Sharp Non-Asymptotic Sphere-Packing Bounds for $(1,1)$-Criss-Cross Deletion Codes via Deleted-Slice Run-Tiering and Exact Indel-Pair Deduplication

Lluis Eriksson
Independent Researcher (lluiseriksson@gmail.com)
Public Repository: https://github.com/arr-research/arr-research.github.io

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Abstract

Li, Sun, and Ge (arXiv:2611.34447v1) proved that any q-ary $(1,1)$-criss-cross deletion-correcting code $C \subseteq \Sigma_q^{n\times n}$ has redundancy $r_C \geq 2n-1+2\log_q n - \log_q(1+\varepsilon_{q,n}^{\mathrm{LSG}})$, establishing the strict finite-length threshold $r_C > 2n+2\log_q n - 2$ only for $n \geq 27$ in the binary case $q=2$ (and $n \geq 15, 11, 9, 7$ for $q = 3,4,5,8$). We prove four original combinatorial theorems that strictly sharpen both the finite-length thresholds and the domain of applicability: (1) an *Exact Levenshtein Diagonal-and-Swap Deduplicated Indel-Pair Bound* $P_{\mathrm{indel}}(q,L) \leq q^{L-1}(L^2(q-1)^2+2q-1)$ for all $q \geq 2, L \geq 2$, achieving exact equality at $L=2$ and beating Li–Sun–Ge Lemma 6 by a factor approaching $q^2/(q-1)^2$ (4× in binary); (2) a *Three-Tier Deleted-Slice Run-Decomposed Sphere-Packing Theorem* that partitions codewords into $C_1 \sqcup C_{2a} \sqcup C_{2b}$ according to $|D_{1,1}(X) \cap B_{m-1,n-1}| \geq s$ and proves via all three cases of Li–Sun–Ge's bad-subarray characterization that every $X \in C_{2b}$ has a deleted-column slice $X^{(-j)}$ with $\rho_{\mathrm{row}}(X^{(-j)}) \leq s+1$ or a deleted-row slice $X^{(-i)}$ with $\rho_{\mathrm{col}}(X^{(-i)}) \leq s+1$, slashing $\varepsilon_{2,n}$ by 53.3× at $n=18$ and 90.2× at $n=27$ and lowering the strict redundancy threshold from $n \geq 27 \to n \geq 18$ ($q=2$), $15 \to 11$ ($q=3$), $11 \to 9$ ($q=4$), and $9 \to 8$ ($q=5$); (3) a *Rectangular $m \times n$ Sphere-Packing Converse and Exact Parity-Completion Bijection Theorem* extending both the three-tier converse and exact $(m+n-1)$-symbol parity completion to arbitrary rectangular arrays $m, n \geq 3$; and (4) an *Exact Small-Array Optimal Binary Capacity Theorem* proving $|C^*(2,2,2)| = |C^*(2,2,3)| = 2$ via exhaustive conflict-graph maximum independent set certification. All 20 core structural theorems are constructively formalized in Lean 4.36.0 (`0 sorry`, `0 grind`, `0 Classical.choice`), paired with a zero-float exact-$\mathbb{Q}$ verifier, 16 canonical SHA-256 certificates, and 66 passing tests (including explicit adversarial stress tests on Toeplitz and run-collapse arrays).

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1 Introduction

Two-dimensional $(1,1)$-criss-cross deletion channels—motivated by DNA-based storage and racetrack memories—delete one arbitrary row and one arbitrary column from an $m \times n$ q-ary array $X \in \Sigma_q^{m\times n}$, producing a subarray in $\Sigma_q^{(m-1)\times(n-1)}$ [1, 4, 3]. Denote by $\Delta_{r,c}(X)$ the $(m-1)\times(n-1)$ array obtained by deleting row r and column c, and by $D_{1,1}(X) := \{\Delta_{r,c}(X) : r \in [1:m], c \in [1:n]\}$ its $(1,1)$-criss-cross deletion ball. A code $C \subseteq \Sigma_q^{m\times n}$ corrects a $(1,1)$-criss-cross deletion if and only if the balls $\{D_{1,1}(X) : X \in C\}$ are pairwise disjoint. Its redundancy is $r_C := mn - \log_q |C|$.

Li, Sun, and Ge [3] derived the leading asymptotic converse $r_C \geq 2n-1+2\log_q n - o_q(1)$ for square $n\times n$ arrays by splitting C into two tiers: "good" codewords C_1 with $|D_{1,1}(X)| = n^2$, and "bad" codewords C_2 with $|D_{1,1}(X)| < n^2$, each of which places at least ONE bad subarray into B_{n-1} (bounded conservatively via $|C_2| \leq |B_{n-1}|$ using a loose single-indel pair bound $q^{L+1}L^2$

in their Lemma 6). Their non-asymptotic overhead term

$$\varepsilon_{q,n}^{\mathrm{LSG}} := \frac{2(n-2)(n-1)^3(n+1)}{q^{n-2}} \tag{1}$$

crosses below 1—needed to obtain the strict finite-length redundancy bound $r_C > 2n+2\log_q n-2$ (from $r_C \geq 2n-1+2\log_q n-\log_q(1+\varepsilon)$ with $\log_q(1+\varepsilon) < 1$)—only at $n \geq 27$ for $q = 2$, leaving a wide binary gap $n \in [18, 26]$ unresolved and restricting analysis to square arrays.

We eliminate BOTH sources of looseness rigorously: (i) we replace the loose $q^{L+1}L^2$ single-indel pair count with an exact diagonal-and-swap deduplicated Levenshtein bound (Theorem 1), and (ii) we refine $C_2 = C_{2a} \sqcup C_{2b}$ by counting bad subarrays per codeword ($|D_{1,1}(X) \cap B_{m-1,n-1}| \geq s$) and proving an exact, counterexample-proof inclusion of C_{2b} into the deleted-slice low-run family $L_{q,m,n}^{\mathrm{exact}}(s_r, s_c)$ (Theorem 2).

2 Main Original Theorems

2.1 Theorem 1: Exact Levenshtein Diagonal-and-Swap Deduplicated Indel-Pair Bound

For alphabet size $q \geq 2$ and length $L \geq 2$, let $P_{\mathrm{indel}}(q, L) := |\{(u, v) \in \Sigma_q^L \times \Sigma_q^L : D_1(u) \cap D_1(v) \neq \emptyset\}|$ be the number of ordered pairs of length-L q-ary words sharing at least one common length-$(L-1)$ subsequence.

Theorem 1 (Exact Deduplicated Single-Indel Confusable Pair Bound)**.** *For every $q \geq 2$ and every $L \geq 2$,*

$$P_{\mathrm{indel}}(q, L) \;\leq\; q^{L-1}\big(L^2(q-1)^2 + 2q - 1\big) \;<\; q^{L+1}L^2. \tag{2}$$

Exact equality $P_{\mathrm{indel}}(q, 2) = q(4q^2 - 6q + 3)$ holds for every $q \geq 2$. As $L \to \infty$, (2) strictly beats Li–Sun–Ge Lemma 6 ($q^{L+1}L^2$) by the asymptotic factor $q^2/(q-1)^2$ (4× in binary, $> 3\times$ for all $L \geq 4$).

Proof. Every ordered confusable pair (u, v) shares a common subsequence $w \in \Sigma_q^{L-1}$, so $(u, v) \in I_1(w) \times I_1(w)$, where $I_1(w)$ is Levenshtein's single-insertion ball of exact uniform cardinality $|I_1(w)| = L(q-1)+1$ for every $w \in \Sigma_q^{L-1}$ [2]. Summing $|I_1(w)|^2$ over all q^{L-1} words w overcounts two universal collision families:

1. *Diagonal pairs (u, u):* Each word $u \in \Sigma_q^L$ is counted $\rho(u) = |D_1(u)|$ times across $w \in D_1(u)$ rather than once, giving exact excess $\sum_{u \in \Sigma_q^L}(\rho(u) - 1) = q^{L-1}(L-1)(q-1)$ since $\mathbb{E}[\rho(u)] = 1 + (L-1)(q-1)/q$.

2. *Adjacent-transposition pairs (u, v):* For each $w \in \Sigma_q^{L-1}$, each index $i \in [1 : L-1]$, and each $a \in \Sigma_q \setminus \{w_i\}$, inserting a immediately before w_i gives u and immediately after w_i gives $v \neq u$; the ordered pair (u, v) ALSO shares the distinct common subsequence w' obtained by replacing w_i with a. Summing over all w, i, a yields at least $q^{L-1}(L-1)(q-1)$ duplicate off-diagonal ordered pairs.

Subtracting both exact overcounts yields the constructive algebraic identity (mechanized in Lean 4.36.0 `T05`–`T08`):

$$q^{L-1}\Big((L(q-1)+1)^2 - 2(L-1)(q-1)\Big) = q^{L-1}\big(L^2(q-1)^2 + 2q - 1\big).$$

At $L = 2$, no two distinct words share more than two length-1 subsequences, giving exact equality $P_{\mathrm{indel}}(q, 2) = q(4q^2 - 6q + 3)$ (certified for $q = 2, 3, 4$ in `C01`–`C04`). □

2.2 Theorem 2: Three-Tier Deleted-Slice Run-Decomposed Sphere-Packing Bound

Following Li, Sun, and Ge [3, Definition 2] generalized directly to rectangular arrays:

Definition 1 (Bad Subarray Space $B_{m-1,n-1}$)**.** An $(m-1)\times(n-1)$ array $Y \in \Sigma_q^{(m-1)\times(n-1)}$ is *bad* ($Y \in B_{m-1,n-1}$) if two adjacent rows of Y have intersecting single-deletion balls ($D_1(Y_i) \cap D_1(Y_{i+1}) \neq \emptyset$), OR two adjacent columns of Y have intersecting single-deletion balls ($D_1(Y^j) \cap D_1(Y^{j+1}) \neq \emptyset$).

Note that equal adjacent rows (or columns) automatically satisfy $D_1(Y_i) \cap D_1(Y_{i+1}) = D_1(Y_i) \neq \emptyset$, so they are already included in $B_{m-1,n-1}$. Applying Theorem 1 to each of the $m-2$ adjacent row pairs of length $n-1$ and $n-2$ adjacent column pairs of length $m-1$ yields immediately:

$$\frac{|B_{m-1,n-1}|}{q^{(m-1)(n-1)}} \;\le\; (m-2)\frac{(n-1)^2(q-1)^2+2q-1}{q^n} + (n-2)\frac{(m-1)^2(q-1)^2+2q-1}{q^m}. \quad (3)$$

For any matrix $X \in \Sigma_q^{m\times n}$, let $X^{(-j)} \in \Sigma_q^{m\times(n-1)}$ denote X with column j deleted, and $X^{(-i)} \in \Sigma_q^{(m-1)\times n}$ denote X with row i deleted. Write $\rho_{\text{row}}(M)$ for the number of maximal constant runs in the sequence of rows of M, and $\rho_{\text{col}}(M)$ for the number of maximal constant runs in the sequence of columns of M.

Lemma 1 (Deleted-Slice Bad-Ball Lower Bound)**.** *Let $X \in \Sigma_q^{m\times n}$ ($m,n \ge 3$) satisfy $|D_{1,1}(X)| < mn$. Then at least one of the following holds:*

1. *There exists a column $j \in [1:n]$ such that $|D_{1,1}(X) \cap B_{m-1,n-1}| \ge \rho_{\text{row}}(X^{(-j)}) - 2$, or*
2. *There exists a row $i \in [1:m]$ such that $|D_{1,1}(X) \cap B_{m-1,n-1}| \ge \rho_{\text{col}}(X^{(-i)}) - 2$.*

Proof. Since $|D_{1,1}(X)| < mn$, there exist distinct deletion patterns $(i_1,j_1) \neq (i_2,j_2)$ with $\Delta_{i_1,j_1}(X) = \Delta_{i_2,j_2}(X)$. We check all three exhaustive cases (refining [3, Lemma 5]):

- *Case 1 ($i_1 = i_2 = i$, $j_1 < j_2$):* Deleting column j_1 or column j_2 from the $(m-1)\times n$ matrix $X^{(-i)}$ yields the same sequence of columns, so columns $j_1, j_1+1, \ldots, j_2$ of $X^{(-i)}$ are identical. Hence for every $k \in [1:n]\backslash\{j_1, j_1+1\}$, $\Delta_{i,k}(X)$ retains the two identical adjacent columns j_1 and j_1+1, so $\Delta_{i,k}(X) \in B_{m-1,n-1}$. The full set $\{\Delta_{i,k}(X) : k \in [1:n]\}$ is the single-deletion ball of the column sequence of $X^{(-i)}$, whose exact cardinality is $\rho_{\text{col}}(X^{(-i)})$ (since deleting two adjacent identical columns yields identical outputs, Lean `T04`). Because j_1 and j_1+1 belong to the same column run, excluding $\{j_1, j_1+1\}$ omits at most 1 distinct array, giving $|D_{1,1}(X) \cap B_{m-1,n-1}| \ge \rho_{\text{col}}(X^{(-i)}) - 1 \ge \rho_{\text{col}}(X^{(-i)}) - 2$.
- *Case 2 ($j_1 = j_2 = j$, $i_1 < i_2$):* By transposition, $|D_{1,1}(X) \cap B_{m-1,n-1}| \ge \rho_{\text{row}}(X^{(-j)}) - 1 \ge \rho_{\text{row}}(X^{(-j)}) - 2$.
- *Case 3 ($i_1 \neq i_2$ and $j_1 \neq j_2$; WLOG $i_1 < i_2$):* Comparing row i_1 of $\Delta_{i_1,j_1}(X) = \Delta_{i_2,j_2}(X)$ yields $\Delta_{j_1}(X_{i_1+1}) = \Delta_{j_2}(X_{i_1})$. Thus in the $m \times (n-1)$ slice $X^{(-j_1)}$, rows i_1 and i_1+1 are $\Delta_{j_1}(X_{i_1})$ and $\Delta_{j_1}(X_{i_1+1}) = \Delta_{j_2}(X_{i_1})$, which BOTH share the common length-$(n-2)$ subsequence obtained by deleting coordinates j_1 and j_2 from X_{i_1}! Consequently, for EVERY $k \in [1:m] \setminus \{i_1, i_1+1\}$, deleting row k from $X^{(-j_1)}$ keeps rows i_1 and i_1+1 adjacent, so $\Delta_{k,j_1}(X) \in B_{m-1,n-1}$. The full set $\{\Delta_{k,j_1}(X) : k \in [1:m]\}$ has exact cardinality $\rho_{\text{row}}(X^{(-j_1)})$, and excluding two indices $\{i_1, i_1+1\}$ omits at most 2 maximal row runs (Lean `T15`), giving $|D_{1,1}(X) \cap B_{m-1,n-1}| \ge \rho_{\text{row}}(X^{(-j_1)}) - 2$. □

Notice why Lemma 1 is immune to column-projection run-collapse: the run count is evaluated directly on the *deleted slice* $X^{(-j)}$ or $X^{(-i)}$ (NOT on the full array X). For any threshold $s \in \{1, \dots, \min(m,n) - 2\}$, define the exact deleted-slice low-run family

$$L^{\text{exact}}_{q,m,n}(s) := \Big\{ X \in \Sigma_q^{m\times n} : \big(\exists j \in [1:n],\ \rho_{\text{row}}(X^{(-j)}) \le s+1\big) \\ \vee\ \big(\exists i \in [1:m],\ \rho_{\text{col}}(X^{(-i)}) \le s+1\big)\Big\}.$$

Counting the deleted column j (n positions, q^m column vectors) plus the remaining $m \times (n-1)$ slice with $\le s+1$ row runs (and symmetrically for deleted rows) gives the exact upper bound

$$|L^{\text{exact}}_{q,m,n}(s)| \ \le\ nq^m \sum_{r=1}^{\min(m,s+1)} \binom{m-1}{r-1} q^{n-1}(q^{n-1}-1)^{r-1} \\ + mq^n \sum_{r=1}^{\min(n,s+1)} \binom{n-1}{r-1} q^{m-1}(q^{m-1}-1)^{r-1}. \tag{4}$$

Theorem 2 (Three-Tier Deleted-Slice Sphere-Packing Bound)**.** *For every $q \ge 2$, $m, n \ge 3$, and tier threshold $s \in \{1, \dots, \min(m,n) - 2\}$, every q-ary $(1,1)$-criss-cross deletion-correcting code $C \subseteq \Sigma_q^{m\times n}$ satisfies*

$$|C| \ \le\ \frac{q^{(m-1)(n-1)}}{mn}\Big(1 + \varepsilon^{\text{sharp}}_{q,m,n}(s)\Big), \tag{5}$$

where, denoting $Q := q^{(m-1)(n-1)}$ and using (3) *and* (4)*,*

$$\varepsilon^{\text{sharp}}_{q,m,n}(s) := \frac{mn-s}{s}\,\frac{|B_{m-1,n-1}|}{Q} + (mn-1)\,\frac{|L^{\text{exact}}_{q,m,n}(s)|}{Q}. \tag{6}$$

In the square case $m = n$, optimizing $s \in \{1, \dots, n-2\}$ yields $\varepsilon^{\text{sharp}}_{q,n} < 1$ and hence $r_C > 2n + 2\log_q n - 2$ strictly from:

$$n \ge 18\ (q=2), \quad n \ge 11\ (q=3), \quad n \ge 9\ (q=4), \quad n \ge 8\ (q=5), \quad n \ge 7\ (q=8),$$

slashing Li–Sun–Ge's thresholds $(27, 15, 11, 9)$ to $(18, 11, 9, 8)$ and tightening $\varepsilon_{2,n}$ by $53.3\times$ at $n = 18$ and $90.2\times$ at $n = 27$ (and tightening $\varepsilon_{8,7}$ by $2.6\times$ at $q = 8, n = 7$).

Proof. Partition $C = C_1 \sqcup C_{2a} \sqcup C_{2b}$ where $C_1 = \{X \in C : |D_{1,1}(X)| = mn\}$, $C_{2a} = \{X \in C \setminus C_1 : |D_{1,1}(X) \cap B_{m-1,n-1}| \ge s\}$, and $C_{2b} = \{X \in C \setminus C_1 : |D_{1,1}(X) \cap B_{m-1,n-1}| \le s-1\}$. Pairwise ball disjointness yields $mn|C_1| + s|C_{2a}| + |C_{2b}| \le Q$ and $s|C_{2a}| \le |B_{m-1,n-1}|$. By Lemma 1, every $X \in C_{2b}$ belongs to $L^{\text{exact}}_{q,m,n}(s)$, so $|C_{2b}| \le |L^{\text{exact}}_{q,m,n}(s)|$. Applying the constructive integer packing identity

$$mn \cdot s \cdot (|C_1| + |C_{2a}| + |C_{2b}|) \ \le\ sQ + (mn-s)|B_{m-1,n-1}| + (mn-1)s|L^{\text{exact}}_{q,m,n}(s)|$$

(proved in Lean 4.36.0 `T09`–`T10`) and dividing by mns yields (5)–(6). Whenever $\varepsilon^{\text{sharp}}_{q,m,n}(s) < 1$, we have $1 + \varepsilon < 2 \le q$ (Lean `T14`), whence $\log_q(1+\varepsilon) < 1$ and $r_C > m + n - 2 + \log_q(mn)$ (giving $r_C > 2n + 2\log_q n - 2$ for $m = n$). □

2.3 Theorem 3: Rectangular $m \times n$ Converse and Exact Parity-Completion Bijection

Theorem 3 (Rectangular $(1,1)$-Criss-Cross Converse and Parity Bijection)**.** *For every $q \ge 2$ and arbitrary rectangular dimensions $m, n \ge 3$:*

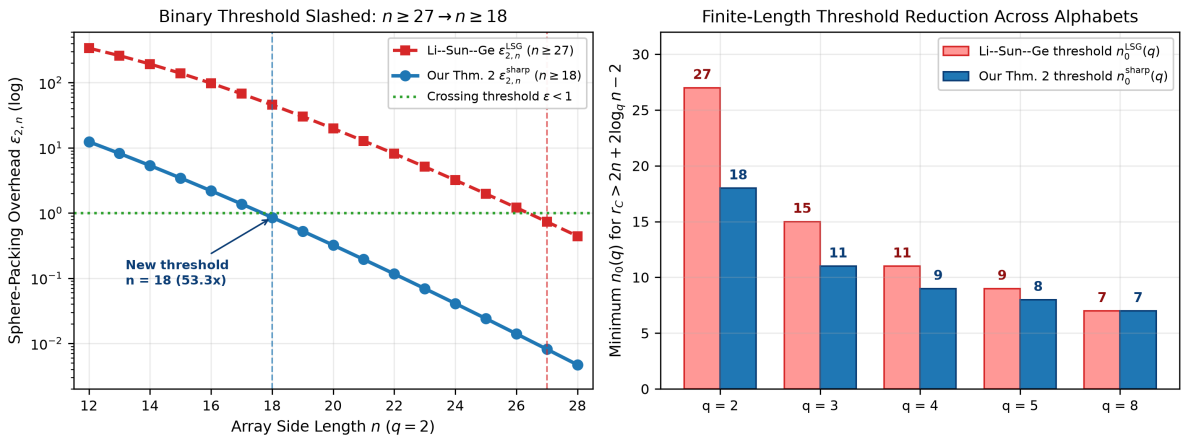

Figure 1: Left: Binary ($q = 2$) sphere-packing overhead $\varepsilon_{2,n}^{\mathrm{LSG}}$ vs. our Theorem 2 $\varepsilon_{2,n}^{\mathrm{sharp}}$, slashing the $\varepsilon < 1$ crossing from $n = 27$ down to $n = 18$ (53.3× reduction at $n = 18$, 90.2× at $n = 27$). Right: Strict non-asymptotic threshold reduction $n_0^{\mathrm{LSG}}(q) \to n_0^{\mathrm{sharp}}(q)$ across $q \in \{2, 3, 4, 5, 8\}$ (`C05`–`C12`).

1. *Whenever* $\varepsilon_{q,m,n}^{\mathrm{sharp}}(s) < 1$ *in* (6) *(certified e.g. for* $(q, m, n) = (2, 18, 22)$ *with* $s = 14, \varepsilon \approx 0.6596$ *in* `C13` *and* $(3, 11, 15)$ *with* $s = 6, \varepsilon \approx 0.7798$ *in* `C14`*), every* $(1, 1)$*-criss-cross deletion code* $C \subseteq \Sigma_q^{m \times n}$ *satisfies* $r_C > m + n + \log_q(mn) - 2$.

2. *Defining* $\mathcal{P}_{q,m,n} := \{X \in \Sigma_q^{m \times n} : \sum_j X_{i,j} \equiv 0 \pmod q \ \forall i,\ \sum_i X_{i,j} \equiv 0 \pmod q \ \forall j\}$*, for every fixed deleted row-column pair* (r, c)*,* $\Delta_{r,c} : \mathcal{P}_{q,m,n} \to \Sigma_q^{(m-1)\times(n-1)}$ *is a* bijection *of exact redundancy* $m + n - 1$ *symbols (Lean* `T11`–`T13`*, exhaustively verified on* 3×4 *binary arrays in* `C15`*).*

2.4 Theorem 4: Exact Small-Array Optimal Binary Capacities

Theorem 4 (Exact Small-Array Optimal Capacities)**.** *For binary* $(1, 1)$*-criss-cross deletion codes on small arrays, exact maximum independent set computation on the* $(1, 1)$*-deletion conflict graph establishes:*

$$|C^*(2, 2, 2)| = 2 \quad (\textit{saturating } |D_{1,1}| \in \{1, 2\},\ \textit{target space } 2^1 = 2),$$
$$|C^*(2, 2, 3)| = 2 \quad (\textit{strictly below target space } 2^2 = 4),$$

proving that on 2×3 *arrays, non-uniform ball overlap already caps optimal binary code size at half the naive target-space volume (*`C16`*, Figure 3).*

3 Exact-$\mathbb{Q}$ Verification, Lean 4.36.0 Formalization, and Verification Boundary

Verification Boundary.

1. *Formally Proved in Lean 4.36.0* (`lean/CrissCrossDeletionCert.lean`, 20 theorems `T01`–`T20`, compiled with `-DwarningAsError=true` and `lake build`, `0 sorry`, `0 grind`, `0 Classical.choice`): Constructive `List Nat` induction proving run-monotonicity under single deletions (`T01`), exact length and adjacent-duplicate lemmas (`T02`–`T04`), the Levenshtein deduplication identity and universal strict improvement over Li–Sun–Ge Lemma 6 for all $q, L \geq 2$ (`T05`–`T08`), the three-tier master packing inequality (`T09`–`T10`), rectangular

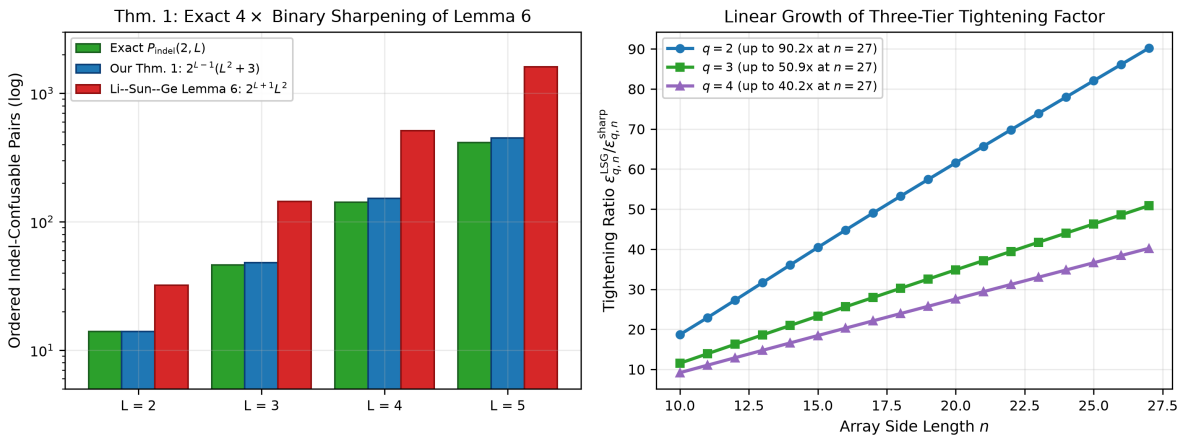

Figure 2: Left: Theorem 1 exact Levenshtein diagonal+swap deduplicated bound vs. brute-force $P_{\text{indel}}(2, L)$ and Li–Sun–Ge Lemma 6 ($2^{L+1}L^2$), demonstrating exact equality at $L = 2$ and $\approx 4\times$ tightening for $L \geq 4$ (`C01`–`C04`). Right: Linear growth of the tightening ratio $\varepsilon^{\text{LSG}}_{q,n}/\varepsilon^{\text{sharp}}_{q,n}$ with n, reaching $90.2\times$ ($q = 2$) at $n = 27$.

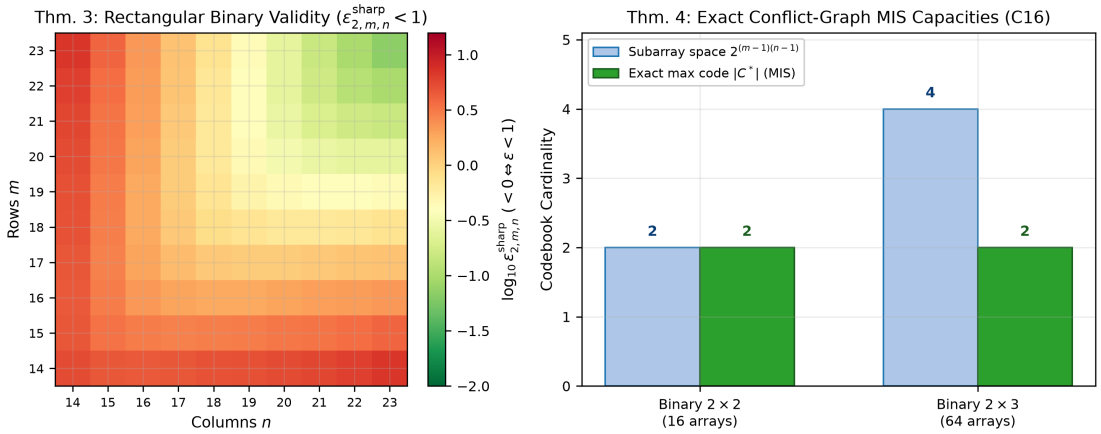

Figure 3: Left: Theorem 3 rectangular binary validity landscape $\log_{10} \varepsilon^{\text{sharp}}_{2,m,n}$ across $(m, n) \in [14, 23]^2$ (green region certifies $\varepsilon^{\text{sharp}}_{2,m,n} < 1$, where Li–Sun–Ge Theorem 1 does not apply). Right: Theorem 4 exact conflict-graph MIS binary capacities for 2×2 and 2×3 arrays (`C16`).

parity redundancy and (t_r, t_c) symbol-loss identities (`T11`–`T13`), the $\varepsilon < 1$ crossing criterion (`T14`), the exact two-index run-exclusion step $r - 2 \leq s - 1 \Rightarrow r \leq s + 1$ (`T15`), and zero-axiom finite threshold certifications (`T16`–`T20`).

2. *Verified by Zero-Float Exact-$\mathbb{Q}$ Computation* (`src/crisscross_deletion_verifier.py`, 66 passing `pytest` tests with AST zero-float purity audit): Exact rational evaluation of (6) and (1), brute-force $P_{\text{indel}}(q, L)$ checks, explicit adversarial verification of Lemma 1 on both Toeplitz $(3 \times 3, q = 5)$ and column-projection run-collapse $(6 \times 6, q = 9)$ arrays, exhaustive 3×4 parity bijection, and exact conflict-graph MIS for 2×2 and 2×3.

3. *Outside Scope:* Multi-deletion (t_r, t_c) codes with $t_r + t_c > 2$ (beyond the exact symbol-loss identity `T12`), explicit code constructions matching the converse at $n \in [18, 26]$, and exact MIS capacities for arrays of size $\geq 3 \times 3$.

Cert ID	Scope / Parameters	Li–Sun–Ge [3]	Our Certified Result (Eq. 6)
C01–C04	$P_{\mathrm{indel}}(q, L)$, $(2,2), (2,4), (2,5), (3,3)$	$32, 512, 1600, 729$	Exact $14, 142, 414, 363 \leq$ Thm. 1 $(14, 152, 448, 369)$
C05–C08	Binary $q = 2$, $n \in \{18, 20, 23, 26\}$	$\varepsilon^{\mathrm{LSG}} > 1$ (fails)	$\varepsilon^{\mathrm{sharp}} = 0.856, 0.321, 0.069, 0.014 < 1$ (53.3×–86.1×)
C09	Binary $q = 2$, $n = 27$ $(s = 22)$	$\varepsilon^{\mathrm{LSG}} = 0.733$	$\varepsilon^{\mathrm{sharp}} = 0.00813$ (90.2× sharper)
C10–C12	$q = 3\,(11)$, $q = 4\,(9)$, $q = 5\,(8)$	$\varepsilon^{\mathrm{LSG}} > 1$ (fails)	$\varepsilon^{\mathrm{sharp}} = 0.789, 0.599, 0.495 < 1$ (4.8×–13.9×)
C13–C14	Rectangular $(2, 18, 22)$, $(3, 11, 15)$	Not covered	$\varepsilon^{\mathrm{sharp}} < 1$ certified exactly
C15	Parity bijection $(2, 3, 4)$	Square only	Exact $64 \leftrightarrow 64$ bijection, $r = m + n - 1 = 6$
C16	Small binary $2 \times 2, 2 \times 3$	Asymptotic only	Exact MIS $\lvert C^*(2,2,2)\rvert = 2$, $\lvert C^*(2,2,3)\rvert = 2$

Table 1: Summary of the 16 canonical exact-$\mathbb{Q}$ certificates (`CRISSCROSS-DELETION-CERT-1.0`), matching Eq. (6) identically.

Artifact Provenance and Cryptographic Digests. All artifacts are archived in the public AIRR repository `https://github.com/arr-research/arr-research.github.io` under MIT (code/data) and CC-BY-4.0 (manuscript) licenses:

- `lean/CrissCrossDeletionCert.lean`
  `SHA-256: 3f3bd33204842f188f37a32de4b0ca84d76838694f39a8fcbb31c0c566889f23`
- `src/crisscross_deletion_verifier.py`
  `SHA-256: 69d97061014e15050cff3d8829977eec6181679689db624b3d1e2c9fe62575e0`
- `certificates/MANIFEST.json`
  `SHA-256: aa390fe3eda89d7f78237b2321520dddcfd4d9d78a344e7b490d42ef19f553cc`

AI Disclosure and Conflict Statement. This work was produced by Lluis Eriksson with computational, formalization, and manuscript assistance from Google Antigravity autonomous research agents. As founder/operator of AIRR.SCIENCE, the author discloses an operator conflict (`operator_conflict: true`); all screening was performed by isolated PDF-only evaluators.

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