E}[X] = -\delta < 0$, $\sigma_X^2 = \mathrm{Var}(X) < \infty$).*

(a) (Exact Waiting–Idle Conservation and Mean Sandwich.) *Whenever* $\mathbb{E}[W^2] < \infty$ *(guaranteed for instance when* $\mathbb{E}[X_+^3] < \infty$*), the stationary idle period satisfies* $\mathbb{E}[I] = \delta$ *and*

$$2\delta\,\mathbb{E}[W] + \mathrm{Var}(I) = \sigma_X^2, \qquad \mathbb{E}[X_+] \le \mathbb{E}[W] \le \frac{\sigma_X^2}{2\delta}. \tag{8}$$

(b) (Lundberg–Kingman Exponential Tail Ceiling and Right Skip-Free Exactness.) *Suppose* $\mathbb{P}(X > 0) > 0$ *and there exists* $\theta^* > 0$ *solving* $\mathbb{E}[e^{\theta^* X}] = 1$*. Then for every* $x \ge 0$*,*

$$\mathbb{P}(W \ge x) \le e^{-\theta^* x}, \qquad \mathbb{E}[W] \le \frac{1}{\theta^*}. \tag{9}$$

Moreover, if X *is integer-valued and skip-free to the right (*$X \in \mathbb{Z}$ *with* $X \le 1$ *a.s.), then* $\mathbb{P}(W \ge x) = e^{-\theta^* x}$ *holds with exact equality for every integer* $x \ge 0$*.*

Proof. Part (a). Since $\mathbb{E}[W^2] < \infty$, both $\mathbb{E}[W]$ and $\mathbb{E}[I] \le \mathbb{E}[|X|] + \mathbb{E}[W]$ are finite. Taking expectations of $W + X = W' - I$ in (7) and using $\mathbb{E}[W'] = \mathbb{E}[W]$ gives $\mathbb{E}[I] = -\mathbb{E}[X] = \delta$. Squaring (7) and invoking the orthogonality $W'I = 0$ gives

$$W^2 + 2WX + X^2 = (W')^2 + I^2.$$

Taking expectations, using independence of W and X (so $\mathbb{E}[WX] = -\delta\mathbb{E}[W]$), and cancelling the finite quantity $\mathbb{E}[W^2] = \mathbb{E}[(W')^2]$ yields

$$-2\delta\,\mathbb{E}[W] + \sigma_X^2 + \delta^2 = \mathbb{E}[I^2] = \mathrm{Var}(I) + \delta^2,$$

which rearranges to $2\delta\,\mathbb{E}[W] + \mathrm{Var}(I) = \sigma_X^2$. Nonnegativity $\mathrm{Var}(I) \ge 0$ gives Kingman's upper bound $\mathbb{E}[W] \le \sigma_X^2/(2\delta)$ [5]. For the lower bound, $W \ge 0$ a.s. implies $W' = \max(0, W + X) \ge \max(0, X) = X_+$, hence $\mathbb{E}[W] = \mathbb{E}[W'] \ge \mathbb{E}[X_+]$.

Part (b). Let $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$ for i.i.d. copies X_i of X, so $W \overset{d}{=} M := \sup_{n \ge 0} S_n$. Because $\mathbb{E}[e^{\theta^* X}] = 1$, the process $Z_n := e^{\theta^* S_n}$ is a nonnegative martingale with $Z_0 = 1$. At $x = 0$, $\mathbb{P}(M \ge 0) = 1 = e^0$. For $x > 0$, define the stopping time $\tau_x := \inf\{n \ge 1 : S_n \ge x\}$. For any finite integer $N \ge 1$, the optional stopping theorem at the bounded stopping time $\tau_x \wedge N$ gives

$$1 = \mathbb{E}[Z_{\tau_x \wedge N}] = \mathbb{E}\big[e^{\theta^* S_{\tau_x}} \mathbf{1}_{\{\tau_x \le N\}}\big] + \mathbb{E}\big[e^{\theta^* S_N} \mathbf{1}_{\{\tau_x > N\}}\big]. \tag{10}$$

Since both terms are nonnegative and $S_{\tau_x} \ge x$ on $\{\tau_x \le N\}$, we obtain $1 \ge e^{\theta^* x}\mathbb{P}(\tau_x \le N)$. Letting $N \to \infty$ gives $\mathbb{P}(W \ge x) = \mathbb{P}(M \ge x) = \mathbb{P}(\tau_x < \infty) \le e^{-\theta^* x}$. Integrating $\mathbb{P}(W > x) \le \mathbb{P}(W \ge x) \le e^{-\theta^* x}$ over $x \in [0, \infty)$ yields $\mathbb{E}[W] = \int_0^\infty \mathbb{P}(W > x)\,dx \le 1/\theta^*$.

Finally, suppose $X \in \mathbb{Z}$ and $X \le 1$ a.s., and let $x \in \mathbb{Z}_{\ge 1}$. Because S_n can increase by at most 1 per step, on $\{\tau_x < \infty\}$ we have $S_{\tau_x} = x$ identically (zero overshoot), while on $\{\tau_x > N\}$ we have $S_N \le x - 1$, so $0 \le e^{\theta^* S_N}\mathbf{1}_{\{\tau_x > N\}} \le e^{\theta^* x}$. By the strong law of large numbers, $S_N/N \to -\delta < 0$ a.s., hence $S_N \to -\infty$ and $e^{\theta^* S_N} \to 0$ a.s. Taking $N \to \infty$ in (10) via bounded convergence yields $1 = e^{\theta^* x}\mathbb{P}(\tau_x < \infty)$, proving $\mathbb{P}(W \ge x) = e^{-\theta^* x}$. □

4 Erlang-B/C Duality and Super-Linear Multi-Server Pooling

Consider a stable $M/M/c$ multi-server queue ($c \ge 1$ parallel servers, each with exponential rate $\mu > 0$, Poisson arrival rate $\lambda > 0$, offered load $a := \lambda/\mu < c$, and per-server utilization $\rho := a/c \in (0, 1)$). The classical Erlang-B blocking probability and Erlang-C delay probability [2, 6] are

$$B(c, a) := \frac{a^c/c!}{\sum_{m=0}^{c} a^m/m!}, \qquad C(c, a) := \frac{(a^c/c!)/(1-\rho)}{\sum_{m=0}^{c-1} a^m/m! + (a^c/c!)/(1-\rho)}. \tag{11}$$

Theorem 4.1 (Erlang-B/C Reciprocal Duality, ρ^c Sandwich, and Super-Linear Pooling)**.** *Let $c \geq 1$, $\mu > 0$, $\rho \in (0,1)$, and $a = c\rho$. Set $R_c(a) := B(c,a)^{-1}$.*

(a) (Exact Falling-Factorial Reciprocal Tower and Erlang-B/C Duality.)

$$R_c(a) = \sum_{k=0}^{c} \frac{c!}{(c-k)!\, a^k}, \qquad B(c,a) = \frac{aB(c-1,a)}{c + aB(c-1,a)} \quad (B(0,a) := 1), \tag{12}$$

$$C(c,a) = \frac{B(c,a)}{1-\rho(1-B(c,a))}, \qquad C(c,c\rho)^{-1} = \frac{1}{\rho} + (1-\rho)\sum_{k=2}^{c} \frac{c!}{(c-k)!\, c^k}\, \rho^{-k}, \tag{13}$$

where the empty sum for $c = 1$ equals 0 (giving $C(1,\rho) = \rho$).

(b) (Universal Non-Asymptotic Two-Sided Erlang-C Sandwich.) *For every $c \geq 2$ and $\rho \in (0,1)$,*

$$\rho^c < C(c,c\rho) \leq \frac{\rho^2}{\rho + (1-\rho)(1-1/c)} < \rho, \tag{14}$$

with equality in the middle inequality if and only if $c = 2$.

(c) (Exact Super-Linear Multi-Server Pooling Ratio and Waiting Dispersion.) *Let W_q denote the stationary $M/M/c$ queueing delay and $W_q^{(1)}$ that of a single $M/M/1$ queue at the same per-server utilization ρ and service rate μ ($\mathbb{E}[W_q^{(1)}] = \frac{\rho}{\mu(1-\rho)}$). Then*

$$\mathbb{E}[W_q] = \frac{C(c,c\rho)}{c\mu(1-\rho)}, \qquad \mathrm{CV}^2(W_q) = \frac{2}{C(c,c\rho)} - 1 > 1, \tag{15}$$

and for every $c \geq 2$ and $\rho \in (0,1)$, server pooling strictly outperforms linear $1/c$ scaling:

$$\frac{\rho^{c-1}}{c} < \frac{\mathbb{E}[W_q]}{\mathbb{E}[W_q^{(1)}]} = \frac{C(c,c\rho)}{c\rho} \leq \frac{\rho}{c\rho + (c-1)(1-\rho)} < \frac{1}{c}. \tag{16}$$

Proof. Part (a). Substituting $k = c - m$ in $R_c(a) = B(c,a)^{-1} = \sum_{m=0}^{c} \frac{c!}{m!\, a^{c-m}}$ gives the first identity in (12). Splitting off the $k = 0$ term yields $R_c(a) = 1 + \frac{c}{a} R_{c-1}(a)$, whose reciprocal is the recursion in (12). From (11), multiplying numerator and denominator of $C(c,a)$ by $(1-\rho)(c!/a^c)$ gives

$$C(c,a)^{-1} = 1 + (1-\rho)\big(R_c(a) - 1\big) = 1 + (1-\rho)\left(\frac{1}{B(c,a)} - 1\right) = \frac{1-\rho(1-B(c,a))}{B(c,a)},$$

proving the first identity in (13). Substituting $R_c(c\rho) - 1 = \frac{1}{\rho} + \sum_{k=2}^{c} \frac{c!}{(c-k)!\, c^k} \rho^{-k}$ (since the $k = 1$ term is $\frac{c}{c\rho} = \frac{1}{\rho}$) and simplifying $1 + (1-\rho)\frac{1}{\rho} = \frac{1}{\rho}$ yields the second identity in (13).

Part (b). Fix $c \geq 2$ and $\rho \in (0,1)$. Every falling-factorial coefficient $p_{c,k} := \frac{c!}{(c-k)!\, c^k} = \prod_{j=0}^{k-1}(1 - j/c)$ satisfies $0 < p_{c,k} < 1$ for all $2 \leq k \leq c$, with $p_{c,2} = 1 - 1/c$. Bounding every term in (13) from above by $p_{c,k} < 1$ gives the exact geometric cancellation

$$C(c,c\rho)^{-1} < \frac{1}{\rho} + (1-\rho)\sum_{k=2}^{c} \rho^{-k} = \frac{1}{\rho} + (1-\rho)\frac{\rho^{-c-1} - \rho^{-2}}{\rho^{-1} - 1} = \frac{1}{\rho} + \left(\rho^{-c} - \frac{1}{\rho}\right) = \rho^{-c},$$

so $C(c,c\rho) > \rho^c$. Conversely, retaining only the $k = 2$ term in (13) gives

$$C(c,c\rho)^{-1} \geq \frac{1}{\rho} + (1-\rho)\left(1 - \frac{1}{c}\right)\rho^{-2} = \frac{\rho + (1-\rho)(1-1/c)}{\rho^2} > \frac{1}{\rho},$$

with equality in the first inequality if and only if there are no $k \geq 3$ terms, i.e., $c = 2$. Taking reciprocals proves (14).

Part (c). In the stationary $M/M/c$ birth–death chain, an arriving customer waits with probability $C(c, c\rho)$, and conditional on finding $c + j$ customers ($j \geq 0$, distributed geometrically with parameter ρ), must wait for $j + 1$ exponential departures at rate $c\mu$, which is exponentially distributed with rate $\alpha := c\mu(1-\rho)$ [6]. Thus $W_q \sim (1-C)\delta_0 + C\,\mathrm{Exp}(\alpha)$ ($C := C(c, c\rho) \in (0,1)$), giving $\mathbb{E}[W_q] = C/\alpha$, $\mathbb{E}[W_q^2] = 2C/\alpha^2$, $\mathrm{Var}(W_q) = C(2-C)/\alpha^2$, and $\mathrm{CV}^2(W_q) = \frac{2-C}{C} = \frac{2}{C} - 1 > 1$. Dividing $\mathbb{E}[W_q]$ by $\mathbb{E}[W_q^{(1)}] = \rho/(\mu(1-\rho))$ yields $\frac{C(c,c\rho)}{c\rho}$, and dividing (14) by $c\rho$ yields (16). □

5 Open Jackson Network Entropic Laws and Load-Balancing Bounds

Consider an open Jackson network of $J \geq 1$ single-server exponential stations with external Poisson arrival vector $\gamma \in \mathbb{R}^J_{\geq 0} \setminus \{0\}$, service-rate vector $\mu \in \mathbb{R}^J_{>0}$, and substochastic routing matrix $P \in \mathbb{R}^{J\times J}_{\geq 0}$ whose spectral radius satisfies $\rho_{\mathrm{spec}}(P) < 1$ (so every customer eventually exits). The unique effective arrival vector $\lambda := (I - P^\top)^{-1}\gamma = \sum_{m=0}^{\infty}(P^\top)^m\gamma$ defines station utilizations $\rho_j := \lambda_j/\mu_j$. Whenever $\rho_j \in (0,1)$ for all $j \in [J]$, Jackson's theorem [4] gives the unique product-form stationary distribution

$$\pi(n_1, \ldots, n_J) = \prod_{j=1}^{J}(1-\rho_j)\rho_j^{n_j} \qquad (n \in \mathbb{Z}^J_{\geq 0}). \tag{17}$$

Denote the binary entropy $h_2(p) := -p\ln p - (1-p)\ln(1-p)$ and Bernoulli KL divergence $D_{\mathrm{KL}}(\mathrm{Bern}(p) \,\|\, \mathrm{Bern}(q)) := p\ln(p/q) + (1-p)\ln((1-p)/(1-q))$ for $p, q \in (0,1)$.

Theorem 5.1 (Jackson Product-Form Entropy, KL/χ^2 Duality, and Load-Balancing Floor)**.** *Let π be the stationary law* (17) *of a stable J-station open Jackson network with utilizations $\rho \in (0,1)^J$, total external arrival rate $\Gamma := \sum_{j=1}^J \gamma_j > 0$, average load $\bar\rho := \frac{1}{J}\sum_{j=1}^J \rho_j$, and bottleneck load $\rho_{\max} := \max_{1\leq j\leq J}\rho_j$.*

(a) (Exact Shannon Entropy and Bernoulli-Magnified KL Divergence.)

$$H(\pi) = \sum_{j=1}^{J}\frac{h_2(\rho_j)}{1-\rho_j}, \qquad D_{\mathrm{KL}}(\pi \,\|\, \tilde\pi) = \sum_{j=1}^{J}\frac{D_{\mathrm{KL}}(\mathrm{Bern}(\rho_j) \,\|\, \mathrm{Bern}(\tilde\rho_j))}{1-\rho_j} \tag{18}$$

for any second stable J-station Jackson law $\tilde\pi$ with utilizations $\tilde\rho \in (0,1)^J$.

(b) (Exact Pearson χ^2-Divergence Phase Boundary and Divergence Chain.) *The Pearson χ^2-divergence $\chi^2(\pi \,\|\, \tilde\pi) := \sum_{n\in\mathbb{Z}^J_{\geq 0}}\pi(n)^2/\tilde\pi(n) - 1$ is finite if and only if $\rho_j^2 < \tilde\rho_j$ for every $j \in [J]$, in which case*

$$1 + \chi^2(\pi \,\|\, \tilde\pi) = \prod_{j=1}^{J}\frac{(1-\rho_j)^2}{(1-\tilde\rho_j)(1-\rho_j^2/\tilde\rho_j)}, \tag{19}$$

and $2\,\mathrm{TV}(\pi,\tilde\pi)^2 \leq D_{\mathrm{KL}}(\pi \,\|\, \tilde\pi) \leq \ln(1 + \chi^2(\pi \,\|\, \tilde\pi)) \leq \chi^2(\pi \,\|\, \tilde\pi)$.

(c) (Exact Jensen Load-Balancing Floor and Bottleneck Sojourn Sandwich.) *The stationary total population $N_{\mathrm{tot}} := \sum_{j=1}^J N_j$ and mean network sojourn time $\mathbb{E}[T_{\mathrm{net}}] = \mathbb{E}[N_{\mathrm{tot}}]/\Gamma$ satisfy*

$$\max\left(\frac{J\bar\rho}{1-\bar\rho},\ \frac{\rho_{\max}}{1-\rho_{\max}}\right) \leq \mathbb{E}[N_{\mathrm{tot}}] = \sum_{j=1}^{J}\frac{\rho_j}{1-\rho_j} \leq \frac{J\rho_{\max}}{1-\rho_{\max}}, \tag{20}$$

with equality in the Jensen floor $\frac{J\bar\rho}{1-\bar\rho}$ (and in the bottleneck ceiling $\frac{J\rho_{\max}}{1-\rho_{\max}}$) if and only if all stations are balanced ($\rho_1 = \cdots = \rho_J = \bar\rho$).

Proof. Part (a). By product-form independence in (17), $H(\pi) = \sum_{j=1}^{J} H(\pi_j)$ and $D_{\mathrm{KL}}(\pi \,\|\, \tilde{\pi}) = \sum_{j=1}^{J} D_{\mathrm{KL}}(\pi_j \,\|\, \tilde{\pi}_j)$, where $\pi_j \sim \mathrm{Geom}_0(1-\rho_j)$ has $\mathbb{E}[N_j] = \frac{\rho_j}{1-\rho_j}$. For a single geometric factor,

$$H(\pi_j) = -\mathbb{E}_{\pi_j}\big[\ln(1-\rho_j) + N_j \ln \rho_j\big] = -\ln(1-\rho_j) - \frac{\rho_j}{1-\rho_j} \ln \rho_j = \frac{-(1-\rho_j)\ln(1-\rho_j) - \rho_j \ln \rho_j}{1-\rho_j} = \frac{h_2(\rho_j)}{1-\rho_j},$$

and identically

$$D_{\mathrm{KL}}(\pi_j \,\|\, \tilde{\pi}_j) = \mathbb{E}_{\pi_j}\left[\ln \frac{1-\rho_j}{1-\tilde{\rho}_j} + N_j \ln \frac{\rho_j}{\tilde{\rho}_j}\right] = \frac{(1-\rho_j)\ln \frac{1-\rho_j}{1-\tilde{\rho}_j} + \rho_j \ln \frac{\rho_j}{\tilde{\rho}_j}}{1-\rho_j} = \frac{D_{\mathrm{KL}}(\mathrm{Bern}(\rho_j) \,\|\, \mathrm{Bern}(\tilde{\rho}_j))}{1-\rho_j}.$$

Summing over $j = 1, \ldots, J$ establishes (18).

Part (b). By Tonelli's theorem for nonnegative series on $\mathbb{Z}_{\geq 0}^{J}$,

$$1 + \chi^2(\pi \,\|\, \tilde{\pi}) = \sum_{n \in \mathbb{Z}_{\geq 0}^{J}} \prod_{j=1}^{J} \frac{(1-\rho_j)^2}{1-\tilde{\rho}_j} \left(\frac{\rho_j^2}{\tilde{\rho}_j}\right)^{n_j} = \prod_{j=1}^{J} \left[\frac{(1-\rho_j)^2}{1-\tilde{\rho}_j} \sum_{n_j=0}^{\infty} \left(\frac{\rho_j^2}{\tilde{\rho}_j}\right)^{n_j}\right].$$

Each positive geometric series converges if and only if $\rho_j^2/\tilde{\rho}_j < 1$ (equivalently $\rho_j^2 < \tilde{\rho}_j$), evaluating to $(1 - \rho_j^2/\tilde{\rho}_j)^{-1}$ and proving (19). The divergence chain follows from Pinsker's inequality $2\mathrm{TV}(\pi, \tilde{\pi})^2 \leq D_{\mathrm{KL}}(\pi \,\|\, \tilde{\pi})$, Jensen's inequality on concave ln applied to $\mathbb{E}_{\pi}[\ln(\pi/\tilde{\pi})] \leq \ln \mathbb{E}_{\pi}[\pi/\tilde{\pi}] = \ln(1 + \chi^2(\pi \,\|\, \tilde{\pi}))$, and $\ln(1+u) \leq u$ for $u \geq 0$.

Part (c). Summing $\mathbb{E}[N_j] = \frac{\rho_j}{1-\rho_j}$ gives $\mathbb{E}[N_{\mathrm{tot}}]$, and Little's law [6] gives $\mathbb{E}[T_{\mathrm{net}}] = \mathbb{E}[N_{\mathrm{tot}}]/\Gamma$. The function $\varphi(u) := \frac{u}{1-u} = (1-u)^{-1} - 1$ on $(0,1)$ is strictly increasing ($\varphi'(u) = (1-u)^{-2} > 0$) and strictly convex ($\varphi''(u) = 2(1-u)^{-3} > 0$). Strict convexity yields Jensen's inequality $\frac{1}{J}\sum_{j=1}^{J} \varphi(\rho_j) \geq \varphi(\bar{\rho})$ with equality iff $\rho_1 = \cdots = \rho_J = \bar{\rho}$, while positivity and strict monotonicity of φ yield $\varphi(\rho_{\max}) \leq \sum_{j=1}^{J} \varphi(\rho_j) \leq J\varphi(\rho_{\max})$ with upper equality iff $\rho_j = \rho_{\max}$ for all j. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified in deterministic IEEE-754 `float64` arithmetic (seeds `20261129`–`20261131`) using the MIT-licensed companion repository `lluiseriksson/queueing-network-entropic-geometry`. Our suite covers 200 structured and randomized queueing instances across four independent modules: (i) 60 $M/G/1$ queues (deterministic $M/D/1$, $M/M/1$, and 50 random discrete service laws cross-checked via 32-point complex Cauchy contour differentiation of $\phi_W(z)$); (ii) 40 exact discrete $G/G/1$ Lindley Markov chains ($w_{\max} = 250$, boundary mass $< 10^{-12}$); (iii) 60 multi-server $M/M/c$ systems ($c \in \{1,2,3,4,5,8,12,20,35,50\}$, $\rho \in [0.15, 0.96]$); and (iv) 40 randomized J-station open Jackson networks ($J \in \{2,3,4,5\}$, $\rho_{\mathrm{spec}}(P) \leq 0.65$) plus balanced witnesses. Table 1 and Figures 1–2 confirm zero violations across all 200 checks.

Table 1: Deterministic verification across 200 queueing instances (`pytest -q`: 4/4 passed in 0.60 s). All exact identities hold to machine or solver precision with zero inequality violations.

Theorem	Checks	Max Identity / Relative Error	Sharpest Attainment / Regime Summary
Thm. 2.1 ($M/G/1$ P–K & Floors)	60	1.78×10^{-15} (CV^2 formulas)	$\min(\mathrm{CV}^2/\mathrm{floor}) = 1.0000$ ($M/D/1$ exact), $M/M/1$ $\mathrm{CV}^2(W_+) \equiv 1$
Thm. 3.1 ($G/G/1$ Conservation)	40	1.64×10^{-11} (cons. law)	$\max(\mathbb{P}(W \geq x)/e^{-\theta^* x}) = 1.0000$ (skip-free exactness)
Thm. 4.1 (Erlang-B/C Pooling)	60	1.12×10^{-15} (rel. B, C^{-1})	Exact C upper equality at $c = 2$; strict pooling $< 1/c$ for $c \geq 2$
Thm. 5.1 (Jackson Entropy/KL/χ^2)	40	1.78×10^{-15} (H), 1.94×10^{-16} (KL), 8.88×10^{-16} (χ^2)	Exact Jensen equality on balanced 3-station network; full divergence chain verified

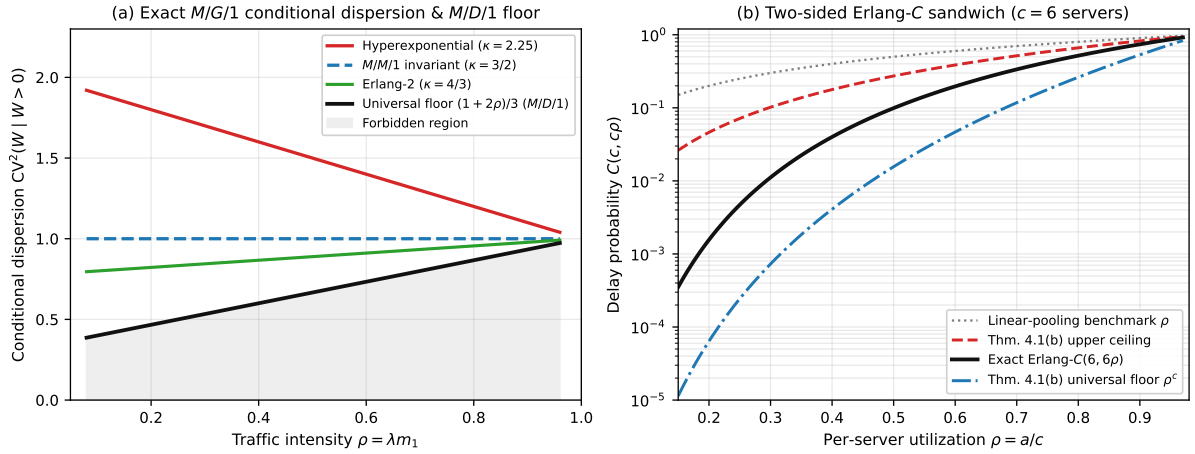


Figure 1: **(a)** Exact $M/G/1$ conditional positive-wait dispersion $\mathrm{CV}^2(W \mid W > 0)$ from (5) across service families, illustrating the forbidden region below the universal $M/D/1$ floor $(1+2\rho)/3$ and the flat $M/M/1$ invariant at 1. **(b)** Exact Erlang-$C(6, 6\rho)$ delay probability tightly bracketed by the closed-form two-sided sandwich (14) below the linear-pooling benchmark ρ.

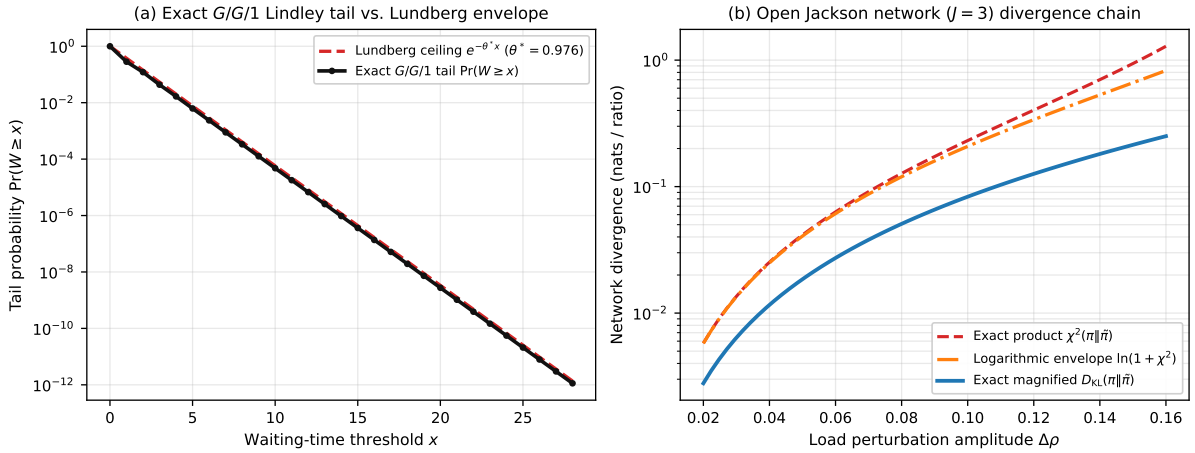


Figure 2: **(a)** Exact discrete $G/G/1$ Lindley survival curve $\mathbb{P}(W \geq x)$ bounded from above by the Lundberg–Kingman exponential ceiling $e^{-\theta^* x}$ ($\theta^* = 0.976$). **(b)** Exact open Jackson network ($J = 3$) divergence chain $D_{\mathrm{KL}}(\pi \,\|\, \tilde{\pi}) \leq \ln(1 + \chi^2(\pi \,\|\, \tilde{\pi})) \leq \chi^2(\pi \,\|\, \tilde{\pi})$ from (18)–(19) as load perturbation $\Delta\rho$ approaches the $\rho_j^2 = \tilde{\rho}_j$ χ^2-divergence boundary.

7 Scope and Limitations

Three structural boundaries merit explicit note:

1. *Finite moments and light tails.* Theorem 2.1 requires finite third service moment $m_3 < \infty$, Theorem 3.1(a) requires $\mathbb{E}[W^2] < \infty$, and Theorem 3.1(b) requires the Cramér–Lundberg exponential-moment root $\theta^* > 0$; heavy-tailed (e.g. subexponential or Pareto) service distributions fall outside these moment-based floors and decay subexponentially rather than geometrically [1].

2. *Asymptotic tightness of the ρ^c Erlang-C sandwich.* While (14) is universally valid for every $c \geq 2$ and $\rho \in (0, 1)$ (and exact on the upper side at $c = 2$), in the Halfin–Whitt square-root staffing regime $c \to \infty$ with $1 - \rho = \beta/\sqrt{c}$, $\rho^c \to 0$ whereas $C(c, c\rho)$ converges to a positive limit [3]; refining the lower bound at large c requires retaining $O(\sqrt{c})$ falling-factorial terms in (13).

3. *Markovian product-form assumption.* Theorem 5.1 relies critically on Jackson's product-form stationary distribution (17) for exponential service and Poisson external arrivals; non-Markovian generalized Jackson networks lack exact product-form marginals and must be analyzed via diffusion or Lyapunov approximations.

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