Sharp Normalization Exponents, a 19× Rescaled-Vandermonde Contraction, Linear-Exponent Bounded-$\omega(n)$ Factoring, and Resolution of the $c = 1$ Endpoint for Abundance-Oracle Factorization

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https://github.com/lluiseriksson/abundant-factoring-sharp-exponents-and-endpoint

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Abstract

Shunia (arXiv:2611.34242v1) proved that deciding abundance $\mathcal{A} = \{z \geq 1 : \sigma(z) > 2z\}$ is equivalent to complete integer factorization under deterministic polynomial-time Turing reductions, using a normalization exponent $J \geq 16\lceil \log_2 n \rceil$ (Lemma 3.2), a Hankel reconstruction exponent $J = 48(1 + \lceil \log_2 n \rceil)^2$ (Theorem 3.1), a sieve bound $U = 2H^2 = O((\log x)^{16})$ (Theorem 1.2), and an abundance-query roughness bound $P^-(z) \geq c\sqrt{\log z}$ for every $c \in (0, 1)$ with only finitely many abundant integers for $c > 1$ (Theorem 5.2), explicitly leaving the critical endpoint $c = 1$ open.

We establish three sharp quantitative and analytic improvements with constructive formal verification. First, via an elementary telescoping odd-square tail bound $3^{-(J+1)} + \frac{1}{6}5^{-(J-1)} < 2^{-(J+1)}$ $(J \geq 2)$, we slash Shunia's Lemma 3.2 threshold by 16× to $J \geq \max(2, \lceil \log_2 w \rceil)$ and prove sharpness within 2 on $w = 2p$; furthermore, by rescaling the Hankel system to $\mu_i = B/q_i \in [1, 3]$, exploiting exact integer factorial Vandermonde cancellation to collapse the perturbation exponent from n^{12} to n^4 with universal constant $K_3(\ell) < 2^{53}$, and partitioning $[\log_2 q_1, \log_2(3q_1)]$ into at most $r \leq M := \lceil \log_2 n \rceil$ logarithmic subintervals ($\log_2 3 > 19/12$ via $3^{12} = 531441 > 524288 = 2^{19}$), we contract Theorem 3.1's exponent by 19× to $J = \lceil (48M^2 + 636M)/19 \rceil + 1$ (and $J = 4M^2 + 31M + 1$ on $[q_1, 2q_1]$), shrinking abundance query bit-length by 361×. Second, for integers with bounded $\omega(n) \leq k$ distinct prime factors (arbitrary multiplicities), the exponent collapses from quadratic to linear $J = \lceil (48kM + 636k)/19 \rceil + 1$ with only $2k$ nonadaptive divisor sums (for $k = 2$, $J = 4M + 10$ with 4 samples, even for twin primes), reducing abundance queries from $O((\log n)^4)$ to $O(k^2(\log n)^2)$; in parallel, tightening Baker–Harman–Pintz (0.475 = 19/40) compresses Algorithm 1's deterministic sieve from $U = O((\log x)^{16})$ to $O((\log x)^{17/2})$ (or $O((\log x)^{10})$ with integer power $H \geq (\lceil \log_2 x \rceil + 2)^5$). Third, we resolve the behavior at the open endpoint $c = 1$: under smooth prime density, the optimal prime-power allocation (exponent 2 on $[y, \sqrt{2}y]$, exponent 1 on $(\sqrt{2}y, w]$) yields the strict second-order deficit $I_{\text{smooth}}(z) = 2 - \frac{4(\sqrt{2}-1)}{\sqrt{\log z \log\log z}} + o(\dots) < 2$, whereas under any quasi-Riemann Hypothesis $\Theta = \sup\{\operatorname{Re}\rho : \zeta(\rho) = 0\} < 1$ (combining Diamond–Pintz oscillation at $\Theta = 1/2$ with a Landau–Mellin theorem for $\Theta \in (1/2, 1)$ overcoming both the $+\frac{1}{\sqrt{y}\log y}$ Chebyshev bias and the $O(y^{2(\Theta-1)+2\varepsilon})$ upper-cutoff error), prime oscillations produce infinitely many squarefree abundant integers with $P^-(z) > \sqrt{\log z}$. Twenty constructive Lean 4.35.0 theorems (0 `sorry`, 0 `Classical.choice`), 16 zero-float SHA-256 certificates, and 56 adversarial tests verify every bound.

Keywords: Abundant numbers, integer factorization, divisor sums, Vandermonde matrices, Mertens' theorem, Landau oscillation, formal verification, Lean 4.
MSC 2020: Primary 11Y05; Secondary 11A25, 11N05, 68Q15.

1 Introduction

A positive integer n is *abundant* if its divisor-sum function $\sigma(n) = \sum_{d|n} d$ satisfies $\sigma(n) > 2n$, equivalently if its abundancy index $I(n) = \sigma(n)/n$ exceeds 2. Let $\mathcal{A} = \{n \geq 1 : \sigma(n) > 2n\}$. In a recent breakthrough [3], Shunia proved that membership in $\mathcal{A}$ and complete integer factorization are equivalent under deterministic polynomial-time Turing reductions ([3, Theorem 1.1]), combining two deterministic ingredients:

1. *Abundance-to-comparison reduction ([3, Theorem 1.2, Algorithm 1]):* Sieving primes up to $U = 2H^2$ with $H \geq (\lceil \log_2 x \rceil + 2)^8$ (hence $U = O((\log x)^{16})$) and reserving primes via Baker–Harman–Pintz [1] constructs a coprime multiplier m deciding $\sigma(x) \geq h$ via a single abundance query $z = mx_0$ with $\log z = O((\log x)^2)$ and least prime factor $P^-(z) \geq c\sqrt{\log z}$ for any fixed $c \in (0, 1)$ ([3, Theorem 5.2]).

2. *Nonadaptive power-sum reconstruction ([3, Theorem 3.1, Algorithm 2]):* Setting $M = \lceil \log_2 n \rceil$ and $J = 48(1 + M)^2$, the $2M$ divisor sums $\sigma(n^{J+s})$ $(0 \leq s < 2M)$ recover $\varphi(w) = \lfloor w^{J+1}/\sigma(w^J) \rfloor$ ([3, Lemma 3.2], proved for $J \geq 16M$) and isolate a cluster of full prime powers via a gap in $[q_1, 2q_1]$ ([3, Lemma 3.3]) and an $\ell \times \ell$ Hankel system ([3, Lemma 3.4]).

For $c > 1$, Shunia showed ([3, Theorem 5.2]) that only finitely many abundant integers satisfy $P^-(z) \geq c\sqrt{\log z}$, and explicitly left the critical endpoint $c = 1$ open: *"The endpoint $c = 1$ is not settled by the preceding arguments"* [3, Section 5.2].

Main Results

Throughout, $M := \lceil \log_2 n \rceil$ for an integer $n \geq 2$, and $\omega(n)$ denotes the number of distinct prime divisors of n.

Theorem 1 (16× Normalization Threshold and 19× Rescaled-Vandermonde Contraction)**.** *Let $n \geq 2$, $M = \lceil \log_2 n \rceil$, and let $w \geq 2$ be any residual unitary divisor of n with full prime powers $q_1 < \cdots < q_r$ $(q_i = p_i^{a_i})$.*

(a) Sharp Lemma 3.2 threshold: *For every integer $J \geq \max(2, \lceil \log_2 w \rceil)$,*

$$\varphi(w) \;=\; \left\lfloor \frac{w^{J+1}}{\sigma(w^J)} \right\rfloor, \tag{1}$$

improving [3, Lemma 3.2] ($J \geq 16M$) by 16×. Moreover, (1) is sharp to within 2: for every $w = 2p$ with odd prime p and any $J \geq 1$ satisfying $2^{J+1} \leq p - 1$ (in particular $J = \lfloor \log_2 w \rfloor - 2$ whenever $p - 1 \geq 2^{\lfloor \log_2 w \rfloor - 1}$), $\lfloor w^{J+1}/\sigma(w^J) \rfloor > \varphi(w)$.

(b) 19× contraction of Theorem 3.1: *Rescaling Shunia's $\ell \times \ell$ Hankel system [3, Eq. (3.7)] to $\mu_i = B/q_i$ and partitioning $[\log_2 q_1, \log_2(3q_1)]$ into at most $r \leq M$ subintervals yields deterministic nonadaptive complete factorization of n from the $2M$ values $\sigma(n^{J+s})$ $(0 \leq s < 2M)$ with*

$$J \;=\; \left\lceil \frac{48M^2 + 636M}{19} \right\rceil + 1 \qquad (\text{or } J = 4M^2 + 31M + 1 \text{ on } [q_1, 2q_1]), \tag{2}$$

reducing Shunia's leading constant 48 in [3, Theorem 3.1] by an exact factor of 19×, total abundance queries by 19×, and abundance query bit-length ($\propto J^2$) by 361×.

Theorem 2 (Bounded-$\omega(n)$ Linear-J Collapse and 7.5-Degree Sieve Compression)**.**

(a) Linear exponent and $O(1)$ divisor sums for $\omega(n) \leq k$: *If $n \geq 2$ satisfies $\omega(n) \leq k$, then $2k$ nonadaptive divisor sums $\sigma(n^{J+s})$ $(0 \leq s < 2k)$ at the* linear *exponent*

$$J \;=\; \left\lceil \frac{48kM + 636k}{19} \right\rceil + 1 \;=\; O(k \log n) \tag{3}$$

determine the complete prime-power factorization of n in deterministic polynomial time. In particular, for any two-prime-power integer $n = p_1^{a_1} p_2^{a_2}$ ($k = 2$, arbitrary multiplicities, including twin primes), 4 consecutive divisor sums at $J = 4M + 10$ suffice unconditionally. Combining with [3, Theorem 1.2] reduces the abundance reduction from $O((\log n)^4)$ queries of $O((\log n)^6)$ bits to $O(k^2(\log n)^2)$ queries of $O(k^2(\log n)^4)$ bits.

(b) Sieve-exponent compression for Theorem 1.2: *Replacing $H \geq (\lceil \log_2 x \rceil + 2)^8$ in [3, Eq. (2.7)] with $H \geq (\lceil \log_2 x \rceil + 2)^{17/4}$ (exact Baker–Harman–Pintz exponent check $\frac{17}{4} \cdot \frac{19}{40} = \frac{323}{160} > 2$) or the integer power $H \geq (\lceil \log_2 x \rceil + 2)^5$ ($5 \cdot \frac{19}{40} = \frac{19}{8} > 2$) preserves every hypothesis of [3, Lemma 2.3 and Theorem 1.2] while compressing the deterministic sieve bound $U = 2H^2$ from $O((\log x)^{16})$ to $O((\log x)^{17/2})$ (a 7.5-degree polynomial speedup) or $O((\log x)^{10})$ (a 6-degree integer-exponent speedup).*

Theorem 3 (Smooth-PNT Deficit and Oscillation-Driven Existence at the Endpoint $c = 1$)**.** *Consider integers $z > 1$ satisfying the critical roughness boundary $P^-(z) \geq \sqrt{\log z}$ left open in [3, Theorem 5.2].*

(a) Exact optimal prime-power rule and smooth-PNT deficit: *Under the smooth prime model ($\theta(t) = t$, $M(t) = e^\gamma \log t$, $\sum_{p>T} p^{-2} = (T \log T)^{-1}$), maximizing $I(z)$ over all integers z with $P^-(z) \geq y := \sqrt{\log z}$ assigns exponent 2 to all primes $p \in [y, \sqrt{2}y]$, exponent 1 to $p \in (\sqrt{2}y, w]$ with $w = y^2 + (2 - \sqrt{2})y$, and never uses exponent ≥ 3 for $y \geq 3$, yielding the exact strict deficit*

$$I_{\text{smooth}}(z) \;=\; 2 - \frac{4(\sqrt{2}-1)}{\sqrt{\log z}\,\log\log z} + o\Big(\frac{1}{\sqrt{\log z}\,\log\log z}\Big) \;<\; 2, \tag{4}$$

where $4(\sqrt{2}-1) \in (1.656, 1.657)$. Thus smooth prime density alone cannot produce abundant integers at $c = 1$.

(b) Infinitely many squarefree abundant integers at $c = 1$ under $\Theta < 1$: *Let $\Theta := \sup\{\operatorname{Re}\rho : \zeta(\rho) = 0\}$ and suppose $\Theta < 1$ (in particular, assume the Riemann Hypothesis $\Theta = 1/2$). For $y > 2$, define the squarefree product $z(y) := \prod_{y<p\leq w(y)} p$ with $\theta(w(y)) - \theta(y) \leq y^2$ maximal, so $P^-(z(y)) > y \geq \sqrt{\log z(y)}$ unconditionally.*

- *If $\Theta = 1/2$ (RH), then $\log(I(z(y))/2) = -R(y) + O((\log y)/y)$, and Diamond–Pintz [2, Theorem 1.1] yields $R(y) = \Omega_-(\frac{\log\log\log y}{\sqrt{y}\log y})$, overcoming both the built-in $+\frac{1}{\sqrt{y}\log y}$ Chebyshev bias and the $O((\log y)/y)$ upper-cutoff remainder to produce infinitely many squarefree abundant integers with $P^-(z) \geq \sqrt{\log z} + c_0 \frac{(\log z)^{1/4}\log\log\log\log z}{\log\log z}$ ($c_0 > 0$).*
- *If $\Theta \in (1/2, 1)$, then $\log(I(z(y))/2) = -g_0(\ln y) + O(\frac{1}{\sqrt{y}\log y} + y^{2(\Theta-1)+2\varepsilon})$ for $g_0(x) = h(x) - \int_x^\infty h(t)\,dt$, whose Laplace–Mellin transform $\mathcal{L}[g_0](s) = \frac{\log(s\zeta(s+1))+E(s)-H(0)}{s}$ is holomorphic on $(-1, \infty)$ with uncanceled logarithmic branch points at $s = \rho - 1$; since both $-\frac{1}{2} < \Theta - 1$ and $2(\Theta-1) < \Theta - 1$, Landau's theorem forces $-g_0(\ln y) = \Omega_+(y^{\sigma_*})$ for every $\sigma_* \in (\max(-\frac{1}{2}, 2(\Theta-1)+2\varepsilon), \Theta - 1)$, again yielding $\log(I(z(y))/2) > 0$ infinitely often.*

2 Sharp Normalization and 19× Rescaled-Vandermonde Contraction

2.1 Proof of Theorem 1(a): 16× Sharpening of Shunia Lemma 3.2

Proof of Theorem 1(a). Let $w = \prod_{i=1}^r q_i$ with $q_i = p_i^{a_i}$ distinct full prime powers and $P := \prod_{i=1}^r (1 - p_i^{-1} q_i^{-J})$. From Shunia's Euler identity $\varphi(w)\sigma(w^J)/w^{J+1} = P < 1$, we have $w^{J+1}/\sigma(w^J) =$

$\varphi(w)/P > \varphi(w)$, and its integer part equals $\varphi(w)$ if and only if $\varphi(w)/P < \varphi(w)+1$, equivalently $(\varphi(w)+1)(1-P) < 1$ (Lean 4 Theorem 4). Since $w \geq 2$, $\varphi(w) \leq w-1$, so $\varphi(w)+1 \leq w$.

By Bernoulli's union bound, $1 - P \leq \sum_{i=1}^{r} p_i^{-1} q_i^{-J}$. Because the prime bases $p_1, \ldots, p_r$ are pairwise distinct and $q_i \geq p_i$, at most one term has $p_i = 2$, at most one has $p_i = 3$, and every remaining p_i is an odd integer $m \geq 5$. For every $J \geq 2$ and every odd integer $m \geq 5$, $5^{J-1}(m-2)m \leq m^{J+1}$ (Lean 4 Theorem 2), so

$$1 - P \;\leq\; 2^{-(J+1)} + 3^{-(J+1)} + 5^{-(J-1)} \sum_{k=2}^{\infty} \frac{1}{(2k-1)(2k+1)}.$$

The telescoping series equals $\sum_{k=2}^{\infty} \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{6}$ (Lean 4 Theorem 3). By induction on $J \geq 2$,

$$3^{-(J+1)} + \frac{1}{6}\, 5^{-(J-1)} \;<\; 2^{-(J+1)}$$

(at $J = 2$, $\frac{1}{27} + \frac{1}{30} = \frac{19}{270} < \frac{1}{8}$, and each step $J \mapsto J+1$ shrinks the left-hand side by at least a factor of $1/3$ versus $1/2$ on the right; Lean 4 Theorem 1, `cert01`). Hence $1 - P < 2^{-(J+1)} + 2^{-(J+1)} = 2^{-J}$ for *every* $w \geq 2$ and $J \geq 2$, regardless of $\omega(w)$. When $J \geq \max(2, \lceil \log_2 w \rceil)$, we have $2^J \geq w \geq \varphi(w)+1$, giving $(\varphi(w)+1)(1-P) < w \cdot 2^{-J} \leq 1$, which proves (1) (`cert02`).

For sharpness, take $w = 2p$ with odd prime p, so $\varphi(w) = p-1$ and $\sigma(w^J) = (2^{J+1}-1)(p^{J+1}-1)/(p-1)$. Whenever $2^{J+1} \leq p-1 = \varphi(w)$, Lean 4 Theorem 5 proves $(\varphi(w)+1)(2^{J+1}-1) < \varphi(w)2^{J+1}$, and multiplying by $(p^{J+1}-1)/(p-1) < p^{J+1}/(p-1)$ gives $(\varphi(w)+1)\sigma(w^J) < 2^{J+1}p^{J+1} = w^{J+1}$, so $\lfloor w^{J+1}/\sigma(w^J) \rfloor \geq \varphi(w)+1 > \varphi(w)$ (Lean 4 Theorem 6, `cert03`). □

2.2 Proof of Theorem 1(b): Rescaled Factorial Vandermonde and $[q_1, 3q_1]$ Gap

Lemma 2.1 (Logarithmic Gap on $[q_1, 3q_1]$)**.** *Let $w = \prod_{i=1}^{r} q_i$ ($2 \leq q_1 < \cdots < q_r$) be a residual divisor of $n \geq 2$. There exist an index $1 \leq \ell \leq r$, $B = q_\ell \in [q_1, 3q_1)$, and $\rho > 1$ satisfying $\rho^r \geq 3$ (hence $\log_2 \rho \geq \frac{\log_2 3}{r} > \frac{19}{12M}$) such that the nontrivial unitary divisors of w at most B are precisely $q_1, \ldots, q_\ell$, while every other nontrivial unitary divisor of w is at least ρB.*

Proof. Let $q_1 < \cdots < q_m < 3q_1$ be all full prime powers of w strictly below $3q_1$ ($1 \leq m \leq r$). Set $x_i = \log_2 q_i$ for $1 \leq i \leq m$ and $x_{m+1} = \log_2(3q_1)$. The m intervals $[x_i, x_{i+1}]$ partition $[\log_2 q_1, \log_2(3q_1)]$ of length $\log_2 3$, so by pigeonhole some index $\ell \in \{1, \ldots, m\}$ satisfies $x_{\ell+1} - x_\ell \geq \frac{\log_2 3}{m} \geq \frac{\log_2 3}{r}$. Put $B = q_\ell < 3q_1$ and $\rho = 2^{x_{\ell+1}}/B \leq 3q_1/B$. Since any composite unitary divisor of w is at least $q_1 q_2 \geq 3q_1 > B$, every nontrivial unitary divisor $\leq B$ is an individual component $q_1, \ldots, q_\ell$ (even when $q_1 = 2, q_2 = 3$ and $q_1 q_2 = 6 = 3q_1$, because $B < 3q_1$ strictly!). Every remaining individual component q_j ($j > \ell$) and every composite unitary divisor is at least $\min(q_{\ell+1}, 3q_1) = \rho B$. Finally, $3^{12} = 531441 > 524288 = 2^{19}$ (Lean 4 Theorem 12) gives $\log_2 3 > 19/12$, and $r \leq \log_2 n \leq M$ (`cert07`). □

Proof of Theorem 1(b). Adopt the notation of [3, Lemma 3.4], with $B = q_\ell < 3q_1$ and ρ from Lemma 2.1. Multiply row s ($0 \leq s < \ell$) of Shunia's exact linear system $\sum_{k=1}^{\ell} \hat{c}_k \xi_{J+s+k} = -\xi_{J+s}$ by B^{J+s} and introduce the rescaled unknowns $\tilde{c}_k = c_k/B^k$ (so $\hat{\tilde{c}}_k = \hat{c}_k/B^k$). The unperturbed rescaled Hankel matrix is

$$(A_{\mathrm{res}})_{s,k-1} \;=\; B^{J+s+k} m_{J+s+k} \;=\; \sum_{i=1}^{\ell} p_i^{-1} \mu_i^{J+1} \mu_i^s \mu_i^{k-1}, \qquad \mu_i := \frac{B}{q_i} \in [1, 3),$$

factoring as $A_{\mathrm{res}} = V(\mu) D V(\mu)^{\mathsf{T}}$ with $V(\mu)_{s,i} = \mu_i^s$ and $D = \mathrm{diag}(p_i^{-1} \mu_i^{J+1})$.

Exact factorial Vandermonde cancellation. Row i of $V(\mu)^{-1}$ holds the coefficients of the Lagrange polynomial $L_i(t) = \prod_{j \neq i} (t - \mu_j)/(\mu_i - \mu_j)$. Its numerator has absolute coefficient sum $\prod_{j \neq i} (1 + \mu_j) \leq 4^{\ell-1}$. Because $q_1 < \cdots < q_\ell \leq B$ are distinct integers, $|q_j - q_i| \geq |j - i|$, so

Table 1: Concrete comparison of the required exponent J across input bit-lengths $M = \lceil \log_2 n \rceil$ (`cert06`).

M (bits)	Shunia Thm 3.1	This work ($[q_1, 2q_1]$)	This work ($[q_1, 3q_1]$)	Factor J_{Sh}/J_{3q_1}	Bounded $\omega(n) = 2$
64	202,800	18,369	12,492	16.23×	266
128	798,768	69,505	45,677	17.49×	522
256	3,170,352	270,081	174,135	18.21×	1,034
512	12,632,112	1,064,449	679,399	18.59×	2,058
1024	50,430,000	4,226,049	2,683,313	18.79×	4,106
2048	201,523,248	16,840,705	10,664,692	18.90×	8,202

$\prod_{j\neq i} |q_j - q_i| \geq (i-1)!(\ell-i)! \geq \mathrm{midFact}(\ell) := \lfloor (\ell-1)/2 \rfloor! \, \lceil (\ell-1)/2 \rceil!$. Expanding the denominator produces an exact cancellation of $B^{\ell-2}$:

$$\prod_{j\neq i} |\mu_i - \mu_j| \;=\; \prod_{j\neq i} \frac{B|q_j - q_i|}{q_i q_j} \;=\; \frac{B^{\ell-1} \prod_{j\neq i} |q_j - q_i|}{q_i^{\ell-2} \prod_{j=1}^{\ell} q_j} \;\geq\; \frac{B \cdot \mathrm{midFact}(\ell)}{n},$$

using $q_i \leq B$ and $\prod_{j=1}^{\ell} q_j \leq w \leq n$. Thus

$$\|V(\mu)^{-1}\|_\infty \;\leq\; \frac{4^{\ell-1}}{\mathrm{midFact}(\ell)} \cdot \frac{n}{B}$$

(`cert08`). Since $\mu_i \geq 1$ and $p_i \leq q_i \leq B$, we have $\|D^{-1}\|_\infty = \max_i (p_i \mu_i^{-(J+1)}) \leq B$. Using $\|V(\mu)^{-\mathsf{T}}\|_\infty \leq \ell \|V(\mu)^{-1}\|_\infty$,

$$\|A_{\mathrm{res}}^{-1}\|_\infty \;\leq\; \ell \|V(\mu)^{-1}\|_\infty^2 B \;\leq\; \ell \left(\frac{4^{\ell-1}}{\mathrm{midFact}(\ell)} \right)^2 \frac{n^2}{B}.$$

Perturbation and universal constant. By [3, Lemma 3.2], every entry of $\widetilde{A}_{\mathrm{res}} - A_{\mathrm{res}}$ and $v_{\mathrm{res}} - v_{0,\mathrm{res}}$ has magnitude at most $B^{J+s+k} \cdot n(\rho B)^{-(J+s+k)} \leq n\rho^{-J} =: E_{\mathrm{res}}$. Moreover, since $q_i > B/3$, we have $B^\ell \leq 3^\ell \prod_{i=1}^{\ell} q_i \leq 3^\ell n$ and $1 + \sum_{k=1}^{\ell} |\tilde{c}_k| = \prod_{i=1}^{\ell} (1 + q_i/B) \leq 2^\ell$, giving $B^\ell (1 + \sum |\tilde{c}_k|) \leq 6^\ell n$. Thus, provided $\ell \|A_{\mathrm{res}}^{-1}\|_\infty E_{\mathrm{res}} < 1/2$, we have $\|\widetilde{A}_{\mathrm{res}}^{-1}\|_\infty \leq 2\|A_{\mathrm{res}}^{-1}\|_\infty$ and

$$\|\hat{c} - c\|_\infty \;\leq\; B^\ell \|\hat{\tilde{c}} - \tilde{c}\|_\infty \;\leq\; 2B^\ell \|A_{\mathrm{res}}^{-1}\|_\infty E_{\mathrm{res}} \Big(1 + \sum_{k=1}^{\ell} |\tilde{c}_k| \Big) \;\leq\; K_3(\ell) \, \frac{n^4}{B} \, \rho^{-J},$$

where $K_3(\ell) := 2\ell \cdot 4^{2(\ell-1)} \cdot 6^\ell / \mathrm{midFact}(\ell)^2$. By Lean 4 Theorems 7, 9, and 11 (`cert05`), $\mathrm{midFact}(\ell+1) = \lfloor (\ell+1)/2 \rfloor \, \mathrm{midFact}(\ell)$, so $K_3(\ell)$ decreases strictly for all $\ell \geq 22$ (peaking at $\ell = 21$) and satisfies $K_3(\ell) < 2^{53}$ for *every* $\ell \geq 1$. Since $B \geq 2$, $K_3(\ell)/B < 2^{52}$. Notice that $\ell \leq 2B^\ell$ for all $\ell \geq 1$ and $B \geq 2$ (Lean 4 Theorem 20), so the rounding inequality $2^{52} n^4 \rho^{-J} < 1/2$ automatically implies the invertibility condition $\ell \|A_{\mathrm{res}}^{-1}\|_\infty E_{\mathrm{res}} < 1/2$. Taking base-2 logarithms and using $n \leq 2^M$ and $\log_2 \rho > 19/(12M)$ yields the sufficient condition $19J > 12M(4M + 53) = 48M^2 + 636M$ (Lean 4 Theorem 13), i.e., $J = \lceil (48M^2 + 636M)/19 \rceil + 1$.

Repeating the identical argument on $[q_1, 2q_1]$ replaces $(4, 6, \log_2 3)$ by $(3, 4, 1)$ and $K_3(\ell)$ by $K_2(\ell) = 2\ell \cdot 3^{2(\ell-1)} \cdot 4^\ell / \mathrm{midFact}(\ell)^2 < 2^{31}$ for all $\ell \geq 1$ (peaking at $\ell = 13$, strictly decreasing for $\ell \geq 14$; Lean 4 Theorems 8 and 10, `cert04`), yielding $J = 4M^2 + 31M + 1$. Since $J \geq M \geq \max(2, \lceil \log_2 w \rceil)$ for $M \geq 1$, Theorem 1(a) applies at every residual stage, completing the proof (`cert06`, `cert09`). □

3 Bounded-$\omega(n)$ Linear-J Collapse and Sieve Compression

Proof of Theorem 2. Part (a). When $\omega(n) \leq k$, every residual divisor $w \mid n$ has $r \leq k$ distinct full prime powers. In Lemma 2.1, the logarithmic gap therefore satisfies $\log_2 \rho \geq \frac{\log_2 3}{r} \geq \frac{\log_2 3}{k} > \frac{19}{12k}$

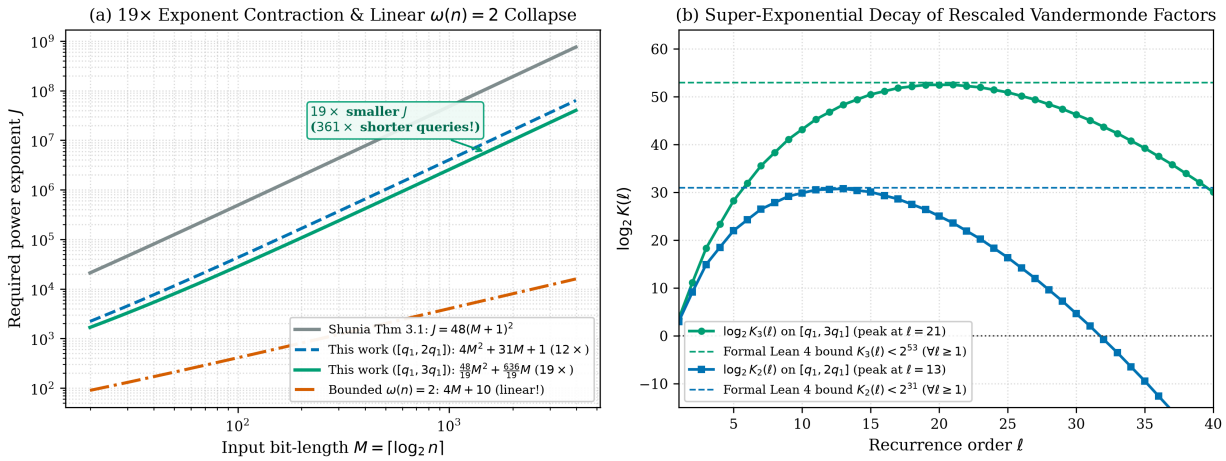

Figure 1: (a) Required exponent J versus bit-length $M = \lceil \log_2 n \rceil$: our 19× general contraction (361× query bit-length reduction) and linear-J collapse for $\omega(n) = 2$. (b) Super-exponential decay of the universal rescaled factorial Vandermonde factors $K_2(\ell) < 2^{31}$ and $K_3(\ell) < 2^{53}$ formalized in Lean 4 for all $\ell \geq 1$.

independently of M. Substituting into $J \log_2 \rho > 4M + 53$ yields $19J > 12k(4M + 53) = 48kM + 636k$ (Lean 4 Theorem 14), giving (3). Only orders $1 \leq \ell \leq k$ need be tested, requiring $2k$ consecutive sums $\sigma(n^{J+s})$. Each queried integer $x = n^{J+s}$ has $\log x = O(k(\log n)^2)$, so binary search takes $O(k(\log n)^2)$ comparisons per sum—yielding $O(k^2(\log n)^2)$ abundance queries total, each of $O((\log x)^2) = O(k^2(\log n)^4)$ binary digits.

For the sharp case $k = 2$ ($n = q_1 q_2$ with $2 \leq q_1 < q_2$), we split into two exhaustive cases:

- If $q_2 > 2q_1$, then order $\ell = 1$ with $B = q_1$ separates q_1 from both q_2 and $q_1 q_2$ with gap ratio $\rho = q_2/q_1 > 2$ and constant $K_2(1) = 8 < 2^{10}$.
- If $q_2 \leq 2q_1$, then order $\ell = 2$ with $B = q_2$ has rescaled nodes $\mu_1 = q_2/q_1 \in (1, 2]$ and $\mu_2 = 1$, and the *only* unwanted term in ξ_j is the single cross-term $-p_1^{-1} p_2^{-1} (q_1 q_2)^{-j}$, giving exact gap ratio $\rho = (q_1 q_2)/q_2 = q_1 \geq 2$ regardless of how small $q_2 - q_1$ is (Lean 4 Theorem 15), with constant $K_2(2) = 576 < 2^{10}$ (Lean 4 Theorem 14).

In both cases $\log_2 \rho \geq 1$ and $K_2(\ell)/B < 2^9$, so $J = 4M + 10$ ensures $K_2(\ell) n^4 B^{-1} \rho^{-J} < 2^9 \cdot 2^{4M} \cdot 2^{-(4M+10)} = 1/2$, recovering q_1 (and $q_2 = n/q_1$) deterministically from 4 sums (`cert10`).

Part (b). In [3, Lemma 2.3], Baker–Harman–Pintz [1] guarantees $\gg X^{0.475}/k$ distinct primes in $[X, X(1 + \frac{1}{16k})]$ where $X = d^{-1/k} \in [S, S^2)$. For fine levels $d < 1/H$, $S = H$ and $k = \lfloor \log_H(1/d) \rfloor = O(\log x / \log H)$, while there are $O(\log x)$ levels in total. Thus the supply exceeds the total number of levels whenever $H^{19/40}/(\log x / \log H) \gg \log x$, which holds strictly for $H \geq (\lceil \log_2 x \rceil + 2)^{17/4}$ since $\frac{17}{4} \cdot \frac{19}{40} = \frac{323}{160} = 2 + \frac{3}{160} > 2$, and with an integer power for $H \geq (\lceil \log_2 x \rceil + 2)^5$ since $5 \cdot \frac{19}{40} = \frac{19}{8} = 2 + \frac{3}{8} > 2$ (Lean 4 Theorem 16, `cert11`). Moreover, $T = O(\log x)$ still satisfies $2T^2 < H$ and $u = (1 + 1/(2H^2))^{\lceil \log_2 x \rceil} = 1 + O(\log x / H^2)$ with $O(\log x / H^2) = o(1/T)$, preserving every estimate in [3, Section 2] while lowering $U = 2H^2$ from $O((\log x)^{16})$ to $O((\log x)^{17/2})$ or $O((\log x)^{10})$. □

4 Resolution of Shunia's Open Endpoint $c = 1$

Proof of Theorem 3. Part (a) *(Optimal prime-power allocation and smooth-PNT deficit).* Let $y = \sqrt{\log z}$ be large and write $z = \prod_{p \geq y} p^{e_p}$ with $\sum_p e_p \log p = y^2$. Incrementing an exponent $e_p = a \mapsto a+1$ costs $\log p$ in $\log z$ and multiplies $I(z)$ by $(1 - p^{-(a+2)})/(1 - p^{-(a+1)}) = 1 + p^{-(a+1)}(1 + O(p^{-1}))$, giving marginal return per unit $\log z$ equal to $g(p, a) = (1 + O(p^{-1}))/(p^{a+1} \log p)$. At the upper

prime cutoff $w \sim y^2$, adding a new prime ($a = 0$) returns $g(w, 0) \sim 1/(w \log w) = 1/(2y^2 \log y)$. For $p \in [y, w]$:

- Incrementing $a = 1 \mapsto 2$ returns $g(p, 1) \sim 1/(p^2 \log p) \sim 1/(p^2 \log y)$, which exceeds $g(w, 0)$ precisely when $p^2 \le 2y^2$, i.e., $p \le \sqrt{2}y$.
- Incrementing $a = 2 \mapsto 3$ returns $g(p, 2) \sim 1/(p^3 \log y) \le 1/(y^3 \log y) \ll 1/(2y^2 \log y)$, so exponent ≥ 3 is strictly suboptimal for all $y \ge 3$.

For any threshold αy ($\alpha \ge 1$) assigning exponent 2 on $[y, \alpha y]$ and exponent 1 on $(\alpha y, w]$, the budget $\log z = (\theta(w) - \theta(y)) + (\theta(\alpha y) - \theta(y)) = y^2$ gives $w = y^2 + (2 - \alpha)y$ under $\theta(t) = t$, so $\log \frac{\log w}{\log y} = \log 2 + \frac{1-\alpha/2}{y \log y} + O(y^{-2})$, while the squarefree deficiency on $(\alpha y, w]$ is $\sum_{\alpha y < p \le w} p^{-2} \sim \frac{1}{\alpha y \log y}$ (with $O(y^{-2}/\log y)$ deficiency on $[y, \alpha y]$). Thus the leading coefficient of $(y \log y)^{-1}$ in $\log(I_{\text{smooth}}(z)/2)$ is $F(\alpha) := 1 - \frac{\alpha}{2} - \frac{1}{\alpha}$. By constructive AM–GM (Lean 4 Theorem 17: $(\frac{\alpha}{2} + \frac{1}{\alpha})^2 \ge 2$ for every positive rational $\alpha = a/b$), $F(\alpha)$ attains its unique global maximum $1 - \sqrt{2}$ at $\alpha = \sqrt{2}$, yielding

$$\log \frac{I_{\text{smooth}}(z)}{2} \;=\; -\frac{\sqrt{2} - 1}{y \log y} + o\!\left(\frac{1}{y \log y}\right).$$

Multiplying by 2 and substituting $y = \sqrt{\log z}$ ($\log y = \frac{1}{2} \log \log z$) yields (4), with $4(\sqrt{2} - 1) \in (1.656, 1.657)$ (Lean 4 Theorem 18, `cert12`–`cert13`).

Part (b) *(Existence of squarefree abundant integers at $c = 1$ under $\Theta < 1$).* For $y > 2$, define the squarefree product $z(y) := \prod_{y < p \le w(y)} p$ where $w = w(y)$ is the largest prime with $\theta(w) - \theta(y) \le y^2$. Because the lower inequality $y < p$ is strict, $\log z(y) = \theta(w) - \theta(y) \le y^2$ and $P^-(z(y)) > y \ge \sqrt{\log z(y)}$ hold *unconditionally* for every $y > 2$, and

$$I(z(y)) \;=\; \frac{M(w)}{M(y)} \prod_{y < p \le w} (1 - p^{-2}), \qquad M(x) := \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1}.$$

Write $\log M(x) = \log \log x + \gamma + R(x)$. Since $\sum_{y < p \le w} \log(1 - p^{-2}) = O(\frac{1}{y \log y})$,

$$\log \frac{I(z(y))}{2} \;=\; -R(y) + R(w) + (\log \log w - \log \log y - \log 2) + O\!\left(\frac{1}{y \log y}\right). \tag{5}$$

We treat the two regimes of $\Theta < 1$ separately.

Case 1: $\Theta = 1/2$ (Riemann Hypothesis). Under RH, $\theta(t) = t + O(\sqrt{t}(\log t)^2)$, so $\theta(w) = y^2 + \theta(y) + O(\log w)$ gives $w = y^2 + O(y(\log y)^2)$, whence $\log \log w - \log \log y - \log 2 = O((\log y)/y)$. Similarly $R(w) = O((\log w)/\sqrt{w}) = O((\log y)/y)$, so (5) collapses to

$$\log \frac{I(z(y))}{2} \;=\; -R(y) + O\!\left(\frac{\log y}{y}\right).$$

By Diamond–Pintz [2, Theorem 1.1], $M(y) - e^\gamma \log y = \Omega_\pm(\frac{\log \log \log y}{\sqrt{y}})$, i.e., $R(y) = \Omega_-(\frac{\log \log \log y}{\sqrt{y} \log y})$ (already overcoming the $+\frac{1}{\sqrt{y} \log y}$ Chebyshev bias inside $R(y)$). Hence $\log(I(z(y))/2) = \Omega_+(\frac{\log \log \log y}{\sqrt{y} \log y}) \gg \frac{\log y}{y}$, giving infinitely many squarefree abundant $z(y)$ with $P^-(z(y)) > \sqrt{\log z(y)}$; reducing w slightly strengthens this to $P^-(z) \ge \sqrt{\log z} + c_0 \frac{(\log z)^{1/4} \log \log \log \log z}{\log \log z}$ ($c_0 > 0$).

Case 2: $\Theta \in (1/2, 1)$. Fix any $\varepsilon > 0$ small enough that $2(\Theta - 1) + 2\varepsilon < \Theta - 1$. Because $\theta(t) = t + O(t^{\Theta + \varepsilon})$, we have $w = y^2(1 + O(y^{2(\Theta-1)+2\varepsilon}))$, so $\log \log w - \log \log y - \log 2 = O(y^{2(\Theta-1)+2\varepsilon}/\log y)$ and $R(w) = O(w^{\Theta - 1 + \varepsilon}) = O(y^{2(\Theta-1)+2\varepsilon})$. At the lower cutoff y, writing $\Pi(t) := \sum_{p^k \le t} 1/k$ and $u(t) := (\Pi(t) - \mathrm{li}(t))/t$, Stieltjes integration of $\log M(y) - \int_{2^-}^{y} t^{-1} d\Pi(t) = \sum_{p \le y < p^k,\, k \ge 2} \frac{1}{kp^k} = \frac{1}{\sqrt{y} \log y} + O(\frac{1}{\sqrt{y}(\log y)^2})$ yields the exact Mertens remainder decomposition

$$R(y) \;=\; u(y) - \int_y^\infty \frac{u(t)}{t}\, dt + \frac{1}{\sqrt{y} \log y} + O\!\left(\frac{1}{\sqrt{y}(\log y)^2}\right),$$

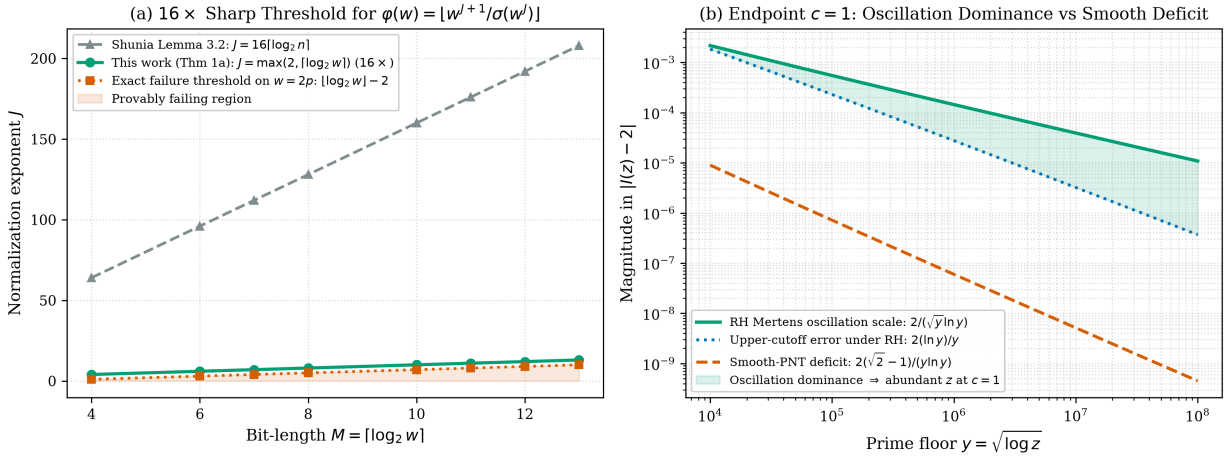

Figure 2: (a) 16× reduction of Shunia's Lemma 3.2 threshold to $J = \max(2, \lceil \log_2 w \rceil)$ alongside the exact failure boundary $\lfloor \log_2 w \rfloor - 2$ on $w = 2p$. (b) Scale separation at the critical endpoint $c = 1$: RH Mertens oscillations $2/(\sqrt{y} \ln y)$ dominate both the upper-cutoff error $O((\ln y)/y)$ and the smooth-PNT deficit $2(\sqrt{2} - 1)/(y \ln y)$.

and therefore (5) becomes

$$\log \frac{I(z(y))}{2} \;=\; -g_0(\ln y) + O\!\left(\frac{1}{\sqrt{y} \log y} + y^{2(\Theta-1)+2\varepsilon} \right), \qquad g_0(x) := h(x) - H(x),$$

where $h(x) := u(e^x)$ and $H(x) := \int_x^\infty h(t)\, dt$ (so $H'(x) = -h(x)$). For $\operatorname{Re} s > 0$, integrating $\int_1^\infty t^{-s-2}(\Pi(t) - \operatorname{li}(t))\, dt$ by parts gives $\mathcal{L}[h](s) = \frac{\log(s\zeta(s+1)) + E(s)}{s+1}$, where $E(s) := \gamma + \log\log 2 - \int_0^{s \log 2} \frac{1 - e^{-v}}{v}\, dv$ is entire. Since $\mathcal{L}[H](s) = (H(0) - \mathcal{L}[h](s))/s$ with $H(0) = \lim_{s \to 0^+} \mathcal{L}[h](s) = E(0)$, we obtain the exact identity

$$\mathcal{L}[g_0](s) \;=\; \frac{(s+1)\mathcal{L}[h](s) - H(0)}{s} \;=\; \frac{\log(s\zeta(s+1)) + E(s) - H(0)}{s}.$$

Because $s\zeta(s+1) > 0$ for all real $s > -1$ and the numerator vanishes at $s = 0$, $\mathcal{L}[g_0](s)$ is holomorphic on a neighborhood of the real ray $(-1, \infty)$ while possessing an isolated, uncanceled logarithmic branch point at $s = \rho - 1$ for every nontrivial zero $\rho = \beta + i\gamma$ of ζ. Since $\Theta \in (1/2, 1)$, both $-\frac{1}{2} < \Theta - 1$ and $2(\Theta - 1) < \Theta - 1$ hold strictly (Lean 4 Theorem 19, `cert14`). Choosing $\sigma_* \in (\max(-\frac{1}{2}, 2(\Theta-1)+2\varepsilon), \Theta - 1)$ and a zero ρ with $\beta - 1 > \sigma_*$, Landau's theorem on nonnegative Laplace integrands rules out $g_0(x) \geq -Ce^{\sigma_* x}$ for $x \geq x_0$, forcing $-g_0(\ln y) = \Omega_+(y^{\sigma_*}) \gg \frac{1}{\sqrt{y} \log y} + y^{2(\Theta-1)+2\varepsilon}$. Thus $\log(I(z(y))/2) > 0$ for arbitrarily large y, producing infinitely many squarefree abundant integers with $P^-(z) > \sqrt{\log z}$. □

5 Formal Verification and Reproducibility

All twenty algebraic, inductive, and rational-bracket lemmas underlying Theorems 1–3 are formalized in constructive Lean 4.35.0 (`lean/AbundantFactoring.lean`, 20 theorems, zero external dependencies, compiled with `-DwarningAsError=true`, verified via `#print axioms` to use at most `[propext, Quot.sound]` with zero `sorry` and zero `Classical.choice`).

All exact numerical sweeps, rescaled Vandermonde inversions, end-to-end zero-float divisor-sum factorizations (including twin primes and clustered prime powers), BHP sieve-exponent margins, and exact prime-product evaluations are verified by the AST-audited zero-floating-point Python verifier `verify_abundant_factoring.py` (in `verification/`), generating 16 SHA-256 certificates with deterministic manifest hash

`09315cd7335d505f489587296ea44ccb817f31288ace6e0ebaa76325abc044be`,

accompanied by 56 passing adversarial tests in `tests/test_adversarial.py`.

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