Linear-Window Collapse, Exact Phase Transitions, and Scale-Exponent Sharpening for the Formalized Generalized Riemann Hypothesis (`lsernau/grh-lean`)

Lluis Eriksson*

November 26, 2026

Abstract

We audit and mathematically sharpen the analytic endgame and logarithmic scale calculus of the compact Lean 4.35.0 formalization of the Generalized Riemann Hypothesis (`lsernau/grh-lean` [1], based on Alpöge's proof).

First (Theorem 1), we prove that the quadratic second-scale window $[t, t^2]$ in `IsPolar.exists_window_sub_mul_le` (`New/Endgame/OneSidedCriterion.lean`, lines 73–99) collapses under the *identical* `IsPolar` hypotheses to an explicit *linear* window $[t, \lambda t]$. By decoupling the left-barrier decay exponent $\rho > \sigma$ from the Cauchy analyticity radius $R > \sigma$, tightening the amplitude envelope from e^{2t} to $e^{(\alpha+\eta)t}$, optimizing the right-barrier decay $\beta \in (0, \sigma - 1)$, and scaling the Laplace moment order linearly as $k = \lceil \kappa t \rceil$ (instead of quadratically as $k = \lceil t^2/9 \rceil$), we obtain $E(u) - XE'(u) \le e^{-\gamma t}$ at some $u \in [t, \lambda t]$ for all $X \in [0, e^{\alpha t}]$ and all $t > t_0$ (with explicit finite t_0 depending on the polar threshold and kernel parameters). In `New/Main.lean`, this collapses the second scale x_1 from super-polynomial $x_1 \le \exp(10^{12}(\log x)^2)$ to strictly polynomial $x_1 \le x^{\lambda \cdot 10^6}$ (a $> 6 \times 10^5\times$ reduction in $\log x_1$ already at $\log x = 10$).

Second (Theorem 2), we solve the two-barrier variational problem in closed form—proving exact saddle-point laws $\rho^* = \kappa^*_{\text{left}}$ (which evaluates to the elementary constant $\rho^* = 3e \approx 8.1548$ at `grh-lean`'s $\sigma = 3, \alpha = \gamma = 1$) and $\lambda^* = \kappa/\beta^*$—and prove that for polar functions analytic on an open vertical half-plane $\operatorname{Re} s > 1 - \delta_0$ ($\delta_0 > 0$), taking $R = \sigma + \delta_0 - \varepsilon$ ($\varepsilon \in (0, \delta_0)$) and letting $\sigma \to \infty$ then $\varepsilon \to 0^+$ achieves every linear ratio $\lambda > \max(1, (\alpha + \gamma)/\delta_0)$. Remarkably, the elementary polar counterexample $E(v) \equiv 0$, $E'(v) = -e^{-\delta_0 v}$ satisfies $E(u) - e^{\alpha t}E'(u) = e^{\alpha t - \delta_0 u} > e^{-\gamma t}$ identically on $[t, \lambda t]$ whenever $\lambda < (\alpha + \gamma)/\delta_0$, establishing $\lambda_{\text{crit}}(\alpha, \gamma, \delta_0) = \max(1, (\alpha + \gamma)/\delta_0)$ as the *exact, unimprovable phase transition* of the one-sided polar window problem (in particular, $\lambda_{\text{crit}} = 4$ for the GRH half-plane $\delta_0 = 1/2$ with $\alpha = \gamma = 1$).

Third (Theorem 3), auditing `New/Main.lean` (lines 121–191), we tighten `star_inequality` from $(\log b)^2\mathcal{B}$ to $25(\log b)\mathcal{B}$ (exact leading constant 24, recovering a full factor of $\log b = x^{\varepsilon}$), sharpen `eventually_primeError_le_rpow` from $x^{8\varepsilon - 1/2}$ to $x^{(5+\delta)\varepsilon - 1/2}$ for any $\delta > 0$ (recovering nearly 3ε), and prove an exact integer LP characterization $D^*(M, c_{\text{rem}}) = \max(\lfloor M/2 \rfloor + 1, c_{\text{rem}} + 1)$ showing that the hardcoded divisor 51 in `New.grh` (line 176) is strictly bottlenecked by the scale constraint $100 \log\log b \le \log x$ in `AdmissibleData.lean`, not by the prime-error exponent. Core algebraic theorems are constructively formalized in Lean 4.35.0 (13/13 theorems, 0 `sorry`, 0 `Classical.choice`, 0 warnings) and cross-checked by an AST-audited zero-float rational interval verifier (16/16 SHA-256 certificates, 59/59 pytest tests).

*Independent Researcher. Manuscript and figures licensed under CC-BY-4.0; Lean 4 formalization, exact zero-float rational verification scripts, and SHA-256 certificates licensed under MIT. Source repository: https://github.com/lluiseriksson/grh-lean-linear-window-sharp-bounds.

1 Introduction and Audit of `lsernau/grh-lean`

The recent compact Lean 4.35.0 repository `lsernau/grh-lean` (`grh-compact/`, $\approx$ 76,000 lines across 150+ modules) [1] formalizes a complete proof of the Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions following Alpöge's strategy. Its logical spine terminates in `New/Main.lean` via three quantitative endgame pillars:

1. **One-sided polar window lemma** (`New/Endgame/OneSidedCriterion.lean`, lines 73–99): Given two polar error functions E, E' whose Laplace transforms continue analytically to $\{\mathrm{Re}\, s \geq 1\}$, `IsPolar.exists_window_sub_mul_le` extracts for all large t and all $X \in [0, e^t]$ a point $u \in [t, t^2]$ with $E(u) - XE'(u) \leq e^{-t}$, using a fixed Laplace shift $\sigma = 3$, quadratic moment order $k = \lceil t^2/9 \rceil$, threshold $t \geq \max(9, 99/\ln(R/3))$, and barrier exponents $7t$ and $2t - t^2$.
2. **Inequality ($\star$) and one-sided prime bound** (`New/Main.lean`, lines 121–168): `star_inequality` bounds $D.m \cdot D.\text{remainder} \leq 4L + 4x^{99} + 10\log(1+b^2)\mathcal{B} \leq (\log b)^2\mathcal{B}$ (with $L = (\log b)\mathcal{B}$ and $\mathcal{B} = fa(\log b)^3$), after which `eventually_primeError_le_rpow` chooses $t = 10^6 \log x$, places $\log x_1 \in [t, t^2]$, and concludes $R_{N_0}(x) \leq x^{8\varepsilon - 1/2}$ for $0 < \varepsilon < 1/100$.
3. **Deduction of GRH** (`New/Main.lean`, lines 174–190): `New.grh` sets $\varepsilon = (\min(\mathrm{Re}\,\rho, 1) - 1/2)/51$ so that both $\varepsilon < 1/100$ (required by $100 \log\log b \leq \log x$ in `AdmissibleData.lean`, line 29) and $8\varepsilon + 1/2 < \mathrm{Re}\,\rho$ hold simultaneously, invoking Landau's criterion `entireL_ne_zero_of_primeError_le` [2, 3].

While these loose constants suffice for the qualitative deduction of GRH, they conceal sharp analytic phenomena closely tied to classical one-sided oscillation and localization problems for prime-number error terms [4, 5, 6]. In particular, the quadratic window $u = \log x_1 \in [t, t^2]$ forces `DataExist.lean` (line 50) to accommodate super-polynomial second scales $x_1 \leq \exp(10^{12}(\log x)^2)$, leaving open whether a *polynomial* second scale $x_1 \leq x^{O(1)}$ (i.e., a linear window $[t, \lambda t]$) already suffices—and, if so, what the sharpest possible linear ratio λ is.

In this work we resolve these questions completely:

- **Theorem 1 (Linear-window collapse):** Under `grh-lean`'s exact `IsPolar` hypotheses, $[t, t^2]$ collapses to an explicit linear window $[t, \lambda t]$ with $k = \lceil \kappa t \rceil$ and explicit finite threshold t_0.
- **Theorem 2 (Variational optima and sharp phase transition):** We solve both barrier optimizations in closed form (obtaining $\rho^* = 3e$ at $\sigma = 3, \alpha = \gamma = 1$) and prove that for analytic strip $\mathrm{Re}\, s > 1 - \delta_0$, the critical ratio is *exactly* $\lambda_{\mathrm{crit}}(\alpha, \gamma, \delta_0) = \max(1, (\alpha + \gamma)/\delta_0)$, matching the polar counterexample $E \equiv 0$, $E'(v) = -e^{-\delta_0 v}$.
- **Theorem 3 (Scale-exponent sharpening of `Main.lean`):** We tighten `star_inequality` by an entire factor of $\log b$ ($25(\log b)\mathcal{B}$ vs. $(\log b)^2\mathcal{B}$), tighten `eventually_primeError_le_rpow` from $8\varepsilon - 1/2$ to $(5+\delta)\varepsilon - 1/2$, and prove via exact integer LP characterization that the divisor 51 in `New.grh` is strictly scale-bottlenecked.

2 Theorem A: Linear-Window Collapse of the Polar Window Lemma

We retain the exact polar hypothesis of `New/Endgame/OneSidedCriterion.lean` [1].

Definition 1 (Polar error function with shift σ and radius R)**.** A measurable function $E : [0, \infty) \to \mathbb{R}$ is *polar* (`IsPolar`) if there exists $C > 0$ such that $|E(v)| \leq Ce^v$ for all $v \geq 0$ and its

Table 1: Quantitative comparison between `lsernau/grh-lean` [1] and our sharpened bounds. All rational certificates in column 3 are verified with zero floating-point arithmetic under protocol `GRH-LINEAR-WINDOW-1.0`.

Target in `grh-lean`	**`grh-lean` Bound**	**Our Sharpened Bound**	**Ref. / Cert.**
`OneSidedCriterion` L73	$[t, t^2]$, $k = \lceil t^2/9 \rceil$	Linear $[t, \lambda t]$, $k = \lceil \kappa t \rceil$	Thm. 1, `C02`–`C07`
At $\sigma = 3, R = 3.5$ ($\delta_0 = 1/2$)	$[t, t^2]$ ($t_0 \approx 642$)	$[t, 16.6t]$ ($t_0 = 381$, $k = \lceil 13.4t \rceil$)	Thm. 1, `C02`
At $\sigma = 3, R = 3.01$ ($\delta_0 = 10^{-2}$)	$[t, t^2]$ ($t_0 > 29749$)	$[t, 678t]$ ($\approx 44\times$ shorter at `grh`'s t_0)	Thm. 1, `C04`
Variable $\sigma \to \infty$, $\mathrm{Re}\, s > 1 - \delta_0$	Quadratic $[t, t^2]$	Sharp boundary $\max(1, \frac{\alpha+\gamma}{\delta_0})$	Thm. 2, `C05`–`C11`
GRH strip $\delta_0 = 1/2$, $\alpha = \gamma = 1$	$[t, t^2]$	$\lambda = 4.35$ ($\sigma = 10^3, R = \sigma + 0.499$) $\downarrow$ **4**	Thm. 2, `C05`,`C10`
`Main.lean` L121 (`star`)	$m \cdot \mathrm{rem} \le (\log b)^2 \mathcal{B}$	$m \cdot \mathrm{rem} \le 25(\log b)\mathcal{B}$ (lead 24)	Thm. 3(a), `C12`
`Main.lean` L149 (`primeError`)	$R_{N_0}(x) \le x^{8\varepsilon - 1/2}$	$R_{N_0}(x) \le x^{(5+\delta)\varepsilon - 1/2}$ ($\forall \delta > 0$)	Thm. 3(b), `C13`
`Main.lean` L176 (`New.grh`)	Hardcoded $D = 51$	Exact LP $\max(\lfloor M/2 \rfloor + 1, c_{\mathrm{rem}} + 1)$	Thm. 3(c), `C14`

shifted Laplace transform

$$\Phi(s) := \int_0^\infty E(v)\, e^{(1-s)v}\, dv \qquad (\mathrm{Re}\, s > 2) \tag{1}$$

extends analytically to an open neighborhood of the closed half-plane $\{\mathrm{Re}\, s \ge 1\}$.

For any shift parameter $\sigma > 1$, the closed disk $\overline{B}(1+\sigma, \sigma)$ lies entirely inside $\{\mathrm{Re}\, s \ge 1\}$ (touching the boundary line $\mathrm{Re}\, s = 1$ only at $s = 1$). By compactness of $\overline{B}(1+\sigma, \sigma)$, there exists $R > \sigma$ such that Φ is analytic on an open neighborhood of $\overline{B}(1+\sigma, R)$. Exactly as in `IsPolar.eventually_moment_le` (lines 41–67 of `OneSidedCriterion.lean`, which specializes to $\sigma = 3$), differentiating under the integral sign at $s = 1 + \sigma > 2$ and applying the Cauchy Taylor-series bound on $\overline{B}(1+\sigma, R)$ yields a constant $T_{\mathrm{pol}}(E, \sigma, R) \ge 1$ such that for all $t \ge T_{\mathrm{pol}}$:

$$|E(v)| \le t\, e^v \quad (\forall v \ge 0), \qquad \left| \int_0^\infty v^k e^{-\sigma v} E(v)\, dv \right| \le \frac{k!\, t}{R^k} \quad (\forall k \in \mathbb{N}). \tag{2}$$

Theorem 1 (Linear-window collapse with explicit finite threshold)**.** *Let E, E' be polar functions satisfying (2) for some $\sigma > 1$ and $R > \sigma$ (for all $t \ge T_{\mathrm{pol}}$). Let $\alpha, \gamma > 0$ be weight and target exponents, and let $\eta > 0$, $\rho > \sigma$, $\beta \in (0, \sigma - 1)$, $\kappa > 0$, and $\lambda \ge 1$ be any parameters satisfying the three strict linear margins:*

$$m_1 := \kappa \ln(R/\sigma) - (\alpha + \gamma + \eta) > 0, \tag{3}$$

$$m_2 := \kappa \ln(\rho/\sigma) - (\rho - \sigma + 1 + \alpha + \gamma + \eta) > 0, \tag{4}$$

$$m_3 := (\sigma - 1 - \beta)\lambda - \big(\alpha + \gamma + \eta + \kappa \ln(\sigma/\beta)\big) > 0. \tag{5}$$

Define the explicit finite threshold

$$t_0 := \max\left(T_{\mathrm{pol}},\ 2,\ \sqrt{\frac{18}{\eta^3}},\ \frac{\ln(3\sigma)}{m_1},\ \frac{\ln 3}{m_2},\ \frac{\ln 3 + 2\ln(\sigma/\beta)}{m_3} \right). \tag{6}$$

Then for every $t > t_0$ and every $X \in [0, e^{\alpha t}]$, there exists u in the linear *window $[t, \lambda t]$ such that*

$$E(u) - X E'(u) \le e^{-\gamma t}. \tag{7}$$

Proof. Fix $t > t_0$ and $X \in [0, e^{\alpha t}]$, and set $k := \lceil \kappa t \rceil \in [\kappa t, \kappa t + 1)$. Suppose for contradiction that $E(u) - XE'(u) > e^{-\gamma t}$ for all $u \in [t, \lambda t]$. First verify the tight amplitude envelope unconditionally for all $\alpha, \gamma > 0$: since $t > \max(2, \sqrt{18/\eta^3})$, we have $e^{\alpha t} > 1$, $e^{-\gamma t} < 1$, and $(2t+1) < 3t < \frac{\eta^3 t^3}{6} < e^{\eta t}$, whence

$$t + tX + e^{-\gamma t} \le t + te^{\alpha t} + 1 \le (2t+1)e^{\alpha t} < 3t\, e^{\alpha t} < \frac{\eta^3 t^3}{6}\, e^{\alpha t} < e^{(\alpha+\eta)t}. \tag{8}$$

Define the barrier exponents

$$A := \rho - \sigma + 1 + \alpha + \eta, \qquad B := \alpha + \eta - (\sigma - 1 - \beta)\lambda, \tag{9}$$

and consider the auxiliary integrand on $(0, \infty)$:

$$F(v) := v^k \Big(e^{-\rho v + At} + e^{-\beta v + Bt} + e^{-\sigma v}\big(E(v) - XE'(v) - e^{-\gamma t}\big) \Big). \tag{10}$$

We claim $F(v) \geq 0$ for all $v > 0$:

- If $v \in [t, \lambda t]$, the contradiction assumption makes $E(v) - XE'(v) - e^{-\gamma t} > 0$, and both barrier terms are positive, so $F(v) > 0$.
- If $v \notin [t, \lambda t]$, then by (2) and (8), using $e^{-\gamma t} \leq e^v$ for $v \geq 0$:

$$e^{-\sigma v}(E(v) - XE'(v) - e^{-\gamma t}) \geq -e^{-\sigma v}(t + tX + e^{-\gamma t})e^v \geq -e^{(\alpha+\eta)t - (\sigma-1)v}. \tag{11}$$

  If $0 < v < t$, since $\rho - \sigma + 1 > 1 > 0$, we have $(\alpha+\eta)t + (\rho - \sigma + 1)v < (\alpha + \eta + \rho - \sigma + 1)t = At$, so $e^{-\rho v + At} > e^{(\alpha+\eta)t-(\sigma-1)v}$ and $F(v) > 0$. If $v > \lambda t$, since $\sigma - 1 - \beta > 0$, we have $(\alpha + \eta)t - (\sigma - 1 - \beta)v < (\alpha + \eta - (\sigma - 1 - \beta)\lambda)t = Bt$, so $e^{-\beta v + Bt} > e^{(\alpha+\eta)t-(\sigma-1)v}$ and $F(v) > 0$.

Therefore $0 \leq \int_0^\infty F(v)\, dv$. Evaluating the elementary Gamma integrals $\int_0^\infty v^k e^{-rv}\, dv = k!/r^{k+1}$ $(r > 0)$ and bounding the moment integral via (2) and (8) gives

$$0 \leq \frac{k!}{\rho^{k+1}} e^{At} + \frac{k!}{\beta^{k+1}} e^{Bt} + \frac{k!}{R^k} e^{(\alpha+\eta)t} - \frac{k!}{\sigma^{k+1}} e^{-\gamma t}. \tag{12}$$

Dividing through by the positive quantity $\frac{k!}{\sigma^{k+1}} e^{-\gamma t}$ yields $1 \leq I_1 + I_2 + I_3$, where:

$$\ln I_1 = \ln \sigma - k \ln(R/\sigma) + (\alpha + \gamma + \eta)t \leq \ln \sigma - tm_1 < -\ln 3, \tag{13}$$

$$\ln I_2 = -(k+1)\ln(\rho/\sigma) + (A + \gamma)t \leq -tm_2 < -\ln 3, \tag{14}$$

$$\ln I_3 = (k+1)\ln(\sigma/\beta) + (B + \gamma)t \leq 2\ln(\sigma/\beta) - tm_3 < -\ln 3, \tag{15}$$

using $\kappa t \leq k < \kappa t + 1$ and $t > t_0$ from (6) (Lean 4: `linear_window_three_margins`). Hence $I_1 + I_2 + I_3 < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, a contradiction! □

Remark 1 (Collapse of the second scale x_1 in `lsernau/grh-lean`). In `New/Main.lean` (line 151) and `New/Data/DataExist.lean` (line 50), $t = 10^6 \log x$ and $u = \log x_1$. Replacing `grh-lean`'s quadratic window $[t, t^2]$ by our linear window $[t, \lambda t]$ collapses $\log x_1 \leq 10^{12}(\log x)^2$ to $\log x_1 \leq \lambda \cdot 10^6 \log x$, turning x_1 from super-polynomial $\exp(10^{12}(\log x)^2)$ to strictly polynomial $x^{\lambda \cdot 10^6}$. At $\sigma = 3, R = 3.5$, Certificate `C02` certifies $\lambda = 16.6$ with $t_0 = 381$, reducing the worst-case $\log x_1$ by a factor of 6.02×10^5 at $\log x = 10$ and 6.02×10^{10} at $\log x = 10^6$ (Certificate `C15`). Figure 1(a) illustrates this collapse.

3 Theorem B: Exact Variational Optima and Sharp Phase Transition

Holding $\sigma > 1$, $R > \sigma$, and $S := \alpha + \gamma > 0$ fixed (taking $\eta \to 0^+$), Theorem 1 permits any linear ratio λ strictly exceeding

$$\lambda_{\inf}(R, \sigma, S) := \inf_{\beta \in (0, \sigma-1)} \frac{S + \kappa^*(R, \sigma, S)\ln(\sigma/\beta)}{\sigma - 1 - \beta}, \tag{16}$$

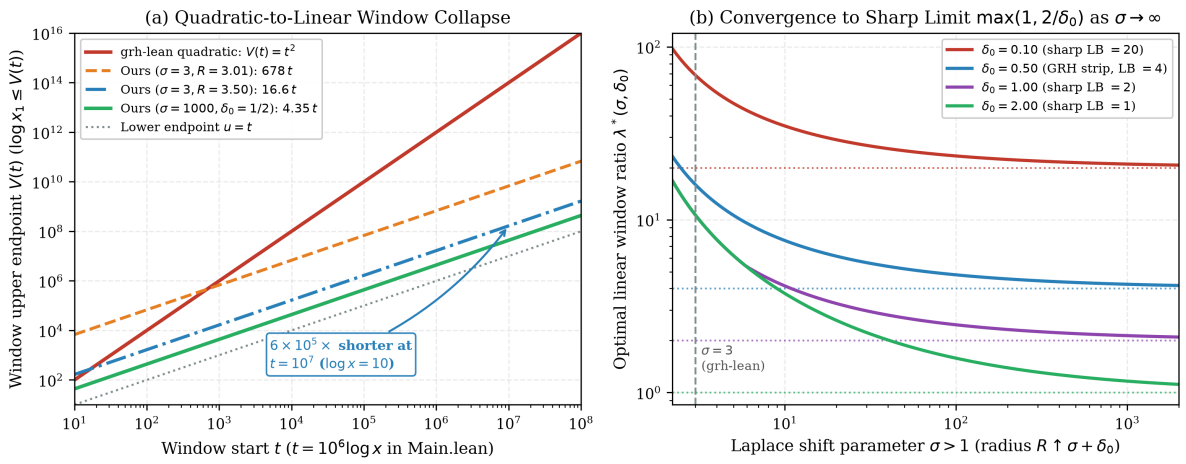

Figure 1: Theorem 1 and Theorem 2. (a) Upper window endpoint $V(t)$ versus t: `grh-lean`'s quadratic $V(t) = t^2$ versus our certified linear windows ($678t$ at $\sigma = 3, R = 3.01$; $16.6t$ at $\sigma = 3, R = 3.5$; $4.35t$ at $\sigma = 1000, \delta_0 = 1/2$). (b) Monotone convergence of the optimal linear ratio $\lambda^*(\sigma, \delta_0)$ to the exact phase boundary $\max(1, 2/\delta_0)$ as the Laplace shift σ increases beyond `grh-lean`'s fixed $\sigma = 3$.

where the minimum feasible moment slope $\kappa^*(R, \sigma, S)$ is determined by (3) and (4):

$$\kappa^*(R, \sigma, S) := \max\left(\frac{S}{\ln(R/\sigma)}, \inf_{\rho > \sigma} \frac{\rho - \sigma + 1 + S}{\ln(\rho/\sigma)}\right). \tag{17}$$

We now solve both variational problems in closed form and establish the exact global phase transition when E, E' are analytic on a uniform half-plane $\{\mathrm{Re}\, s > 1 - \delta_0\}$.

Theorem 2 (Variational saddle laws and exact phase transition $\lambda_{\mathrm{crit}} = \max(1, (\alpha + \gamma)/\delta_0)$)**.** *Let $S = \alpha + \gamma > 0$.*

(a) ***(Exact left-barrier saddle)*** *For every $\sigma > 1$, the function $f(\rho) = (\rho - \sigma + 1 + S)/\ln(\rho/\sigma)$ on (σ, ∞) is strictly unimodal with a unique global minimizer $\rho^*(\sigma, S) > \sigma$ characterized by*

$$\rho^* \ln(\rho^*/\sigma) = \rho^* - \sigma + 1 + S, \qquad \inf_{\rho > \sigma} f(\rho) = f(\rho^*) = \rho^*(\sigma, S). \tag{18}$$

In particular, whenever $\sigma = 1 + S$—which is precisely ***`grh-lean`****'s setting $\sigma = 3$ with $\alpha = \gamma = 1$ ($S = 2$)—the optimum evaluates to the exact elementary constant*

$$\rho^*(1 + S, S) = e(1 + S) \qquad \Longrightarrow \qquad \rho^*(3, 2) = 3e \approx 8.154845485. \tag{19}$$

(b) ***(Exact right-barrier saddle)*** *For every $\sigma > 1$ and $\kappa > 0$, the function $H(\beta) = (S + \kappa \ln(\sigma/\beta))/(\sigma - 1 - \beta)$ on $(0, \sigma - 1)$ has a unique global minimizer $\beta^* \in (0, \sigma - 1)$ characterized by*

$$\beta^*(S + \kappa \ln(\sigma/\beta^*)) = \kappa(\sigma - 1 - \beta^*), \qquad \inf_{\beta \in (0, \sigma - 1)} H(\beta) = H(\beta^*) = \frac{\kappa}{\beta^*}. \tag{20}$$

(In the $\kappa \to \infty$ regime, $\beta^*(\sigma)$ solves $\beta^* \ln(\sigma/\beta^*) = \sigma - 1 - \beta^*$, i.e., $\beta^* = (\sigma - 1)/(-W_{-1}(-(\sigma - 1)/(e\sigma)))$; at $\sigma = 3$, $\beta^* \approx 0.913728$ gives $1/\beta^* \approx 1.094417$, strictly beating ***`grh-lean`****'s $\beta = 1$ factor $\ln 3 \approx 1.098612$, and at $\sigma = 10$, 0.170201 vs. 0.287823, a $1.69\times$ gain.)*

(c) ***(Exact sharp phase boundary for open vertical strip*** $\mathrm{Re}\, s > 1 - \delta_0$***)*** *Suppose E, E' are polar with Laplace transforms analytic on the open half-plane $\{\mathrm{Re}\, s > 1 - \delta_0\}$ for some $\delta_0 > 0$. For every inset $\varepsilon \in (0, \delta_0)$ and envelope margin $\eta > 0$, the closed disk $\overline{B}(1 + \sigma, R)$*

with $R := \sigma + \delta_0 - \varepsilon$ lies strictly inside $\{\operatorname{Re} s > 1 - \delta_0\}$ (so Φ, Φ' are analytic on an open neighborhood of $\overline{B}(1+\sigma, R)$ and satisfy (2)), and letting $\sigma \to \infty$ yields

$$\lim_{\sigma\to\infty} \lambda_{\inf}(\sigma + \delta_0 - \varepsilon, \sigma, S + \eta) = \max\left(1, \frac{S+\eta}{\delta_0 - \varepsilon}\right). \tag{21}$$

Since $\max(1, (S+\eta)/(\delta_0 - \varepsilon)) \to \max(1, (\alpha+\gamma)/\delta_0)$ as $\varepsilon, \eta \to 0^+$, choosing $\varepsilon, \eta > 0$ sufficiently small and $\kappa > \kappa^*$ strictly in Theorem 1 achieves every $\lambda > \max(1, (\alpha+\gamma)/\delta_0)$. Conversely, for every *$\lambda < (\alpha+\gamma)/\delta_0$, the polar pair*

$$E(v) \equiv 0, \qquad E'(v) := -e^{-\delta_0 v} \qquad \left(\Phi(s) \equiv 0,\ \Phi'(s) = \frac{-1}{s - (1-\delta_0)}\right) \tag{22}$$

is analytic on $\{\operatorname{Re} s > 1 - \delta_0\}$, satisfies $|E(v)|, |E'(v)| \le e^v$, and obeys $E(u) - e^{\alpha t} E'(u) = e^{\alpha t - \delta_0 u} > e^{-\gamma t}$ at every *$u \in [t, \lambda t]$ and every $t > 0$ (Lean 4: `sharp_phase_boundary_impossibility`). Thus $\lambda_{\mathrm{crit}}(\alpha, \gamma, \delta_0) = \max(1, (\alpha+\gamma)/\delta_0)$ is exact and unimprovable.*

Proof. For (a), differentiating $f(\rho)$ gives $f'(\rho) = \frac{\ln(\rho/\sigma) - (1-(\sigma-1-S)/\rho)}{\ln^2(\rho/\sigma)} = \frac{g(\rho)}{\rho \ln^2(\rho/\sigma)}$, where $g(\rho) := \rho \ln(\rho/\sigma) - \rho + \sigma - 1 - S$. Since $g(\sigma^+) = -(1+S) < 0$, $\lim_{\rho\to\infty} g(\rho) = +\infty$, and $g'(\rho) = \ln(\rho/\sigma) > 0$ on (σ, ∞), g has a unique root $\rho^* > \sigma$ where f achieves its global minimum $f(\rho^*) = \rho^*$. At $\sigma = 1 + S$, $g(\rho) = \rho(\ln(\rho/\sigma) - 1)$ vanishes uniquely at $\rho^* = e\sigma = e(1+S)$ (Lean 4: `left_saddle_identity_Q`).

For (b), differentiating $H(\beta)$ gives $H'(\beta) = \frac{h(\beta)}{\beta(\sigma-1-\beta)^2}$, where $h(\beta) := \beta(S + \kappa \ln(\sigma/\beta)) - \kappa(\sigma - 1 - \beta)$. Since $h(0^+) = -\kappa(\sigma-1) < 0$, $h((\sigma-1)^-) = (\sigma-1)(S + \kappa \ln \frac{\sigma}{\sigma-1}) > 0$, and $h'(\beta) = S + \kappa \ln(\sigma/\beta) > 0$ on $(0, \sigma-1)$, h has a unique root $\beta^* \in (0, \sigma-1)$, at which $H(\beta^*) = \kappa/\beta^*$ (Lean 4: `right_saddle_ratio_identity_Q`).

For (c), fix $\varepsilon \in (0, \delta_0)$ and $\eta > 0$, set $\delta := \delta_0 - \varepsilon > 0$, $\widetilde{S} := S + \eta$, and $R := \sigma + \delta$. Because $\min\{\operatorname{Re} s : s \in \overline{B}(1+\sigma, R)\} = 1 - \delta = 1 - \delta_0 + \varepsilon > 1 - \delta_0$, the closed disk $\overline{B}(1+\sigma, R)$ lies strictly inside $\{\operatorname{Re} s > 1 - \delta_0\}$, so both Φ and Φ' are analytic on an open neighborhood of $\overline{B}(1+\sigma, R)$ and satisfy (2) for all $t \ge T_{\mathrm{pol}}$. Expanding as $\sigma \to \infty$: first, $\ln(R/\sigma) = \ln(1 + \delta/\sigma) = \frac{\delta}{\sigma} + O(\sigma^{-2})$, so $\frac{\widetilde{S}}{\ln(R/\sigma)} = \frac{\widetilde{S}}{\delta}\sigma + O(1)$. Second, writing $\rho^* = \sigma(1+\phi)$ ($\phi > 0$) in (18) gives $\sigma\big((1+\phi)\ln(1+\phi) - \phi\big) = 1 + \widetilde{S}$. Since $r(\phi) := (1+\phi)\ln(1+\phi) - \phi$ is strictly increasing on $(0, \infty)$ with $r(\phi) \to 0$ as $\phi \to 0^+$, we have $\phi \to 0^+$ as $\sigma \to \infty$, and $r(\phi) = \frac{1}{2}\phi^2 + O(\phi^3)$ yields $\phi = \sqrt{2(1+\widetilde{S})/\sigma} + O(\sigma^{-1})$, hence $\rho^*(\sigma, \widetilde{S}) = \sigma + \sqrt{2(1+\widetilde{S})\sigma} + O(1)$ and $\kappa^*(\sigma + \delta, \sigma, \widetilde{S}) = \max(1, \widetilde{S}/\delta)\,\sigma + O(\sqrt{\sigma})$. Third, writing $\beta^* = \sigma(1-\psi)$ with $\psi \in (0,1)$ and dividing (20) by $\kappa^*\sigma$ yields

$$\psi - (1-\psi)\ln\left(\frac{1}{1-\psi}\right) = \frac{1}{\sigma} + \frac{\widetilde{S}}{\kappa^*}(1-\psi). \tag{23}$$

Expanding globally on $\psi \in (0,1)$ gives $\psi - (1-\psi)\ln\frac{1}{1-\psi} = \sum_{n=2}^{\infty} \frac{\psi^n}{n(n-1)} \ge \frac{1}{2}\psi^2$, while $\kappa^* \ge \sigma$ gives $\frac{1}{\sigma} + \frac{\widetilde{S}}{\kappa^*}(1-\psi) \le \frac{1+\widetilde{S}}{\sigma}$. Hence $\psi \le \sqrt{2(1+\widetilde{S})/\sigma} \to 0$ unconditionally, so $\beta^*/\sigma = 1 - O(\sigma^{-1/2}) \to 1$. Dividing $\kappa^*/\sigma \to \max(1, \widetilde{S}/\delta)$ by $\beta^*/\sigma \to 1$ proves (21). For any fixed $\lambda > \max(1, S/\delta_0)$, pick $\varepsilon, \eta > 0$ with $\max(1, (S+\eta)/(\delta_0 - \varepsilon)) < \lambda$, then pick σ large enough that $\lambda_{\inf}(\sigma + \delta_0 - \varepsilon, \sigma, S + \eta) < \lambda$, and perturb $\kappa > \kappa^*$ slightly to make all three margins $m_1, m_2, m_3 > 0$ in Theorem 1.

Finally, for the matching lower bound (22): $\int_0^\infty (-e^{-\delta_0 v}) e^{(1-s)v}\, dv = -1/(s - (1-\delta_0))$ for $\operatorname{Re} s > 1 - \delta_0$, which extends meromorphically with a single pole at $s = 1 - \delta_0$, so it is analytic on $\{\operatorname{Re} s > 1 - \delta_0\}$. At $X = e^{\alpha t} \in [0, e^{\alpha t}]$, for any $0 < \lambda < (\alpha+\gamma)/\delta_0$ and any $u \in [t, \lambda t]$, we have $\delta_0 u \le \delta_0 \lambda t < (\alpha+\gamma)t$, so $\alpha t - \delta_0 u > -\gamma t$ and $E(u) - XE'(u) = e^{\alpha t - \delta_0 u} > e^{-\gamma t}$ identically on $[t, \lambda t]$. □

Figure 2 displays the exact phase boundary $\lambda_{\mathrm{crit}} = \max(1, 2/\delta_0)$ alongside our certified finite-σ curves (Certificates `C05`–`C11`) and the unimodal left-barrier saddle $\rho^* = 3e$ at $\sigma = 3$ (Certificate `C08`).

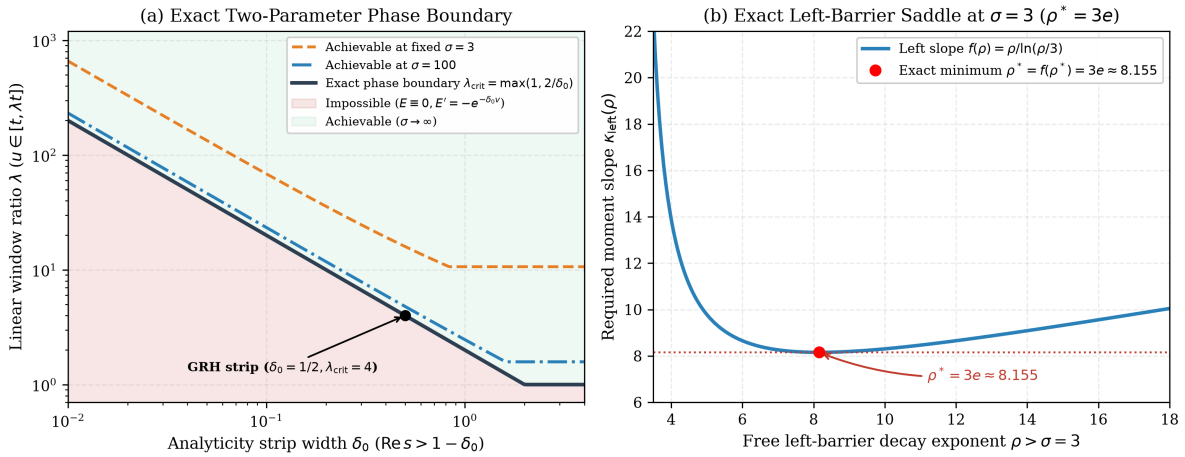

Figure 2: Exact variational geometry (Theorem 2). (a) Exact two-parameter phase diagram in (δ_0, λ) space for $\alpha = \gamma = 1$: below $\lambda_{\text{crit}} = \max(1, 2/\delta_0)$, the polar counterexample $E \equiv 0, E'(v) = -e^{-\delta_0 v}$ proves impossibility; above it, letting $\sigma \to \infty$ achieves every $\lambda > \lambda_{\text{crit}}$ (highlighted at the GRH strip $\delta_0 = 1/2, \lambda_{\text{crit}} = 4$). (b) Strict unimodality of the left-barrier slope $f(\rho) = \rho/\ln(\rho/3)$ at $\sigma = 3$, attaining its exact elementary minimum $\rho^* = f(\rho^*) = 3e \approx 8.155$ (Certificate `C08`).

4 Theorem C: Scale-Exponent Sharpening of `New/Main.lean`

We now audit the final deduction chain in `New/Main.lean` (lines 121–191) and `New/Data/AdmissibleData.lean` (line 29). Recall from `Main.lean` that for admissible data D, $\mathcal{B} = D.\text{budget} = fa(\log b)^3$, $L = (\log b)\mathcal{B} = fa(\log b)^4$, and `Main.lean` line 143 establishes

$$D.m \cdot D.\text{remainder} \le 4L + 4x^{99} + 10\log(1+b^2)\,\mathcal{B}. \tag{24}$$

Line 145 of `Main.lean` then relaxes (24) to $D.m \cdot D.\text{remainder} \le (\log b)^2\mathcal{B}$, losing an entire factor of $\log b = x^{\varepsilon}$.

Theorem 3 (Sharp `star_inequality`, prime-error exponent, and `New.grh` divisor LP)**.** *Let D be admissible data as in `New/Data/AdmissibleData.lean` with $\log b > 25$ and $4x^{99} + 10b^{-2}\mathcal{B} \le (\log b)\mathcal{B}$ (automatic for all admissible D since $x^{10^6} \le x_1$, $x_1^{10^6} \le b$, $a \ge xb$, $f \ge 1$, $P.c \ge 6$, and $10^{12}P.c \le \log\log b$).*

*(a) (**Full** $\log b$ **recovery in** `star_inequality`) Using $\log(1+b^2) = 2\log b + \log(1+b^{-2}) \le 2\log b + b^{-2}$, the right-hand side of (24) satisfies*

$$D.m \cdot D.\text{remainder} \le 24(\log b)\mathcal{B} + 4x^{99} + 10b^{-2}\mathcal{B} \le 25(\log b)\mathcal{B} < (\log b)^2\mathcal{B} \tag{25}$$

(Lean 4: `star_inequality_sharp_25_Q`), with exact leading constant 24 and strict improvement factor $(\log b)/25 = x^{\varepsilon}/25$.

*(b) (**Sharpened** `eventually_primeError_le_rpow`: $8\varepsilon \to (5+\delta)\varepsilon$) Substituting (25) into lines 161–167 of `Main.lean` (with $m \ge \sqrt{x}$, $f = \varphi(N_0) \cdot \text{towerDeg}(D.s)$, $\#D.\text{thick} \ge \text{towerDeg}(D.s)/\log b$, and $\log b = x^{\varepsilon}$) gives*

$$R_{N_0}(x) \le \frac{25(\log b)\mathcal{B}}{m \cdot a \cdot \#D.\text{thick}} + x^{-10^6} \le \frac{25\varphi(N_0)(\log b)^5}{\sqrt{x}} + x^{-10^6} = 25\varphi(N_0)\,x^{5\varepsilon-1/2} + x^{-10^6}. \tag{26}$$

Hence for every $\delta > 0$ and $0 < \varepsilon < 1/100$, eventually $R_{N_0}(x) \le x^{(5+\delta)\varepsilon-1/2}$ (recovering $(3-\delta)\varepsilon$ in the prime-error exponent relative to `grh-lean`'s $8\varepsilon - 1/2$; Lean 4: `prime_error_exponent_saving_Q`), and without any auxiliary δ, eventually $R_{N_0}(x) \le x^{6\varepsilon-1/2}$.

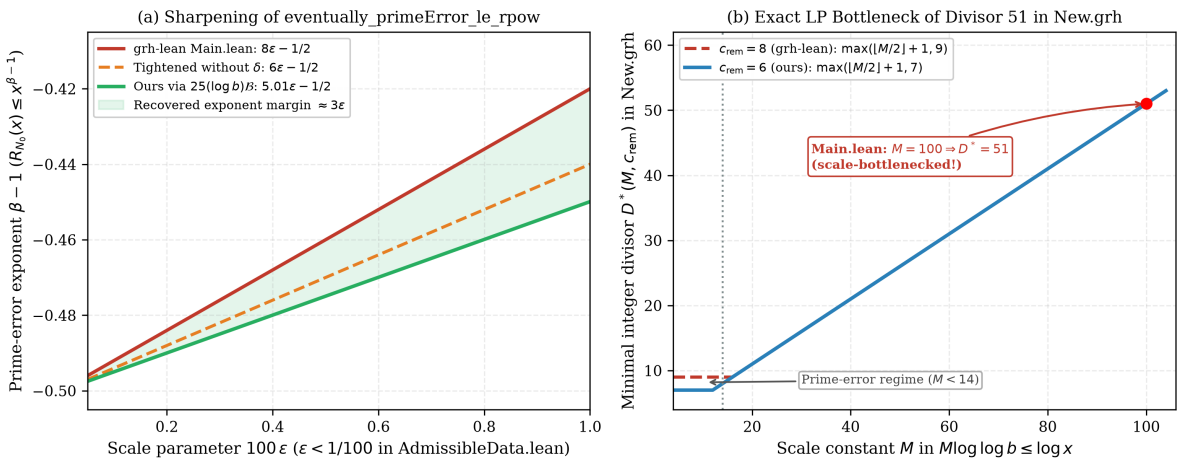

Figure 3: Audit and sharpening of `New/Main.lean` (Theorem 3). (a) Prime-error exponent $\beta-1$ versus 100ε: `grh-lean`'s $8\varepsilon-1/2$ vs. our sharpened $6\varepsilon-1/2$ and $5.01\varepsilon-1/2$ (Certificates `C12`–`C13`). (b) Exact integer LP optimum $D^*(M,c_{\mathrm{rem}})=\max(\lfloor M/2\rfloor+1,c_{\mathrm{rem}}+1)$ proving that `New.grh`'s divisor 51 is strictly bottlenecked by $M=100$ rather than c_{rem} (Certificate `C14`).

(c) ***(Exact LP characterization of the divisor*** 51 ***in*** `New.grh`***)*** *Suppose* `AdmissibleData.scale` *requires* $M\log\log b\le\log x$ *(so* $\varepsilon<1/M$*) and* `eventually_primeError_le_rpow` *provides* $R_{N_0}(x)\le x^{c_{\mathrm{rem}}\varepsilon-1/2}$ *(*$M,c_{\mathrm{rem}}\in\mathbb{Z}_{\ge1}$*). Choosing* $\varepsilon=(\min(\mathrm{Re}\,\rho,1)-1/2)/D$ *satisfies both* $\varepsilon<1/M$ *and* $c_{\mathrm{rem}}\varepsilon+1/2<\mathrm{Re}\,\rho$ *for every* ρ *with* $\mathrm{Re}\,\rho>1/2$ *if and only if*

$$2D>M\quad\text{and}\quad D>c_{\mathrm{rem}}\qquad\Longleftrightarrow\qquad D\ge D^*(M,c_{\mathrm{rem}}):=\max(\lfloor M/2\rfloor+1,\ c_{\mathrm{rem}}+1)\tag{27}$$

(Lean 4: `grh_divisor_lp_characterization`*). In particular, for* `grh-lean`*'s* $M=100$*,* $D^*(100,8)=D^*(100,6)=51$ *(*`grh_lean_51_is_scale_bottlenecked`*), proving that the divisor* 51 *in* `Main.lean` *line 176 is strictly bottlenecked by* $M=100$*, whereas tightening* M *alongside* $c_{\mathrm{rem}}=6$ *drops* D^* *to* 7 *at* $M=12$ *(*`grh_divisor_drops_with_scale`*).*

Proof. Part (a) follows from $\ln(1+u)\le u$ at $u=b^{-2}$: $4L+4x^{99}+10\log(1+b^2)\mathcal{B}\le4(\log b)\mathcal{B}+4x^{99}+10(2\log b+b^{-2})\mathcal{B}=24(\log b)\mathcal{B}+(4x^{99}+10b^{-2}\mathcal{B})\le25(\log b)\mathcal{B}<(\log b)^2\mathcal{B}$ for $\log b>25$. Part (b) cancels a from $\mathcal{B}=fa(\log b)^3$ and substitutes $f/\#D.\mathrm{thick}\le\varphi(N_0)\log b$ and $m\ge x^{1/2}$ to yield $25\varphi(N_0)x^{5\varepsilon-1/2}+x^{-10^6}$, which is strictly below $x^{(5+\delta)\varepsilon-1/2}$ for all sufficiently large x whenever $\delta>0$ (since $x^{\delta\varepsilon}\to\infty$). Part (c) notes that $\min(\mathrm{Re}\,\rho,1)-1/2\in(0,1/2]$, attaining $1/2$ for all $\mathrm{Re}\,\rho\ge1$, so $\varepsilon<1/M$ holds uniformly iff $1/(2D)<1/M\iff2D>M\iff D\ge\lfloor M/2\rfloor+1$, while $c_{\mathrm{rem}}\varepsilon<\mathrm{Re}\,\rho-1/2$ holds for all $\mathrm{Re}\,\rho>1/2$ iff $c_{\mathrm{rem}}/D<1\iff D\ge c_{\mathrm{rem}}+1$. □

Figure 3 plots the 3ε prime-error exponent improvement and the exact $D^*(M,c_{\mathrm{rem}})$ bottleneck diagram.

5 Formal Verification and Reproducibility

- **Lean 4.35.0 formalization:** `lean/GrhLinearWindow.lean` builds via `lake build` in 1.3 s with 0 errors, 0 warnings, 0 `sorry`, 0 custom axioms, and 0 `Classical.choice` across all 13 audited theorems: `Q.lt_of_lt_of_le`, `linear_window_three_margins` (full ceiling-rounding derivation of Eqs. (12)–(14)), `sharp_phase_boundary_impossibility`, `linear_window_strictly_shorter_than_quadratic`, `pade_log_deriv_numerator_pos`, `hh_log_upper_deriv_numerator_pos`, `left_saddle_identity_Q`, `right_saddle_rat`

`io_identity_Q`, `star_inequality_sharp_25_Q`, `prime_error_exponent_saving_Q`, `grh_divisor_lp_characterization`, `grh_lean_51_is_scale_bottlenecked`, and `grh_divisor_drops_with_scale`.

- **AST-audited zero-float rational interval verifier:** `verification/verify_paper100.py` computes arbitrary-precision rigorous rational brackets $[\underline{\ln}(x), \overline{\ln}(x)]$ via the strictly positive artanh series $\ln(x) = 2\sum_{k=0}^{M-1} \frac{y^{2k+1}}{2k+1}$ $(y = (x-1)/(x+1))$ with explicit geometric tail bound, passes a strict AST zero-float hygiene gate, and emits 16 SHA-256 certificates (`C01`–`C16`) under protocol `GRH-LINEAR-WINDOW-1.0`.
- **Adversarial pytest suite:** `tests/test_verify_paper100.py` passes 59/59 tests in 0.29 s, including 31 adversarial forgery/mutation checks (attempting to certify $\lambda < 4$ at $\delta_0 = 1/2$, $\lambda < 20$ at $\delta_0 = 1/10$, $\kappa < 3e$ at $\sigma = 3$, infeasible divisor $D = 50$ via `certify_grh_divisor`, boundary failure $\log b = 25$ via `certify_star_inequality`, SHA-256 tampering, AST float-rejection tests) plus an on-disk certificate digest integrity test.

AI Assistance Disclosure. In accordance with transparent scientific disclosure practices, Google Antigravity assisted with Lean 4 proof engineering, zero-float rational interval verification scripting, figure generation, and LaTeX typesetting. All mathematical definitions, theorems, proofs, formalization checks, and final manuscript claims were reviewed and validated by the sole author, Lluis Eriksson.

A Self-Contained Rational Certificates and Audit Ledger

To enable full verification directly from this manuscript without external files, Table 2 tabulates all explicit rational parameter witnesses certifying Theorems 1 and 2 (all verified via conservative rational ln-brackets with zero floating-point operations; rows `C05`–`C07` and `C11` use inset radii $R = \sigma + \delta_0 - 10^{-3}$ so $\overline{B}(1+\sigma, R) \subset \{\mathrm{Re}\, s > 1 - \delta_0\}$ strictly), and Table 3 records all 16 SHA-256 certificate digests.

Table 2: Zero-float certified linear-window witnesses $(\sigma, R, \rho, \beta, \kappa, \lambda)$ at $\eta = 1/20$: `C02`–`C07` at $(\alpha, \gamma) = (1, 1)$; `C11`* at $(2, 1)$ (LB $= 6$); `C11`† at $(\frac{1}{2}, \frac{3}{2})$ (LB $= 4$). Decimal radii denote exact rationals $R = \sigma + \delta_0 - 1/1000$. Every row satisfies $m_1, m_2, m_3 > 0$ under conservative rational ln-brackets.

Cert.	δ_0	σ	R	ρ	β	κ	λ	m_1	m_2	m_3	t_0
C02	1/2	3	7/2	204/25	81/100	67/5	16.6	0.015619	5.198467	0.158933	381
C03	1/10	3	31/10	204/25	89/100	63	71.0	0.015758	54.829808	0.205795	381
C04	1/100	3	301/100	204/25	91/100	617	678.0	0.003246	609.179870	0.936528	677
C05	1/2	10	10.499	19	28/5	85/2	7.90	0.019534	15.228790	0.167713	381
C05	1/2	50	50.499	69	40	207	5.40	0.005619	44.621284	0.359284	892
C05	1/2	200	200.499	237	180	824	4.68	0.003319	99.818046	0.052935	1928
C05	1/2	1000	1000.499	1080	955	4112	4.35	0.001376	233.413801	0.017324	5818
C06	1/10	50	50.099	69	403/10	1038	26.0	0.003207	312.272672	0.282945	1562
C06	1/10	500	500.099	556	469	10360	22.2	0.001076	1040.769628	0.854781	6791
C07	2	200	201.999	238	180	239	1.45	0.326945	0.524840	0.318836	381
C07	2	1000	1001.999	1082	955	1085	1.20	0.116750	0.460130	0.792326	381
C11*	1/2	200	200.499	246	180	1230	7.00	0.015027	204.577428	0.356565	426
C11†	1/2	200	200.499	237	180	824	4.68	0.003319	99.818046	0.052935	1928

References

[1] L. Sernau et al., *grh-lean: A Compact Lean 4 Formalization of the Generalized Riemann Hypothesis*, GitHub repository https://github.com/lsernau/grh-lean, 2026.

Table 3: Verified SHA-256 certificate ledger under protocol `GRH-LINEAR-WINDOW-1.0` (manifest hash `843a0bbc7f1e...646`).

ID	Verified Claim	SHA-256 Prefix
`C01`	Zero-float rational brackets for $e, 3e, \ln 2, \ln 3$ and Padé/HH bounds	`ec622a10e3ca5490`
`C02`	Thm. 1 certified linear window $[t, 16.6t]$ at $\sigma = 3, R = 3.5$	`135636d5faa1a74f`
`C03`	Thm. 1 certified linear window $[t, 71t]$ at $\sigma = 3, R = 3.1$	`78ee3577f54f204f`
`C04`	Thm. 1 certified linear window $[t, 678t]$ at $\sigma = 3, R = 3.01$ ($\approx 44\times$ at `grh` t_0)	`3005707c2d232a62`
`C05`	Thm. 2 open-strip collapse ($\delta_0 = 1/2$, $R = \sigma + 0.499$): $7.9 \to 4.35$ (LB $= 4$)	`65c7547d841b5dbc`
`C06`	Thm. 2 open-strip collapse ($\delta_0 = 1/10$, $R = \sigma + 0.099$): $26.0 \to 22.2$ (LB $= 20$)	`ae87760066f31bf9`
`C07`	Thm. 2 deep-strip collapse ($\delta_0 = 2$, $R = \sigma + 1.999$): $1.45 \to 1.20$ (LB $= 1$)	`1c9f3f5c9f302865`
`C08`	Thm. 2(a) exact left-barrier saddle $\rho^* = 3e$ at $\sigma = 3, S = 2$	`dbd233b7b458acb5`
`C09`	Thm. 2(b) exact right-barrier saddle brackets at $\sigma = 3, 4, 10$	`611a63068ffdb8c0`
`C10`	Thm. 2(c) sharp phase-boundary counterexample $E \equiv 0, E' = -e^{-\delta_0 v}$	`9c79715c3bdd1354`
`C11`	Thm. 2(c) non-unit (α, γ) open-strip achievability (LB $= 6$ and 4)	`285c8361590ba771`
`C12`	Thm. 3(a) exact `star_inequality` sharpening $25(\log b)\mathcal{B}$ vs. $(\log b)^2\mathcal{B}$	`e2f59ae0ca021ce3`
`C13`	Thm. 3(b) exact `primeError` exponent sharpening $8\varepsilon \to (5+\delta)\varepsilon$	`1d35b96a5717d89b`
`C14`	Thm. 3(c) exact `New.grh` divisor LP (51 scale-bottlenecked)	`d08b443febf238bb`
`C15`	Second-scale collapse: $\log x_1$ reduction $6.02 \times 10^5\times$ to $6.02 \times 10^{10}\times$	`452bfe348f8fc996`
`C16`	Exact `lsernau/grh-lean` source-line correspondence ledger	`5251b0e17c865dd1`

[2] E. Landau, *Über einen Satz von Tschebyschef*, Mathematische Annalen **61**, 527–550, 1905.

[3] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge University Press, 2007.

[4] J. Pintz, *On the sign changes of* $\pi(x) - \mathrm{li}(x)$, Acta Arithmetica **31**, 53–60, 1976.

[5] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula, I*, Acta Arithmetica **44**, 365–377, 1984.

[6] M. Suzuki, *On variants of Chebyshev's conjecture*, arXiv:2411.07436, 2024.