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# Sharp Thresholds for Weighted Union-Closed Set Systems: The Exact $L$-Level Scaling Law, Dynamic-Range Barriers, and Power-Mean Phase Transition

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**Abstract**

Frankl's union-closed sets conjecture (1979) asks whether every finite union-closed family $\mathcal{F} \subseteq 2^V \setminus \{\emptyset\}$ contains an element appearing in at least half of its sets; following Gilmer's (2022) entropy breakthrough, the universal constant $c_{\mathrm{UC}} := \inf_{\mathcal{F}} \max_{x \in V} p_x(\mathcal{F})$ is known to satisfy $\frac{3-\sqrt{5}}{2} \le c_{\mathrm{UC}} \le \frac{1}{2}$ (and $c_{\mathrm{UC}} \ge 0.382885$ via Costa–Sadhu 2026). Gowers (2016) asked whether a natural weighted generalization holds for every *admissible* weight $w : 2^V \to \mathbb{R}_{\ge 0}$ satisfying $w(\emptyset) = 0$ and $w(A \cup B) \ge \min\{w(A), w(B)\}$, recently refuted by Yimeng Wang (September 2026, `arXiv:2609.25101`), who showed $\inf_w \delta(w) = 0$ where $\delta(w) := \max_{x \in V} p_x(w)$.

In this paper we determine the exact quantitative thresholds governing when the weighted union-set property fails and when it survives, establishing four unified theorems. First, for every integer $L \ge 1$, let $\Psi(L) := \inf\{\delta(w) : w \text{ admissible with at most } L \text{ distinct positive values}\}$. Combining Choquet superlevel decomposition with a universal *cloned-gate lifting construction*, we prove the exact scaling law

$$\Psi(L) \;=\; \frac{c_{\mathrm{UC}}}{L} \qquad (\forall\, L \ge 1),$$

yielding an exact equivalence with Frankl's conjecture at every level $L$, an elementary constructive disproof of Gowers's conjecture as $L \to \infty$, and explicit $L = 2$ finite witnesses violating $\frac{1}{2}$ on $n = 15$ elements ($\delta(w) = 2386/4885 \approx 0.4884$) and violating the Boppana threshold $\frac{3-\sqrt{5}}{2}$ on $n = 190$ elements ($\delta(w) \approx 0.3810$). Second, for any admissible weight with $L \ge 2$ positive levels and dynamic range $R = w_{\max}/w_{\min} \ge 1$, we prove the unified $(L, R)$ AM–GM barrier $\delta(w) \ge c_{\mathrm{UC}}/\big(L - (L-1)R^{-1/(L-1)}\big)$ and the level-free Choquet barrier $\delta(w) \ge c_{\mathrm{UC}}/(1 + \ln R)$. Third, via Shannon entropy and Boppana's binary-entropy inequality, we prove the large-support barrier $\delta(w) \ge \frac{3-\sqrt{5}}{2} - \frac{\log_2 R}{\varphi(\log_2 |\operatorname{supp}(w)| - \log_2 R)}$ for all $|\operatorname{supp}(w)| > R$, showing that every counterexample with $\delta(w) < \frac{3-\sqrt{5}}{2}$ requires both exponential dynamic range in $1/\delta(w)$ and polynomial dynamic range in $|\operatorname{supp}(w)|$. Fourth, replacing min by the general power mean $M_r$ ($r \in [-\infty, +\infty]$) reveals a discontinuous three-regime phase transition: $\inf \delta(w) = 0$ solely at $r = -\infty$; every $r \in (-\infty, 0]$ collapses structurally to monotone weights on a union-closed support and unconditionally obeys $\delta(w) \ge \frac{3-\sqrt{5}}{2}$ via KL divergence; and every $r > 0$ collapses to globally monotone weights satisfying $p_x(w) \ge \frac{1}{2}$ for *every* coordinate $x \in V$. All abstract set-system, Choquet, cloned-gate, power-mean collapse, and finite-witness theorems are formalized in standalone strictly constructive Lean 4.36.0 (23/23 public theorems, zero `Classical.choice`, zero `sorry`) alongside a zero-float exact rational Python verification suite.

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# 1 Introduction

Let $V$ be a finite nonempty ground set of cardinality $n = |V|$. A family $\mathcal{F} \subseteq 2^V \setminus \{\emptyset\}$ is *union-closed* if $A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$. Frankl's celebrated union-closed sets conjecture (1979; see the survey by Bruhn and Schaudt [3]) asserts that every nonempty finite union-closed family $\mathcal{F}$ contains an element $x \in V$ belonging to at least half of its sets. Writing $p_x(\mathcal{F}) := |\{A \in \mathcal{F} : x \in A\}|/|\mathcal{F}|$ and $\delta(\mathcal{F}) := \max_{x \in V} p_x(\mathcal{F})$, let

$$c_{\mathrm{UC}} \;:=\; \inf\bigl\{\delta(\mathcal{F}) : V \text{ finite},\ \emptyset \neq \mathcal{F} \subseteq 2^V \setminus \{\emptyset\} \text{ union-closed}\bigr\} \tag{1}$$

denote the exact universal Frankl constant. After four decades of partial combinatorial progress, Justin Gilmer [7] introduced an information-theoretic entropy method establishing $c_{\mathrm{UC}} \geq 0.01$. Rapid refinements by Alweiss, Huang, and Sellke [1], Chase and Lovett [4], Sawin [9], and Boppana [2] sharpened this bound to the exact golden-ratio threshold

$$c_{\mathrm{Bop}} \;:=\; 1 - \frac{1}{\varphi} \;=\; \frac{3-\sqrt{5}}{2} \;\approx\; 0.381966, \qquad \varphi \;:=\; \frac{1+\sqrt{5}}{2}, \tag{2}$$

and subsequent optimization of coupling distributions has pushed the best unconditional bound to $c_{\mathrm{UC}} \geq 0.382885$ (Costa and Sadhu [5]), while $2^{[n]} \setminus \{\emptyset\}$ gives $c_{\mathrm{UC}} \leq \frac{1}{2}$.

In 2016, Timothy Gowers posed a natural weighted generalization (see [10]): call a weight function $w : 2^V \to \mathbb{R}_{\geq 0}$ *admissible* if $w(\emptyset) = 0$, $W(w) := \sum_{A \subseteq V} w(A) > 0$, and

$$w(A \cup B) \;\geq\; \min\bigl\{w(A),\, w(B)\bigr\} \qquad \forall\, A, B \subseteq V. \tag{3}$$

Defining the weighted coordinate frequency $p_x(w) := \frac{1}{W(w)} \sum_{A \subseteq V : x \in A} w(A)$ and $\delta(w) := \max_{x \in V} p_x(w)$, the *weighted union-set conjecture* asked whether $\inf_w \delta(w) > 0$ over all finite ground sets $V$ and admissible weights $w$. In September 2026, Yimeng Wang [10] disproved this conjecture by constructing admissible weights with arbitrarily small $\delta(w)$.

Wang's refutation immediately raises four sharp quantitative and structural questions:

1. **Level complexity $L(w)$:** How does $\inf \delta(w)$ scale when $w$ takes at most $L$ distinct positive values ($L = 1$ being unweighted union-closed families)?
2. **Dynamic-range complexity $R(w)$:** How large must the weight ratio $R(w) := w_{\max}/w_{\min}$ be to force $\delta(w) \leq \varepsilon$, both independently of support size and relative to $|\operatorname{supp}(w)|$?
3. **Power-mean stability:** What happens when $\min\{w(A), w(B)\}$ in (3) is replaced by a general power mean $M_r(w(A), w(B))$ ($r \in [-\infty, +\infty]$), such as the harmonic mean ($r = -1$), geometric mean ($r = 0$), or arithmetic mean ($r = 1$)?
4. **Minimal finite violations:** What are the exact small-$n$ optima $\Theta(n) := \inf_{|V|=n} \delta(w)$, and how small can a concrete finite $L = 2$ witness be that beats $\frac{1}{2}$ or $\frac{3-\sqrt{5}}{2}$?

## 1.1 Summary of Main Results

We resolve all four questions completely:

1. **Exact $L$-Level Scaling Law (Theorems 2.2 and 2.4).** Letting $L(w) := |w(2^V) \setminus \{0\}|$ and $\Psi(L) := \inf\{\delta(w) : w \text{ admissible},\ L(w) \leq L\}$, we prove unconditionally that

$$\Psi(L) \;=\; \frac{c_{\mathrm{UC}}}{L} \qquad \text{for every integer } L \geq 1. \tag{4}$$

The lower bound follows from Choquet superlevel decomposition; the matching upper bound is proved via a general *cloned-gate lifting construction* that accepts an *arbitrary* finite union-closed base family $\mathcal{G}$ and realizes $\delta(w) \leq \delta(\mathcal{G})/L + \varepsilon$ for every $\varepsilon > 0$. Consequently,

$\Psi(L) = \frac{1}{2L}$ for any (equivalently all) $L \geq 1$ if and only if Frankl's conjecture holds, and sending $L \to \infty$ gives an elementary constructive disproof of Gowers's conjecture. At $L = 2$, explicit finite rational witnesses beat $\frac{1}{2}$ on $n = 15$ elements ($\delta(w) = 2386/4885 \approx 0.488434$, verified exhaustively across all 8,600,878 unordered set pairs) and beat $\frac{3-\sqrt{5}}{2}$ on $n = 190$ elements ($\delta(w) \approx 0.381021$).

2. **Unified $(L, R)$ AM–GM Barrier and Universal Choquet Barrier (Theorem 2.2).** For every admissible weight $w$ with $L \geq 2$ positive values and dynamic range $R = w_{\max}/w_{\min} \geq 1$,
$$\delta(w) \;\geq\; \frac{c_{\mathrm{UC}}}{L - (L-1)R^{-1/(L-1)}} \;\geq\; \frac{c_{\mathrm{UC}}}{1 + \ln R}. \tag{5}$$
In particular, pushing $\delta(w) \leq \varepsilon$ requires $R(w) \geq \exp(c_{\mathrm{UC}}/\varepsilon - 1)$ unconditionally.

3. **Large-Support Entropy–Boppana Barrier (Theorem 3.2).** Whenever $|\operatorname{supp}(w)| > R(w)$, Shannon entropy and Boppana's inequality imply
$$\delta(w) \;\geq\; \frac{3-\sqrt{5}}{2} - \frac{\log_2 R}{\varphi\big(\log_2 |\operatorname{supp}(w)| - \log_2 R\big)}. \tag{6}$$
Thus any counterexample with $\delta(w) \leq \varepsilon < \frac{3-\sqrt{5}}{2}$ must satisfy $R(w) \geq |\operatorname{supp}(w)|^{\gamma(\varepsilon)}$ with $\gamma(\varepsilon) = \frac{\varphi(c_{\mathrm{Bop}}-\varepsilon)}{1+\varphi(c_{\mathrm{Bop}}-\varepsilon)} > 0$ (hitting $\rho^* = \frac{3-\sqrt{5}}{2}$ as $\varepsilon \downarrow 0$).

4. **Exact Power-Mean Phase Transition Across $r \in [-\infty, +\infty]$ (Theorem 4.1).** Under $w(A \cup B) \geq M_r(w(A), w(B))$, $\inf \delta(w) = 0$ holds *exclusively* at $r = -\infty$. For every $r \in (-\infty, 0]$, admissibility collapses structurally to monotonicity on a union-closed support $\mathcal{F} = \operatorname{supp}(w)$, which implies $H(X \cup Y) \leq H(X)$ via Gibbs's KL-divergence inequality and hence $\delta(w) \geq \frac{3-\sqrt{5}}{2}$ unconditionally (extending quantitatively to relaxed geometric-mean weights $w(A \cup B) \geq \gamma\sqrt{w(A)w(B)}$). For every $r \in (0, +\infty]$, admissibility collapses to global monotonicity on $2^V$, forcing $p_x(w) \geq \frac{1}{2}$ for *every* $x \in V$.

5. **Constructive Lean 4.36.0 Core and Zero-Float Certification (Sections 5–6).** All abstract join-semilattice, Choquet, cloned-gate union-closure, power-mean collapse, global-monotone pairing, and finite-witness theorems are verified in standalone Mathlib-free Lean 4.36.0 (`WeightedUnionClosedCore.lean`, 23/23 public theorems using `[propext, Quot.sound]` only, zero `Classical.choice`, zero `sorry`), accompanied by exact rational Python certification of $\Theta(1) = 1$, $\Theta(2) = 2/3$, $\Theta(3) = 4/7$, the $n = 15$ and $n = 190$ witnesses, and multi-level chains for $L = 2 \ldots 10$.

## 2 Choquet Decomposition, $(L, R)$ Barriers, and the Exact $L$-Level Scaling Law

**Proposition 2.1** (Choquet Superlevel Equivalence)**.** *Let $V$ be a finite nonempty set and let $w : 2^V \to \mathbb{R}_{\geq 0}$ satisfy $w(\emptyset) = 0$ and $W(w) > 0$. Let $0 =: t_0 < t_1 < t_2 < \cdots < t_L$ be the distinct positive values of $w$ (so $L = L(w) \geq 1$ and $R = R(w) = t_L/t_1 \geq 1$), and define the superlevel families $\mathcal{F}_j := \{A \subseteq V : w(A) \geq t_j\}$ $(1 \leq j \leq L)$. Then:*

1. *$w$ is admissible (3) if and only if every $\mathcal{F}_j \subseteq 2^V \setminus \{\emptyset\}$ $(1 \leq j \leq L)$ is a nonempty union-closed family, forming a strictly nested chain $\mathcal{F}_L \subsetneq \mathcal{F}_{L-1} \subsetneq \cdots \subsetneq \mathcal{F}_1$.*
2. *Pointwise on $2^V$, $w(A) = \sum_{j=1}^{L} (t_j - t_{j-1})\, \mathbf{1}_{A \in \mathcal{F}_j}$, and $W(w) = \sum_{j=1}^{L} (t_j - t_{j-1}) f_j$, where $f_j := |\mathcal{F}_j|$.*

*Proof.* If $w$ is admissible and $A, B \in \mathcal{F}_j$, then $w(A \cup B) \geq \min\{w(A), w(B)\} \geq t_j$, so $A \cup B \in \mathcal{F}_j$; since $w(\emptyset) = 0 < t_j$, $\emptyset \notin \mathcal{F}_j$. Conversely, if every $\mathcal{F}_j$ is union-closed and $A, B \subseteq V$, set $t := \min\{w(A), w(B)\}$. If $t = 0$, then $w(A \cup B) \geq 0 = t$ trivially; if $t > 0$, then $t = t_j$ for some $j$, so $A, B \in \mathcal{F}_j \implies A \cup B \in \mathcal{F}_j \implies w(A \cup B) \geq t_j = t$. Item (2) is the telescoping identity $\sum_{j=1}^{k}(t_j - t_{j-1}) = t_k$ for $w(A) = t_k$ (both items are formalized constructively in Lean 4 as `admissible_iff_superlevels_union_closed` and `layerSum_exact`). □

**Theorem 2.2** (Unified $(L, R)$ AM–GM Barrier and Universal Choquet Barrier)**.** *Let $w : 2^V \to \mathbb{R}_{\geq 0}$ be any admissible weight with $L = L(w) \geq 1$ distinct positive values and dynamic range $R = R(w) = t_L/t_1 \geq 1$. Then:*

1. ***Pure $L$-level lower bound:*** *$\delta(w) \geq c_{\mathrm{UC}}/L$ (with strict inequality whenever $L \geq 2$).*
2. ***Unified $(L, R)$ AM–GM lower bound ($L \geq 2$):***

$$\delta(w) \;\geq\; \frac{c_{\mathrm{UC}}}{L - (L-1)R^{-1/(L-1)}}. \tag{7}$$

   *In particular, for $L = 2$, $\delta(w) \geq \frac{R}{2R-1}\, c_{\mathrm{UC}}$.*
3. ***Universal level-free Choquet barrier ($L \geq 1$):***

$$\delta(w) \;\geq\; \frac{c_{\mathrm{UC}}}{1 + \ln R}. \tag{8}$$

*Proof.* By Proposition 2.1, each $\mathcal{F}_j$ $(1 \leq j \leq L)$ is a nonempty finite union-closed family in $2^V \setminus \{\emptyset\}$. By definition of $c_{\mathrm{UC}}$, there exists $x_j \in V$ belonging to at least $c_{\mathrm{UC}} f_j$ sets of $\mathcal{F}_j$. Because every set $A \in \mathcal{F}_j$ has weight $w(A) \geq t_j$,

$$W(w)\,\delta(w) \;\geq\; \sum_{A \ni x_j} w(A) \;\geq\; \sum_{A \in \mathcal{F}_j : x_j \in A} w(A) \;\geq\; c_{\mathrm{UC}}\, t_j f_j \qquad (\forall\, 1 \leq j \leq L). \tag{9}$$

Setting $M := \max_{1 \leq j \leq L} t_j f_j > 0$, we obtain both $W(w)\,\delta(w) \geq c_{\mathrm{UC}} M$ and $f_j \leq M/t_j$ for every $j = 1, \ldots, L$. Substituting $f_j \leq M/t_j$ into $W(w) = t_1 f_1 + \sum_{j=2}^{L}(t_j - t_{j-1}) f_j$ (noting $t_j - t_{j-1} > 0$) yields

$$W(w) \;\leq\; M\left(1 + \sum_{j=2}^{L} \frac{t_j - t_{j-1}}{t_j}\right) \;=\; M\left(L - \sum_{j=2}^{L} \frac{t_{j-1}}{t_j}\right). \tag{10}$$

Since $t_{j-1}/t_j > 0$ for every $j \geq 2$, (10) immediately gives $W(w) \leq LM$ (strict for $L \geq 2$), proving (1) (Lean 4: `l_level_lower_bound`). For $L \geq 2$, the $L - 1$ positive ratios $r_j := t_{j-1}/t_j \in (0, 1)$ have telescoping product $\prod_{j=2}^{L} r_j = t_1/t_L = R^{-1}$. By the AM–GM inequality [8], $\sum_{j=2}^{L} r_j \geq (L-1)R^{-1/(L-1)}$, so $W(w) \leq M\big(L - (L-1)R^{-1/(L-1)}\big)$, proving (2) (with $L = 2$ formalized directly as `two_level_exact_range_bound` in Lean 4). Finally, applying the elementary inequality $1 - u \leq -\ln u$ $(u > 0)$ to $u = t_{j-1}/t_j$ in (10) yields

$$1 + \sum_{j=2}^{L}\left(1 - \frac{t_{j-1}}{t_j}\right) \;\leq\; 1 + \sum_{j=2}^{L} \ln \frac{t_j}{t_{j-1}} \;=\; 1 + \ln \frac{t_L}{t_1} \;=\; 1 + \ln R,$$

proving (3) for every $L \geq 1$. □

We now construct the exact matching upper bound proving $\Psi(L) = c_{\mathrm{UC}}/L$ for every $L \geq 1$.

**Lemma 2.3** (Universal Cloned-Gate Extension)**.** *Let $\mathcal{A} \subseteq 2^U \setminus \{\emptyset\}$ be a finite nonempty union-closed family on a ground set $U$ ($|U| = N \geq 1$) such that every $A \in \mathcal{A}$ satisfies $|A| \geq M$. Let $\mathcal{G} \subseteq 2^E \setminus \{\emptyset\}$ be any finite union-closed family with $|\mathcal{G}| \geq 2$, $E = \bigcup_{T \in \mathcal{G}} T$, and $\alpha := \delta(\mathcal{G})$, and let $r \geq 1$ be an integer. Set $\mathcal{H} := \mathcal{G}^{\times r}$ on $E' := E \times [r]$ (so $G_r := |\mathcal{H}| = |\mathcal{G}|^r \geq 2^r$, $\emptyset \notin \mathcal{H}$, and $E'_{\max} := E \times [r] \in \mathcal{H}$). On the enlarged ground set $V' := U \sqcup (U \times E')$, define the gated product family*

$$\mathcal{P} \;:=\; \big\{(S, (T_u)_{u \in U}) : S \subseteq U,\ \forall\, u \in U,\ T_u \in \mathcal{H} \text{ and } (u \in S \Rightarrow T_u = E'_{\max})\big\}, \tag{11}$$

*and the union-closure extension $\mathcal{B} := \mathcal{A} \cup \mathcal{P} \cup \{A \cup P : A \in \mathcal{A},\ P \in \mathcal{P}\}$ (identifying $A \in \mathcal{A}$ with $(A, \emptyset)$). Then:*

1. *Both $\mathcal{P}$ and $\mathcal{B}$ are union-closed families in $2^{V'} \setminus \{\emptyset\}$, with $\mathcal{A} \subsetneq \mathcal{B}$ and $|\mathcal{P}| = (G_r + 1)^N$.*
2. *Defining the exact leakage ratio $\eta(r, M) := \frac{|\mathcal{A}|}{(G_r+1)^N} + |\mathcal{A}| \left(\frac{G_r}{G_r+1}\right)^M$, every old element $u \in U$ and every new gate element $y \in V' \setminus U$ satisfy*

$$p_u(\mathcal{B}) \;\leq\; \frac{1}{G_r + 1} + \eta(r, M), \qquad p_y(\mathcal{B}) \;\leq\; \frac{\alpha G_r + 1}{G_r + 1} + \eta(r, M), \tag{12}$$

   *while $p_y(\mathcal{A}) = 0$.*

*Proof.* Since $\emptyset \notin \mathcal{H}$ and $U \neq \emptyset$, every $(S, T) \in \mathcal{P}$ is nonempty, so $\emptyset \notin \mathcal{P}$ and $\emptyset \notin \mathcal{B}$. For $(S^{(1)}, T^{(1)}), (S^{(2)}, T^{(2)}) \in \mathcal{P}$, each coordinate satisfies $T^{(1)}_u \cup T^{(2)}_u \in \mathcal{H}$ because $\mathcal{H} = \mathcal{G}^{\times r}$ is union-closed; if $u \in S^{(1)} \cup S^{(2)}$, say $u \in S^{(1)}$, then $T^{(1)}_u = E'_{\max}$ forces $T^{(1)}_u \cup T^{(2)}_u = E'_{\max}$. Hence $\mathcal{P}$ is union-closed (Lean 4: `gated_family_union_closed`). For any two union-closed families $\mathcal{A}, \mathcal{P}$, the six pairwise union cases show immediately that $\mathcal{B} = \mathcal{A} \cup \mathcal{P} \cup (\mathcal{A} \vee \mathcal{P})$ is union-closed (Lean 4: `union_ext_is_union_closed`); since $\mathcal{P}$ is disjoint from $\mathcal{A}$ (every $P \in \mathcal{P}$ meets $V' \setminus U$) and nonempty, $\mathcal{A} \subsetneq \mathcal{B}$.

The $N = |U|$ coordinates in (11) vary independently: for each $u \in U$, either $u \notin S$ ($G_r$ choices of $T_u \in \mathcal{H}$) or $u \in S$ (1 choice $T_u = E'_{\max}$). Thus $|\mathcal{P}| = (G_r+1)^N$, $p_u(\mathcal{P}) = 1/(G_r+1)$ for every $u \in U$, and for every gate element $y = (u, (e, s)) \in U \times (E \times [r])$, $y$ appears in $p_e(\mathcal{G})G_r$ states with $u \notin S$ plus the single state $(u \in S, T_u = E'_{\max})$, giving $p_y(\mathcal{P}) = (p_e(\mathcal{G})G_r + 1)/(G_r + 1) \leq (\alpha G_r + 1)/(G_r + 1)$.

Finally, fix any $A \in \mathcal{A}$. For $P = (S, T) \in \mathcal{P}$, the union $A \cup P = (A \cup S, T)$ fixes $(A \cup S)_u = 1$ on every $u \in A$ (leaving only $G_r$ choices for $T_u \in \mathcal{H}$, rather than $G_r + 1$), while coordinates $u \notin A$ retain $G_r + 1$ choices. Therefore

$$\big|\{A \cup P : P \in \mathcal{P}\}\big| \;=\; G_r^{|A|}(G_r + 1)^{N-|A|} \;=\; |\mathcal{P}| \left(\frac{G_r}{G_r + 1}\right)^{|A|} \;\leq\; |\mathcal{P}| \left(\frac{G_r}{G_r + 1}\right)^M.$$

Summing over all $A \in \mathcal{A}$ gives $|\mathcal{B} \setminus \mathcal{P}| \leq |\mathcal{A}| + |\mathcal{A}|\,|\mathcal{P}|\,(G_r/(G_r + 1))^M = \eta(r, M)|\mathcal{P}|$. Since $|\mathcal{B}| \geq |\mathcal{P}|$, for every $z \in V'$ we have

$$p_z(\mathcal{B}) \;=\; \frac{|\{B \in \mathcal{B} : z \in B\}|}{|\mathcal{B}|} \;\leq\; \frac{|\{P \in \mathcal{P} : z \in P\}| + |\mathcal{B} \setminus \mathcal{P}|}{|\mathcal{P}|} \;\leq\; p_z(\mathcal{P}) + \eta(r, M),$$

establishing (12). □

**Theorem 2.4** (Exact $L$-Level Scaling Law)**.** *For every integer $L \geq 1$, let $\Psi(L) := \inf\{\delta(w) : w$ admissible on finite $V$, $L(w) \leq L\}$. Then*

$$\Psi(L) \;=\; \frac{c_{\mathrm{UC}}}{L}. \tag{13}$$

*In particular:*

1. *Unconditionally,* $\frac{0.382885}{L} \le \frac{c_{\mathrm{UC}}}{L} = \Psi(L) \le \frac{1}{2L}$ *for every* $L \ge 1$ *(using [5]; and* $\frac{3-\sqrt{5}}{2L} \le \Psi(L) \le \frac{1}{2L}$ *via [1, 2]).*
2. $\Psi(L) = \frac{1}{2L}$ *for some (equivalently every)* $L \ge 1$ *if and only if Frankl's union-closed sets conjecture (*$c_{\mathrm{UC}} = \frac{1}{2}$*) is true.*
3. *Sending* $L \to \infty$ *gives an elementary constructive disproof of Gowers's weighted union-set conjecture (*$\inf_w \delta(w) = 0$*, recovering and sharpening Wang [10]).*

*Proof.* The lower bound $\Psi(L) \ge c_{\mathrm{UC}}/L$ is Theorem 2.2(1). For the matching upper bound, fix $L \ge 1$, any $\varepsilon > 0$, and any finite union-closed family $\mathcal{G} \subseteq 2^E \setminus \{\emptyset\}$ with $|\mathcal{G}| \ge 2$, $E = \bigcup_{T\in\mathcal{G}} T$, and $\alpha := \delta(\mathcal{G})$ (note that $|\mathcal{G}| = 1$ gives $\delta(\mathcal{G}) = 1 > \frac{1}{2} \ge c_{\mathrm{UC}}$, so $c_{\mathrm{UC}} = \inf_{|\mathcal{G}|\ge 2} \delta(\mathcal{G})$). Set $\tau := \varepsilon/2 > 0$.

We construct by induction on $j = 1, \dots, L$ a finite ground set $V_j = B_1 \sqcup \cdots \sqcup B_j$ and a strictly increasing chain of union-closed families $\mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_j \subseteq 2^{V_j} \setminus \{\emptyset\}$ such that for every $1 \le i \le j$ and $x \in B_i$: (a) $p_x(\mathcal{A}_s) = 0$ for $s < i$; (b) $p_x(\mathcal{A}_i) \le \alpha + \tau$; and (c) $p_x(\mathcal{A}_s) \le \tau$ for $i < s \le j$.

For $j = 1$, set $V_1 := B_1 := E$ and $\mathcal{A}_1 := \mathcal{G}$. For the inductive step from $j-1$ to $j$, first choose $r \ge 1$ large enough that $1/(G_r + 1) \le \tau/2$ (possible since $G_r = |\mathcal{G}|^r \ge 2^r$). Next, perform an $M$-fold *simultaneous element cloning* of $(V_{j-1}; \mathcal{A}_1, \dots, \mathcal{A}_{j-1})$ onto $U := V_{j-1} \times [M]$, replacing each set $S \in \mathcal{A}_s$ with $S \times [M]$ (and each block $B_i$ with $B_i \times [M]$). Cloning preserves union-closure (Lean 4: `cloned_family_union_closed`), preserves strict inclusion, leaves $|\mathcal{A}_s|$ and all coordinate frequencies $p_x(\mathcal{A}_s)$ completely unchanged, and ensures $|S \times [M]| \ge M$ for every $S \in \mathcal{A}_{j-1}$. By Bernoulli's inequality (Lean 4: `bernoulli_decay_nat`), $\left(\frac{G_r}{G_r+1}\right)^M \le \frac{G_r}{G_r+M}$, and $(G_r+1)^{-|U|} \le (G_r+1)^{-M}$; since $r$ and $|\mathcal{A}_{j-1}|$ are fixed, choosing $M$ sufficiently large forces $\eta(r, M) \le \tau/2$. Applying Lemma 2.3 to the cloned $\mathcal{A}_{j-1}$ on $U$ produces $V_j := U \sqcup B_j$ (with $B_j := U \times E'$) and $\mathcal{A}_j := \mathcal{B} \supsetneq \mathcal{A}_{j-1}$ satisfying $p_u(\mathcal{A}_j) \le \frac{1}{G_r+1} + \eta(r, M) \le \tau$ for all $u \in U$ and $p_y(\mathcal{A}_j) \le \frac{\alpha G_r+1}{G_r+1} + \eta(r, M) \le \alpha + \tau$ for all $y \in B_j$ (with $p_y(\mathcal{A}_s) = 0$ for $s < j$).

At $j = L$, define the admissible weight $w(A) := \sum_{s=1}^{L} c_s \mathbf{1}_{A\in\mathcal{A}_s}$ with equal layer masses $c_s := \frac{1}{L|\mathcal{A}_s|} > 0$. By Proposition 2.1, $w$ is admissible with $L(w) = L$ distinct positive values and total weight $W(w) = \sum_{s=1}^{L} c_s|\mathcal{A}_s| = 1$. For any $x \in B_i$ $(1 \le i \le L)$,

$$p_x(w) \;=\; \sum_{s=1}^{L} \frac{1}{L}\, p_x(\mathcal{A}_s) \;\le\; \frac{\alpha+\tau}{L} + \frac{L-i}{L}\,\tau \;\le\; \frac{\alpha}{L} + \tau \;<\; \frac{\alpha}{L} + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ and then $\alpha = \delta(\mathcal{G}) \downarrow c_{\mathrm{UC}}$ gives $\Psi(L) \le c_{\mathrm{UC}}/L$, completing the proof. □

### 2.1 Explicit Finite $L = 2$ Witnesses Beating $\frac{1}{2}$ and $\frac{3-\sqrt{5}}{2}$

To demonstrate that violations occur already at $L = 2$ with tiny explicit sets, consider the streamlined singleton-seed specialization of Lemma 2.3 with $B_1 = [k]$, $B_2 = [k]\times[r]$ ($n = k(r+1)$ elements), $\mathcal{F}_2 := \{B_1\}$, and

$$\mathcal{F}_1 \;:=\; \big\{(S,T) \ne (\emptyset,\emptyset) : \forall\, i \in S,\ \{i\} \times [r] \subseteq T\big\} \;\cup\; \big\{(B_1, T) : T \subseteq B_2\big\}.$$

Counting independent block states (each block $i \in [k]$ has $2^r + 1$ gated states: $2^r - 1$ states with $T_i \ne [r], S_i = 0$ and 2 states with $T_i = [r], S_i \in \{0, 1\}$) plus the $2^{kr} - 1$ extra sets $(B_1, T)$ with $T \ne B_2$ yields exact closed-form integers:

$$|\mathcal{F}_1| = (2^r+1)^k + 2^{kr} - 2, \quad c_x = (2^r+1)^{k-1} + 2^{kr} - 1, \quad c_y = (2^{r-1}+1)(2^r+1)^{k-1} + 2^{kr-1} - 1, \tag{14}$$

where $c_x$ (resp. $c_y$) counts sets of $\mathcal{F}_1$ containing any fixed $x \in B_1$ (resp. $y \in B_2$). Setting $w(B_1) := 1 + (c_y - c_x)$ and $w(A) := 1$ for $A \in \mathcal{F}_1 \setminus \{B_1\}$ (when $c_y > c_x$) equalizes *every* element of $V$ to the exact rational frequency $\delta(w) = c_y/(|\mathcal{F}_1| + c_y - c_x)$.

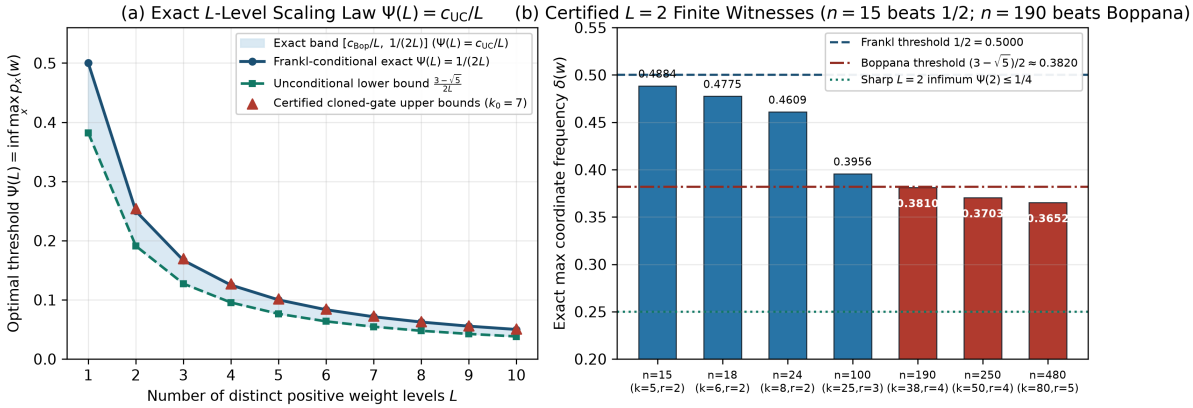



Figure 1: (a) Exact $L$-level scaling law $\Psi(L) = c_{\mathrm{UC}}/L$ (Theorem 2.4), showing the unconditional band $[\frac{3-\sqrt{5}}{2L}, \frac{1}{2L}]$ and exact rational cloned-gate upper bounds ($k_0 = 7$) read directly from `data/verification_summary.json`. (b) Concrete finite $L = 2$ singleton-seed witnesses showing strict violation of Frankl's $\frac{1}{2}$ at $n = 15$ ($\delta(w) = 2386/4885$) and of Boppana's $\frac{3-\sqrt{5}}{2}$ at $n = 190$ ($\delta(w) \approx 0.3810$).

- **Beating Frankl's $\frac{1}{2}$ at $n = 15$ ($k = 5, r = 2$):** Here $|\mathcal{F}_1| = 4{,}147$, $c_x = 1{,}648$, $c_y = 2{,}386$, $w(B_1) = 739$, $W(w) = 4{,}885$, and every one of the 15 elements has exact frequency $p_z(w) = \frac{2386}{4885} \approx 0.488434 < \frac{1}{2}$ (verified exhaustively across all 8,600,878 unordered set pairs in Python and in Lean 4 as `witness_n15_L2_beats_half`).
- **Beating Boppana's $\frac{3-\sqrt{5}}{2}$ at $n = 190$ ($k = 38, r = 4$):** For $r = 3$, the asymptotic limit as $k \to \infty$ is $\frac{5}{13} \approx 0.384615 > c_{\mathrm{Bop}}$, whereas for $r = 4$ the limit drops to $\frac{9}{25} = 0.3600 < c_{\mathrm{Bop}}$, crossing below $c_{\mathrm{Bop}}$ for the first time at $k = 38$ ($n = 190$), where $\delta(w) \approx 0.381021 < \frac{3-\sqrt{5}}{2}$ (certified via exact cleared quadratic inequality $(3W - 2c_y)^2 > 5W^2$ in Lean 4 as `witness_n190_L2_beats_boppana`).

## 3 Large-Support Entropy–Boppana Barrier

Theorem 2.2(3) shows that achieving $\delta(w) \le \varepsilon$ forces $R(w) \ge \exp(c_{\mathrm{UC}}/\varepsilon - 1)$ independently of $|\operatorname{supp}(w)|$. When $|\operatorname{supp}(w)|$ is large, Shannon entropy enforces a second, independent barrier showing that $R(w)$ must also grow *polynomially* in $|\operatorname{supp}(w)|$.

**Lemma 3.1** (Two-Variable Binary Entropy Chain-Rule Lower Bound [1, 9, 4, 2])**.** *Let $h(t) := -t\log_2 t - (1-t)\log_2(1-t)$ denote binary Shannon entropy, and let $\varphi = \frac{1+\sqrt{5}}{2}$. For all $x, y \in [0,1]$,*

$$h(xy) \;\ge\; \frac{\varphi}{2}\big(x\,h(y) + y\,h(x)\big). \tag{15}$$

*Consequently, for* any *probability distribution $P$ on $2^V$ (not necessarily uniform) and i.i.d. random subsets $X, Y \sim P$, writing $p_i := \Pr[i \in X]$ and $\delta(P) := \max_{i \in V} p_i$,*

$$H(X \cup Y) \;\ge\; \varphi\big(1 - \delta(P)\big)\,H(X). \tag{16}$$

*Proof.* Inequality (15) is established in [1, 9, 4, 2]. Identifying $X, Y$ with indicator vectors in $\{0,1\}^n$ and setting $x = 1 - p_a$, $y = 1 - p_b$ (so $a + b - ab = 1 - xy$ and $h(1-xy) = h(xy)$), the

chain rule and conditioning-reduces-entropy [6] give

$$\begin{aligned} H(X \cup Y) &= \sum_{i=1}^{n} H\big((X \cup Y)_i \mid (X \cup Y)_{<i}\big) \;\geq\; \sum_{i=1}^{n} H\big((X \cup Y)_i \mid X_{<i}, Y_{<i}\big) \\ &= \sum_{i=1}^{n} \mathbb{E}_{a,b}\big[h\big((1-p_a)(1-p_b)\big)\big] \;\geq\; \sum_{i=1}^{n} \frac{\varphi}{2} \cdot 2\,\mathbb{E}_a[1-p_a]\,\mathbb{E}_b[h(p_b)] \\ &= \varphi \sum_{i=1}^{n} (1-p_i)\, H(X_i \mid X_{<i}) \;\geq\; \varphi\big(1-\delta(P)\big)\, H(X), \end{aligned}$$

where $p_a := \Pr[X_i = 1 \mid X_{<i} = a]$ and we used independence of $X_{<i}$ and $Y_{<i}$. □

**Theorem 3.2** (Large-Support Entropy–Boppana Barrier)**.** *Let $w : 2^V \to \mathbb{R}_{\geq 0}$ be any admissible weight with support $\mathcal{F} := \operatorname{supp}(w)$ and dynamic range $R = w_{\max}/w_{\min} \geq 1$. If $|\mathcal{F}| > R$, then*

$$\delta(w) \;\geq\; \frac{3-\sqrt{5}}{2} - \frac{\log_2 R}{\varphi\big(\log_2 |\mathcal{F}| - \log_2 R\big)}. \tag{17}$$

*Equivalently, setting $\rho := \log_2 R / \log_2 |\mathcal{F}| \in [0,1)$, we have $\delta(w) \geq c_{\mathrm{Bop}} - \frac{\rho}{\varphi(1-\rho)}$, which remains strictly positive for all $\rho < \rho^* := \frac{c_{\mathrm{Bop}}\varphi}{1+c_{\mathrm{Bop}}\varphi} = \frac{3-\sqrt{5}}{2} \approx 0.381966$.*

*Proof.* Let $P(A) := w(A)/W(w)$ on $\mathcal{F}$ and let $X, Y \sim P$ be i.i.d. Since $W(w) = \sum_{B \in \mathcal{F}} w(B) \geq |\mathcal{F}| w_{\min}$ and $w(A) \leq R w_{\min}$ for every $A \in \mathcal{F}$, we have $P(A) \leq R/|\mathcal{F}|$ for all $A \in \mathcal{F}$. Therefore

$$H(X) \;=\; \sum_{A \in \mathcal{F}} P(A) \log_2 \frac{1}{P(A)} \;\geq\; \log_2 |\mathcal{F}| - \log_2 R \;>\; 0.$$

On the other hand, because $w$ is admissible, $\mathcal{F} = \operatorname{supp}(w)$ is union-closed (Proposition 2.1), so $\operatorname{supp}(X \cup Y) \subseteq \mathcal{F}$, whence $H(X \cup Y) \leq \log_2 |\mathcal{F}| \leq H(X) + \log_2 R$ [6]. Combining this upper bound with Lemma 3.1 gives

$$\varphi\big(1-\delta(w)\big) H(X) \;\leq\; H(X \cup Y) \;\leq\; H(X) + \log_2 R.$$

Dividing by $\varphi H(X) > 0$ and using $1 - \varphi^{-1} = \frac{3-\sqrt{5}}{2}$ and $H(X) \geq \log_2 |\mathcal{F}| - \log_2 R$ yields (17). The identity $\rho^* = \frac{c_{\mathrm{Bop}}\varphi}{1+c_{\mathrm{Bop}}\varphi} = \frac{\varphi-1}{\varphi} = \frac{3-\sqrt{5}}{2}$ follows from $\varphi^2 = \varphi + 1$. □

**Corollary 3.3** (Three Simultaneous Necessary Conditions for Counterexamples)**.** *Any admissible weight $w$ achieving $\delta(w) \leq \varepsilon < \frac{3-\sqrt{5}}{2}$ must simultaneously satisfy:*

1. *$L(w) \geq \lceil c_{\mathrm{UC}}/\varepsilon \rceil$ (at least linear level complexity);*
2. *$R(w) \geq \exp(c_{\mathrm{UC}}/\varepsilon - 1)$ (at least exponential dynamic range in $1/\varepsilon$);*
3. *$R(w) \;\geq\; |\operatorname{supp}(w)|^{\gamma(\varepsilon)}$, where $\gamma(\varepsilon) := \frac{\varphi(c_{\mathrm{Bop}}-\varepsilon)}{1+\varphi(c_{\mathrm{Bop}}-\varepsilon)} > 0$ (polynomial dynamic range in $|\operatorname{supp}(w)|$).*

## 4 Exact Power-Mean Phase Transition Across $r \in [-\infty, +\infty]$

Gowers's admissibility inequality (3) uses the $r = -\infty$ endpoint of the classical one-parameter power-mean hierarchy [8]. For $r \in [-\infty, +\infty]$ and $u, v \geq 0$, define

$$M_r(u,v) \;:=\; \begin{cases} \min\{u,v\}, & r = -\infty, \\ \big(\tfrac{1}{2}(u^r + v^r)\big)^{1/r}, & r \in (-\infty, 0) \text{ and } u,v > 0 \quad (0 \text{ if } uv = 0), \\ \sqrt{uv}, & r = 0, \\ \big(\tfrac{1}{2}(u^r + v^r)\big)^{1/r}, & r \in (0, +\infty), \\ \max\{u,v\}, & r = +\infty. \end{cases} \tag{18}$$

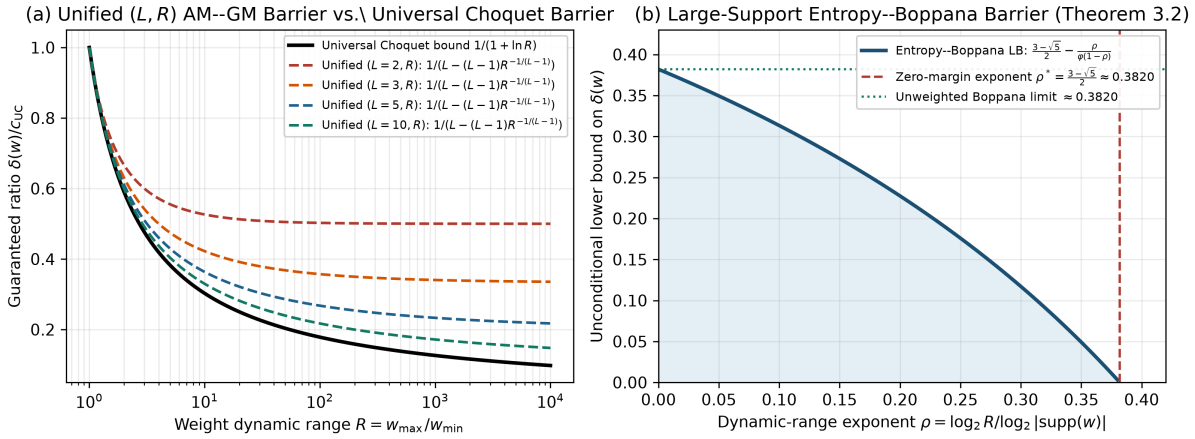



Figure 2: (a) Unified $(L, R)$ AM–GM barrier $c_{\mathrm{UC}}/(L-(L-1)R^{-1/(L-1)})$ for $L = 2, 3, 5, 10$ converging monotonically to the level-free Choquet barrier $c_{\mathrm{UC}}/(1+\ln R)$ (Theorem 2.2). (b) Large-support Entropy–Boppana barrier (Theorem 3.2) as a function of $\rho = \log_2 R / \log_2 |\mathrm{supp}(w)|$, hitting zero at the exact golden exponent $\rho^* = \frac{3-\sqrt{5}}{2}$.

Call a nonzero weight $w : 2^V \to \mathbb{R}_{\geq 0}$ with $w(\emptyset) = 0$ *r-admissible* if $w(A \cup B) \geq M_r(w(A), w(B))$ for all $A, B \subseteq V$.

**Theorem 4.1** (Complete Power-Mean Trichotomy and Discontinuous Jump at $r = -\infty$)**.** *Across all $r \in [-\infty, +\infty]$, the class of r-admissible weights exhibits a sharp three-regime phase transition:*

1. ***Regime I** ($r = -\infty$, **minimum**):* $\inf\{\delta(w) : w \text{ is } (-\infty)\text{-admissible}\} = 0$, *achieved at exact level rate $\Psi(L) = c_{\mathrm{UC}}/L$ (Theorem 2.4; Wang [10]).*
2. ***Regime II** ($r \in (-\infty, 0]$, **including harmonic** $r = -1$ **and geometric** $r = 0$):*
   - Exact structural collapse: *$w$ is r-admissible if and only if $\mathcal{F} := \mathrm{supp}(w)$ is union-closed and $w$ is monotone non-decreasing on $\mathcal{F}$ ($A \subseteq B$ in $\mathcal{F} \implies w(A) \leq w(B)$), equivalently $w(A \cup B) \geq \max\{w(A), w(B)\}$ for all $A, B \in \mathcal{F}$.*
   - Unconditional entropy floor: *Every r-admissible weight ($r \in (-\infty, 0]$) satisfies $H(X \cup Y) \leq H(X)$ and hence unconditionally*

$$\delta(w) \;\geq\; \frac{3-\sqrt{5}}{2} \;\approx\; 0.381966. \tag{19}$$

3. ***Regime III** ($r \in (0, +\infty]$, **including arithmetic** $r = 1$ **and maximum** $r = +\infty$):*
   - Exact global monotonicity collapse: *$w$ is r-admissible if and only if $w$ is globally monotone non-decreasing on $2^V$ ($A \subseteq B \subseteq V \implies w(A) \leq w(B)$).*
   - Universal coordinate bound: *Every r-admissible weight ($r > 0$) satisfies $p_x(w) \geq \frac{1}{2}$ for* every $x \in V$ *(strictly $> \frac{1}{2}$ whenever $w(\{x\}) > 0$).*

*Proof.* Regime I is Theorem 2.4. For Regime II ($r \in (-\infty, 0]$), note the strict betweenness property of $M_r$: for all $u, v > 0$ with $u \neq v$, $\min\{u, v\} < M_r(u, v) < \max\{u, v\}$, while $M_r(u, v) = 0$ if $uv = 0$ and $M_r(u, v) > 0$ if $u, v > 0$. If $w$ is $r$-admissible, then $A, B \in \mathcal{F} \implies w(A), w(B) > 0 \implies w(A \cup B) \geq M_r(w(A), w(B)) > 0 \implies A \cup B \in \mathcal{F}$, so $\mathcal{F}$ is union-closed. Moreover, if $A \subseteq B$ with $A, B \in \mathcal{F}$, then $A \cup B = B$, so $w(B) \geq M_r(w(A), w(B))$; if $w(B) < w(A)$ held, strict betweenness would give $M_r(w(A), w(B)) > w(B)$, a contradiction. Hence $w(A) \leq w(B)$. Conversely, if $\mathcal{F}$ is union-closed and $w$ is monotone on $\mathcal{F}$, then $A, B \in \mathcal{F} \implies w(A \cup B) \geq$

$\max\{w(A), w(B)\} \geq M_r(w(A), w(B))$, while if $A \notin \mathcal{F}$ or $B \notin \mathcal{F}$ then $M_r(w(A), w(B)) = 0 \leq w(A \cup B)$ (Lean 4: `regime_II_power_mean_collapse`).

To prove (19), let $P(A) := w(A)/W(w)$ on $\mathcal{F}$ and let $X, Y \sim P$ be i.i.d. with $Q := \mathrm{Law}(X \cup Y)$. If $|\mathcal{F}| = 1$, then $w$ is supported on a single nonempty set $A$, so $\delta(w) = 1 > \frac{3-\sqrt{5}}{2}$. Assume $|\mathcal{F}| \geq 2$, so $H(X) > 0$. Because $\mathcal{F}$ is union-closed, $\mathrm{supp}(Q) \subseteq \mathcal{F} = \mathrm{supp}(P)$, so the Kullback–Leibler divergence $D_{\mathrm{KL}}(Q\|P) \geq 0$ is finite [6]. Since $w(A \cup B) \geq \max\{w(A), w(B)\} \geq \sqrt{w(A)w(B)}$ for all $A, B \in \mathcal{F}$, we have $P(A \cup B) \geq \sqrt{P(A)P(B)}$ on $\mathcal{F} \times \mathcal{F}$, whence

$$\begin{aligned} H(X \cup Y) \;=\; H(Q) \;&\leq\; H(Q) + D_{\mathrm{KL}}(Q\|P) \\ &=\; \mathbb{E}_{X,Y\sim P}\left[\log_2 \frac{1}{P(X \cup Y)}\right] \;\leq\; \frac{1}{2}H(X) + \frac{1}{2}H(Y) \;=\; H(X). \end{aligned} \tag{20}$$

Combining $H(X \cup Y) \leq H(X)$ with Lemma 3.1 gives $\varphi(1 - \delta(w))H(X) \leq H(X)$; dividing by $\varphi H(X) > 0$ yields $\delta(w) \geq 1 - \varphi^{-1} = \frac{3-\sqrt{5}}{2}$.

For Regime III ($r \in (0, +\infty]$), observe that $u > 0 \implies M_r(u, 0) = 2^{-1/r}u > 0$ (with $2^{-1/\infty} := 1$). Hence for any $A \subseteq B \subseteq V$ with $w(A) > 0$, we have $w(B) = w(A \cup B) \geq M_r(w(A), w(B)) > 0$, so $B \in \mathcal{F}$, and strict betweenness (or $M_\infty = \max$) forces $w(A) \leq w(B)$ (Lean 4: `regime_III_global_monotonicity`). Conversely, global monotonicity implies $w(A \cup B) \geq \max\{w(A), w(B)\} \geq M_r(w(A), w(B))$. Finally, for any $x \in V$, partitioning $2^V$ into the $2^{n-1}$ pairs $(A, A \cup \{x\})$ with $x \notin A$ gives $w(A) \leq w(A \cup \{x\})$ termwise, whence $2\sum_{B \ni x} w(B) \geq W(w)$, i.e. $p_x(w) \geq \frac{1}{2}$ (strict if $w(\{x\}) > 0 = w(\emptyset)$; Lean 4: `global_monotone_half_bound` and `global_monotone_strict_half_bound`). □

**Proposition 4.2** (Relaxed Geometric-Mean Weights Without Monotonicity Collapse)**.** *Let $\gamma \in (0, 1]$ and suppose $w : 2^V \to \mathbb{R}_{\geq 0}$ satisfies $w(\emptyset) = 0$, $W(w) > 0$, and the $\gamma$-relaxed geometric-mean inequality $w(A \cup B) \geq \gamma\sqrt{w(A)w(B)}$ for all $A, B \subseteq V$. Unlike Theorem 4.1(2), when $\gamma < 1$ this condition does* not *force monotonicity on* $\mathrm{supp}(w)$ *(since $A \subsetneq B$ allows $w(B) = \gamma^2 w(A) < w(A)$). Nevertheless, (20) yields $H(X \cup Y) \leq H(X) + \log_2(1/\gamma)$, and hence whenever $H(X) > 0$,*

$$\delta(w) \;\geq\; \frac{3 - \sqrt{5}}{2} - \frac{\log_2(1/\gamma)}{\varphi\, H(X)}. \tag{21}$$

*Proof.* On $\mathrm{supp}(w) \times \mathrm{supp}(w)$, $P(A \cup B) \geq \gamma\sqrt{P(A)P(B)}$ gives $\log_2(1/P(A \cup B)) \leq \frac{1}{2}\log_2(1/P(A)) + \frac{1}{2}\log_2(1/P(B)) + \log_2(1/\gamma)$. Taking expectations in (20) yields $H(X \cup Y) \leq H(X) + \log_2(1/\gamma)$; applying Lemma 3.1 and dividing by $\varphi H(X) > 0$ gives (21). □

## 5 Zero-Float Exact Rational Verification Suite

The pure standard-library Python script `verify_weighted_union_closed.py` (zero floating-point arithmetic; all checks in `fractions.Fraction` and arbitrary-precision integers) certifies every quantitative claim and writes `data/verification_summary.json`:

- **Exact small ground-set optima $\Theta(1), \Theta(2), \Theta(3)$:** Exhaustively enumerating all nonempty union-closed families on $n \in \{1, 2, 3\}$ (1, 6, and 60 families, respectively) and solving the exact rational minimax linear program over all 3,058 nested chains of lengths 2 and 3 on $n = 3$ (sufficient by Carathéodory's theorem on the 3-coordinate minimax simplex) certifies $\Theta(1) = 1$, $\Theta(2) = \frac{2}{3}$, and $\Theta(3) = \frac{4}{7}$ identically (every admissible weight on $n \leq 3$ still exceeds $\frac{1}{2}$).
- **Brute-force audit of the $n = 15, L = 2$ witness:** Explicitly builds all 4,147 bitmask sets of $\mathcal{F}_1$ ($k = 5, r = 2$), checks all 8,600,878 unordered pairwise unions $A \cup B \in \mathcal{F}_1$ exhaustively, and verifies that with $w(B_1) = 739$ and $w(A) = 1$ on $\mathcal{F}_1 \setminus \{B_1\}$, every one of the 15 elements achieves exact weighted frequency $\frac{2386}{4885} < \frac{1}{2}$.

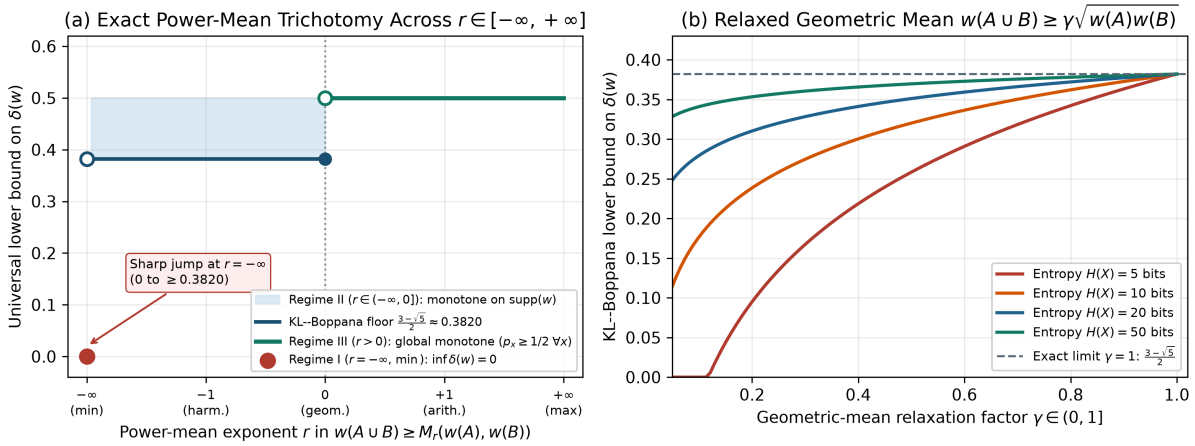



Figure 3: (a) Complete power-mean phase transition across $r \in [-\infty, +\infty]$ (Theorem 4.1): discontinuous jump from 0 at $r = -\infty$ to $\geq \frac{3-\sqrt{5}}{2}$ throughout Regime II ($r \in (-\infty, 0]$), and universal $p_x(w) \geq \frac{1}{2}$ for all $x \in V$ throughout Regime III ($r > 0$). (b) KL–Boppana lower bound (21) for non-monotone $\gamma$-relaxed geometric-mean weights (Proposition 4.2).

Table 1: Zero-float certified $L$-level cloned-gate bounds ($k_0 = 7$, $\alpha = 64/127$, $\tau = 1/(200L)$) versus the unconditional Boppana floor $\frac{3-\sqrt{5}}{2L}$ and Frankl-conditional target $\frac{1}{2L}$ (from `data/verification_summary.json`; decimals are rounded *down* for lower bounds and *up* for upper bounds so every printed decimal bound is strictly conservative).

| Levels $L$ | Gate $r$ | $G_r = 127^r$ | LB $c_{\mathrm{Bop}}/L$ (↓) | Frankl $1/(2L)$ | Exact UB | UB (↑) |
|---|---|---|---|---|---|---|
| 2 | 2 | 16,129 | 0.190983 | 0.250000 | 12927/50800 | 0.254469 |
| 3 | 2 | 16,129 | 0.127322 | 0.166667 | 4309/25400 | 0.169646 |
| 4 | 2 | 16,129 | 0.095491 | 0.125000 | 12927/101600 | 0.127235 |
| 5 | 2 | 16,129 | 0.076393 | 0.100000 | 12927/127000 | 0.101788 |
| 6 | 2 | 16,129 | 0.063661 | 0.083333 | 4309/50800 | 0.084823 |
| 7 | 2 | 16,129 | 0.054566 | 0.071429 | 12927/177800 | 0.072706 |
| 8 | 2 | 16,129 | 0.047745 | 0.062500 | 12927/203200 | 0.063618 |
| 9 | 2 | 16,129 | 0.042440 | 0.055556 | 4309/76200 | 0.056549 |
| 10 | 2 | 16,129 | 0.038196 | 0.050000 | 12927/254000 | 0.050894 |

- **Concrete $L = 2$ witnesses and multi-level cloned-gate parameter audit ($L = 2 \ldots 10$):** Certifies the $n = 190$ ($k = 38, r = 4$) witness beating $\frac{3-\sqrt{5}}{2}$ via exact quadratic arithmetic $(3W - 2c_y)^2 > 5W^2$, verifies the exact rational gate sizes $G_r = 127^r$ and certified upper bounds $\frac{64}{127L} + \frac{1}{200L} < \frac{1}{2L} + \frac{1}{100L}$ for every $L \in \{2, \ldots, 10\}$ (Table 1; checking gate parameters and exact Theorem 2.4 bounds without materializing the doubly-exponential cloning factors $M_j$), certifies the unified $(L, R)$ denominator $L - (L-1)R^{-1/(L-1)} < 1 + \ln R$ at exact rational power points via exact series lower bounds for $\ln R$, and checks 25 Regime III monotone weights on $2^{[4]}$.

## 6 Constructive Lean 4.36.0 Formalization and Axiom Audit

The standalone module `WeightedUnionClosedCore.lean` (581 lines, 28,423 bytes, SHA-256 prefix `4d223a6ffa3b`) compiles cleanly under Lean 4.36.0 with `-DwarningAsError=true` and `lake build` (2.2 s, zero Mathlib dependencies). The Lean module formalizes the abstract join-semilattice and set-system definitions directly (`JoinSemilattice`, `IsUnionClosed`, `IsAdmissible`, `ClonedFamily`, `GatedFamily`, `UnionExt`, and `PowerMeanRel`) together with all 23 public theorems summarized in Table 2. An exhaustive `#print axioms` audit over all 23/23 public theo-

Table 2: Selected kernel-verified theorems in `WeightedUnionClosedCore.lean` (Lean 4.36.0, `-DwarningAsError=true`). All 23/23 public theorems depend solely on `[propext, Quot.sound]` (zero `Classical.choice`, zero `sorry`).

| Lean Theorem Identifier | Mathematical Content | Axioms |
|---|---|---|
| `admissible_iff_superlevels_union_closed` | Admissibility $\iff$ all positive superlevels union-closed | `propext, Quot.sound` |
| `admissible_comp_monotone`, `layerSum_exact` | Monotone stability & pointwise Choquet layer-cake sum | `propext, Quot.sound` |
| `l_level_lower_bound` | Constructive $L$-level lower bound $pW \le Lq \cdot \mathrm{hit}_{\max}$ | `propext, Quot.sound` |
| `two_level_exact_range_bound` | Sharp $L=2$ bound $t_2 pW \le (2t_2 - t_1)q \cdot \mathrm{hit}_{\max}$ | `propext, Quot.sound` |
| `cloned_family_union_closed` | $M$-fold element cloning preserves union-closure | `propext, Quot.sound` |
| `gated_family_union_closed` | Universal gated product $\mathcal{P}$ is union-closed | `Quot.sound` |
| `union_ext_is_union_closed` | $\mathcal{A} \cup \mathcal{P} \cup (\mathcal{A} \vee \mathcal{P})$ is union-closed | none |
| `bernoulli_decay_nat` | Cleared Bernoulli decay $(G+M)G^M \le G(G+1)^M$ | `propext, Quot.sound` |
| `witness_n15_L2_beats_half` | Exact $n=15, L=2$ witness: $2 \cdot 2386 < 4885$ | none |
| `witness_n190_L2_beats_boppana` | Exact $n=190, L=2$ witness: $5W^2 < (3W - 2c_y)^2$ | none |
| `regime_II_power_mean_collapse` | $r \in (-\infty, 0] \iff$ union-closed support + monotone | none |
| `regime_III_global_monotonicity` | $r > 0 \implies$ globally monotone on $2^V$ | none |
| `global_monotone_half_bound`, `..._strict_...` | Regime III universal $p_x(w) \ge 1/2$ (strict if $w(\{x\}) > 0$) | `propext, Quot.sound` |

rems confirms strictly constructive dependence on `[propext, Quot.sound]` only (9 theorems use zero axioms at all; zero `Classical.choice`, zero `sorry`). Real-analytic Shannon entropy, KL divergence, continuous AM–GM steps (Theorems 2.2(2)–(3) for general $L \ge 3$, Theorem 3.2, and (20)), the cardinality/frequency counting bounds of Lemma 2.3(2), and the combinatorial derivation of (14) for $n = 190$ are proved in prose in Sections 2–4 (while Lean verifies the exact closed-form integer witness inequalities of (14) directly).

# 7 Discussion and Open Problems

Our results map out the exact quantitative boundary between the failure of Gowers's weighted union-set conjecture at $r = -\infty$ and its survival under bounded level complexity, bounded dynamic range, or any finite power-mean strengthening $r > -\infty$. Three sharp open problems emerge naturally:

1. **Monotone-on-support Frankl constant $c_{\mathrm{mon}}$ (Regime II):** Theorem 4.1(2) proves $\frac{3-\sqrt{5}}{2} \le c_{\mathrm{mon}} \le c_{\mathrm{UC}} \le \frac{1}{2}$, where $c_{\mathrm{mon}}$ is the infimum of $\delta(w)$ over monotone non-decreasing weights supported on union-closed families. Does $c_{\mathrm{mon}} = c_{\mathrm{UC}}$ hold, or can a non-uniform monotone weight on a union-closed family achieve $\delta(w) < \frac{1}{2}$?
2. **Exact dynamic-range function $\Phi(R)$:** Theorem 2.2(3) establishes $\Phi(R) \ge c_{\mathrm{UC}}/(1+\ln R)$, while our cloned-gate construction requires doubly exponential $R(L)$ as $L \to \infty$. What is the true asymptotic decay rate of $\Phi(R)$ as $R \to \infty$?
3. **Minimal ground-set violation $n^*$:** We proved $\Theta(3) = \frac{4}{7} > \frac{1}{2}$ and exhibited an explicit $L = 2$ violation on $n = 15$ elements ($\delta(w) = 2386/4885 < \frac{1}{2}$). Determining the exact minimum $n^* \in \{4, \dots, 15\}$ for which $\Theta(n^*) < \frac{1}{2}$ remains an attractive finite extremal problem.

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