Orthogonal Spectral Fan-Out Costs $k(L-1)$ States: A Global Memory Law Beyond Pairwise Routing

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Abstract

A passive lossless network may route the same k-dimensional input channel to different output subspaces at different frequencies. Every two-frequency restriction can be cheap while the joint task is not. We prove an exact global law. Let $L \geq 2$, let $\zeta_1, \ldots, \zeta_L$ be distinct boundary frequencies and let the L prescribed output k-planes be mutually orthogonal. Among square finite rational-inner transfer matrices regular at the nodes, the minimum McMillan degree is

$$d_{\min} = k(L-1).$$

Each pair alone has degree k, so the largest two-node minimum underestimates the global memory by the unbounded factor $L-1$. The result is independent of node spacing. Necessity has two proofs: a zero budget for one analytic minor, and a Pick–Stein displacement identity whose negative inertia cannot exceed realization rank. Sufficiency follows from an explicit positive completion

$$P_0 = \big(C - \lambda_{\min}(C) I_L\big) \otimes I_k, \qquad C_{ij} = \frac{1}{1 - \bar{\zeta}_i \zeta_j} \quad (i \neq j),$$

followed by a lurking-isometry colligation. For regular-polygon nodes the completion spectrum is exactly $\{0, 1, \ldots, L-1\}$, each repeated k times.

The displacement proof yields a more general certificate: for calibrated target-frame Gram matrix G, every interpolant satisfies $\deg_{\mathrm{McM}} S \geq \nu_-(\mathbf{1}\mathbf{1}^* \otimes I_k - G)$. Consequently the full $k(L-1)$ bound survives whenever $\|G - I\| < 1$, and noisy estimates give a fail-closed eigenvalue count. Summing endpoint-only Grassmann speed limits certifies only $\lceil kL/2 \rceil$ states; the exact global law is asymptotically twice as large. Deterministic code tests arbitrary nodes, synthesizes minimal multichannel colligations, and is replayed by an independent verifier.

Keywords: rational inner matrix; McMillan degree; boundary Nevanlinna–Pick interpolation; Stein identity; matrix inertia; Wigner–Smith delay; spectral routing; passive network.

1 A global task hidden from every pair

Consider one fixed input subspace observed at several frequencies. At each frequency a lossless transfer sends it to a prescribed output subspace. The two-node problem is governed by canonical angles, proper-delay action and a finite rational degree [10, 9]. It is tempting to replace a multi-node task by its largest two-node restriction. This paper gives an exact family for which that surrogate loses an arbitrarily large factor.

The phenomenon is not geometric distance alone. Mutually orthogonal output planes are pairwise maximally separated, but each pair is realizable with only k states. The global obstruction comes from requiring all pairwise kernel blocks to belong to one positive Gram matrix. Its size is measured by the negative inertia of a Stein displacement before the unknown diagonal blocks are chosen.

Boundary matrix interpolation, minimal rational interpolation, Pick matrices, lurking isometries and their rank–degree correspondence are classical [3, 5, 8, 6, 4, 13]. Subspace all-pass interpolation and nodewise group-delay design are also established [7]. We compared the theorem statements and construction objectives of the closest sources. Bolotnikov–Dym supply the boundary Schur and Stein framework; Ball–Kim and Gombani–Michaletzky analyze stability and degree for general rational

Source	Established ingredient	Added here
Bolotnikov–Dym [8]	Boundary matrix Schur interpolation, Stein identity and realizations	Exact $k(L-1)$ orthogonal fan-out rank and robust inertia reading
Ball–Kim; Gombani–Michaletzky [6, 13]	McMillan-degree questions for rational interpolants	Node-independent closed formula for this subspace family
Bharath et al. [7]	Subspace all-pass synthesis and delay optimization	Unbounded separation between the largest pair task and the joint task
Preceding resource laws [11, 10, 9]	Exact or robust one-arc action and degree costs	Global compatibility cost invisible to every one-arc diamond

Table 1: Closest precedents and theorem-level scope.

interpolation; Bharath et al. construct boundary subspace all-pass filters by choosing positive group-delay blocks. None of these comparisons yielded the orthogonal free-diagonal rank formula, the exact pairwise/global ratio, or the operator-norm robust inertia certificate. Our priority claim is restricted to those explicit specializations. The rank–degree proposition below is a finite-dimensional corollary of classical realization theory, not a new general Pick theorem.

Contributions. The paper proves four linked statements.

1. Orthogonal L-way fan-out has exact minimum degree $k(L-1)$ at arbitrary distinct boundary nodes.
2. A general Pick–Stein inertia lower bound holds beyond orthogonality; its gauge-independent span corollary is
$$\deg_{\mathrm{McM}} S \geq \dim \operatorname{span}(\mathcal{Y}_1, \ldots, \mathcal{Y}_L) - k.$$
3. The orthogonal lower bound persists on the open neighborhood $\|G - I\| < 1$ and admits a fail-closed noisy certificate.
4. Summed endpoint-only Grassmann speed limits certify only $\lceil kL/2 \rceil$ states, exposing a global analytic memory overhead.

2 Setup and the boundary Pick rank

Let $L \geq 2$ and let $\zeta_1, \ldots, \zeta_L \in \mathbb{T}$ be distinct. Let $X \in \mathbb{C}^{N\times k}$ be isometric and let $\mathcal{Y}_i \subset \mathbb{C}^N$ be k-dimensional target subspaces. We seek an N-by-N rational inner matrix S, regular at all nodes, such that
$$S(\zeta_i) \operatorname{ran} X = \mathcal{Y}_i, \qquad i = 1, \ldots, L. \tag{1}$$
Its McMillan degree $n = \deg_{\mathrm{McM}} S$ is the dimension of a minimal conservative realization [15, 14]. For square rational-inner matrices it also equals the scalar degree of $\det S$; equivalently, S has exactly n rank-one factors in a reduced Blaschke–Potapov factorization.

Choose isometric target frames Y_i and gauges $V_i \in U(k)$, and put $W_i = Y_i V_i$. The frame interpolation is $S(\zeta_i)X = W_i$. Boundary tangential Nevanlinna–Pick theory leaves the diagonal blocks free and fixes
$$P_{ij} = \frac{I_k - W_i^* W_j}{1 - \bar{\zeta}_i \zeta_j}, \qquad i \neq j. \tag{2}$$
The statement needed below is a finite-dimensional corollary of the classical boundary interpolation framework [8]. We state it in the present frame-quotient form and include the lurking-isometry proof.

Proposition 2.1 (Boundary Pick rank after frame quotient)**.** *The minimum degree in* (1) *is*

$$\min\{\operatorname{rank} P : P \succeq 0,\ P_{ij} \text{ obeys } (2),\ V_i \in U(k)\}, \tag{3}$$

where the Hermitian diagonal blocks are unconstrained.

Proof. We prove both rank–degree inequalities, including the boundary regularity step. Fix gauges and let P be feasible with rank r. Factor $P_{ij} = F_i^* F_j$, where $F_i \in \mathbb{C}^{r\times k}$. For $u, v \in \mathbb{C}^k$, the block identity (2), together with $X^*X = W_i^* W_i = I_k$, shows that the two indexed families

$$\begin{pmatrix} \zeta_i F_i u \\ X u \end{pmatrix}, \qquad \begin{pmatrix} F_i u \\ W_i u \end{pmatrix} \tag{4}$$

have identical Gram matrices, including on the diagonal. Their linear correspondence is therefore a well-defined isometry between two subspaces of $\mathbb{C}^r \oplus \mathbb{C}^N$. The subspaces have equal dimension, so this isometry extends to a unitary

$$\mathcal{U} = \begin{pmatrix} A & B \\ C_0 & D \end{pmatrix} \quad \text{on } \mathbb{C}^r \oplus \mathbb{C}^N.$$

The relation defined by (4) gives

$$(I - \zeta_i A)F_i = BX, \qquad W_i = \zeta_i C_0 F_i + DX.$$

We record why this produces a transfer regular at the interpolation nodes. Since A is a contraction, its eigenspaces with unimodular eigenvalues form a reducing subspace E. The unitary colligation identities imply $C_0 E = 0$ and $B^* E = 0$, so E is completely decoupled from the external channels. Delete this summand and project each F_i onto $E^\perp$. On the remaining state space A has spectral radius strictly below one, and hence $I - zA$ is invertible for every $z \in \overline{\mathbb{D}}$. Consequently

$$S(z) = D + zC_0(I - zA)^{-1}B \tag{5}$$

is rational inner, regular on $\mathbb{T}$, and the preceding two identities give $S(\zeta_i)X = W_i$. Its state dimension, and therefore its McMillan degree after controllable/observable reduction, is at most r. Thus the minimum degree is no larger than the minimum in (3).

Conversely, let S be any degree-n rational-inner interpolant, regular at the nodes, and use a minimal unitary realization of dimension n in the form (5). Its kernel identity is

$$\frac{I - S(z)^* S(w)}{1 - \overline{z}w} = B^*(I - \overline{z}A^*)^{-1}(I - wA)^{-1}B. \tag{6}$$

Taking nontangential limits at $(z, w) = (\zeta_i, \zeta_j)$ and compressing by X gives a positive block matrix

$$P_{ij} = X^* B^* (I - \overline{\zeta}_i A^*)^{-1}(I - \zeta_j A)^{-1} BX.$$

It has rank at most n, and for $i \neq j$ equation (6) is exactly (2). Hence every degree-n interpolant supplies an admissible completion of rank at most n. Taking minima proves (3). □

3 The exact orthogonal fan-out law

We now impose the global routing task

$$Y_i^* Y_j = 0 \quad (i \neq j). \tag{7}$$

It requires $N \geq Lk$. The gauges disappear from every off-diagonal target overlap.

Theorem 3.1 (Exact orthogonal fan-out memory)**.** *Let $L \geq 2$, let $\zeta_1, \ldots, \zeta_L \in \mathbb{T}$ be distinct and let $\mathcal{Y}_1, \ldots, \mathcal{Y}_L$ be mutually orthogonal k-planes. Then*

$$\boxed{\min\{\deg_{\mathrm{McM}} S : S \textit{ satisfies } (1)\} = k(L-1).} \tag{8}$$

The value is independent of the spacings between the nodes.

We give two necessity arguments. The first is an elementary zero count and connects the theorem to topological calibration laws [12].

Lemma 3.2 (One minor has only one degree budget)**.** *Let S be finite rational inner of degree n, and let U, V be fixed isometries with k columns. If $f(z) = \det(U^*S(z)V)$ is not identically zero, its zeros have total multiplicity at most n.*

Proof. Factor S, up to constant unitaries, into n rank-one Blaschke–Potapov factors [16, 1]. Holding all other factors fixed, the compressed matrix depends on the scalar Blaschke coordinate of one factor through a rank-one affine perturbation. Its determinant is therefore affine in that coordinate. After clearing the n scalar denominators, the numerator of f has degree at most n. Regularity on $\mathbb{T}$ ensures that boundary zeros are counted by that numerator. □

First proof of necessity in Theorem 3.1. Fix i and set

$$f_i(z) = \det(Y_i^* S(z) X).$$

At $z = \zeta_i$, the compressed matrix is unitary up to an endpoint gauge, so $|f_i(\zeta_i)| = 1$. At every other node ζ_j, orthogonality gives the zero k-by-k matrix. Every entry then vanishes to first order or higher, so f_i has a zero of multiplicity at least k at each of the $L-1$ other nodes. By Theorem 3.2, $n \geq k(L-1)$. □

The second proof will extend robustly. For the upper bound define the scalar Hermitian matrix

$$C_{ii} = 0, \qquad C_{ij} = \frac{1}{1 - \bar{\zeta}_i \zeta_j} \quad (i \neq j), \tag{9}$$

and put $\lambda_* = \lambda_{\min}(C) < 0$.

Pick–Stein proof and sufficiency. For any feasible Pick completion set

$$Z = \operatorname{diag}(\zeta_1 I_k, \ldots, \zeta_L I_k).$$

Equation (2) and orthogonality give

$$P - Z^* P Z = (\mathbf{1}\mathbf{1}^* - I_L) \otimes I_k. \tag{10}$$

The right side has k positive and $k(L-1)$ negative eigenvalues. If $A, B \succeq 0$, then $\nu_-(A-B) \leq \operatorname{rank} B$: on $\ker B$ the quadratic form of $A - B$ is nonnegative. Applying this with $A = P$ and $B = Z^*PZ$ gives

$$\operatorname{rank} P \geq k(L-1). \tag{11}$$

Conversely,

$$P_0 = (C - \lambda_* I_L) \otimes I_k \tag{12}$$

is positive semidefinite, singular, and has exactly the required off-diagonal blocks. Hence $\operatorname{rank} P_0 \leq k(L-1)$. The lower bound (11) forces equality. Proposition 2.1 now produces a regular rational-inner interpolant of degree $k(L-1)$. □

The construction also proves a small spectral fact: for any distinct points on the unit circle, the smallest eigenvalue of the Hermitian matrix C in (9) is simple. Otherwise the canonical completion would have rank below $L-1$, contradicting (10) with $k = 1$.

Corollary 3.3 (Unbounded failure of pairwise memory)**.** *Every two-node restriction of* (7) *has minimum degree k, while the joint problem has minimum degree $k(L-1)$. Thus*

$$\frac{d_{\min}^{(L\ nodes)}}{\max_{i\neq j} d_{\min}^{(i,j)}} = L-1. \tag{13}$$

In particular, no universal constant times the largest two-node minimum controls the joint minimum as L varies. This does not preclude a global aggregation rule, such as summing edge costs along a spanning tree.

4 A general Pick–Stein inertia certificate

Orthogonality makes the exact formula transparent, but the displacement identity contains a broader lower bound. Stack the calibrated target frames and form their block Gram matrix

$$G = [W_i^* W_j]_{i,j=1}^{L} \succeq 0. \tag{14}$$

Let $J_L = \mathbf{1}\mathbf{1}^*$.

Theorem 4.1 (Global inertia obstruction)**.** *If a finite rational-inner S of degree n satisfies the calibrated frame conditions $S(\zeta_i)X = W_i$, then*

$$\boxed{n \geq \nu_-(J_L \otimes I_k - G).} \tag{15}$$

For subspace-only data, the right side may be minimized over the endpoint gauges. Independently of gauges,

$$n \geq \operatorname{rank} G - k = \dim \operatorname{span}(\mathcal{Y}_1, \ldots, \mathcal{Y}_L) - k. \tag{16}$$

Proof. Every Pick completion obeys the exact Stein identity

$$P - Z^* P Z = J_L \otimes I_k - G. \tag{17}$$

The negative-inertia lemma used above yields $\nu_-(J_L \otimes I_k - G) \leq \operatorname{rank} P \leq n$ for the completion induced by S. For (16), the matrix $-G$ has $\operatorname{rank} G$ negative eigenvalues and adding the positive semidefinite rank-k matrix $J_L \otimes I_k$ can remove at most k of them. Finally, $\operatorname{rank} G$ is the dimension of the span of the target frames and is unchanged by block-diagonal frame gauges. □

The inertia bound uses cross-frequency coherences, not just canonical angles. It can exceed every two-node rank and every sum formed from a disjoint matching. Unlike a generic rank minimization, it is obtained by one Hermitian eigendecomposition before any diagonal Pick blocks are chosen.

5 Quantized robustness and noisy data

Exact orthogonality is not a knife-edge hypothesis. Let G be the Gram matrix of any choice of isometric frames for the target planes. Replacing those frames by $W_i U_i$ conjugates $G - I$ by a block-diagonal unitary, so $\|G - I\|$ is a subspace invariant.

Theorem 5.1 (Open robust fan-out region)**.** *If*

$$\|G - I_{Lk}\| < 1, \tag{18}$$

then every rational-inner realization of the subspace data has degree at least $k(L-1)$. In particular, if

$$\mu := \max_{i\neq j} \|Y_i^* Y_j\| < \frac{1}{L-1}, \tag{19}$$

the same conclusion holds.

Proof. For every gauge, $J_L \otimes I_k - G$ differs in operator norm by less than one from $(J_L - I_L) \otimes I_k$. The latter has $k(L-1)$ eigenvalues equal to -1. Weyl's perturbation bound keeps all of them strictly negative, and Theorem 4.1 applies. The block row-sum estimate $\|G - I\| \leq (L-1)\mu$ proves the second statement. $\square$

The strict inequality is important. It produces an open region on which an integer memory conclusion is constant; no rounding of small angles or singular values is involved.

Corollary 5.2 (Fail-closed noisy inertia count)**.** *Suppose a measured Hermitian matrix $\widehat{H}$ satisfies*

$$\|\widehat{H} - (J_L \otimes I_k - G)\| \leq \eta.$$

Then every compatible interpolant obeys

$$\deg_{\mathrm{McM}} S \geq \#\{j : \lambda_j(\widehat{H}) < -\eta\}. \tag{20}$$

Here G is the Gram matrix of coherently calibrated frames. For purely subspace-valued observations, the matrix $J_L \otimes I_k - G$ depends on the endpoint gauges and the count must be minimized over gauges; the robust norm condition in Theorem 5.1, by contrast, is gauge invariant.

Only eigenvalues separated from zero by more than the declared error are counted. Therefore (20) cannot overstate degree when the error model holds. Uncertain eigenvalues are reported as indeterminate.

6 What endpoint-only action bounds cannot certify

For a rational inner matrix, the boundary transfer is unitary and the proper delay

$$Q(\theta) = -\mathrm{i}S(\mathrm{e}^{\mathrm{i}\theta})^* \partial_\theta S(\mathrm{e}^{\mathrm{i}\theta}) \tag{21}$$

is positive semidefinite. Its total trace action is quantized [19, 18, 16]:

$$\int_0^{2\pi} \operatorname{tr} Q(\theta)\, \mathrm{d}\theta = 2\pi \deg_{\mathrm{McM}} S. \tag{22}$$

Order the L nodes cyclically and include the closing arc. On every adjacent arc the input k-plane moves between orthogonal outputs, so all k principal angles are $\pi/2$. The positive-generator Grassmann speed limit [17, 2, 10] gives at least $k\pi$ trace action on each arc. Summing these endpoint-only geodesic minima and using (22) yields only

$$\deg_{\mathrm{McM}} S \geq \left\lceil \frac{kL}{2} \right\rceil. \tag{23}$$

Corollary 6.1 (Global analytic overhead)**.** *For orthogonal fan-out,*

$$k(L-1) = d_{\min}^{\mathrm{global}} \geq \left\lceil \frac{kL}{2} \right\rceil = d_{\mathrm{lower}}^{\mathrm{local\ action}}. \tag{24}$$

The ratio of the exact global degree to the unrounded action count tends to 2 as $L \to \infty$. At minimum degree the total winding action exceeds the sum of all local geodesic minima by

$$2\pi k(L-1) - \pi kL = \pi k(L-2). \tag{25}$$

The extra budget is not forced by the endpoint geometry of any individual arc. It is the cost of placing all local transports inside one analytic, periodic, lossless transfer. If the full $Q(\theta)$ is measured, then (22) recovers the exact degree; the limitation is specific to endpoint-only arc bounds, not to Wigner–Smith delay itself.

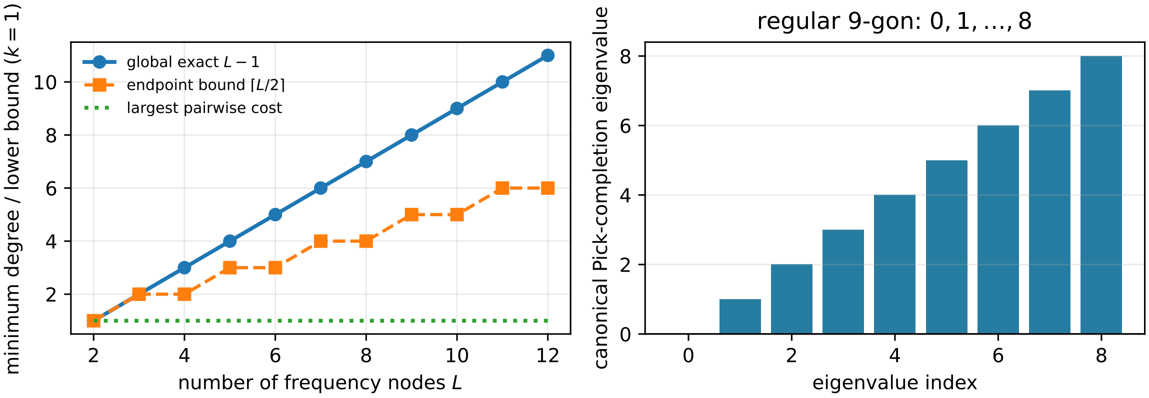

Figure 1: Left: for one channel, the exact global degree $L-1$ grows above both the largest pairwise cost and the degree inferred by summing endpoint speed-limit bounds. Right: at nine equally spaced nodes, the canonical Pick completion has the exact spectrum $0, 1, \ldots, 8$.

7 Explicit regular-polygon completion

Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/L}$ and $\zeta_j = \omega^j$. The matrix C in (9) is circulant. A finite geometric-series calculation on the Fourier vectors gives

$$\operatorname{spec} C = \left\{-\frac{L-1}{2}, -\frac{L-3}{2}, \ldots, \frac{L-3}{2}, \frac{L-1}{2}\right\}. \tag{26}$$

Consequently

$$P_0 = \left(C + \frac{L-1}{2} I_L\right) \otimes I_k, \qquad \operatorname{spec} P_0 = \{0, 1, \ldots, L-1\} \text{ repeated } k \text{ times}. \tag{27}$$

This is an exact certificate of rank $k(L-1)$.

Factor $P_0 = F^*F$ and use the lurking pairs from Theorem 2.1. Extending their isometry gives a unitary colligation

$$\mathcal{U} = \begin{pmatrix} A & B \\ C_0 & D \end{pmatrix} \quad \text{on } \mathbb{C}^{k(L-1)} \oplus \mathbb{C}^{Lk}$$

and the transfer

$$S(z) = D + zC_0(I - zA)^{-1}B. \tag{28}$$

Here the ambient output space has been chosen minimally as $N = Lk$; for $N > Lk$, replace the second summand by $\mathbb{C}^N$. After deleting decoupled unit-circle state modes, each $I - \zeta_i A$ is invertible. The lower bound prevents any loss of state dimension, so the controllable and observable part has exactly $k(L-1)$ states.

For arbitrary distinct nodes the same algorithm uses $C - \lambda_{\min}(C)I$. No optimization over diagonal blocks is required: one eigenvalue computation, one Gram factorization and one unitary completion produce a minimum-degree solution.

8 Reproducible implementation audit

The proofs are analytic; computation targets the interfaces most likely to hide a sign, rank or regularity error. The producer and verifier are separate programs, and the verifier imports no producer functions. The frozen campaign contains:

- the exact regular-polygon spectrum for $L = 2, \ldots, 24$;
- 384 arbitrary-node completions with $3 \le L \le 12$, plus four conditioning fixtures with node gaps down to 10^{-9};

(L,k)	pairwise degree	endpoint-bound count	exact degree	max interpolation error
$(3,1)$	1	2	2	3.2×10^{-15}
$(4,1)$	1	2	3	7.7×10^{-15}
$(5,1)$	1	3	4	9.0×10^{-15}
$(4,2)$	2	4	6	1.1×10^{-14}
$(5,2)$	2	5	8	9.4×10^{-15}

Table 2: Minimal lurking-isometry realizations in the frozen artifact.

- five explicit minimal colligations, including $(L,k) = (5,2)$ with eight controllable and observable states;
- 1,024 noisy inertia trials, half exact fan-out alternatives and half null controls;
- 512 independently perturbed near-orthogonal frame families, plus strict-threshold fixtures up to $\|G - I\| = 0.999999$ and equality controls.

Across the arbitrary-node campaign there are zero rank failures. The maximum Stein residual is below 1.2×10^{-14} and the maximum numerical PSD violation below 6.1×10^{-14}. The largest interpolation error among the synthesized colligations is below 1.1×10^{-14}; all prescribed state dimensions are both controllable and observable. The noisy campaign has zero false certificates and zero missed full certificates. In the near-orthogonal random campaign, the largest observed $\|G - I\|$ is $0.111 < 1$ and all 512 cases retain the exact negative-inertia count. Deterministic fixtures retain all $k(L-1)$ negative directions at $\|G - I\| = 0.999999$; at equality, where two target planes coincide, exactly k compulsory directions disappear. The noisy trials are implementation checks of the fail-closed Weyl count, not statistical evidence. These numbers audit the implementation and do not replace the proofs.

9 Scope, falsifiers and physical reading

The theorem concerns square finite rational-inner transfers, distinct boundary nodes and a fixed input subspace. Loss, gain, time variation, singular inner factors and nonrational continuous-time propagation delays are outside the class. Repeated nodes require confluent interpolation. McMillan degree counts dynamical state dimension; converting it into fabrication volume, thermodynamic work or monetary cost requires an additional physical model.

The main result is falsifiable at three levels. A positive completion of rank below $k(L-1)$ would refute necessity. Failure of the canonical completion to produce a regular degree-$k(L-1)$ inner interpolant would refute sufficiency. A compatible noisy instance for which (20) overcounts the true degree would refute the robust certificate. The released tests audit the algebraic identities, the sharp completion, independent colligation synthesis and the fail-closed count.

Operationally, the theorem says that a passive device cannot reuse one small state space independently for arbitrarily many orthogonal frequency routes. Each new orthogonal destination adds exactly k states except the first. The conclusion is architecture independent inside the rational-inner class: it does not depend on a cascade, lattice, cavity or delay-line implementation. Near-orthogonal outputs require at least the same number of states while their collective Gram perturbation remains below the explicit threshold; exact attainability away from orthogonality is not claimed.

10 Conclusion

Pairwise spectral routing is not compositional. At L distinct frequencies, routing one k-channel input to L mutually orthogonal outputs costs exactly $k(L-1)$ internal states, although every pair costs only k. The law holds at arbitrary node spacings, is attained by a closed Pick completion, and has both an elementary minor-zero converse and a robust inertia converse.

The general identity

$$P - Z^* P Z = J_L \otimes I_k - G$$

turns measured cross-frequency coherence into a direct lower bound on hidden memory. It also explains why summing endpoint-only Grassmann speed limits cannot close the multi-node problem: geometry prices each arc, while Pick inertia prices the compatibility of all arcs inside one analytic network. The resulting largest-pair/global gap grows without bound, and the endpoint-bound/global gap approaches a factor of two.

Two next questions are now sharply posed: characterize exact attainability for nonorthogonal Gram geometries, and combine the inertia coordinate with a full multi-arc action region. Neither is needed for the present fan-out law.

Data and code availability

The manuscript, deterministic certificate, independent verifier, figure and hash manifest are archived in the immutable GitHub release. The main entry points are `research/global_fanout_memory_certificate.py` and `verification/verify_global_fanout_memory.py`. Seeds, tolerances and all reported diagnostics are frozen in the JSON certificate.

A Proof details for the finite-zero budget

Write a rank-one Potapov factor as

$$F_\ell(z) = I + (b_\ell(z) - 1)p_\ell,$$

where p_ℓ has rank one and b_ℓ is scalar Blaschke of degree one. For fixed values of the other scalar coordinates, the full product is affine in b_ℓ through a rank-one matrix. If $M(t) = M_0 + tuv^*$, every term in the Leibniz expansion of $\det M(t)$ containing two entries from uv^* vanishes, so $\det M(t)$ is affine in t. Thus $\det(U^* S(z) V)$ is multiaffine in the n scalar Blaschke coordinates. Putting the coordinates over their common denominator yields a scalar numerator of degree at most n.

If an analytic k-by-k matrix H satisfies $H(z_0) = 0$, write $H(z) = (z - z_0)\widetilde{H}(z)$. Its determinant contains the factor $(z - z_0)^k$. This proves the multiplicity statement used in the first converse without a genericity or transversality assumption.

B The inertia lemma and its noisy form

Let $A, B \succeq 0$ on an m-dimensional space. Any subspace on which $A - B$ is negative definite intersects $\ker B$ trivially, because $x^*(A - B)x = x^* A x \geq 0$ on $\ker B$. Hence its dimension is at most $\operatorname{codim} \ker B = \operatorname{rank} B$, proving

$$\nu_-(A - B) \leq \operatorname{rank} B.$$

For a Hermitian perturbation $\widehat{H} = H + E$ with $\|E\| \leq \eta$, Weyl's inequality gives $\lambda_j(H) \leq \lambda_j(\widehat{H}) + \eta$. Every observed eigenvalue strictly below $-\eta$ therefore corresponds to a truly negative eigenvalue, which proves Theorem 5.2.

C Fourier diagonalization at equally spaced nodes

For $\omega = \mathrm{e}^{2\pi\mathrm{i}/L}$, the first row of C is $0, (1-\omega)^{-1}, \ldots, (1 - \omega^{L-1})^{-1}$. Testing the Fourier vector $(1, \omega^{-m}, \ldots, \omega^{-(L-1)m})$ and pairing the terms j and $L - j$ gives

$$\sum_{j=1}^{L-1} \frac{\omega^{-mj}}{1 - \omega^j} = \frac{L-1}{2} - m, \qquad m = 0, \ldots, L-1,$$

with the opposite Fourier convention reversing the order. This proves (26) and (27).

D Verification contract

Interface	Independent invariant	Frozen tolerance
Regular polygon	spectrum $0, 1, \ldots, L-1$	2×10^{-11}
Arbitrary nodes	PSD, rank and Stein residual	2×10^{-10}
Colligation	unitarity, interpolation and minimality	3×10^{-7}
Noisy inertia	alternatives and null controls	zero count failures
Near orthogonality	$\|G - I\| < 1$ and inertia	zero count failures
Strict boundary	0.999999 and equality controls	exact inertia counts

Table 3: Fail-closed numerical contract.

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