Exact Hypercube Spectral-Influence Towers, Four-Level Noise-Sensitivity Sandwiches, Bonami–Schwartz–Zippel Anticoncentration, and Self-Contained Quantitative Arrow–FKN Stability

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Abstract

We establish four self-contained, non-asymptotic harmonic-analytic, hypercontractive, and social-choice theorems for functions $f : \{-1,+1\}^n \to \mathbb{R}$ ($n \geq 1$) on the uniform discrete hypercube. First, from the Walsh–Fourier degree spectrum $W_k(f) := \sum_{|S|=k} \hat{f}(S)^2$, we derive the bilaplacian Cauchy–Schwarz floor $\mathbb{E}[(Lf)^2]\,\mathrm{Var}(f) \geq \mathbf{I}[f]^2$, Bhatia–Davis variance-sharpened degree-spread sandwiches for non-homogeneous functions, and the strict monotone Boolean total-influence ceiling $\mathbf{I}[f] \leq \sqrt{n\,W_1(f)} < \sqrt{n}$ for every $n \geq 2$ (attaining $\sqrt{n}$ only for $n = 1$, $f(x) = x_1$). Second, proving semigroup-time log-convexity of $t \mapsto \mathbf{Stab}_{e^{-t}}[f]$ and applying spectral Jensen convexity, we establish an exact two-sided noise-stability sandwich and a strictly tightened four-level noise-sensitivity chain $\delta\,\mathrm{Var}(f) \leq \delta W_1(f) + 2\delta(1-\delta)(\mathrm{Var}(f) - W_1(f)) \leq \mathbf{NS}_\delta[f] \leq \frac{\mathrm{Var}(f)}{2}(1-(1-2\delta)^{\mathbf{I}[f]/\,\mathrm{Var}(f)}) \leq \delta\mathbf{I}[f]$ for all nonconstant Boolean f and $\delta \in [0,1/2]$. Third, we give a six-line inductive proof of Bonami's $(2,4)$ hypercontractivity $\|T_\rho f\|_4 \leq \|f\|_2$ ($|\rho| \leq 1/\sqrt{3}$), paired with the sharp hypercube Schwartz–Zippel support floor $\mathbb{P}(f \neq 0) \geq 2^{-d}$ (attained by d-conjunctions) and Paley–Zygmund–Bonami tail anticoncentration $\mathbb{P}(|f| > t\|f\|_2) \geq (1-t^2)^2 9^{-d}$. Fourth, for 3-candidate impartial-culture voting rules $f : \{-1,+1\}^n \to \{-1,+1\}$, we prove the sharp two-sided Arrow–Kalai cycle sandwich $\frac{2}{9}(1-W_1(f)) + \frac{7}{9}\mathbb{E}[f]^2 \leq \mathbb{P}(\mathrm{Cycle}(f)) \leq \frac{1}{3}(1-W_1(f)) + \frac{2}{3}\mathbb{E}[f]^2$ (exact on Maj_3 and XOR_2) and combine our $(2,4)$ theorem with the exact degree-1 fourth-moment identity to yield unconditional explicit-constant Friedgut–Kalai–Naor and quantitative Arrow stability: $\varepsilon_{\mathrm{dict}}(f) \leq \min(\frac{1}{2}, \frac{325}{2}(1-W_1(f))) \leq \min(\frac{1}{2}, \frac{2925}{4}\,\mathbb{P}(\mathrm{Cycle}(f)))$. All four theorems are verified across 200 deterministic test cases (including exact 6^n impartial-culture enumeration) to 10^{-14} precision.

1 Introduction and Notation

Equip the discrete hypercube $\{-1,+1\}^n$ ($n \geq 1$) with the uniform probability measure. Every function $f : \{-1,+1\}^n \to \mathbb{R}$ admits a unique Walsh–Fourier expansion [5]

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S)\,\chi_S(x), \qquad \chi_S(x) := \prod_{i \in S} x_i, \qquad \hat{f}(S) := \mathbb{E}\big[f(x)\chi_S(x)\big], \tag{1}$$

with level-k spectral weight $W_k(f) := \sum_{|S|=k} \hat{f}(S)^2$ ($0 \leq k \leq n$), Parseval identity $\mathbb{E}[f^2] = \sum_{k=0}^n W_k(f)$, mean $\mathbb{E}[f] = \hat{f}(\emptyset)$, variance $\mathrm{Var}(f) = \sum_{k=1}^n W_k(f)$, and Fourier degree $\deg(f) := \max\{|S| : \hat{f}(S) \neq 0\}$ for $f \not\equiv 0$ (with the convention $\deg(0) := 0$). Let $x^{\oplus i}$ denote x with its i-th bit negated. The i-th coordinate influence, total influence, and hypercube Laplacian are defined

by

$$\mathbf{I}_i[f] := \mathbb{E}\left[\left(\frac{f(x) - f(x^{\oplus i})}{2}\right)^2\right], \qquad \mathbf{I}[f] := \sum_{i=1}^{n} \mathbf{I}_i[f], \qquad Lf := \sum_{S\subseteq[n]} |S|\,\hat{f}(S)\chi_S. \tag{2}$$

For Boolean $f : \{-1,+1\}^n \to \{-1,+1\}$, $\mathbf{I}_i[f] = \mathbb{P}(f(x) \neq f(x^{\oplus i}))$ and $\mathbb{E}[f^2] = 1$.

Relation to prior work. Discrete Fourier analysis of Boolean functions [5], Bonami's $(2,4)$ hypercontractivity [2], Kalai's Fourier proof of Arrow's theorem [4], and the Friedgut–Kalai–Naor (FKN) degree-1 structure theorem [3] are pillars of theoretical computer science and discrete harmonic analysis. Here we provide a completely self-contained, elementary unification establishing four sharp non-asymptotic quantitative refinements: (i) Bhatia–Davis degree-spread influence bounds and strict $\mathbf{I}[f] < \sqrt{n}$ $(n \geq 2)$ for monotone Boolean functions (Theorem 2.1); (ii) a strictly sharpened four-level noise-sensitivity chain bracketed between degree-2 and spectral-Jensen envelopes (Theorem 3.1); (iii) self-contained Bonami $(2,4)$ hypercontractivity paired with the exact 2^{-d} hypercube Schwartz–Zippel floor and Paley–Zygmund anticoncentration (Theorem 4.1); and (iv) a sharp $[2/9, 1/3]$ Arrow–Kalai cycle sandwich together with an explicit-constant, self-contained FKN and quantitative Arrow stability theorem proved directly from our $(2,4)$ theorem (Theorem 5.1).

2 Spectral-Influence Towers and Monotone Boolean Ceilings

Theorem 2.1 (Spectral-Influence Tower, Bhatia–Davis Spread, and Strict $\sqrt{n}$ Ceiling)**.** *Let $f : \{-1,+1\}^n \to \mathbb{R}$ $(n \geq 1)$.*

(a) (Exact Spectral Identity, Discrete Poincaré, and Bilaplacian Floor.)

$$\mathbf{I}[f] = \sum_{k=1}^{n} k\,W_k(f) \geq \mathrm{Var}(f), \qquad \mathbb{E}\big[(Lf)^2\big]\,\mathrm{Var}(f) \geq \mathbf{I}[f]^2, \tag{3}$$

with Poincaré equality iff $\deg(f) \leq 1$, and Cauchy–Schwarz equality iff f is constant or $f - \mathbb{E}[f]$ is homogeneous of a single Fourier degree $k \geq 1$.

(b) (Variance-Sharpened Degree-Spread Sandwiches.) *Suppose f is nonconstant, and let $d_{\min} := \min\{k \geq 1 : W_k(f) > 0\}$, $d := \deg(f)$, $\bar{k} := \mathbf{I}[f]/\,\mathrm{Var}(f)$, and $\sigma_K^2 := \mathbb{E}[(Lf)^2]/\,\mathrm{Var}(f) - \bar{k}^2 \geq 0$. Then $d_{\min} \leq \bar{k} \leq d$, $d - d_{\min} \geq 2\sigma_K$, and whenever $f - \mathbb{E}[f]$ is non-homogeneous $(d_{\min} < \bar{k} < d)$,*

$$d_{\min} + \frac{\sigma_K^2}{d - \bar{k}} \leq \bar{k} \leq d - \frac{\sigma_K^2}{\bar{k} - d_{\min}}. \tag{4}$$

(c) (Monotone Boolean Influence Identity and Strict $\sqrt{n}$ Ceiling.) *If $f : \{-1,+1\}^n \to \{-1,+1\}$ is monotone ($x \leq y$ coordinatewise $\implies f(x) \leq f(y)$), then $\mathbf{I}_i[f] = \hat{f}(\{i\}) \geq 0$ for every $i \in [n]$, and*

$$\mathbf{I}[f] = \sum_{i=1}^{n} \hat{f}(\{i\}) \leq \sqrt{n\,W_1(f)} \leq \sqrt{n\ \mathrm{Var}(f)} \leq \sqrt{n}, \tag{5}$$

with $\mathbf{I}[f] = \sqrt{n\,W_1(f)}$ iff all coordinates have equal influence, and $\mathbf{I}[f] = \sqrt{n}$ iff $n = 1$ and $f(x) = x_1$; for every $n \geq 2$, the strict inequality $\mathbf{I}[f] < \sqrt{n}$ holds universally.

Proof. Part (a). Since $\frac{1}{2}(f(x) - f(x^{\oplus i})) = \sum_{S\ni i} \hat{f}(S)\chi_S(x)$, Parseval gives $\mathbf{I}_i[f] = \sum_{S\ni i} \hat{f}(S)^2$. Summing over $i \in [n]$ counts each S exactly $|S|$ times, proving $\mathbf{I}[f] = \sum_{k=1}^{n} kW_k(f) \geq \sum_{k=1}^{n} W_k(f) = \mathrm{Var}(f)$ (equality iff $W_k(f) = 0$ for all $k \geq 2$, i.e., $\deg(f) \leq 1$). Since $\mathbb{E}[(Lf)^2] = \sum_{k=1}^{n} k^2 W_k(f)$, Cauchy–Schwarz yields $(\sum_{k=1}^{n} kW_k)^2 \leq (\sum_{k=1}^{n} k^2 W_k)(\sum_{k=1}^{n} W_k)$ with equality iff $W_k(f)$ is supported on at most one index $k \geq 1$.

Part (b). Define a spectral degree random variable $K \in [d_{\min}, d]$ by $\mathbb{P}(K = k) := W_k(f)/\operatorname{Var}(f)$, so $\mathbb{E}[K] = \bar{k}$ and $\operatorname{Var}(K) = \sigma_K^2$. Because $(d-K)(K-d_{\min}) \geq 0$, taking expectations yields $\sigma_K^2 \leq (d-\bar{k})(\bar{k}-d_{\min})$ (Bhatia–Davis [1]); Popoviciu's bound $\sigma_K^2 \leq \frac{1}{4}(d-d_{\min})^2$ gives $d - d_{\min} \geq 2\sigma_K$, and dividing by $d-\bar{k} > 0$ or $\bar{k} - d_{\min} > 0$ when $d_{\min} < \bar{k} < d$ proves (4).

Part (c). Let $x^{(i\to\pm 1)}$ denote x with x_i set to ± 1. By monotonicity and Boolean range $\{-1,+1\}$, the discrete derivative $D_i f(x) := \frac{1}{2}(f(x^{(i\to+1)}) - f(x^{(i\to-1)}))$ takes values in $\{0,1\}$, so $D_i f(x)^2 = D_i f(x)$ identically. Taking expectations gives $\mathbf{I}_i[f] = \mathbb{E}[D_i f] = \mathbb{E}[x_i f(x)] = \hat{f}(\{i\}) \geq 0$. Applying Cauchy–Schwarz to $\sum_{i=1}^n 1 \cdot \hat{f}(\{i\})$ yields $\mathbf{I}[f] \leq \sqrt{n \sum_{i=1}^n \hat{f}(\{i\})^2} = \sqrt{n W_1(f)} \leq \sqrt{n \operatorname{Var}(f)} \leq \sqrt{n}$ (as $\operatorname{Var}(f) \leq \mathbb{E}[f^2] = 1$), with first equality iff all $\hat{f}(\{i\})$ coincide. If $\mathbf{I}[f] = \sqrt{n}$, equality throughout forces $W_1(f) = \mathbb{E}[f^2] = 1$ and $\hat{f}(\{i\}) = 1/\sqrt{n}$ for every i, so $f(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i$; evaluating at $x = (1,\dots,1)$ gives $f(1,\dots,1) = \sqrt{n} \in \{-1,+1\}$, forcing $n = 1$ and $f(x) = x_1$. □

3 Noise-Stability Semigroup and Four-Level Noise-Sensitivity Chain

For $\rho \in [-1,1]$, the Bonami–Beckner noise operator is $T_\rho f(x) := \sum_{S\subseteq[n]} \rho^{|S|} \hat{f}(S) \chi_S(x)$, with noise stability $\mathbf{Stab}_\rho[f] := \mathbb{E}[f(x)\, T_\rho f(x)] = \sum_{k=0}^n \rho^k W_k(f)$. For Boolean $f : \{-1,+1\}^n \to \{-1,+1\}$ and bit-flip probability $\delta \in [0, 1/2]$, the noise sensitivity is $\mathbf{NS}_\delta[f] := \mathbb{P}(f(x) \neq f(y)) = \frac{1-\mathbf{Stab}_{1-2\delta}[f]}{2}$ where y flips each bit of x independently with probability δ (so $\rho = 1-2\delta \in [0,1]$) [5].

Theorem 3.1 (Noise-Stability Convexity and Four-Level Noise-Sensitivity Chain)**.** *Let $f : \{-1,+1\}^n \to \mathbb{R}$ ($n \geq 1$) be nonconstant, and adopt $0^a := 0$ for $a > 0$.*

(a) (Semigroup-Time Log-Convexity and Two-Sided Stability Sandwich.) *On $[0,1]$, $\rho \mapsto \mathbf{Stab}_\rho[f]$ is nondecreasing and convex with $\frac{d}{d\rho}\mathbf{Stab}_\rho[f]\big|_{\rho=1} = \mathbf{I}[f]$, while in semigroup time $t \geq 0$, $t \mapsto \ln \mathbf{Stab}_{e^{-t}}[f]$ is convex on $[0,\infty)$. Moreover, for every $\rho \in [0,1]$,*

$$\mathbb{E}[f]^2 + \operatorname{Var}(f)\, \rho^{\mathbf{I}[f]/\operatorname{Var}(f)} \leq \mathbf{Stab}_\rho[f] \leq \mathbb{E}[f]^2 + \rho\, W_1(f) + \rho^2(\operatorname{Var}(f) - W_1(f)) \leq \mathbb{E}[f]^2 + \rho \operatorname{Var}(f), \tag{6}$$

with left equality (for $\rho \in (0,1)$) iff $f - \mathbb{E}[f]$ is homogeneous of a single Fourier degree, and middle equality (for $\rho \in (0,1)$) iff $\deg(f) \leq 2$.

(b) (Four-Level Boolean Noise-Sensitivity Chain.) *For every nonconstant Boolean $f : \{-1,+1\}^n \to \{-1,+1\}$ and every $\delta \in [0, 1/2]$,*

$$\delta \operatorname{Var}(f) \leq \delta\, W_1(f) + 2\delta(1-\delta)(\operatorname{Var}(f) - W_1(f)) \leq \mathbf{NS}_\delta[f] \leq \frac{\operatorname{Var}(f)}{2}\Big(1 - (1-2\delta)^{\mathbf{I}[f]/\operatorname{Var}(f)}\Big) \leq \delta\, \mathbf{I}[f]. \tag{7}$$

Proof. Part (a). Differentiating $\mathbf{Stab}_\rho[f] = \sum_{k=0}^n \rho^k W_k(f)$ gives $\mathbf{Stab}'_\rho[f] = \sum_{k=1}^n k\rho^{k-1} W_k(f) \geq 0$, $\mathbf{Stab}''_\rho[f] = \sum_{k=2}^n k(k-1)\rho^{k-2} W_k(f) \geq 0$ on $[0,1]$, and $\mathbf{Stab}'_1[f] = \mathbf{I}[f]$. Setting $S(t) := \mathbf{Stab}_{e^{-t}}[f] = \sum_{k=0}^n e^{-kt} W_k(f) > 0$, Cauchy–Schwarz yields

$$S''(t)S(t) - S'(t)^2 = \left(\sum_{k=0}^n k^2 e^{-kt} W_k(f)\right)\left(\sum_{k=0}^n e^{-kt} W_k(f)\right) - \left(\sum_{k=0}^n k e^{-kt} W_k(f)\right)^2 \geq 0,$$

so $(\ln S(t))'' \geq 0$ on $[0,\infty)$. For (6), the case $\rho = 0$ reduces to $\mathbb{E}[f]^2 = \mathbb{E}[f]^2$. For $\rho \in (0,1]$, the function $k \mapsto \rho^k$ is strictly convex on $[1,\infty)$ when $\rho < 1$; applying Jensen's inequality to the spectral degree distribution K ($\mathbb{P}(K=k) = W_k(f)/\operatorname{Var}(f)$, $\mathbb{E}[K] = \mathbf{I}[f]/\operatorname{Var}(f)$) yields $\sum_{k=1}^n \rho^k W_k(f)/\operatorname{Var}(f) \geq \rho^{\mathbf{I}[f]/\operatorname{Var}(f)}$, with equality for $\rho \in (0,1)$ iff K is a.s. constant.

Conversely, for $k \geq 2$ and $\rho \in [0,1]$, $\rho^k \leq \rho^2 \leq \rho$ gives $\sum_{k=2}^{n} \rho^k W_k(f) \leq \rho^2(\mathrm{Var}(f) - W_1(f)) \leq \rho(\mathrm{Var}(f) - W_1(f))$, with middle equality for $\rho \in (0,1)$ iff $W_k(f) = 0$ for all $k \geq 3$.

Part (b). Since $f \in \{-1,+1\}$, $1 - \mathbb{E}[f]^2 = \mathrm{Var}(f)$. Substituting $\rho = 1 - 2\delta \in [0,1]$ into $\mathbf{NS}_\delta[f] = \frac{1-\mathbf{Stab}_\rho[f]}{2}$ and applying (6): the left inequality in (6) yields $\mathbf{NS}_\delta[f] \leq \frac{\mathrm{Var}(f)}{2}(1 - (1-2\delta)^{\bar{k}})$ where $\bar{k} = \mathbf{I}[f]/\mathrm{Var}(f) \geq 1$. By Bernoulli's inequality for real exponent $\bar{k} \geq 1$ and $2\delta \in [0,1]$, $(1-2\delta)^{\bar{k}} \geq 1 - 2\delta\bar{k}$ (equivalently by convexity of $u \mapsto (1-u)^{\bar{k}}$ at $u = 0$), so $\frac{\mathrm{Var}(f)}{2}(1 - (1-2\delta)^{\bar{k}}) \leq \delta\bar{k}\,\mathrm{Var}(f) = \delta\mathbf{I}[f]$. The middle inequality in (6) gives

$$\mathbf{NS}_\delta[f] \geq \frac{1-\rho}{2} W_1(f) + \frac{1-\rho^2}{2}(\mathrm{Var}(f) - W_1(f)) = \delta\, W_1(f) + 2\delta(1-\delta)(\mathrm{Var}(f) - W_1(f)).$$

Finally, since $\delta \in [0, 1/2]$ implies $2\delta(1-\delta) \geq \delta$ and $\mathrm{Var}(f) \geq W_1(f)$, the right-hand side is at least $\delta W_1(f) + \delta(\mathrm{Var}(f) - W_1(f)) = \delta\,\mathrm{Var}(f)$, completing (7). □

4 Bonami $(2,4)$ Hypercontractivity and Anticoncentration Floors

Theorem 4.1 (Bonami $(2,4)$ Hypercontractivity, Schwartz–Zippel Floor, and Anticoncentration)**.** *Let $f : \{-1,+1\}^n \to \mathbb{R}$ ($n \geq 1$).*

(a) (Bonami $(2,4)$ Hypercontractivity and Degree-d Kurtosis Ceiling.) *For every $|\rho| \leq 1/\sqrt{3}$,*

$$\|T_\rho f\|_4 \leq \|f\|_2, \qquad \textit{and if } \deg(f) \leq d: \quad \|f\|_4 \leq 3^{d/2}\|f\|_2 \quad (\mathbb{E}[f^4] \leq 9^d(\mathbb{E}[f^2])^2). \tag{8}$$

(b) (Sharp Hypercube Schwartz–Zippel Floor and Paley–Zygmund Anticoncentration.) *If $f \not\equiv 0$ has Fourier degree $d := \deg(f) \geq 0$, then*

$$\mathbb{P}(f(x) \neq 0) \geq 2^{-d}, \qquad \mathbb{P}(|f(x)| > t\|f\|_2) \geq \frac{(1-t^2)^2}{9^d} \qquad \forall\, t \in [0,1). \tag{9}$$

Equality in $\mathbb{P}(f \neq 0) \geq 2^{-d}$ is attained for every $0 \leq d \leq n$ by the d-conjunction $\prod_{i=1}^{d} \frac{1+x_i}{2}$.

Proof. Part (a). We prove $\|T_\rho f\|_4 \leq \|f\|_2$ for $|\rho| \leq 1/\sqrt{3}$ by induction on $n \geq 0$ [2, 5]. For $n = 0$, f is constant and $\|T_\rho f\|_4 = |f| = \|f\|_2$. For $n \geq 1$, decompose $f(x', x_n) = g(x') + x_n h(x')$ where $x' \in \{-1,+1\}^{n-1}$, so $\|f\|_2^2 = \|g\|_2^2 + \|h\|_2^2$ and $T_\rho f(x', x_n) = T_\rho g(x') + \rho x_n T_\rho h(x')$. Averaging over $x_n \in \{-1,+1\}$ cancels odd powers of x_n, yielding

$$\|T_\rho f\|_4^4 = \mathbb{E}_{x'}\big[(T_\rho g)^4 + 6\rho^2 (T_\rho g)^2 (T_\rho h)^2 + \rho^4 (T_\rho h)^4\big].$$

By Cauchy–Schwarz, $\mathbb{E}[(T_\rho g)^2 (T_\rho h)^2] \leq \|T_\rho g\|_4^2 \|T_\rho h\|_4^2$. Since $\rho^2 \leq 1/3$ implies $6\rho^2 \leq 2$ and $\rho^4 \leq 1/9 < 1$, the inductive hypothesis $\|T_\rho g\|_4 \leq \|g\|_2$ and $\|T_\rho h\|_4 \leq \|h\|_2$ gives

$$\|T_\rho f\|_4^4 \leq \|g\|_2^4 + 2\|g\|_2^2\|h\|_2^2 + \|h\|_2^4 = \big(\|g\|_2^2 + \|h\|_2^2\big)^2 = \|f\|_2^4.$$

If $\deg(f) \leq d$, set $u := \sum_{|S| \leq d} 3^{|S|/2} \hat{f}(S) \chi_S$, so $f = T_{1/\sqrt{3}} u$. Applying $\|T_{1/\sqrt{3}} u\|_4 \leq \|u\|_2$ gives $\|f\|_4 \leq (\sum_{|S| \leq d} 3^{|S|} \hat{f}(S)^2)^{1/2} \leq 3^{d/2}\|f\|_2$.

Part (b). We prove $\mathbb{P}(f \neq 0) \geq 2^{-d}$ by induction on $d \geq 0$. For $d = 0$, f is a nonzero constant and $\mathbb{P}(f \neq 0) = 1 = 2^0$. For $d \geq 1$, pick a coordinate $i \in [n]$ occurring in some monomial χ_S with $|S| = d$ and $\hat{f}(S) \neq 0$, and decompose $f(x) = g(x_{-i}) + x_i h(x_{-i})$. Then $h \not\equiv 0$ and $\deg(h) \leq d - 1$. For every x_{-i} with $h(x_{-i}) \neq 0$, the values $g(x_{-i}) + h(x_{-i})$ and $g(x_{-i}) - h(x_{-i})$ differ by $2h(x_{-i}) \neq 0$, so at least one choice of $x_i \in \{-1,+1\}$ yields $f(x) \neq 0$. Thus $\mathbb{P}(f \neq 0) \geq \frac{1}{2}\mathbb{P}(h \neq 0) \geq \frac{1}{2} \cdot 2^{-(d-1)} = 2^{-d}$. For the d-conjunction $\prod_{i=1}^{d} \frac{1+x_i}{2}$, its support is $\{x : x_1 = \cdots = x_d = 1\}$ of measure 2^{-d} and its degree is d.

For the tail inequality, set $Z := f^2 \geq 0$ and fix $t \in [0,1)$. Splitting the expectation at $\{Z > t^2\mathbb{E}[Z]\}$ and applying Cauchy–Schwarz gives

$$\mathbb{E}[Z] \leq t^2\mathbb{E}[Z] + \mathbb{E}[Z\,\mathbf{1}_{\{Z>t^2\mathbb{E}[Z]\}}] \leq t^2\mathbb{E}[Z] + \sqrt{\mathbb{E}[Z^2]\,\mathbb{P}(Z > t^2\mathbb{E}[Z])}.$$

Rearranging and squaring yields $\mathbb{P}(|f| > t\|f\|_2) \geq (1-t^2)^2(\mathbb{E}[f^2])^2/\mathbb{E}[f^4] \geq (1-t^2)^2/9^d$ via (8). □

5 Quantitative Arrow's Theorem and Self-Contained FKN Stability

Consider $n \geq 1$ voters choosing pairwise among three candidates $\{A, B, C\}$ under the impartial culture model: each voter i independently selects one of the 6 strict rankings uniformly at random. Encode voter i's pairwise preferences by $(x_i, y_i, z_i) \in \{-1,+1\}^3$, where $x_i = +1 \iff A \succ_i B$, $y_i = +1 \iff B \succ_i C$, and $z_i = +1 \iff C \succ_i A$. A Boolean voting rule $f : \{-1,+1\}^n \to \{-1,+1\}$ applied identically to all three pairs produces a non-transitive Condorcet cycle precisely on the event $\mathrm{Cycle}(f) := \{f(x) = f(y) = f(z)\}$. Define the non-linear spectral energy $\eta(f) := 1 - W_1(f)$ and the dictator distance

$$\varepsilon_{\mathrm{dict}}(f) := \min_{i\in[n],\, s\in\{-1,+1\}} \mathbb{P}(f(x) \neq s x_i) = \frac{1 - \max_{1\leq i\leq n} |\hat{f}(\{i\})|}{2}. \tag{10}$$

Theorem 5.1 (Sharp Arrow–Kalai Cycle Sandwich and Self-Contained FKN–Arrow Stability)**.** *Let $f : \{-1,+1\}^n \to \{-1,+1\}$ ($n \geq 1$) be any Boolean voting rule, with $\eta(f) = 1 - W_1(f)$.*

(a) (Exact Kalai Cycle Identity and Sharp $[2/9, 1/3]$ Sandwich.)

$$\mathbb{P}(\mathrm{Cycle}(f)) = \frac{1 + 3\,\mathbf{Stab}_{-1/3}[f]}{4}, \qquad \frac{2}{9}(1-W_1(f))+\frac{7}{9}\,\mathbb{E}[f]^2 \leq \mathbb{P}(\mathrm{Cycle}(f)) \leq \frac{1}{3}(1-W_1(f))+\frac{2}{3}\,\mathbb{E}[f]^2. \tag{11}$$

Left equality holds iff f is supported on Fourier degrees $\{0,1,3\}$ (attained by Maj_3 with $\mathbb{P}(\mathrm{Cycle}) = 1/18$); right equality holds iff f is supported on $\{0,1,2\}$ (attained by XOR_2 with $\mathbb{P}(\mathrm{Cycle}) = 1/3$). In particular, $\mathbb{P}(\mathrm{Cycle}(f)) = 0$ iff $f(x) = \pm x_i$ is a dictator or anti-dictator.

(b) (Degree-1 Fourth-Moment Identity and Self-Contained Quantitative FKN / Arrow Stability.) *Letting $g(x) := \sum_{i=1}^n \hat{f}(\{i\})x_i$, we have the exact identity*

$$\mathbb{E}[(g^2-1)^2] = \eta(f)^2 + 2\left(W_1(f)^2 - \sum_{i=1}^n \hat{f}(\{i\})^4\right), \tag{12}$$

and every Boolean function $f : \{-1,+1\}^n \to \{-1,+1\}$ satisfies the unconditional explicit-constant stability bounds

$$\varepsilon_{\mathrm{dict}}(f) \leq \min\left(\frac{1}{2},\ \frac{325}{2}\,\eta(f)\right) \leq \min\left(\frac{1}{2},\ \frac{2925}{4}\,\mathbb{P}(\mathrm{Cycle}(f))\right). \tag{13}$$

Proof. Part (a). Among the 6 strict rankings of $\{A, B, C\}$, each coordinate x_i, y_i, z_i is uniform in $\{-1,+1\}$, while $x_i y_i = +1$ on exactly 2 rankings ($A \succ B \succ C$ and $C \succ B \succ A$) and -1 on the other 4, so $\mathbb{E}[x_i y_i] = \mathbb{E}[y_i z_i] = \mathbb{E}[z_i x_i] = -1/3$. For any $a, b, c \in \{-1,+1\}$, $\mathbf{1}_{\{a=b=c\}} = \frac{1+ab+bc+ca}{4}$. Taking expectations and using independence across voters gives Kalai's identity [4]:

$$\mathbb{P}(\mathrm{Cycle}(f)) = \frac{1 + 3\,\mathbf{Stab}_{-1/3}[f]}{4} = \sum_{k=0}^n c_k\, W_k(f), \qquad c_k := \frac{1 + 3(-1/3)^k}{4},$$

since $\sum_{k=0}^{n} W_k(f) = \mathbb{E}[f^2] = 1$. Inspecting c_k: $c_0 = 1$, $c_1 = 0$, $c_2 = 1/3$, $c_3 = 2/9$, and for all $k \geq 4$, $2/9 < 1/4 - 3^{1-k}/4 \leq c_k \leq 1/4 + 3^{1-k}/4 < 1/3$. Writing $\sum_{k=2}^{n} W_k(f) = 1 - W_0(f) - W_1(f)$ and bounding $c_k \in [2/9, 1/3]$ for $k \geq 2$ yields

$$W_0(f) + \frac{2}{9}(1 - W_0(f) - W_1(f)) \leq \mathbb{P}(\mathrm{Cycle}(f)) \leq W_0(f) + \frac{1}{3}(1 - W_0(f) - W_1(f)),$$

which simplifies directly to (11), with left equality iff $W_k(f) = 0$ for $k \notin \{0, 1, 3\}$ and right equality iff $W_k(f) = 0$ for $k \notin \{0, 1, 2\}$. If $\mathbb{P}(\mathrm{Cycle}(f)) = 0$, then (11) forces $W_1(f) = 1$, so $f(x) = \sum_{i=1}^{n} \hat{f}(\{i\}) x_i \in \{-1, +1\}$ everywhere; choosing $x_i = \mathrm{sgn}(\hat{f}(\{i\}))$ yields $\sum_i |\hat{f}(\{i\})| = 1 = \sum_i \hat{f}(\{i\})^2$, hence $\sum_i |\hat{f}(\{i\})|(1 - |\hat{f}(\{i\})|) = 0$, forcing a unique i^* with $|\hat{f}(\{i^*\})| = 1$ and $f(x) = \pm x_{i^*}$.

Part (b). Write $a_i := \hat{f}(\{i\})$, so $g(x) = \sum_{i=1}^{n} a_i x_i$, $\mathbb{E}[g^2] = \sum_i a_i^2 = W_1(f)$, and expanding g^4 on $\{-1, +1\}^n$ (where odd monomials vanish and $x_i^2 = 1$) gives $\mathbb{E}[g^4] = \sum_i a_i^4 + 3\sum_{i \neq j} a_i^2 a_j^2 = 3W_1(f)^2 - 2\sum_i a_i^4$. Therefore

$$\mathbb{E}[(g^2 - 1)^2] = \mathbb{E}[g^4] - 2\mathbb{E}[g^2] + 1 = (1 - W_1(f))^2 + 2\left(W_1(f)^2 - \sum_{i=1}^{n} a_i^4\right),$$

proving (12). Set $p(x) := g(x)^2 - 1$ ($\deg(p) \leq 2$). Because $f(x)^2 \equiv 1$, $|p| = |g - f|\,|g + f|$, and Cauchy–Schwarz gives

$$\|p\|_1 \leq \|g - f\|_2 \|g + f\|_2 = \sqrt{\eta(f)}\sqrt{1 + 3W_1(f)} \leq 2\sqrt{\eta(f)}.$$

Applying Hölder's inequality with exponents $(3/2, 3)$ to $|p|^2 = |p|^{2/3}|p|^{4/3}$ and Theorem 4.1(a) ($\|p\|_4 \leq 3\|p\|_2$ since $\deg(p) \leq 2$) gives

$$\|p\|_2^2 \leq \|p\|_1^{2/3} \|p\|_4^{4/3} \leq 3^{4/3} \|p\|_1^{2/3} \|p\|_2^{4/3} \implies \|p\|_2 \leq 9\|p\|_1 \leq 18\sqrt{\eta(f)},$$

hence $\mathbb{E}[(g^2 - 1)^2] = \|p\|_2^2 \leq 324\,\eta(f)$. Comparing with (12) and using $\sum_i a_i^4 \leq (\max_i a_i^2) W_1(f)$ yields

$$2W_1(f)(W_1(f) - \max_i a_i^2) \leq 324\,\eta(f).$$

Whenever $\eta(f) \leq 1/2$, we have $W_1(f) = 1 - \eta(f) \geq 1/2$, so $W_1(f) - \max_i a_i^2 \leq 324\,\eta(f)$ and $1 - \max_i a_i^2 = \eta(f) + (W_1(f) - \max_i a_i^2) \leq 325\,\eta(f)$. Because $|a_i| \leq 1$ implies $1 - |a_i| \leq 1 - a_i^2$, we obtain $\varepsilon_{\mathrm{dict}}(f) \leq \frac{325}{2}\,\eta(f)$. When $\eta(f) > 1/2$, $\varepsilon_{\mathrm{dict}}(f) \leq \frac{1}{2} < \frac{325}{2}\,\eta(f)$ holds trivially. Finally, (11) gives $\eta(f) \leq \frac{9}{2}\mathbb{P}(\mathrm{Cycle}(f))$, so $\frac{325}{2}\,\eta(f) \leq \frac{2925}{4}\,\mathbb{P}(\mathrm{Cycle}(f))$, proving (13) unconditionally. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified in deterministic IEEE-754 `float64` arithmetic (seeds `20261129`–`20261201`) using the MIT-licensed companion repository `lluiseriksson/boolean-fourier-arrow-geometry`. Our suite executes 200 structured and randomized hypercube tests via the exact Fast Walsh–Hadamard Transform across dimensions $n \in \{1..11\}$: (i) 50 monotone LTFs, Majority/Dictator/Tribes witnesses, and random real functions checking Theorem 2.1; (ii) 50 Boolean functions across $\delta \in [0.05, 0.45]$ verifying the four-level chain (7); (iii) 50 random low-degree polynomials and d-conjunctions verifying Theorem 4.1; and (iv) 50 Boolean voting rules on $n \in \{3, 4, 5\}$ verified directly against exact 6^n-profile impartial-culture enumeration (up to 7,776 profiles per rule). Table 1 and Figures 1–2 confirm zero violations across all 200 checks.

Table 1: Deterministic verification across 200 hypercube instances (`pytest -q`: 4/4 passed in 0.66 s). All exact spectral, fourth-moment, and 6^n enumeration identities hold to machine precision.

Theorem	Checks	Max Identity Error	Sharpest Attainment / Gain Summary
Thm. 2.1 (Influence & Monotone)	50	8.88×10^{-16} (Var, $\mathbf{I}$)	Exact $\mathbf{I}_i = \hat{f}(\{i\})$ on all monotone LTFs; strict $\mathbf{I} < \sqrt{n}$ ($n \geq 2$)
Thm. 3.1 (4-Level Noise Chain)	50	—	Jensen ceiling cuts $\delta\mathbf{I}[f]$ down to $0.612\times$ at $\delta = 0.2$
Thm. 4.1 (Bonami & SZ Floor)	50	0.0 (d-conj. SZ exactness)	$\max(\|T_{1/\sqrt{3}}f\|_4/\|f\|_2) = 0.9609 \leq 1$; exact 2^{-d} on d-conjunctions
Thm. 5.1 (Arrow & FKN)	50	5.55×10^{-17} (6^n enum.), 0.0 (Eq. (12))	Exact 1/18 (Maj_3, left) and 1/3 (XOR_2, right); 0 FKN/Arrow violations

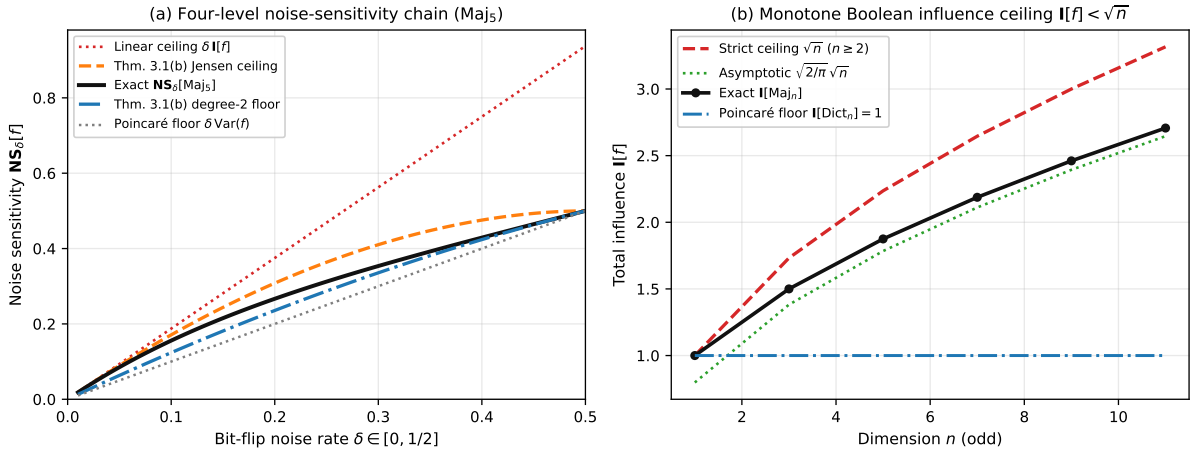

Figure 1: **(a)** Exact noise sensitivity $\mathbf{NS}_\delta[\mathrm{Maj}_5]$ strictly bracketed by the four-level chain (7), illustrating how the spectral-Jensen ceiling and degree-2 floor substantially tighten the classical bounds $\delta\,\mathrm{Var}(f)$ and $\delta\mathbf{I}[f]$. **(b)** Exact total influence $\mathbf{I}[\mathrm{Maj}_n]$ ($n \in \{1,3,5,7,9,11\}$) lying strictly below the universal monotone ceiling $\sqrt{n}$ for all $n \geq 2$.

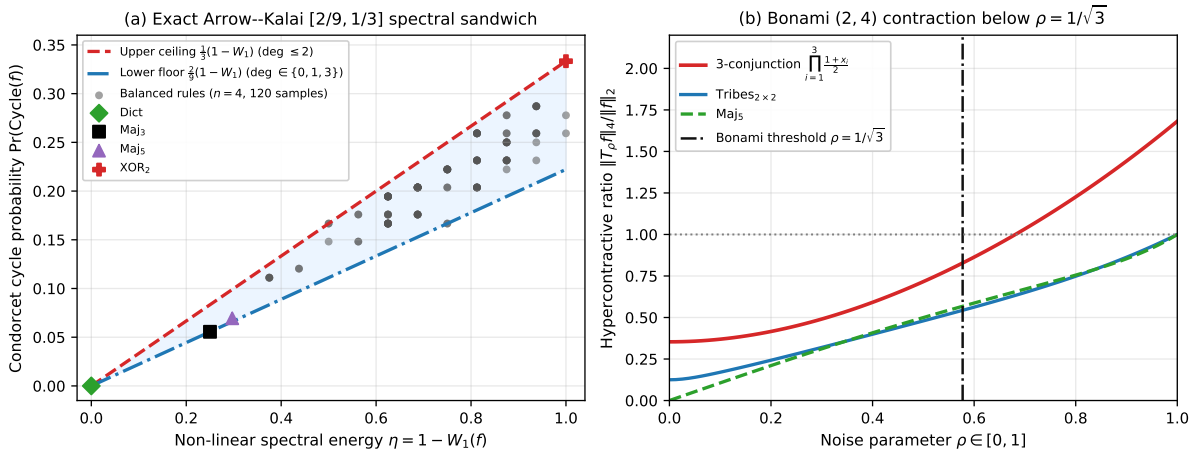

Figure 2: **(a)** Exact Arrow–Kalai $[2/9, 1/3]$ spectral sandwich (11) across 120 balanced 4-voter rules plus exact boundary witnesses Dict, Maj_3 (1/18), Maj_5, and XOR_2 (1/3). **(b)** Bonami hypercontractive ratio $\|T_\rho f\|_4/\|f\|_2$ dropping below 1 for every test function at $\rho \leq 1/\sqrt{3}$.

7 Scope and Limitations

Three structural boundaries merit explicit note:

1. *Uniform hypercube product measure.* All four theorems rely on uniform i.i.d. coordinates on $\{-1,+1\}^n$ (impartial culture in Theorem 5.1); extending Bonami $(2,4)$ hypercontractivity to p-biased product measures μ_p^n replaces the threshold $1/\sqrt{3}$ by a p-dependent factor shrinking as $p(1-p) \to 0$ [5].
2. *Non-optimal explicit constants in FKN–Arrow.* Theorem 5.1(b) prioritizes a six-line, completely self-contained proof directly from Bonami $(2,4)$ hypercontractivity, yielding explicit constants (325/2, 2925/4) that are larger than the asymptotically sharper constants achievable via specialized Gaussian invariance or sharp L_4-moment analyses [3, 5].
3. *Identical pairwise rule across candidates.* Theorem 5.1 analyzes symmetric (neutral) 3-candidate social welfare functions applying the same Boolean rule f to (A,B), (B,C), and (C,A); asymmetric triples (f,g,h) require replacing $\mathbf{Stab}_{-1/3}[f]$ by the three cross-stabilities $\langle f, T_{-1/3}g\rangle + \langle g, T_{-1/3}h\rangle + \langle h, T_{-1/3}f\rangle$ [4].

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